MATHEMATICAL PROPERTIES OF THE BANZHAF POWER INDEX

by

Pradeep Dubey
and
Lloyd S. Shapley

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ABSTRACT

The Banzhaf index of power in a voting situation depends on the number of ways in which each voter can effect a "swing" in the outcome. It is comparable--but not actually equivalent--to the better-known Shapley-Shubik index, which depends on the number of alignments or "orders of support" in which each voter is pivotal. This paper investigates some properties of the Banzhaf index, the main topics being its derivation from axioms and its behavior in weighted-voting models when the number of small voters tends to infinity. These matters have previously been studied from the Shapley-Shubik viewpoint, but the present work reveals some striking difference between the two indices. The paper also attempts to promote better communication and less duplication of mathematical effort in this field by identifying and describing several other theories, formally equivalent to Banzhaf's, that are found in fields ranging from sociology to electrical engineering. An extensive bibliography is provided.

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CONTENTS

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1. INTRODUCTION AND BACKGROUND

The use of game theory to study the distribution of power in voting systems can be traced back to the invention of "simple games" by John von Neumann and Oskar Morgenstern in their 1944 classic, Theory of Games and Economic Behavior [73]. Speaking intuitively, a simple game is a cooperative/competitive enterprise in which the only goal is "winning" and the only rule is a specification of which coalitions are capable of doing so. This abstract definition covers most of the familiar examples of constitutional political machinery, among them direct majority rule, weighted voting, direct or indirect election of a President, bicameral or multicameral legislatures, committees and subcommittees, veto situations etc.* Moreover, simple games often make for clearer modelling and neater mathematical proofs than some of the more restricted classes of voting rules commonly considered by political scientists, who have sometimes adopted an unfortunately narrow view of the logical possibilities for systems of representation and governance.**

After some exploration of the mathematical structure of simple games,*** von Neumann and Morgenstern in [73] proceeded to apply to them the solution concept that they had already developed for a more general class of coalitional games. The logic of that solution concept led them to construct a little economic vote-selling model, in which

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*See Shapley [63].

**Note 4 in Section 12 below.

***In [73], the term "simple game" refers only to what we would now call a decisive simple game; see Section 2 below.

the equilibrium prices describe the share of the spoils that each player might expect to receive if he ends up on the winning side.* Only a small (but significant) class of simple games ultimately proved to be solvable in this way, but the N-M price vectors ought nevertheless to be regarded as an early form of <u>power index</u>, representing an important step forward in the quantitative analysis of the power of voters in abstract political systems.

The next step was taken in a 1954 paper by Shapley and Shubik [66], who proceeded to specialize another general solution concept, the so-called "Shapley value" [62], to the case of simple games. This approach yielded numerical indices which can be interpreted directly in terms of the <u>a priori</u> ability of the players to affect the outcome; moreover they have the advantage over the N-M equilibrium prices of being well defined for all simple games. The Shapley-Shubik power index has become widely known and applied in game theory and political science.**

An unexpected practical turn was given to the problem of measuring voting power when the U. S. Supreme Court in the 1960's handed down a series of "one person one vote" decisions, setting forth new standards of constitutional fairness for systems of electoral

*Op. cit., pp. 435-443. For some later work on these "main simple solutions," as they are called, see Gurk and Isbell [22].

**A partial bibliography: David, Goldman, and Bain [15]; Dubey [17, 18]; Junn [30]; Krislov [33]; Lucas [35]; MacRae and Price [36]; Mann and Shapley [37]; Merrill [38]; Miller [39]; Milnor and Shapley [40]; Monjardet [41]; Nozick [45]; Owen [46, 47, 49]; Riker [54, 55]; Riker and Niemi [56]; Riker and Ordeshook [57]; Riker and Shapley [58]; Shapiro and Shapley [61]; Shapley [64, 65]; Spatt [69].

representation at the state and local levels.* As a result, many existing voting systems had to be revised or at least re-examined, and the advocates or opponents of proposed reforms had frequent occasion to invoke either the Shapley-Shubik index or a rather similarly-conceived index due to John F. Banzhaf III, a young lawyer and reformer with a mathematical background [4, 5, 6, 7]. Laborious calculations using real data were carried out on the computers of the day and presented as evidence in the courtroom or at legislative hearings.** The main ideas underlying the game-theoretic approach to power eventually found wide legal acceptance; indeed, in New York State today, some of the county supervisorial boards are constituted according to a form of Banzhaf's index, in an attempt to equalize the representation of citizens living in municipalities of different size.***

The actual numerical values that issue from the Banzhaf (Bz) and Shapley-Shubik (S-S) models are often quite similar, and the two can be regarded as equivalent for many practical purposes if we grant that law and politics are far from being exact sciences.**** Nevertheless, there are significant differentiating features that have not yet been explored mathematically to any great depth. The Bz index

*Baker v. Carr, 369 U.S. 186 (1962); Grey v. Sanders, 372 U.S. 368 (1963); Wesberry v. Sanders, 376 U.S. 1 (1964); Reynolds v. Sims, 3377 U.S. 533 (1964). The late Justice Frankfurter, dissenting in Baker v. Carr, cautioned the Court against dragging the law into "political thickets and mathematical quagmires." We wonder how he would view our present work!

**One court went so far as to find that the very expense and complexity of the power computations made the question of the fairness of a certain "weighted voting" proposal legally undecidable. (Ianucci v. Board of Supervisors of Washington County, N. Y. State Court of Appeals, 1967.)

***Imrie [27]; Lucas [35].

****See e.g., Owen [49]; Shapley [65]; Straffin [71].

seems to have had the greater appeal to the legal mind, perhaps because of its more straightforward verbal definition. But up to now the S-S index has attracted the lion's share of attention from game theorists, partly because of a certain perceived naturalness in its mathematical foundations and partly because of the continuing research into its parent solution concept, the "Shapley value" for general cooperative games.* With the idea of redressing this imbalance, we have undertaken here to investigate the Bz index from a mathematical standpoint. Much of our work parallels earlier studies of the S-S index, but the conclusions reached are often quite different.

Let us briefly outline the contents. Section 2 defines the Bz index and relates it to an explanatory probability model, which is then compared with the corresponding S-S model with the help of a small example. Section 3 then shows how to derive the Bz index from a set of axioms, comparable to a set Dubey has recently given for the S-S case [17]. The Bz axioms make special use of a certain number, denoted $\bar{\eta}$, which has no counterpart in the S-S theory; intuitively it represents the "total power" available in the game. Section 4 derives some bounds for this number under various assumptions on the set of winning coalitions.

The most substantial portion of the paper, mathematically speaking, deals with the asymptotic properties of <u>weighted majority games</u> when there are many small voters. Section 5 prepares for this by describing some of the elementary properties of these games and

*See [62], or [2] where further references will be found. The Bz index can also be related to a value-type solution for general cooperative games in characteristic function form; see Note 8 in Section 12 below and also Owen [48, 50] and Roth [59].

their power indices. Section 6 then formulates the limit model and the next five sections investigate it in detail. This is a problem that was studied many years ago from the S-S standpoint [61], but the Bz version reveals several surprising new features. For a further discussion of these results, see the introductory paragraphs in Section 6.

Finally, in Section 12 we present a series of supplementary notes, dealing with various extensions and applications of the Bz index. Included are brief summaries of several other theories, both in and out of political science, which in their mathematical aspects can be shown to be equivalent to the Banzhaf theory; they are associated with the names of Coleman, Rae, Dahl, Chow, and others. For the most part, these theories have been developed independently, without reference either to each other or to the game-theory literature. While these supplementary notes do add to the length of an already long paper, we feel that they promote the cause of better communication and less duplication between several parallel lines of research having a common mathematical core. In the same spirit, we have assembled an extensive set of references from many sources.

Most of Sections 3, 6, 7, 8, and 11 are based on Chapters I and II of Dubey's doctoral dissertation, Some Results on Values of Finite and Infinite Games, Cornell University, 1975.

2. PRELIMINARIES

A <u>game</u>, or <u>game on N</u>, is a real-valued function v defined on the subsets of a nonempty, finite set, N, and vanishing on the empty set. The elements of N are called "players," and we shall often identify them with the integers 1, 2, ..., n, where $n = |N|$. The symbol $\mathcal{G}(N)$ will denote the set of all games on N, and $\mathcal{G}_{Sa}(N)$ the set of <u>superadditive</u> games on N, i.e., those obeying the condition

$$v(S \cup T) \geqq v(S) + v(T) \quad \text{whenever} \quad S \cap T = \emptyset.$$

Superadditivity is quite important in most applications since it permits one to regard the number v(S) as the total amount of something--of money, say, or "transferable utility"--that the members of S can be sure of getting if they form a coalition. Non-superadditive games are sometimes called <u>improper</u> because they pose problems in interpretation, but they are nevertheless useful in the mathematical theory.

A game v is said to be <u>simple</u> if it assumes only the values 0 and 1, obeys the condition of monotonicity:

$$v(S) \geqq v(T) \quad \text{whenever} \quad S \supset T,$$

and is not identically 0. Thus, a simple game on N always has $v(N) = 1$. The symbol $\mathcal{C}(N)$ will denote the set of all simple games on N and $\mathcal{C}_{Sa}(N)$ the set of superadditive or proper simple games on N. In the context of simple games, propriety is equivalent to the condition

$$v(S) + v(N - S) \leqq 1 \qquad \text{for all } S.$$

If equality holds here, the game is said to be decisive; this term refers to the interpretation of v as a political or group-decision rule.* Pursuant to this interpretation, sets S with $v(S) = 1$ are called winning coalitions and sets with $v(S) = 0$ losing coalitions. Sets whose complements lose are called blocking coalitions. It is not hard to see that in a proper game winning implies blocking, while in a decisive game winning and blocking are equivalent. In an improper game, however, there will be at least two winning coalitions that do not block. In other words there will be at least one pair of nonintersecting winning coalitions. The number of simple games on a fixed N is finite, of course, but it grows very rapidly with increasing n since we are dealing with sets of sets. (There are already 180 5-person simple games, not counting permutations.) Indeed, every family of pairwise independent subsets of N can serve as the set of minimal winning coalitions defining a simple game.**

We now define a swing, or swing for player i; this is a pair of sets of the form $(S, S - \{i\})$ such that S is winning and $S - \{i\}$ is not. For each $i \in N$, we denote by $\eta_i(v)$ the number of swings for i in the game $v \in \mathcal{C}(N)$. We shall write $\bar{\eta}(v)$ for the total number of

*See [63].

**Two subsets are independent if neither contains the other. Families of independent subsets are sometimes called "Sperner families," "coherent systems," or "clutters," and their enumeration and classification have occupied mathematicians since Dedekind in the 19th century. An account of this work will be found in [23], pp. 1030-1032, or see [67], pp. 23-24. See also Sperner [70], Isbell [28], Golomb [21], and Note 7 in Sec. 12 below.

swings, i.e., $\overline{\eta}(v) = \Sigma_{i\varepsilon N}\eta_i(v)$. A player with $\eta_i(v) = 0$ is called a dummy because, intuitively, he can never help a coalition to win. At the other end of the scale, a player with $\eta_i(v) = \overline{\eta}(v)$ is called a dictator, for obvious reasons.*

The swing numbers $\eta_i(v)$ are what might be called the "raw" Banzhaf indices. Since the principal interest in these numbers lies in their ratios rather than their magnitudes it is common practice to normalize them to add up to 1:

$$(1) \qquad \beta_i(v) = \eta_i(v)/\overline{\eta}(v), \qquad i = 1, \ldots, n;$$

as we shall see, however, this normalization is not as innocent as it seems. We shall call these numbers the normalized Banzhaf indices. For convenience, we shall write $\eta(v)$, $\beta(v)$, etc. for the vectors $(\eta_1(v), \ldots, \eta_n(v))$, $(\beta_1(v), \ldots, \beta_n(v))$, etc. and we shall sometimes omit the "(v)."

Another normalization is in many respects more natural:

$$(2) \qquad \beta_i'(v) = \eta_i(v)/2^{n-1}, \qquad i = 1, \ldots, n.$$

These numbers we shall call the swing probabilities of the players. This term arises from the following probability model. Suppose that a bill is to be decided by an assembly, and that each member of the

*Equivalently, we could define η_i as the difference $\omega_i - \tilde{\omega}_i$ between the number of winning coalitions containing i and the number not containing i (see Note 6 of Sec. 12, below). Proof: Remove i from a winning coalition. This either produces a swing or leaves the coalition winning. Moreover, every swing for i and every winning coalition not containing i is obtained uniquely in this way. So $\omega_i = \eta_i + \tilde{\omega}_i$.

assembly randomly votes "yea" or "nay" on the toss of a coin.* The set Y of yea-voters is then a random variable, assigning probability $1/2^n$ to each subset of N. Of course the bill passes if and only if Y is a winning coalition, under the prescribed voting rule of the assembly. Call member i a "swinger" if his vote (whether yea or nay) affects the passage of the bill. That is, i is a swinger if the pair $(Y \cup \{i\}, Y - \{i\})$ is a swing for i, as previously defined. If S is any set containing i, there are exactly two sets Y such that $Y \cup \{i\} = S$, so the probability of any particular swing for i is two times $1/2^n$. So the probability that i is a swinger in this model is exactly $\beta_i'(v)$.

The above makes an interesting contrast with the familiar probability model for the S-S index.** There, the bill or issue under consideration is assumed to rank the players in order of their enthusiasm for the proposal, with, say, the most fervid supporter coming first and the most stubborn opponent last. Given any such ordering, there will be a unique marginal player--i.e., one who, by joining with his more enthusiastic colleagues, brings the coalition up to winning strength. This player is called the pivot of the ordering. Intuitively, he is the one whom the others must try to persuade or dissuade, or who perhaps determines how strong a law will be enacted, or how much money will actually be appropriated for some purpose, or how hard a candidate will have to campaign, etc. If we now assume *a priori* that all n! orderings are equiprobable, then the S-S index for each player is precisely his probability of being pivotal.

*The case of probabilities other than 1/2 will be considered in Sec. 12, Note 1.

**See [62], [65], etc.

We see, then, that mathematically the S-S index rests on equiprobable permutations of N while the Bz index rests on equiprobable combinations of N.* Since each permutation produces exactly one pivot, the S-S index is inherently an additive measure of power, applicable to sets as well as to individuals. Thus, if φ_i denotes the S-S index for i, then the sum $\Sigma_{i \in S} \varphi_i$ is a plausible measure of coalitional power, because it represents exactly the probability that S contains the pivot. On the other hand, a single combination only rarely produces exactly one swinger. Typically there will be either many swingers, as in a very close election, or none at all, as when a candidate wins by a comfortable margin. The sum $\Sigma_{i \in S} \beta'_i$ therefore does not represent the probability that S contains a swinger, nor does it even represent the probability that the set as a whole, throwing its weight one way or the other, could swing the outcome. Rather, this sum represents the expected number of swingers in S, and only in this rather strange sense do we get an additive measure of "coalitional power" out of the Banzhaf approach.**

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*The S-S index has the effect of making the probability of a combination depend on its size, with the total probability of each size being the same.

Banzhaf himself, apparently seeking to disarm criticism, asserts that "no assumptions are made as to the relative likelihood of any combination" ([4], page 326). His formal definitions, however, speak otherwise. In another place, he argues that "because a priori all voting combinations are equally possible [sic], any objective measure of voting power must treat them as equally significant" ([5], p. 1316).

**See [65]. There may be a philosophical difficulty in simultaneously considering more than one player as a potential swinger (as we must do if we wish to add up swing probabilities over a set of players). Our basic explanatory model for β'_i gives player i freedom of choice while making behavioristic assumptions about the other players, so we are being asked to accept n different subjective views of the voting process, artificially fused into a single model.

To illustrate these remarks, consider a 9-person tri-cameral assembly having three "chambers" of players, A, B, C, with 1, 3, 5 members respectively. The winning coalitions are those that include a majority of every chamber. The lone member of A thus has veto power, but he will swing only if Y includes majorities of both B and C. This can happen in 64 ways. A typical member of B will swing if and only if Y includes A, exactly one other member of B, and a majority of C; this can happen in 32 ways. A typical member of C will swing if Y includes A, a majority of B, and exactly two other members of C; this can happen in 24 ways. Hence

$$\eta = (64, 32, 32, 32, 24, 24, 24, 24, 24),$$
$$\beta = (\tfrac{8}{35}, \tfrac{4}{35}, \tfrac{4}{35}, \tfrac{4}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}),$$
$$\beta' = (\tfrac{8}{32}, \tfrac{4}{32}, \tfrac{4}{32}, \tfrac{4}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}),$$

$$\bar{\eta} = 280, \quad \bar{\beta} = 1, \quad \bar{\beta}' = 35/32 = 1.09375.$$

For comparison, the S-S indices are

$$\varphi = (\tfrac{32}{84}, \tfrac{9}{84}, \tfrac{9}{84}, \tfrac{9}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84});$$

see [66] for the details of this not-too-difficult calculation.

By adding up these indices within the sets A, B, and C, Brams [10, p. 193] arrived at the interesting observation that "Bz" and "S-S" rank these sets in opposite order. Reduced to lowest terms, the cameral power ratios are respectively

$$\text{Bz}: \ 8 : 12 : 15, \qquad \text{S-S}: \ 32 : 27 : 25.$$

So which chamber is really most powerful? The apparent conflict between the two theories is largely a matter of semantics. What is meant by the "power" of a set of individuals? As noted earlier, in our previous remarks on additivity, the sum $\Sigma_S \varphi_i$ represents the probability that S contains the pivot and $\Sigma_S \beta_i'$ represents the expected number of swingers in S. If we ask instead for the probability that each chamber contain a swinger, Bz gives us the ratio 8 : 6 : 5, which agrees fairly well with the S-S ratio. On the other hand, if we compute the probabilities that each chamber could swing if it votes as a bloc, we get the ratio 1 : 1 : 1, or equal power to each house.*

*The reader is referred to [65] for a more extended discussion of this example and the whole question of the additivity of coalitional power indices.

3. AXIOMS FOR THE BANZHAF INDEX

For any game $v \in \mathcal{G}(N)$, if π is a permutation of N we define πv by

$$(\pi v)(S) = v(\pi^{-1}(S)).$$

For simple games $v, w \in \mathcal{C}(N)$, we define the operations $v \vee w$ and $v \wedge w$ by

$$(v \vee w)(S) = \max\,(v(S), w(S)),$$
$$(v \wedge w)(S) = \min\,(v(S), w(S)).$$

It is clear that $\mathcal{C}(N)$ is closed under the operations π, $\vee$, and $\wedge$.

THEOREM 1. *There is a unique function* $\varphi: \mathcal{C}(N) \to R^n$ *that satisfies the following four axioms*:

A1: *If* i *is a dummy in* v *then* $\varphi_i(v) = 0$.

A2: $\Sigma_{i \in N}\, \varphi_i(v) = \bar{\eta}(v)$.

A3: *For any permutation* π *of* N

$$\varphi_{\pi(i)}(\pi v) = \varphi_i(v).$$

A4: *For any* $v \in \mathcal{C}(N)$ *and* $w \in \mathcal{C}(N)$

$$\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w).$$

Moreover, $\varphi(v) = \eta(v)$ *for all* v in $\mathcal{C}(N)$.

Proof.* For any $S \subset N$, $S \neq \varphi$, define the game v_S by

*This proof follows closely the proof given in [17] for the S-S index.

$$v_S(T) = \begin{cases} 0 \text{ if } T \not\supset S, \\ 1 \text{ if } T \supset S. \end{cases}$$

Each $i \in N - S$ is a dummy in v_S, therefore, by A1, $\varphi_i(v_S) = 0$ for such i. Also if π is the permutation that interchanges i and j (for any $i \in S$ and $j \in S$) and leaves the other players fixed, then $\pi v_S = v_S$ and thus, by A3,

$$\varphi_i(v_S) = \varphi_j(v_S).$$

Therefore $\varphi(v_S)$ is uniquely determined, if φ exists, and is given by

(3) $$\varphi_i(v_S) = \begin{cases} 0 & \text{if } i \in N - S \\ \frac{\overline{\eta}(v_S)}{|S|} & \text{if } i \in S, \end{cases}$$

using A2.

Now every v in $\mathcal{C}(N)$ has a finite number of <u>minimal</u> winning coalitions $S_1, \ldots, S_m$, and they completely determine v, since $v(T) = 1$ if and only if $T \supset S_j$ for at least one $j = 1, \ldots, m$. Clearly, we have

$$v = v_{S_1} \vee v_{S_2} \vee \ldots \vee v_{S_m},$$

where the right hand side is defined associatively. Let $n^1(v)$ denote

the size of the smallest winning coalition of v and let $n^2(v)$ denote the number of winning coalitions of v of that size. The proof of the uniqueness of φ will be by induction on $n^1(v)$ and $n^2(v)$.

For $n^1(v) = n$, we have $n^2(v) = 1$ and $v = v_N$, in which case $\varphi(v)$ is obviously uniquely determined. Suppose that for some positive integers $k < n$ and $\ell \leqq \binom{n}{k}$ it has been shown that $\varphi(v)$ is uniquely determined for all v such that either

$$(4) \qquad n^1(v) > k,$$

or

$$(5) \qquad n^1(v) = k \quad \text{and} \quad n^2(v) < \ell.$$

Then we claim that $\varphi(v)$ is uniquely determined when $n^1(v) = k$ and $n^2(v) = \ell$.

Indeed, let $S_1, \ldots, S_\ell$ be the minimal winning coalitions of v with k players each, and let $S_{\ell+1}, \ldots, S_m$ be the other minimal winning coalitions of v, if any. Then $|S_j| > k$ for $\ell + 1 \leqq j \leqq m$. If $m = 1$ we have $v = v_{S_1}$, and so $\varphi(v)$ is determined by (3). If $m > 1$, define

$$v' = v_{S_2} \vee \ldots \vee v_{S_m}.$$

Then

$$v' \vee v_{S_1} = v.$$

Clearly v' satisfies (4) or (5), depending on whether $\ell = 1$ or $\ell > 1$. Moreover, $v' \wedge v_{S_1}$ satisfies (4). Therefore, $\varphi(v')$ and $\varphi(v \wedge v')$ are uniquely defined by the inductive assumption. Invoke A4. Then

$$\varphi(v) = \varphi(v' \vee v_{S_1}) = \varphi(v') + \varphi(v_{S_1}) - \varphi(v_{S_1} \wedge v'),$$

from which it follows that $\varphi(v)$ is uniquely determined.

We must still prove existence. The foregoing proof of uniqueness has implicit in it a recursive construction of φ that establishes existence; however, it is simpler to check directly that the function η as already defined satisfies A1 - A4. In fact, A1 - A3 are obvious, while A4 follows from the equation

$$\eta_i(v) = \sum_{S: i \in S \subseteq N} [v(S) - v(S - \{i\})],$$

showing that $\eta(v)$ can be extended to a linear function on $\mathcal{Y}(N)$.
Since

$$(v \vee w) + (v \wedge w) \equiv v + w,$$

A4 is now obvious. Q.E.D.

It may be worth pointing out that the proof of Theorem 1 makes no essential use of nonsuperadditive games--the so-called "improper" case. In fact, the same proof shows that η is the unique power index for $\mathcal{C}_{Sa}(N)$, if we understand A1 - A3 to be restricted to v in $\mathcal{C}_{Sa}(N)$ and A4 to be restricted to those v and w in $\mathcal{C}_{Sa}(N)$ such that $v \vee w$ is also in $\mathcal{C}_{Sa}(N)$.

Theorem 1 reveals that the Bz index is fundamentally very similar to the S-S index. By changing A2 to:

A2' $$\sum_{i \in N} \varphi_i(v) = n!,$$

we obtain a "raw" S-S index, while replacing the n! by 1 gives us the more usual normalized form that corresponds to the probability model described in Sec. 2.* The proof of this was given in [17].

Thus, it is only in the second axiom that the difference between the two indices is reflected. In this axiom we make an a priori judgment concerning the meaning of "power," by specifying the extent to which the game is responsive to the combined "powers" of the individual players.**

We do not attempt here a heuristic justification of A4. This axiom can be presented in a variety of mathematical guises. For example, one can adopt a "marginal" viewpoint and consider the effect

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*Changing the right side to $\bar{\eta}(v)/2^{n-1}$ would yield the index $\beta'(v)$. But note that we have no way of getting the normalized index $\beta(v)$ out of these axioms, since A4 is not satisfied. This may be taken as an initial sign of trouble with the normalization (1).

**Cf. Note 4 in Sec. 12.

of changing the status of just one coalition from winning to losing--more precisely, from minimal winning to maximal losing. Then one can replace A4 by the assumption that under such a one-coalition change the power variation for each player is independent of the rest of the game, i.e., is the same for all voting rules in which the same coalition is changed from winning to losing. The proof depends on the fact that the same sequence of one-coalition changes that takes us from v to $v \wedge w$ will take us from $v \vee w$ to w.

Another approach would be to introduce lotteries over sets of simple games. Call two lotteries equivalent if they yield the same probabilities for each coalition to be winning. Then in place of A4 we could assume that equivalent lotteries yield the same expected power index. Formally, let $L(v)$ denote the probability of the game $v \in \mathcal{S}(N)$ in the lottery L. The new axiom would state that if $\Sigma\, L(v)v = \Sigma\, L'(v)v$ then $\Sigma\, L(v)\varphi(v) = \Sigma\, L'(v)\varphi(v)$. To derive A4 from this, let L give probability 1/2 to each of v and w and let L' give probability 1/2 to each of $v \wedge w$ and $v \vee w$; the rest is straightforward.

Other equivalent forms of A4 can also be described. Our purpose in this section has been only to clarify the underlying logical structure of the Bz index. But perhaps in some of the varied interpretations and applications of the index (see e.g., Sec. 12, Notes 4, 5, and 6), it may be possible to attach a "story" to one of these versions of A4 that will bolster its intuitive plausibility.

4. COUNTING SWINGS

We have seen how the total number of swings in a simple game v, denoted $\overline{\eta}(v)$, plays an important role in the formal axiomatization of the Banzhaf power index. Intuitively speaking, $\overline{\eta}(v)$ reflects the "volatility" or "degree of suspense" in the decision rule. It gives an indication of the likelihood of a close decision--i.e., one so close that a single voter could tip the scales. As discussed later,* it is also a kind of democratic participation index, measuring the decision rule's responsiveness to the desires of the "average voter" or to the "public will." The swing total can vary widely. For example, if there are ten voters, then direct majority rule produces a total of 1260 swings while the "consensus" or unanimity rule produces only 10 swings.

We shall now examine some mathematical properties of the swing total. Theorem 2 gives the absolute upper bound, for fixed n. Theorems 3 and 4 and Corollaries 1 and 2 give lower bounds, based on different assumptions about the size and number of winning coalitions.**

THEOREM 2. *Let* N *be a fixed set and define* $n = |N|$ *and* $m = [n/2] + 1$, *i.e.*, m *is the next integer after* n/2. *Then for any simple game* $v \in \mathcal{C}(N)$, *we have*

$$\overline{\eta}(v) \leq m\binom{n}{m}.$$

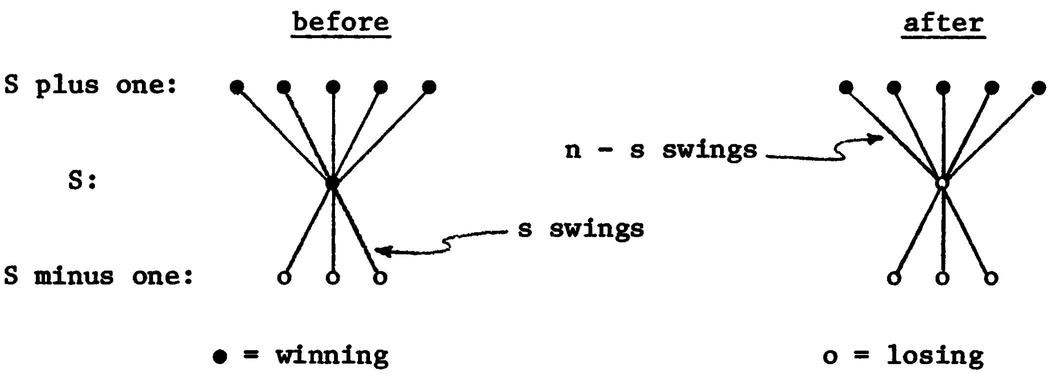
*Note 4 in Sec. 12.

**See also Note 3 in Sec. 12.

Equality holds here if and only if all coalitions with more than $n/2$ *members win and all with less than* $n/2$ *lose.*

In particular, if n is odd then Theorem 2 tells us that the maximum number of swings in $\mathcal{C}(N)$ is achieved *uniquely* by direct majority rule.*

Proof. Let S be a minimal winning coalition of v. There are just $s = |S|$ swings involving S. Changing S from winning to losing would destroy those s swings, but would create $n - s$ new swings, as illustrated here for $n = 8$, $s = 3$:



Thus, $\bar{\eta}$ can be increased if $s < n/2$. Hence, at the maximum, all coalitions smaller than $n/2$ must lose. By the dual argument, all coalitions larger than $n/2$ must win. If n is odd, then the game is determined completely by these conditions and there are precisely $\binom{n}{m}$

*A special case of this theorem was conjectured by Rae [53] and proved by Taylor [72]; see also Curtis [13]. The present simple proof is based on a suggestion of Michael Todd.

marginally winning coalitions, each contributing m swings to the total. If n is even, then we note first that the total number of swings is not changed if we make all the coalitions of size n/2 lose. Then there are again $\binom{n}{m}$ marginally winning coalitions, each contributing m swings. Q.E.D.

THEOREM 3. *Let* v *be a simple game with* n *players, and let* m *be an integer such that* $n/2 < m \leqq n$ *and*

$$|S| \geqq m \Rightarrow v(S) = 1$$
$$|S| \leqq n - m \Rightarrow v(S) = 0.$$

That is, every coalition with at least m *members both wins and blocks. Then we have*

$$\bar{\eta}(v) \geqq m\binom{n}{m}.$$

Proof. Consider the edge-capacitated network whose nodes are the 2^n subsets of N and whose edges are the $n2^{n-1}$ pairs $(S, S - \{i\})$, $i \in S \supset N$, each edge having the same capacity $c > 0$. By the "rank" of a node S we shall mean the integer $|S|$. Let the nodes of rank m be designated *sources* and those of rank n - m *sinks*. Since $m > n/2$, we have $m > n - m$. There are $m\binom{n}{m}$ edges leading from nodes of rank m to nodes of rank m - 1. Similarly, we may count $(m - 1)\binom{n}{m-1}$ edges from rank m - 1 to m - 2, and so on. Finally, there are $(n - m + 1)$, $\binom{n}{n-m+1}$ edges entering the set of sinks, this last number being equal to $m\binom{n}{m}$. Since these counts first increase and then decrease, it is

obviously possible to send a total flow of $c m\binom{n}{m}$ units from the source set to the sink set without exceeding the capacity at any edge. For example, we can distribute the flow evenly among the edges at each level.

Now the swings of the game are just the edges of the network that connects a "winning" node to a "losing" node. Since the sources all win and the sinks all lose, removing all swings from the network necessarily cuts off *all* flow between sources and sinks. The total capacity of the swinging edges is by definition $c\bar{\eta}(v)$. This obviously must be at least as great as the amount $c m\binom{n}{m}$ we were able to send before removing them. Hence $\bar{\eta}(v) \geqq m\binom{n}{m}$. Q.E.D.*

THEOREM 4. Let v *be a simple game with* n *players,* ω *winning coalitions, and* λ *losing coalitions (so that* $\omega + \lambda = 2^n$). *Then*

$$\bar{\eta}(v) \geqq \sum_{k=0}^{\omega-1} (n - 2g(k)) = \sum_{k=0}^{\lambda-1} (n - 2g(k)), \tag{6}$$

where $g(k)$ *is the sum of the digits in the binary representation of the integer* k. *Moreover, for each* n *and* ω *there is a simple game for which equality is attained.*

---

*We are using here the Ford-Fulkerson "max flow equals min cut" theorem [19], but only in the trivial direction which asserts that the capacity of any cut is greater than or equal to the amount of any flow.

This theorem was proved by Sergiu Hart [25]*; for an application see Section 12, Note 4. The following corollary gives a weaker but much less complicated lower bound.

COROLLARY 1. *Under the conditions of Theorem 4 we have*

(7) $$\bar{\eta}(v) \geqq \mu[n - \log_2 \mu],$$

where $\mu = \min(\omega, \lambda)$, *and* $[x]$ *denotes here the greatest integer* $\leqq x$.

Proof. If we write out a list of the integers $0, 1, \ldots, \mu - 1$ in binary form:

$$\begin{array}{r} 0 \\ 1 \\ 10 \\ 11 \\ 100 \\ \cdots \end{array}$$

we see that no more than $\mu/2$ of the digits in each column can be 1's. As the number of columns is $\langle \log_2 \mu \rangle$ where $\langle x \rangle$ denotes the least integer $\geqq x$, we have at once

$$\sum_{k=0}^{\mu-1} g(k) \leqq \frac{\mu}{2} \langle \log_2 \mu \rangle .$$

*An equivalent result was previously established by Bernstein [2], following Harper [24].

With the aid of this inequality, (7) follows directly from (6).
Q.E.D.

COROLLARY 2. *If* v *is a decisive simple game with* n *players, then* $\bar{\eta}(v) \geqq 2^{n-1}$.

Proof. Apply Corollary 1 with $\mu = 2^{n-1}$. Q.E.D.

To see that this bound is sharp, it suffices to consider the game where one player is a dictator. As far as we know, Corollary 2, which seems almost obvious, can only be proved by way of Theorem 2.

5. WEIGHTED MAJORITY GAMES. THE DUALITY PRINCIPLE

We now turn to a special subclass of simple games called weighted majority games. The symbol

$$(8) \qquad [c;\ w_1,\ \dots,\ w_n]$$

will be used, where c and $w_1, \dots, w_n$ are real numbers that are non-negative and satisfy $c \leqq \Sigma_{i\in N}\, w_i$. We may think of w_i as the number of votes, or weight of player i, and c as the threshold or quota needed for a coalition to win. Thus, (8) represents the simple game v defined by

$$(9) \qquad v(S) = \begin{cases} 1 & \text{if } w(S) \geqq c \\ 0 & \text{if } w(S) < c \end{cases}$$

where w(S) means $\Sigma_{i\in S}\, w_i$. The class of all games in $\mathcal{C}(N)$ having such a representation will be denoted $\mathcal{M}(N)$. Obviously the representation (8) is never unique; in fact, the set of vectors (8) that represent any given game in $\mathcal{M}(N)$ forms a full-dimensional convex cone in R^{n+1}.*

In place of (9), we may sometimes prefer the criterion $w(S) > c$ for winning; in this case we shall use the symbol

*Somewhat surprisingly, there is not always a unique minimal representation in integers. Muroga et al. [43] in their exhaustive enumeration of threshold functions uncovered several cases with as few as eight players in which two symmetric players must be given different weights in a minimal integer representation; e.g., [12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 4, 2, 3]. Previously, Isbell [28] had exhibited a 12-player example in which the affected players are not symmetric.

$$\langle c;\ w_1,\ \ldots,\ w_n\rangle. \tag{10}$$

Obviously, any particular weighted majority game expressed in the form (8) can be put into the form (10) and vice versa, by making a small adjustment in the number c.

The <u>dual</u> of any game $\Gamma = (N, v)$ in $\mathcal{S}(N)$ is defined to be the game $\Gamma^* = (N, v^*)$, where

$$v^*(S) = v(N) - v(N - S), \qquad \text{all } S \subset N. \tag{11}$$

Obviously, $\Gamma^{**} = \Gamma$. For $\mathcal{C}(N)$, duality interchanges the ideas of winning and blocking, and a simple game is decisive if and only if it is self-dual: $v = v^*$. For games in $\mathcal{M}(N)$ we have

$$[c;\ w_1,\ \ldots,\ w_n]^* = \langle w(N) - c;\ w_1,\ \ldots,\ w_n\rangle,$$

$$\langle c;\ w_1,\ \ldots,\ w_n\rangle^* = [w(N) - c;\ w_1,\ \ldots,\ w_n].$$

It follows that of any mutually dual pair of weighted majority games, at least one is proper.*

*This is not true for simple games in general. An example is the improper four-person game with minimal winning coalitions {{1, 2}, {2, 3}, {3, 4}}. Its dual, also improper, has minimal winning coalitions {{1, 3}, {2, 3}, {2, 4}} (see [63]). This also illustrates how a game can be isomorphic to its dual without being decisive: we have here $\pi v = v^*$ for a certain permutation π but not $v = v^*$.

It is easily seen that any simple game and its dual have the same raw swing count for each player, since (S, T) is a swing for i in v if and only if (N - T, N - S) is a swing for i in v*. Hence we have

<u>THEOREM 5</u>. <u>If</u> v <u>is any simple game, then</u>

$$\eta(v) = \eta(v^*), \quad \beta(v) = \beta(v^*), \quad \underline{\text{and}} \quad \beta'(v) = \beta'(v^*).$$

As might be expected, the voting weights and the power indices are closely related. Both w and β (or β' or η) induce the same ranking on the players, except that two players with unequal weights may have equal power. Also, it is easily shown that a player's power is a nondecreasing function of his weight if either (a) the rest of the weights and the quota are held fixed, or (b) the rest of the weights are fixed and the quota is kept a fixed fraction of the total weight. Quite often the vectors β and w are even found to be roughly proportional, but the following simple examples from [61] show how "rough" the proportionality can be:

[8; 3, 5, 7]: $\beta = (1/3, 1/3, 1/3)$

[51; 49, 48, 3]: $\beta = (1/3, 1/3, 1/3)$

[4; 2, 2, 2, 1]: $\beta = (1/3, 1/3, 1/3, 0)$

[5; 2, 2, 2, 2, 1]: $\beta = (1/5, 1/5, 1/5, 1/5, 1/5)$.

If we let the quota vary, however, a smooth relationship can be obtained

by considering the average power, as the following theorem reveals.*

THEOREM 6. Let $w_1, \ldots, w_n$ be fixed nonnegative numbers and let c be a random variable uniformly distributed over the interval $[0, w(N)]$. Then for each player i in the weighted majority game (8) (or (10)), we have

$$E\{\beta_i'\} = w_i/w(N). \tag{12}$$

Proof. Fix i and S, with $i \in S$, and note that

$$\mathrm{Prob}\{(S, S - \{i\}) \text{ is a swing}\} = \mathrm{Prob}\{w(S) \geqq c > w(S) - w_i\} = w_i/w(N).$$

The expected number of swings for i is therefore

$$E\{\eta_i\} = \sum_{S \ni i} w_i/w(N) = 2^{n-1} w_i/w(N),$$

and the result follows. Q.E.D.

COROLLARY 3. Choose c only from $(\frac{w}{2}, w]$ (i.e., superadditive case only). Then again (12) holds.

*A similar result holds for the S-S index (see e.g., [64]), as well as for the generalized Bz probability index discussed in Sec. 12, Note 1.

This follows from Theorem 6 with the aid of duality. Of course (12) implies that $E\{\bar{\eta}\}, = 2^{n-1}$.

It is obvious that the weights w_i can always be made integers. If this is done, then it is sufficient in Theorem 6 to average over just the integers c = 1, 2, ..., w(N), or, in Corollary 3, over the integers c = $\langle w(N)/2\rangle$, ..., w(N), except that in the latter case we must give the first term half weight if w(N) is odd. The following table illustrates these remarks by showing the raw swing counts for each of the games [c; 1, 2, 3, 4, 5], c = 1, 2, ..., 15.

c	η_1	η_2	η_3	η_4	η_5	$\bar{\eta}$
1, 15	1	1	1	1	1	5
2, 14	0	2	2	2	2	8
3, 13	1	1	3	3	3	11
4, 12	1	1	3	5	5	15
5, 11	1	3	3	5	7	19
6, 10	2	4	4	6	8	24
7, 9	1	3	5	7	9	25
8	2	2	6	6	10	26
Total	16	32	48	64	80	240

Note that in this set of games there is no choice of quota that makes power exactly proportional to voting weight; indeed there is only one proper quota (c = 9) that manages to make all the powers different. However, the totals are proportional to the weights.

6. PASSAGE TO THE LIMIT

Most of the rest of this paper takes its cue from a 1960 Rand report of Shapiro and Shapley, published recently as [61], which considered the following weighted majority situation: There is a fixed quota, a fixed total weight, and a fixed set of "major" players with fixed individual weights. But there is also a population of "minor" players whose number is allowed to grow to infinity while their individual weights go to zero, keeping the total constant. The question now asked is: What happens to the power indices of the major players in the limit?

In contrast to the result in [61] for the S-S index, we shall find that the normalized Bz indices for the major players do not necessarily converge to a limit if no regularity conditions are imposed on the manner in which the minor weights go to zero. Further, the limiting Bz indices, when we do have convergence, depend rather strangely on the quota c; in fact, for much of the domain $R = [0, w(N)]$ of c, they are identically zero. But there is an interior region in R in which the major players are not "destroyed," and we are pleasantly surprised to find that in this region their limiting indices can be computed easily from another weighted majority game, involving just themselves and having a suitably reduced quota. This region consists of a certain open interval I in R (see below) from which a finite number of points have been deleted; these are the curious "pitfall points" that crop up in I where the major players are all "destroyed" simultaneously and have no power at all, in the "β" sense. At such values of c, the minor-player swings suddenly become so numerous that the relative number of major-player swings goes to zero.

Specifically, let $\{\Gamma^\nu : \nu = 1, 2, \ldots\}$ be a sequence of weighted majority games, as follows:

(13) $$\Gamma^\nu = [c; w_1, \ldots, w_\ell, \alpha_1^\nu, \ldots, \alpha_m^\nu],$$

where ℓ is a fixed positive integer. We require that

$$\sum_{i=1}^{m^\nu} \alpha_i^\nu = \alpha, \qquad \text{for each } \nu,$$

and

$$\alpha_{max}^\nu \longrightarrow 0, \qquad \text{as } \nu \longrightarrow \infty,$$

where α_{max}^ν denotes the maximum of the $\alpha_i^\nu = 1, \ldots, m^\nu$. The set of <u>major players</u>, indexed $1, 2, \ldots, \ell$, is denoted L; the set of <u>minor players</u>, indexed for convenience* $1, 2, \ldots, m^\nu$, is denoted M^ν. Note that necessarily $m^\nu \to \infty$, since $m^\nu \alpha_{max}^\nu \geqq \alpha$. Note also that the minor players do not retain their identity from one game Γ^ν to the next.

Define

$$R = \{c : 0 \leqq c \leqq w(L) + \alpha\},$$

*L and M^ν have no members in common, despite the numbering convention. That is, the j^{th} minor player is really the $(\ell + j)^{th}$ player of the game.

$$I = \{c : \frac{\alpha}{2} < c < w(L) + \frac{\alpha}{2}\},$$
$$Z = R - I,$$
$$P = \{c : c = w(S) + \frac{\alpha}{2} \text{ for some } S \subseteq L\}.$$

Thus, R is the range of possible values for the quota c, while I, Z, and P are various special subsets of that range; the letters chosen are meant to suggest "interior," "zero," and "pitfall," respectively.

Let Γ_0 denote the weighted majority game that would result if the total weight of the minor players were distributed exactly half "yea" and half "nay," thus

$$(14) \qquad \Gamma_0 = [c - \frac{\alpha}{2};\ w_1,\ \ldots,\ w_\ell].$$

The relevance of (14) to the limiting situation should be apparent from the probability model discussed in Sec. 2, since we should expect that the votes of a continuous "ocean" of coin-tossing minor players would be equally divided. Of course, for Γ_0 to be well defined, we must have

$$0 \leqq c - \frac{\alpha}{2} \leqq w(L);$$

in other words, c must lie in the closure of I.

We shall first tackle the symmetric case, where all the minor players have equal weight. The following theorem will be proved in the next two sections:

THEOREM 7. *If* $\alpha_j^\nu \equiv \alpha_{max}^\nu = \alpha/m^\nu$ *for all* $j = 1, \ldots, m^\nu$ *and all* ν, *then for each* $i \in L$,

$$\lim_{\nu\to\infty} \beta_i^\nu = \begin{cases} \beta_i(\Gamma_0) & \text{if } c \in I - P \\ 0 & \text{if } c \in Z \cup P. \end{cases} \tag{15}$$

7. THE SYMMETRIC CASE: ONE MAJOR PLAYER

The argument for Theorem 7 will be best conveyed if we consider first the case where there is only one major player, i.e., $\ell = 1$. To determine the number of swings for that player we must count the number of subsets of minor players having weight less than c but not less than $c - w_1$. Since each minor player has individual weight* α/m, these subsets are just those having r members, where $0 \leqq r \leqq m$ and

$$\frac{(c - w_1)m}{\alpha} \leqq r < \frac{cm}{\alpha} . \tag{16}$$

It is convenient to define r_1, r_0 by**

$$r_1 = \left\langle \frac{(c - w_1)m}{\alpha} \right\rangle - 1, \qquad r_0 = \left\langle \frac{cm}{\alpha} \right\rangle - 1.$$

The number of major swings is then given by

$$\eta_1 = \sum_{r=r_1+1}^{r_0} \binom{m}{r}, \tag{17}$$

it being understood that $\binom{m}{r} = 0$ if $r > m$ or $r < 0$.

Minor swings, on the other hand, can arise in two ways: either from a subset of r_0 minor players (i.e., just short of a winning coalition), or from a subset of r_1 minor players together with the lone

*We are omitting the superscript "ν" until it is needed again.

**Recall that $\langle x \rangle$ means the least integer $\geqq$ x.

major player. The former type of situation produces $m - r_0$ minor swings, the latter $m - r_1$ minor swings. Hence the total number of minor swings is

$$\sum_{j \in M} \eta_j = (m - r_0)\binom{m}{r_0} + (m - r_1)\binom{m}{r_1}. \tag{18}$$

Since we are working with the normalized Banzhaf index, everything depends on the ratio of (17) to (18) as we pass to the limit. If (17) dominates, then the limit of $\beta_1 = \eta_1/\bar{\eta}$ will be 1; if (18) dominates, it will be 0. Only if (17) and (18) are of the same order of magnitude could we obtain intermediate values between 0 and 1. The following lemma gives us the necessary test for dominant terms; the crucial question proves to be whether or not the summation in (17) includes the "center" of the binomial sequence $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$.

<u>LEMMA 1</u>. <u>Let</u> $0 \leqq r < s \leqq 1$. <u>Let</u> $m \longrightarrow \infty$. <u>Then</u>

$$\frac{\sum\limits_{k=\langle rm\rangle}^{\langle sm\rangle} \binom{m}{k}}{m\binom{m}{\langle sm\rangle}} \longrightarrow \begin{cases} \infty & \underline{\text{if}}\ s > 1/2 \\ 0 & \underline{\text{otherwise}}. \end{cases} \tag{19}$$

<u>Proof</u>. Let p and q be fixed real numbers between 0 and 1, and consider the ratio between the binomial coefficients $\binom{m}{\langle pm\rangle}$ and $\binom{m}{\langle qm\rangle}$ as a function of m. If m is large, enabling us to ignore the distinction between $\langle pm\rangle$ and pm, etc., Stirling's approximation gives

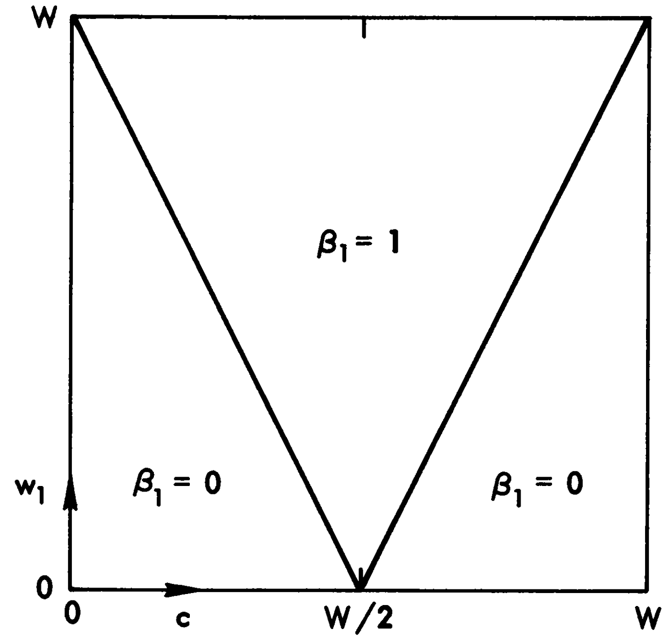
$$\frac{\binom{m}{qm}}{\binom{m}{pm}} \underset{=}{\sim} \sqrt{\frac{p(1-p)}{q(1-q)}} \left(\frac{p^p(1-p)^{1-p}}{q^q(1-q)^{1-q}}\right)^m .$$

So unless $p = q$ or $p = 1 - q$, this ratio is effectively exponential in m. The limit in (19) will therefore be ∞ if the largest term in the numerator of (19) exceeds the $\binom{m}{sm}$ in the denominator. But this happens (for all sufficiently large m) only when $s > 1/2$.

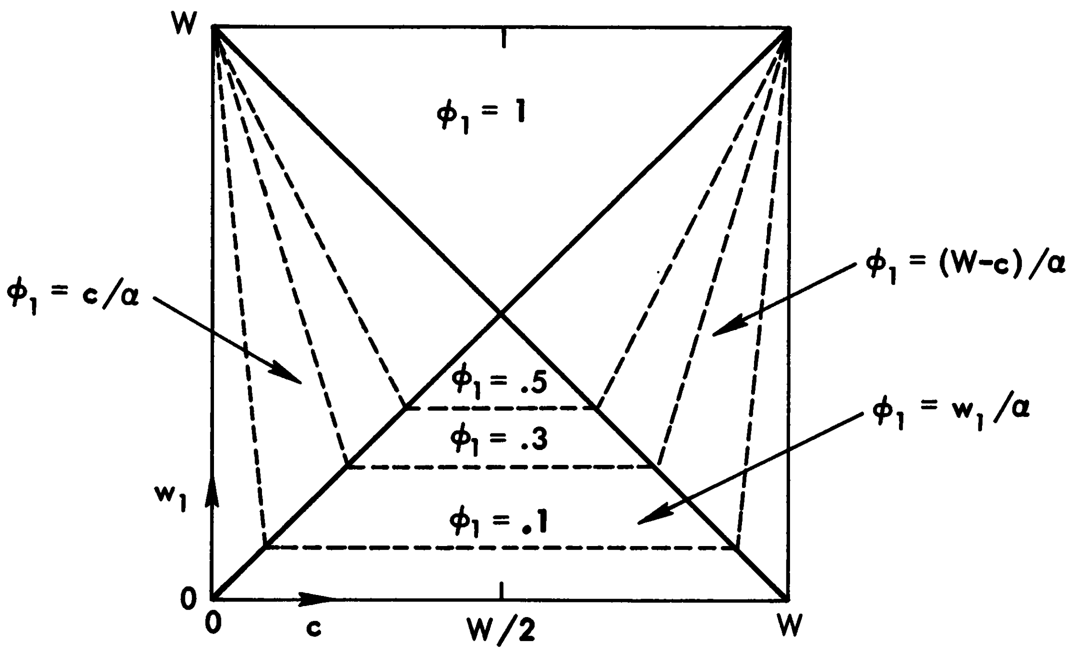
On the other hand, if $s \leqq 1/2$, then the dominating term is $\binom{m}{sm}$, which appears on both top and bottom in (19). In this case the denominator in fact grows faster than the numerator. To show this, let p be a fixed real number satisfying $r < p < s$. Then the numerator is bounded above by a sum consisting of (approximately) $pm - rm$ terms $\binom{m}{pm}$ and $sm - pm$ terms $\binom{m}{sm}$. Since the former are negligible in size compared to the latter, and since there are m terms $\binom{m}{sm}$ in the denominator, the limiting ratio is bounded above by $(sm - pm)/m = s - p$. But p may be chosen as close as we please to s, so the limiting ratio is zero. Q.E.D.

Returning now to the task of comparing (17) and (18), we see that if $r_1 \leqq r_0 \leqq m/2$, then (18) dominates (17) in the limit, and we have $\beta_1^\nu \to 0$. The same happens if $m/2 \leqq r_1 \leqq r_0$. These cases correspond to having $c \leqq \alpha/2$ and $c \geqq \alpha/2 + w_1$, or in other words $c \in Z$. But in the remaining case $r_1 < m/2 < r_0$, (17) contains the dominant term, and so in the limit $\beta_1^\nu \to 1$. This case corresponds to $\alpha/2 < c < \alpha/2 + w_1$, or in other words, to $c \in I$. Since the pitfall points do not matter with only one major player, i.e., since $P \cap I = \emptyset$, we have now verified Theorem 7 for the case $\ell = 1$.

Let us pause to compare the Banzhaf power with the Shapley-Shubik power for the case of one major player and a limiting "ocean" of minor players [40]. There are essentially just two free parameters, since the voting unit is arbitrary. If we hold the total vote W constant, the allowed values of c and w_1 describe a square (see Fig. 1), with the minor weight being given by $\alpha = W - w_1$. The upper diagram (a) (which serves equally well for the probability index β_1'; see Theorem 8) reveals the major player as either virtually a dictator or virtually insignificant. In the lower diagram (b), there is a smaller "dictatorial" region, but there is no region where the major player is totally "wiped out." The sharpest disagreements between the two measures of power occur near $c = W/2$ when w_1 is very small and near $c = 0$ and $c = W$ when w_1 is very large.



(a) "Banzhaf" power β (or β')



(b) "Shapley - Shubik" power ϕ

Fig. 1 — Limiting power indices with one major player

8. THE SYMMETRIC CASE: MANY MAJOR PLAYERS

We now remove the restriction $\ell = 1$, but retain the symmetry of the minor players: $\alpha_j \equiv \alpha_{max} = \alpha/m$. Let i be a major player, let S be a set of major players including i, and let $\eta_{i,S}$ denote the number of swings for i of the form $(T, T - \{i\})$ with $T \cap L = S$. Clearly

$$(20) \qquad \eta_i = \sum_{S: i \in S \subset L} \eta_{i,S}.$$

For any such swing, the number $r = |T - S|$ of minor players in T must satisfy:

$$\frac{(c - w(S))m}{\alpha} \leqq r < \frac{(c - w(S - \{i\}))m}{\alpha},$$

in analogy to (16) above. Accordingly, if we define

$$r_S = \left\langle \frac{(c - w(S))m}{\alpha} \right\rangle - 1,$$

we have

$$(21) \qquad \eta_{i,S} = \sum_{r=r_S+1}^{r_{S-\{i\}}} \binom{m}{r},$$

in analogy to (17) above. With (20), this gives the number of swings for the major player i.

The total number of minor swings, on the other hand, is easily seen to be just the following generalization of (18):

$$\sum_{j \in M} \eta_j = \sum_{S \subset L} (m - r_S)\binom{m}{r_S}. \tag{22}$$

We must now search for the dominant terms among (21) and (22). Note that as m goes to the limit the number of expressions (21), i.e., the range of i and S, is fixed. Also, the number of terms in the summation on S in (22) is fixed.

First, recalling Lemma 1, we observe that (22) will surely dominate in the limit if one of the terms $\binom{m}{r_S}$ is "central" in the sequence of binomial coefficients $\binom{m}{0}, \ldots, \binom{m}{m}$, that is, if $r_S/m \to 1/2$ for at least one S in L. This will happen if and only if we have $(c - w(S))/\alpha = 1/2$ for at least one S in L, and the reader will recognize this as the definition of a "pitfall." Hence we have

$$c \in P \implies \lim_{\nu \to \infty} \beta_i^{\nu} = 0 \quad \text{for each } i \in L. \tag{23}$$

Next, if none of the $\binom{m}{r_S}$ in (22) are "central," Lemma 1 tells us it is still possible for (22) to dominate in the limit if at least one $\binom{m}{r_S}$ is as near to the "center" as any of the terms $\binom{m}{r}$ that appear in the expressions (21). But these terms (considering all values of i and S) cover the entire range of integers r from r_L up to r_ϕ. In order to avoid covering the "center" we must have either $\lim(r_L/m) > 1/2$ or $\lim(r_\phi/m) < 1/2$. These conditions define our region Z, minus the boundary points $c = \alpha/2$ and $c = w(L) + \alpha/2$. Since the latter are members of P, however, we can assert

(24) $$c \in Z_{.} - P \Longrightarrow \lim_{\nu\to\infty} \beta_i^{\nu} = 0 \quad \text{for each } i \in L,$$

which completes the proof of the bottom line of (15).

In all the remaining cases, i.e., whenever $c \in I_{.} - P$, at least one of the sums (21) will include a "central" term and (22) will not. Hence the proportion of minor swings will be negligible in the limit in these cases, and it remains only to establish the relative distribution of swings among the major players. (This task was unncessary in the case $\ell = 1$.)

An easy way to do this is to adopt the probability viewpoint and observe that for $0 \leqq p < q \leqq 1$, the expression

$$\frac{1}{2^m} \sum_{r=\langle pm\rangle}^{\langle qm\rangle - 1} \binom{m}{r}$$

represents the probability that the fraction of successes in m fair coin tosses lies between p and q. The limit of this probability, as $m \to \infty$, is of course 1 if $p < 1/2 < q$ and 0 if $p > 1/2$ or $q < 1/2$. Hence, returning to (21), we see that $\eta_{i,S}/2^m$ converges to 1 if

(25) $$\frac{c - w(S)}{\alpha} < \frac{1}{2} < \frac{c - w(S - \{i\})}{\alpha}$$

and converges to 0 if

$$\frac{c - w(S)}{\alpha} > \frac{1}{2} \quad \text{or} \quad \frac{c - w(S - \{i\})}{\alpha} < \frac{1}{2}.$$

Since we are now assuming $c \in I - P$, we do not have to worry about equality here, so by (20) the relative number of swings for i is proportional in the limit to exactly the number of sets S in L that satisfy (25). But (25), still assuming $c \notin P$, is equivalent to

$$w(S) \geqq c - \frac{\alpha}{2} > w(S - \{i\}),$$

which is just the condition for swinging in the game Γ_0. This, with (23) and (24), completes the proof of Theorem 7.

9. CONVERGENCE OF β'

We return to the general setting described in Sec. 6, in which the behavior of the minor weights is restricted only by the conditions $\Sigma\, \alpha_j^\nu = \alpha$ and $\alpha_{max}^\nu \longrightarrow 0$. First we shall consider the "probability" index β'. Let Γ_0, as before, denote the game

$$[c - \frac{\alpha}{2};\, w_1,\, \dots,\, w_\ell],$$

and let Γ_0' denote the game

$$\langle c - \frac{\alpha}{2};\, w_1,\, \dots,\, w_\ell \rangle;$$

of course these games are well defined only for $c \in I$, and they are different from each other only if $c \in P$. Thus, the distinction between Γ_0 and Γ_0' was not an issue in Theorem 7.

<u>THEOREM 8</u>. <u>If</u> $\Sigma\, a_j^\nu \equiv \alpha > 0$, <u>and if</u> $\alpha_{max}^\nu \to 0$, <u>then for each</u> $i \in L$ <u>we have</u>*

$$\lim_{\nu\to\infty} \beta_i'^{\nu} = \begin{cases} \frac{1}{2}\, \beta_i'(\Gamma_0) + \frac{1}{2}\, \beta_i'(\Gamma_0') & \text{if } c \in I, \\ 0 & \text{if } c \in Z. \end{cases} \tag{26}$$

The following two lemmas form the heart of the proof. Let us write $\alpha^\nu(S)$ for $\Sigma_{i\varepsilon S}\alpha_i^\nu$.

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*Compare (15) in Theorem 7. For generalizations of this result, see Notes 1 and 2 in Section 12.

<u>LEMMA 2.</u> <u>Let</u> $0 \leqq q \leqq \alpha$, <u>and choose a subset</u> S <u>"at random" from</u> M^ν, <u>i.e., with probability</u> $1/2^{m^\nu}$. <u>Then</u>

$$\lim_{\nu\to\infty} \text{Prob}\,\{\alpha^\nu(S) = q\} = 0.$$

<u>Proof</u>. Let $\mathcal{S}^\nu(q)$ denote the collection of $S \subset M^\nu$ such that $\alpha^\nu(S) = q$; we must show that $|\mathcal{S}^\nu(q)|/2^{m^\nu}$ goes to zero as $m^\nu \to \infty$. The members of $\mathcal{S}^\nu(q)$ are pairwise independent, that is, if $S \in \mathcal{S}^\nu(q)$, $T \in \mathcal{S}^\nu(q)$ and $S \subset T$, then $S = T$. Such a collection of sets is called a "clutter" or "Sperner family," and Sperner's well-known lemma [70]* asserts that

$$|\mathcal{S}^\nu(q)| \leqq \binom{m^\nu}{\langle m^\nu/2\rangle}.$$

By Stirling's formula, this bound is approximately $2^{m^\nu+1}/\sqrt{2\pi m^\nu}$, which, when divided by 2^{m^ν}, goes to zero like $1/\sqrt{m^\nu}$. Q.E.D.

<u>LEMMA 3.</u> <u>Let</u> $0 \leqq p < q \leqq \alpha$, <u>and choose</u> $S \subset M^\nu$ <u>as in Lemma 2. Then</u>

$$\text{(27)}\qquad \lim_{\nu\to\infty} \text{Prob}\,\{p \leqq \alpha^\nu(S) < q\} = \begin{cases} 1 & \text{if } p < \frac{\alpha}{2} < q, \\ \frac{1}{2} & \text{if } p = \frac{\alpha}{2} \text{ or } q = \frac{\alpha}{2}, \\ 0 & \text{if } p > \frac{\alpha}{2} \text{ or } q < \frac{\alpha}{2}. \end{cases}$$

---

*Not to be confused with his equally famous lemma on labelled triangulations of the simplex.

<u>Proof</u>. Omit the superscript "ν" until needed. Let H_k denote the random variable $(\alpha(S): |S| = k)$, obtained by summing exactly k of the numbers $\alpha_1, \ldots, \alpha_m$, chosen at random. The mean of H_k is given by $\mu_k = k\alpha/m$, and, as with any random variable,* we have

$$\text{Prob}\,\{H_k \leqq \mu_k - \delta\} \leqq \frac{\sigma_k^2}{\sigma_k^2 + \delta^2}, \tag{28}$$

for any $\delta > 0$, where σ_k^2 denotes the variance of H_k. Also, we have

$$\sigma_k^2 \leqq k\sigma_1^2, \tag{29}$$

since a "sample without replacement" always has lower variance than the corresponding "sample with replacement."** Since we can estimate σ_1^2 by

$$\sigma_1^2 < \frac{1}{m} \Sigma\, \alpha_j^2 \leqq \frac{1}{m} \alpha\alpha_{max},$$

we have from (29) that $\sigma_k^2 < \alpha\alpha_{max}$ and so can conclude that

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*The "worst" case is a distribution concentrated at two points, namely $\mu_k - \delta$ and the unique point on the other side of μ_k that yields the required mean μ_k and variance σ_k^2. For such a distribution it is easy to calculate that $\sigma^2 = \delta^2 P_\delta/(1 - P_\delta)$, where P_δ is the probability of $\mu_k - \delta$; from this (28) follows.

**See Hoeffding [26] or Kemperman [32].

$$\text{(30)} \qquad \text{Prob } \{H_{k'} \leqq \mu_k - \delta\} < \alpha\alpha_{max}/\delta^2$$

for any $\delta > 0$.

Now let $q < \alpha/2$, and consider the sum

$$\text{Prob } \{\alpha(S) < q\} = \sum_{k=0}^{m} \text{Prob } \{|S|' = k\} \cdot \text{Prob } \{H_{k'} < q\}.$$

Choose δ so that $q < q + \delta < \alpha/2$, define $d = (q + \delta)/\alpha$, and split the sum into two parts:

$$\text{(31)} \qquad \sum_{k=0}^{m} = \sum_{k=0}^{\langle dm \rangle} + \sum_{k=\langle dm \rangle + 1}^{m} .$$

We shall show that both parts of (31) go to zero. Taking the second part first, we observe that $\mu_k > d\alpha = q + \delta$, so

$$\text{Prob } \{H_{k'} < q\} \leqq \text{Prob } \{H_{k'} < \mu_k - \delta\} < \alpha\alpha_{max}/\delta^2$$

by (30), and we have

$$\sum_{k=\langle dm \rangle + 1}^{m} \leqq \frac{\alpha\alpha_{max}}{\delta^2} \sum_{k=\langle dm \rangle + 1}^{m} \text{Prob } \{|S|' = k\} \leqq \frac{\alpha\alpha_{max}}{\delta^2},$$

which goes to zero with α_{max}. On the other hand, for the first part of (31) we have, trivially,

$$\sum_{k=0}^{\langle dm \rangle} \leqq \text{Prob } \{|S|' \leqq \langle dm \rangle\},$$

which goes to zero with increasing m by the Central Limit Theorem, since d is a fixed number less than 1/2. Thus, we have established the essential fact*

$$(32) \qquad \lim_{\nu\to\infty} \text{Prob}\,\{\alpha^{\nu}(S) < q\} = 0 \qquad \text{if } q < \alpha/2.$$

This is the key to proving the various assertions in (27).

Indeed, the case $q < \alpha/2$ is immediate from (32). The case $p > \alpha/2$ follows by symmetry, under the reflection $x \leftrightarrow \alpha - x$, since Lemma 2 permits us to ignore any endpoints that don't happen to match. The first case in (27), namely $p < \alpha/2 < q$, follows from the fact that the total probability is 1, for each finite ν and hence in the limit. Finally, the intermediate cases may be deduced from the fact that symmetric open intervals of the form $(r, \alpha/2)$ and $(\alpha/2, \alpha - r)$, with $r < \alpha/2$, must have equal probability in the limit, while their combined probability in the limit must be 1, since the single point between them is negligible by Lemma 2. Hence their separate probabilities in the limit are 1/2, 1/2. This completes the proof of Lemma 3.

Proof of Theorem 8. The definition of $\beta_1'^{\nu}$ may be written

$$\frac{1}{2^{\ell-1}} \sum_{T:\, i\in T\subseteq L} \text{Prob}\left\{ c - w(T) \leqq \alpha^{\nu}(S) < c - w(T - \{i\}) \right\},$$

*This could also have been established by appeal to the theorem of Gnedenko and Kolmogorov cited in Note 1, Sec. 12.

where S is a random subset of M^{ν} as above. As we pass to the limit, the summand approaches 0, 1/2, or 1, by Lemma 3. If $c \in Z$, the limit is always 0. If $c \in I$ the games Γ_0 and Γ_0' are well-defined, and for any i and T for which the limit is 1 we have a swing for i in both games. When the limit is 1/2, we have a swing in exactly one of these games, and when the limit is 0 we have a swing in neither. The proof is completed by the observation that $\beta'(\Gamma_0) = \eta(\Gamma_0)/2^{\ell-1}$ and $\beta'(\Gamma_0') = \eta(\Gamma_0')/2^{\ell-1}$.

10. THE ASYMMETRIC CASE: A COUNTEREXAMPLE

Unfortunately, it is not always true that the normalized Banzhaf indices converge as in Theorem 7 when the minor weights are allowed to go to zero in arbitrary fashion, even if we rule out the "pitfall" quotas $c \in P$. The major problem is caused by the minor swings. We have just seen that the probability indices $\beta_i'^{\nu}$ do converge in general, so the trouble can be traced to the behavior of the normalizing denominator. In fact, the normalized indices β_i^{ν} will converge if and only if the sum of the minor probability indices goes to zero. In other words (taking out the constant factor $2^{\ell-1}$), convergence of β_i^{ν} to $\beta_i(\Gamma_0)$ when c is in I - P requires

$$\lim_{\nu\to\infty} \frac{\sum_{j\in M^{\nu}} \eta_j^{\nu}}{2^{m^{\nu}}} = 0. \tag{33}$$

In this section we shall show by a construction that if the distribution of minor weights is not controlled in some way, the ratio in (33) may instead go to infinity, or perhaps not converge at all. The idea will be to set up two populations of minor players, with one group much more numerous than the first. A "pitfall" type situation will then be created in which there are inordinately many swings by the smaller minor players.

Consider first a weighted majority game with $m = 3m_1 + 2m_2$ players, as follows:

$$\Gamma_{m_1 m_2} = [2\gamma + \delta;\ \underbrace{\frac{\gamma}{m_1}, \ldots, \frac{\gamma}{m_1}}_{3m_1},\ \underbrace{\frac{\delta}{m_2}, \ldots, \frac{\delta}{m_2}}_{2m_2}]. \tag{34}$$

Here γ and δ are positive constants, and the total voting weight is $3\gamma + 2\delta$. We shall focus on the coalitions that have exactly $2m_1$ members of the first type and m_2 of the second. The number of such coalitions is

$$\binom{3m_1}{2m_1}\binom{2m_2}{m_2},$$

which works out by Stirling's approximation to

$$\frac{\sqrt{3}\; 2^m}{2\pi\sqrt{m_1 m_2}}\left(\frac{27}{32}\right)^{m_1}\left(1 - o\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right).$$

Such a coalition is just barely winning and so contributes $2m_1 + m_2$ swings, one for each player in it. We therefore have, for the total number of swings,

$$\text{(35)} \qquad \bar{\eta}(\Gamma_{m_1 m_2}) \geqq (2m_1 + m_2)\,\frac{J2^m}{\sqrt{m_1 m_2}K^{m_1}}\left(1 - o\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right),$$

where J and K are constants with $J > 0$ and $K > 1$.

We now send m_1 and m_2 to infinity in such a way that the second population grows much more rapidly than the first. The idea is to make the ratio of m_2 to $m_1 K^{2m_1}$ go to infinity so that the denominator in (35) will be $o(m_2)$, permitting the m_2 in the numerator to dominate. This can be done by requiring that

$$\text{(36)} \qquad m_1 < H \log m$$

where $H < 1/2 \log K$. Then we have, by direct substitution,

$$\bar{\eta}(\Gamma_{m_1 m_2}) > 2^m J \frac{2m_1 + m_2}{\sqrt{m_1 m_2 m}} m^h \left(1 - O(\frac{1}{m_1} + \frac{1}{m_2})\right),$$

where $h = 1/2 - H \log K > 0$. Hence, using (36), we have

$$\bar{\beta}'(\Gamma_{m_1 m_2}) = \bar{\eta}(\Gamma_{m_1 m_2})/2^{m-1} \geqq O(m^h/\sqrt{\log m}), \tag{37}$$

which yields the desired "explosion" in the expected number of swings.

In order to apply this construction to our general set-up (13) with major and minor players we introduce major weights $w_1, \ldots, w_\ell$, a total minor weight α, and a quota $c \in I - P$. Let R be any coalition of major players satisfying

$$\frac{\alpha}{2} < c - w(R) < \frac{2\alpha}{3}. \tag{38}$$

If R votes "yea" and L - R votes "nay" then the minor players in effect face a reduced quota of $c - w(R)$. Fix the values of γ and δ by

$$\begin{cases} 3\gamma + 2\delta = \alpha \\ 2\gamma + \delta = c - w(R) \end{cases} \quad \text{i.e.,} \quad \begin{cases} \gamma = 2(c - w(R)) - \alpha \\ \delta = 2\alpha - 3(c - w(R)). \end{cases}$$

By (38), γ and δ are positive. If we now group the minor players into two populations as in (34) and let them grow satisfying (36), then the "explosion" (37) will overwhelm the major swings, causing their

proportion of the total to converge to 0 and so wiping out the major players' normalized Bz indices.

Moreover, since some or all of the major players' normalized Bz indices converge to positive numbers in the symmetric setting of Theorem 7, it is clear that a sequence of games could be constructed for which they do not converge at all.

Condition (38) is quite unexceptional, but if the weights w_i are such that it is not satisfied by any $R \subset L$ then some simple modification of the above construction, like changing the "3" in "$3m_1$" to a higher integer or using duality, may do the trick. In fact, it can be shown without much difficulty that the two-population type of example is sufficient to spoil convergence for <u>any</u> set of major weights, provided only that α and c are such that c is in I - P.

11. CONVERGENCE OF β

Section 10 shows that the assignment of votes to the minor players must be restricted somehow if the normalized Bz indices of the major players are to behave well in the limit. Several ways of doing this might be considered; it seems important, however, to make the distribution of minor-player votes reasonably smooth, in some sense, so as to keep the "pitfall" situation under control. One device might be to require that the ratio of α^ν_{max} to α^ν_{min} stay bounded as the number of players increases, or at least that it not grow too rapidly. Another way would be to ignore α^ν_{min} (i.e., permit some players to be very small), but require that the ratio of α^ν_{max} to the mean minor weight α/m^ν stay bounded, or at least not grow too fast. A third way might be to insist that the standard deviation of the minor weights go to zero sufficiently rapidly (cf. the proof of Lemma 3). Of course, these approaches are not entirely independent, being weakly interlinked through certain inequalities.

We shall concentrate on establishing a condition of the second type, i.e., a condition on the asymptotic behavior of α^ν_{max} as a function of m^ν ((39), below). This will yield some weaker results of the first and third types as corollaries. We shall refer repeatedly to the two-population counterexample in Sec. 10 in order to indicate how much room there might be for sharpening these results.

<u>THEOREM 9</u>. <u>If</u> $c \in I - P$, <u>and if</u>

(39) $$\alpha^\nu_{max} = o(1/\sqrt{m^\nu \log m^\nu}),$$

i.e., if $\sqrt{m^\nu \log m^\nu}\,\alpha^\nu_{max} \longrightarrow 0$ *as* $\nu \longrightarrow \infty$, *then* $\beta^\nu_i \longrightarrow \beta_i(\Gamma_0)$ *for each* $i \in L$.

We do not know if (39) can be sharpened. It is easily verified from (36) in Sec. 10 that one can have $\alpha^\nu_{max} = O(1/\log m^\nu)$ without β^ν_i converging, so the uncertainty is reduced to the interval between $1/\sqrt{m^\nu \log m^\nu}$ and $1/\log m^\nu$.

The proof of Theorem 9 is based on the following lemma of Kemperman's, which we quote here directly from [31], p. 113:

LEMMA 4. *Let* $\{Z_k\colon\ k = 0, 1, \ldots\}$ *be a sequence of independent real-valued random variables such that* $|Z_k| \leq 1$ *for all* k. *Let further* c_k *be a sequence of real constants such that*

$$s^2 = \sum_{k=0}^{\infty} c_k^2 < \infty, \qquad (s \geq 0).$$

Then

$$S = \sum_{k=0}^{\infty} c_k(Z_k - m_k), \qquad (m_k = E\{Z_k\}),$$

satisfies

(40) $$\text{Prob}\,\{S > \delta s\} \leq e^{-\delta^2/2}, \qquad \text{Prob}\,\{S < -\delta s\} \leq e^{-\delta^2/2},$$

for each number $\delta > 0$.

This lemma in effect estimates the "tails" of a sum of random variables. To apply it to our problem, fix ν and let Z_k for each $k = 1, \ldots, m^\nu$ be the random variable that takes on the value $\alpha_k^\nu/\alpha_{max}^\nu$ with probability 1/2 and is otherwise 0. For $k > m^\nu$ we let $Z_k \equiv 0$. For the constants c_k, we set the first m^ν of them equal to α_{max}^ν and the rest equal to 0; this makes the "s" of the lemma equal to $\sqrt{m^\nu}\ \alpha_{max}^\nu$. The "S" of the lemma is just the difference between our random variable $\alpha^\nu(T)$ and its mean, $\alpha/2$, where T is a randomly chosen subset of M^ν like the "S" in Lemmas 2 and 3. With these identifications, Kemperman's theorem enables us to conclude that for any positive number ε,

$$(41) \qquad \text{Prob}\ \{\alpha^\nu(T) > \frac{\alpha}{2} + \varepsilon\} = \text{Prob}\ \{S > \varepsilon\} \leqq e^{-\delta^2/2},$$

where δ is an abbreviation for $\varepsilon/s = \varepsilon/(\sqrt{m^\nu}\ \alpha_{max}^\nu)$. For convenience, define $A_\nu = \sqrt{m^\nu \log m^\nu}\ \alpha_{max}^\nu$. Then (41) reduces to

$$(42) \qquad \text{Prob}\ \{\alpha^\nu(T) - \frac{\alpha}{2} > \varepsilon\} \leqq (m^\nu)^{-\varepsilon^2/2A_\nu^2}.$$

Since our hypothesis is that A_ν goes to zero, (42) will enable us to prove Theorem 9.

Indeed, we have for each ν,

$$(43) \qquad \beta_i^\nu = \frac{\beta_i'^\nu}{\sum_{j \in L} \beta_j'^\nu + \sum_{k \in M^\nu} \beta_k'^\nu},$$

and we must show that the second term in the denominator, i.e., the expected number of minor swings, goes to zero. Note that a minor swing cannot occur unless $\alpha^\nu(T)$ is within α^ν_{max} of $c - w(R)$, for some $R \subset L$. But for any fixed quota c in the open set $I - P$, there is an $\epsilon > 0$ such that for every $R \subset L$,

$$|c - w(R) - \frac{\alpha}{2}| > 2\epsilon.$$

For sufficiently large ν we have $\alpha^\nu_{max} < \epsilon$, and so our necessary condition for a minor swing, namely

$$|\alpha^\nu(T) - (c - w(R))| \leqq \alpha^\nu_{max},$$

makes it necessary that:

$$|\alpha^\nu(T) - \frac{\alpha}{2}| > \epsilon.$$

But there are at most m^ν minor swings for any particular $R \subset L$, and at most 2^ℓ possible choices of R. Thus the expected total number of minor swings must satisfy

$$(44) \qquad \sum_{k \in M^\nu} \beta_k'^\nu \leqq 2^\ell m^\nu \text{ Prob } \{|\alpha^\nu(T) - \frac{\alpha}{2}| > \epsilon\}.$$

By (42), we therefore have

$$(45) \qquad \sum_{k \in M^\nu} \beta_k'^\nu \leqq 2^{\ell+1} (m^\nu)^{1-\epsilon^2/2A_\nu^2}.$$

(There is an extra factor of 2 here to take care of the absolute value in (44).) Now, as A_ν approaches zero, the exponent of m^ν eventually becomes negative, so the right hand side of (45) goes to zero. Hence we may pass to the limit in (43) and, with the aid of Theorem 8, obtain

$$\lim_{\nu\to\infty} \beta_i^\nu = \frac{\beta_i'(\Gamma_0)}{\sum_{j\in L} \beta_j'(\Gamma_0)} = \beta_i(\Gamma_0).$$

This completes the proof of Theorem 9.

COROLLARY 4. *If* $c \in I - P$, *and if*

$$\frac{\alpha_{max}^\nu}{\alpha_{min}^\nu} = o(\sqrt{m^\nu/\log m^\nu}), \tag{46}$$

then $\beta_i^\nu \longrightarrow \beta_i(\Gamma_0)$ *for each* $i \in L$.

The proof is immediate from (39) and the inequality $\alpha_{min}^\nu \leq \alpha/m^\nu$. For comparison, in the nonconvergent two-population game of Sec. 10 we had $\alpha_{max}^\nu/\alpha_{min}^\nu = O(m^\nu/\log m^\nu)$.

Corollary 4 can be sharpened with the help of a result of W. Hoeffding [26],* which tells us that the sum $\alpha(\gamma)$ of a randomly-chosen subset of the positive numbers $\alpha_1, \ldots, \alpha_m$ obeys the following inequality:

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*See (4.8) in [32]. We are grateful to J. H. B. Kemperman for pointing out this result and its consequences for our work.

$$\text{(47)} \qquad \text{Prob}\ \{\alpha(Y) > \frac{\alpha(M)}{2} + \varepsilon\} \leqq \exp\{-\frac{8m\varepsilon^2\theta}{\alpha(M)^2(1+\theta)^2}\}$$

for every $\varepsilon > 0$. Here M denotes $\{1, \ldots, m\}$ and θ denotes $\alpha_{max}/\alpha_{min}$. A straightforward argument using (47) permits us to replace (46) by the weaker condition

$$\text{(48)} \qquad \frac{\alpha^{\nu}_{max}}{\alpha^{\nu}_{min}} = o(m^{\nu}/\log m^{\nu}),$$

and thus strengthen the conclusion.

For example, consider a two-population situation, as in Sec. 10, in which the number of "bigger" minor players goes to ∞ no faster than $O(\log m^{\nu})$ (compare (36)). In order for the major Bz indices β_i to converge, (48) tells us that it is sufficient that $\alpha_{max} = o(1/\log m^{\nu})$, since clearly $\alpha_{min} = O(1/m^{\nu})$. On the other hand, our example of non-convergence has $\alpha_{max} = O(1/\log m^{\nu})$, so for this type of situation the gap of uncertainty about convergence is almost closed.

<u>COROLLARY 5</u>. <u>Let</u> $\sigma(\nu)$ <u>denote the standard deviation of the set of numbers</u> $\{\alpha^{\nu}_j : j \in M^{\nu}\}$. <u>If</u> $c \in I - P$, <u>and if</u>

$$\text{(49)} \qquad \sigma(\nu) = o(1/m^{\nu}\sqrt{\log m^{\nu}}),$$

<u>then</u> $\beta^{\nu}_i \longrightarrow \beta_i(\Gamma_0)$ <u>for each</u> $i \in L$.

<u>Proof</u>. By considering the "worst case" (one weight equal to α^{ν}_{max}, all others equal to α^{ν}_{min}), we find that $\alpha^{\nu}_{max} = O(\sqrt{m}\,\sigma(\nu))$; hence (49) implies (39). Q.E.D.

The construction in Sec. 10 shows that it is possible to have nonconvergence with $\sigma(\nu) = O(1/\sqrt{m^{\nu} \log m^{\nu}})$.

12. REMARKS AND EXTENSIONS

This section is a kind of appendix, presenting a miscellany of refinements, extensions, and alternative interpretations of the mathematical theory developed in this paper. The reader's attention is particularly directed to Notes 4 through 7, in which connections are made to the literature of political science and electrical engineering, where several concepts formally equivalent to the Banzhaf index have been introduced.

Note 1. It is only natural to try and extend the probability model in Sec. 2 by introducing parameters p_i, $0 < p_i < 1$, $i \in N$, and making the players vote "yea" and "nay" with probability p_i and $1 - p_i$, respectively, rather than 1/2, 1/2.* The generalized Bz probability index is then given by

$$\beta_i'[p] = \sum_{S: i \in S \subseteq N} P_{S,i}[v(S) - v(S - \{i\})], \tag{50}$$

where $P_{S,i}$ denotes the probability that $Y = S - \{i\}$:

$$P_{S,i} = \left(\prod_{j \in S-\{i\}} p_j\right)\left(\prod_{j \in N-S} (1 - p_j)\right). \tag{51}$$

These indices can of course be normalized, if desired, by defining $\beta_i[p] = \beta_i'[p]/\Sigma_N\, \beta_j'[p]$.

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*Since these probabilities are essentially subjective, existing only in the minds of the other players, it might be better to generalize at once to doubly-indexed parameters p_{ij}, $i \neq j$, representing j's estimate of i's probability of voting "yea." Another generalization, where the voting probabilities are themselves random variables, not necessarily independent, has been considered by Straffin [71]; see also Blair [9] and Dubey [18].

We note that $\beta'[p]$ satisfies Axiom A4 (see Sec. 3), and that it also satisfies Axiom A3 when all the p_i's are equal. It is interesting also that Theorem 6 holds for $\beta'[p]$ (see Sec. 5), although the accompanying Corollary 3, which depends on duality, does not.

Turning to the weighted majority games Γ^ν of Sec. 6, suppose that each major player $i \in L$ belongs to Y with probability p_i^ν and each minor player $j \in M^\nu$ belongs to Y with probability q_j^ν. Then the swing probability for $i \in L$ is given by

$$\beta_i'[p^\nu, q^\nu] = \sum_{S: i \in S \subset L} \sum_{T \subset M^\nu} P^\nu_{S \cup T, i}[v^\nu(S \cup T) - v^\nu((S - \{i\}) \cup T)],$$

where v^ν is the characteristic function of the game Γ^ν and $P_{S \cup T, i}$ (compare (51)) is given by

$$P_{S \cup T, i} = \prod_{k \in S - \{i\}} p_i^\nu \prod_{k \in L - S} (1 - p_i^\nu) \prod_{j \in T} q_j^\nu \prod_{j \in M^\nu - T} (1 - q_j^\nu).$$

In order to make the indices $\beta_i'[p^\nu, q^\nu]$ converge, i.e., in order to generalize Theorem 8, we shall need some conditions on the parameters p^ν, q^ν as $\nu \to \infty$. For each major player i, we obviously must assume that p_i^ν converges to some limit, say p_i^*. For the minor players, what we need to know is that the weight $\alpha^\nu(T)$ of the random subset $T = M^\nu \cap Y$ will converge in probability to its mean, which is simply

$$\mu^\nu = \sum_{j \in M^\nu} q_j^\nu \alpha_j^\nu;$$

moreover this mean must converge in turn to some limit, say μ^*. By a theorem of Gnedenko and Kolmogorov,* a necessary and sufficient condition for the convergence in probability is that

$$(52) \qquad \sum_{j \in M^\nu} \text{Prob}\ \{|\alpha^\nu - \mu^\nu| > 1\} \to 0$$

and

$$(53) \qquad \sum_{j \in M^\nu} q_j^\nu (1 - q_j^\nu)^2 (\alpha_j^\nu)^2 \to 0.$$

Both (52) and (53) are easily seen to be satisfied in our case, because of the assumption that $\alpha_{\max}^\nu \to 0$. Indeed, the first expression is identically zero for $\alpha_{\max} < 1$, while the second expression is bounded by $\Sigma\ (\alpha_j^\nu)^2$, and hence by $\Sigma\ \alpha_j^\nu \alpha_{\max} = \alpha\alpha_{\max}^\nu$.

So we need to assume only that the parameters $p_1^\nu, \ldots, p_\ell^\nu$ and μ^ν, namely, the probability of each major player voting "yea" and the mean weight of the set of yea-voting minor players, all converge to their respective limits as $\nu \to \infty$. The same methods as in Sec. 9 then lead us to the desired generalization of Theorem 8, namely,

$$\lim_{\nu\to\infty} \beta_i'[p^\nu, q^\nu] = \begin{cases} \frac{1}{2}\, \beta_i'[p^*](\Delta_0) + \frac{1}{2}\, \beta_i'[p^*](\Delta_0'), & \text{if } \mu^* < c < w(L) + \mu^*, \\ 0, & \text{if otherwise.} \end{cases}$$

Here Δ_0, Δ_0' are the weighted majority games $[c - \mu^*; w_1, \ldots, w_\ell]$ and $\langle c - \mu^*; w_1, \ldots, w_\ell \rangle$ respectively. Note that the set of "pitfall points" (i.e., values of c for which Δ_0 and Δ_0' are actually different) now depends on μ^*.

*See [20], page 105. This theorem could also have been used in the proof of Lemma 3 (see Sec. 11).

Note 2. There are obvious extensions of our results in Secs. 6-11 to the case where any or all of the parameters c, w_1, ..., w_ℓ, and α also depend on ν, converging to stated limits as $\nu \to \infty$. We omit the details.*

Note 3. Sergiu Hart in [25] has defined the swing probability of a simple game to be the probability of getting a swing when two adjacent coalitions are chosen at random.** If there are n players, each such coalition-pair (i.e., each edge of the n-cube) has proability $1/(n2^{n-1})$, and so the swing probability in our notation is just $\overline{\beta'}/n$. Applying his bound on the total number of swings (our Theorem 4), Hart showed that if the fraction of coalitions that are winning converges to a limit other than 0 or 1,*** in a sequence of n^ν-player games with $n^\nu \to \infty$, then the swing probabilities π^ν of these games satisfy

$$(54) \qquad \liminf_{\nu\to\infty} n^\nu \pi^\nu > 0.$$

In other words, the number $\overline{\beta'}^\nu$, which represents the expected number of swingers, does not go to zero in such a sequence of games. On the other hand this number may go to infinity, as we can see, e.g., by putting $m = [n/2]$ in Theorem 3.

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*Cf. the analogous extension carried out in [61].

**More generally, Hart considers the class of arbitrary (0, 1)-games with $v(\emptyset) = 0$, which he unconventionally calls the simple games.

***Convergence of the winning fraction is not essential here; it is sufficient to assume that 0 and 1 are not limit points.

Note 4. Douglas Rae in a 1969 paper [53] addressed the problem of comparing the responsiveness of different voting systems to the general will of the electorate. His basic idea was to count the number of ways in which the average voter can find his vote in agreement with the outcome of the voting. Assuming symmetry, Rae considered only a single, generic voter, but it is natural to extend his approach by defining an "index of agreement" for each voter:*

$$(55) \qquad \rho_i = \#\{Y : i \in Y \in \mathcal{W} \quad \text{or} \quad i \notin Y \notin \mathcal{W}\}$$

where $\mathcal{W}$ denotes the set of all winning coalitions in a general simple game $(N, \mathcal{W})$. The overall responsiveness of the decision rule represented by the simple game can then be measured by the sum $\bar{\rho}$ (or, if we prefer, by the average $\bar{\rho}/|N|$).

It was not noticed for several years that this index of agreement is nothing but the Banzhaf power index in disguise. In fact, the following identity holds:

$$(56) \qquad \rho_i \equiv 2^{|N|-1} + \eta_i.$$

This may be seen most easily from the following little table in which the possible "yea-voting" sets Y are grouped into six classes according to how Y relates to $\mathcal{W}$ and i:

*The term "satisfaction index" has also been used, but we feel that it unnecessarily injects considerations of utility (not to mention voting sincerity!) into what is otherwise a purely structural analysis. Having his vote agree with the outcome may or may not "satisfy" the voter.

	$Y - \{i\} \in \mathcal{W}$	$Y \cup \{i\} \in \mathcal{W}$	$Y \ni i$	i agrees	i swings
1	true	true	true	✓	×
2	true	true	false	×	×
3	false	true	true	✓	✓
4	false	true	false	✓	✓
5	false	false	true	×	×
6	false	false	false	✓	×

Note that classes 1 and 2 are of equal size, being based on the same collection of subsets of $N - \{i\}$; similarly classes 3 and 4 and classes 5 and 6. Hence, the number of times player i is "in agreement" is just half the total (i.e., $2^{|N|-1}$ times), plus half the number of Y's that make him a swinger. As there are exactly $2\eta_i$ Y's that make i a swinger, (56) is proved.

In view of (56), it is a simple matter to translate results about $\bar{\eta}$ (see Sec. 4) into results about $\bar{\rho}$. Rae originally conjectured, and Taylor [72] proved, that for any fixed number of voters <u>responsiveness is maximized by direct majority rule</u>. However, they considered only the narrow class of symmetric games* $M_{n,k} = [k; 1, 1, \ldots, 1]$.

*These are by no means the only simple games symmetric in the players, since all that is necessary for symmetry is for $\mathcal{W}$ to be invariant under some transitive subgroup of permutations of the players. For example, take any $S \subset N = \{1, 2, \ldots, n\}$ and let the minimal winning coalitions consist of S together with its images under the cyclic permutation (123...n), applied repeatedly. For another example, group the players into equal districts and require a majority of district majorities for victory.

Rae and Taylor do generalize their model to the extent of using different voting probabilities, somewhat in the manner of Note 1 above, and others [3, 13, 60] have followed this lead. But, surprisingly, none of them look beyond $M_{n,k}$ for their underlying voting rule.

Our Theorem 2 shows that the conclusion holds over the much larger domain of all simple games. Moreover, the proof is very easy in the broader context, as we saw in Sec. 4, since we can alter one coalition at a time without leaving the domain. Theorem 2 also identifies other games of maximum responsiveness, obtained from majority rule (when n is even) by designating some of the n/2-player sets to be winning. These could be of practical value in committee design, as they enable the decisiveness of the voting rule to be enhanced without loss of responsiveness, superadditivity, or even symmetry.*

Note 5. James S. Coleman in [12] considered two kinds of power exercised by a member of a "collectivity" (his term for a simple game), namely, a power to prevent action and a power to initiate action. The former he took to be the fraction of all winning coalitions in which the player in question is essential, in that without him the coalition would lose. The latter he took to be the fraction of all losing coalitions which the player in question, by joining, could convert

*For example, one can take any n/2-member set whose complement is not one of its images under the cyclic permutation (123...n), applied repeatedly, and declare all those images to be winning; the result (if n is even and > 4) is a proper, player-symmetric simple game that is more nearly decisive than simple majority rule.

The question of whether full decisiveness can be achieved symmetrically for a given n is more complicated. According to Isbell [29] (we do not know of any more recent work), the simplest conjecture consistent with the known results is that a player-symmetric rule for breaking all ties exists if and only if n is of the form $c2^k$, with c odd and $c > 2^k$. It is known that $c > (k + 4)/2$ is necessary and that, for infinitely many c, $c > 2^k$ is sufficient. The numbers $n \leq 100$ for which the question remains open (at the time of [29]) are 40, 56, 80, and 88.

to winning. In our notation, these two indices come down to

(57) $$\gamma_i = \eta_i/\omega, \qquad \gamma_i^* = \eta_i/\lambda \ .$$

(Recall our use of ω and λ for the total numbers of winning and losing coalitions, respectively.) There is an obvious duality here, given by $\gamma^*(v) = \gamma(v^*)$; that is, the "power to initiate" in any simple game is the same as the "power to prevent" in its dual (see Sec. 5). The two indices obviously give the same relative distribution of power, and in proper games the "prevent" power is always $\geqq$ the "initiate" power, since $\lambda \geqq \omega$. Both indices lie between 0 and 1, and we have $\gamma_i = 1$ if and only if i has a veto, $\gamma_i^* = 1$ if and only if i is a dictator, and $\gamma_i = \gamma_i^* = 0$ if and only if i is a dummy.

Of the three forms of the Banzhaf index defined in Sec. 2 the Coleman indices relate most closely to β'. Indeed, β' is just the harmonic mean of γ_i and γ_i^*:

$$\frac{1}{\beta_i'} = \frac{1}{2}\left(\frac{1}{\gamma_i} + \frac{1}{\gamma_i^*}\right),$$

as the reader may easily verify. It follows that for proper simple games we always have $\gamma_i^* \leqq \beta_i' \leqq \gamma_i$, with equality if and only if the game is decisive.

Note 6. Robert A. Dahl proposed in a well-known 1957 paper [14] to define the power of one individual over another as the extent to which the first can get the other to do something he would not other-

wise do, "minus" the extent that the second can similarly impose his will on the first. Mathematizing this "directed power" concept with the aid of correlations and conditional probabilities, Dahl proceeded to analyze some empirical political data with very interesting results.

More recently, Michael Allingham [1] set out to apply this notion to abstract voting systems. We quote from his explanation:

> "The group decision is binding on its members, so that the acts on which [the directed] power is defined are the favoring of a proposal (by the power applier) and the subjection to the proposal (by the power receiver--or indeed the whole group). Probability is introduced... so that there is equal probability of any player favoring or opposing the proposal."

Member i's power, over anyone or everyone else in the group, is then defined as the probability of a random proposal being carried when i favors it, less the probability of it being carried when i opposes it. The resulting power index can be written

(58) $$\delta_i = \omega_i/2^{n-1} - \tilde{\omega}_i/2^{n-1},$$

where ω_i and $\tilde{\omega}_i = \omega - \omega_i$ denote the numbers of winning coalitions containing and not containing i, respectively. But this, as Allingham was quick to point out, is nothing but a re-definition of the Banzhaf index; indeed, we have $\delta_i \equiv \beta_i'$ by virtue of the identity $\eta_i \equiv \omega_i - \tilde{\omega}_i$ previously noted in Sec. 2. For further discussion of this "Dahlian" interpretation of the Banzhaf index we refer the reader to [1].

Note 7. There is an extensive literature in electrical engineering on the subject of threshold logic and switching functions.* The latter are functions of the form

$$f: \{0, 1\}^n \rightarrow \{0, 1\},$$

which may be thought of as attaching labels "0" and "1" to the vertices of an n-cube. In switching theory, these labels would be interpreted as "off" and "on", just as in logic (where the functions are called Boolean functions) they are read as "false" and "true" and in simple games as "losing" and "winning". The simple games correspond formally to the positive switching functions (PSFs),** characterized by the condition that $x \geqq y$ always implies $f(x) \geqq f(y)$. The weighted majority games correspond to the positive threshold functions (PTFs); characterized geometrically by the existence of a hyperplane with positive normal that separates the "ons" from the "offs."

The electrical engineers seem to have a practical interest in discovering when a given switching function is a threshold function, and they have compiled exhaustive lists of PTFs for this purpose.***

These lists are arranged for the user's convenience according to certain numerical parameters, introduced by C. K. Chow [11], which

*See for example [76] or [77], where more than a hundred references will be found.

**Also known as monotonic Boolean functions; see Golomb [21].

***Winder in [74] gives what amounts to a table of all decisive weighted majority games with eight or fewer players, together with their Banzhaf indices; there are 2470 of them not counting permutations of players. Muroga, Tsuboi, and Baugh in [43] take up the next case; their computations (10 hours on ILIAC II) indicate that there are 172,958 decisive weighted majority games with nine essential players. They do not actually list them in [43], but give a wealth of statistical information and a clear exposition of the methodology. See also [16], [28], [42], [75], and Appendix B of [76].

are well defined for all switching functions. The user is meant to calculate these parameters for the PSF he is interested in and then enter the table in search of a PTF with the same parameters.

The reader will hardly be surprised to learn that the Chow parameters are related to the Banzhaf indices. In their original form [11] they are just the numbers we have denoted by $\omega_1, \ldots, \omega_n$, and ω--i.e., the number of winning coalitions each player belongs to and the total number of winning coalitions. Subsequent authors have found it convenient to redefine them as the differences $\omega_i - \tilde{\omega}_i$, together with ω, and they appear in the tables in that form. As we have seen, these differences are exactly the raw Banzhaf counts η_i, a fact which makes the tables especially convenient for the game theorist or political scientist interested in this area.

In his original paper, Chow [11] established the remarkable properties of his parameters that justify their role in the table look-up scheme. The central lemma can be stated as follows: If $f \in$ PSF and $g \in$ PTF, and if $\omega_i(f) = \omega_i(g)$ for all i, then either $f = g$ or $\omega(f) > \omega(g)$. From this, it follows that the n numbers $\omega_1(g), \ldots, \omega_n(g)$ characterize the function g among all PTFs, while the n + 1 numbers $\omega_1(g), \ldots, \omega_n(g), \omega(g)$ characterize it among all PSFs. In other words, no two PTFs have identical Chow parameters, even if we ignore the parameter ω, and no non-threshold PSF has all of its Chow parameters identical to those of any PTF.*

*As too often happens in this field, Chow's theorem was independently discovered several years later by Lapidot [34]. who used substantially the same proof. Lapidot called $(\omega_1, \ldots, \omega_n)$ the counting vector of the simple game and used it to improve the upper bound on $|\mathcal{M}(N)|$. He did not, however, make the connection either to switching theory or to the Banzhaf index.

Chow's theorem tells us with the aid of the identity

(59) $$\eta_i \equiv 2\omega_i - \omega$$

that the η_i together with ω serve to distinguish any weighted majority game from all other simple games. But the η_i alone, unlike the ω_i alone, do not suffice to distinguish the weighted majority games from all other weighted majority games. Indeed, this would be too much to expect, since we know that simple games and their duals have the same Banzhaf indices (Theorem 5). But this is the only exception: it can be shown that if v and w are weighted majority games having the same raw swing counts η_i, then either v = w or v = w*. (The proof is a simple extension of Chow's argument.) It follows that there is at most one proper weighted majority game with a given set of η_i. The same holds obviously for the β_i', but not for the β_i.

Many other properties of the Chow parameters have been described in the literature,* and they often translate into interesting facts about the Banzhaf indices. For example, it is apparent from (59) that the raw swing counts η_i are either all even or all odd; moreover, in a decisive game with n > 1 they are all even (since decisiveness implies $\omega = 2^{n-1}$). A less obvious result, attributed in [78] to Ichizo Ninomiya, is that in a decisive game the integers $\eta_i/2$ are themselves either all even or all odd; in others words, the differences $\eta_i - \eta_j$ are all divisible by 4.** Such remarks have an obvious bearing on the problem of constructing a simple game having prescribed power indices.***

*See Winder's survey [78].

**The S-S indices have a similar property. Let π_i - $n!\varphi$ denote the raw pivot count for player i (see Sec. 2 above). Then in any simple game the differences $\pi_i - \pi_j$ are divisible by n, and in any dicisive simple game they are divisible by 2n. For some other simple arithmetic properties of the S-S indices, see Nozick [45].

***See Imrie [51].

Note 8. It is natural to try to generalize from the Banzhaf power index to a Banzhaf value, defined for all games representable by numerical characteristic functions $v: 2^N \to R$. The analogous generalization of the S-S index leads to the well-known Shapley value

$$(60) \qquad \varphi_i[v] = \sum_{S:\ i \in S \subset N} \frac{|S - \{i\}|!\,|N - S|!}{|N|!} \left[v(S) - v(S - \{i\})\right],$$

which may be regarded as an average of player i's marginal contribution to all possible coalitions. Note, however, that sets S of different size get unequal weight in (60). The Banzhaf philosophy of regarding all coalitions as equally likely suggests the following variant of (60) as a candidate for a "Banzhaf value"; note that it reduces directly to β_i' in the case of a simple game:

$$(61) \qquad \beta_i'[v] = \sum_{S:\ i \in S \subset N} \frac{1}{2^{|N-\{i\}|}} \left[v(S) - v(S - \{i\})\right].$$

This definition enjoys the symmetry, dummy, and linearity properties that are traditionally used to axiomatize the Shapley value.* Only the "efficiency" axiom fails, since we have in general $\overline{\beta'[v]} \neq v(N)$, just as we had in general $\overline{\beta'} \neq 1$ for simple games.**

When it is only a matter of measuring power, this failure of the indices to add up the "right" total can be tolerated; indeed, the number $\overline{\beta'}$ (or $\overline{\eta}$) tells us something about the "responsiveness" of the voting system, as we saw in Note 5 above. But a value

*See [62], or Appendix A in [2].

**Cf. the discussion of Axioms A2 and A2' in Section 3 above; also Section 5 of [50].

solution is generally supposed to represent some actual or possible outcome of the game, expressed as a vector of players' payoffs or utilities. Since usually $\overline{\beta'[v]} \neq v(N)$, the evaluation (61) will usually be either too pessimistic or too optimistic, i.e., will correspond to an outcome that is either subefficient and possibly infeasible, or hyperefficient and certainly infeasible.

One can of course recover efficiency by a change of scale, i.e., by extending to (61) the normalization (1) that led us to the index β in the case of simple games. But the conversion formula is quite messy:

$$(62) \qquad \beta_i[v] = \frac{\left[v(N) - \Sigma_j v(\{j\})\right] \beta_i'[v] + \left[\overline{\beta'[v]} - v(N)\right] v(\{i\})}{\overline{\beta'[v]} - \Sigma_j v(\{j\})}.$$

Its very typographical appearance* should warn us that the "efficient Banzhaf value" is not a natural mathematical concept. Indeed, the few authors who have tried to work with the Banzhaf value have tended to prefer the β' form (61).**

*Note that (61) must still be substituted into (62). Normalizing $v(\{i\})= 0$ for all i would simplify the appearance of (62), but would not alter the underlying artificiality of the definition.

**See e.g., Owen [48, 50] and Roth [59].

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