ON THE STABILITY OF A CIRCULAR CYLINDER AT HYPERSONIC SPEEDS

J. L. Raymond
T. B. Garber

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The RAND Corporation
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SUMMARY

This paper deals with a single-degree-of-freedom motion of a rigid right circular cylinder, namely, rotation about any axis normal to the longitudinal axis. The two selected space orientations are such that in the first case the translational velocity vector and the longitudinal axis form the plane of rotation (end-over-end motion), and in the second case the longitudinal axis lies in the plane of rotation which is always perpendicular to the translational velocity vector. The aerodynamic static and damping stability of the cylinder are first derived for later use in the equations of motion for each of the above-mentioned cases. Rotation of the first type (which may be coined in-plane rotation) is non-linear in nature and depends upon the aerodynamic damping characteristics as well as the initial spin rate, while rotation of the second type (coined normal rotation) is linear and logarithmically damped.

CONTENTS

SYMBOLS

A	Maximum cross-sectional area, $\pi d^2/4$, ft^2
C_D	Drag coefficient (D/A q), dimensionless
C_M	Pitching moment coefficient about the center of gravity $\frac{M}{qAd}$, dimensionless
C_N	Normal force coefficient $\frac{N}{qA}$, dimensionless
C_N'	Local normal force per caliber $\frac{dC_N}{d(\frac{x}{d})}$, dimensionless
d	Diameter, feet
F_x	Force in the x direction (downstream), lb.
F_y	Force in the y direction, lb.
I	Moment of inertia of a right circular cylinder about its center of gravity and in the plane perpendicular to the longitudinal axis
K, K_1, K_2, K_3	Constants as defined in the text
ℓ	Cylinder length, ft.
M	Pitching moment about the center of gravity, lb. ft.
M_∞	Free stream Mach No., dimensionless
N	Normal force perpendicular to the longitudinal axis of symmetry
p_∞	Free-stream ambient pressure
q	Dynamic pressure $(1/2)\rho V_\infty^2$), lb. $ft.^{-2}$

V_∞	Free-stream velocity, ft. sec.$^{-1}$
x	Coordinate along the longitudinal axis of symmetry, Figs. 1 and 6, or in the free stream direction, Figs. 2 and 3.
y	Coordinate in the direction perpendicular to x and in the plane defined by x and the rotational axis of symmetry of the cylinder, Figs. 2 and 3.
α	Angle of attack measured from the normal, degrees, Figs. 3, 4 and 5.
α_i	Induced angle of attack due to rotation, radians
θ	Pitch angle, radians, Fig. 1
$\dot{\theta}$	Pitch rate, radians sec.$^{-1}$
ψ	Elevation angle, radians, Fig. 2
ϕ	Azimuth angle, radians, Fig. 2, or roll angle, Fig. 6
ρ	Mass density of air, slugs ft.$^{-3}$

I. INTRODUCTION

Recently, attention has been given to the aerodynamic characteristics of right circular cylinders as, for example, an experimental investigation of the aerodynamic characteristics of low-fineness ratio right circular cylinders whose longitudinal axes are at small angles of attack with respect to the free stream.[1] In general, it was shown that a flat-faced right circular cylinder is not statically stable at small angles of attack in the speed range $0 < M_{\infty} \leq 3.2$, approximately. The present investigation does not consider flat-faced cylinders, but rather hemispherically-faced cylinders. It is shown that at hypersonic speeds, $M_{\infty} > 5$, that the aerodynamic and dynamic properties of the right circular cylinder are indeed somewhat peculiar to this shape and should provoke interest in this perhaps neglected aerodynamic shape.

It must be emphasized that only one degree of freedom in rotation, with two space orientations (with respect to the translational velocity) has been analyzed. There are two other degrees of freedom of rotation possible which could be considered in order to complete the analysis. These are more difficult to handle. Finally, the motion is rectilinear, that is, the flight path is always straight.

II. LONGITUDINAL STABILITY - AERODYNAMICS OF IN-PLANE MOTION

DAMPING-IN PITCH FOR IN-PLANE ROTATION

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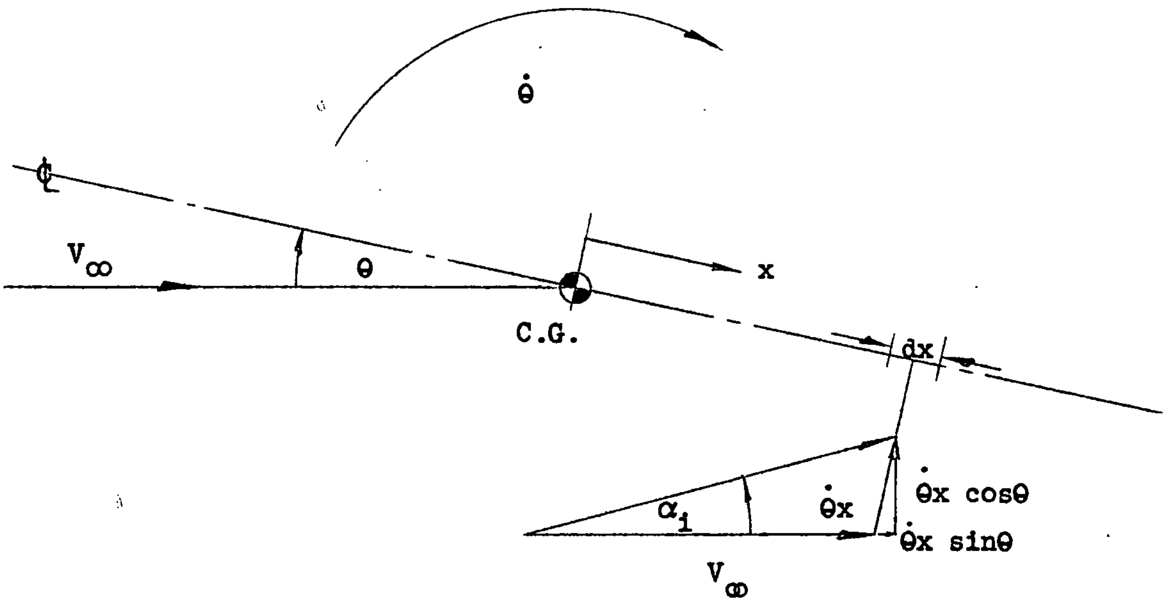

CYLINDRICAL ROD AT ANGLE OF PITCH

Figure 1

The above figure is an idealized sketch of the cylindrical rod whose axis lies at a pitch angle θ and whose pitch rate is $\dot{\theta}$. There is no difference between the usual angle of attack, α, and angle of attack rate $\dot{\alpha}$, and the same parameters in θ, since there is no change in path angle, i.e. the velocity vector does not rotate with time. Accordingly, the development followed is based on the usual pitch and pitch rate concepts but the calculated pitching moment is plotted versus angle of attack measured from the normal since the cylinder is statically stable in the $\theta = 90^{\circ}$ position, see Fig. 5.

The tangent of the induced angle due to steady pitch rate is given by:

$$\tan \alpha_i = \frac{\dot{\theta}\, x \cos \theta}{V_{\infty} + \dot{\theta}\, x \sin \theta} \tag{1}$$

Using the small angle approximation for α_i and recognizing that [redacted] $\dot{\theta}\, x \sin \theta \ll V_{\infty}$, then

$$\alpha_i \doteq \frac{\dot{\theta}\, x \cos \theta}{V_{\infty}} \tag{2}$$

From the Newtonian flow-theory corrected for centrifugal effects, Ref. 2, the local normal force per caliber is

$$C_N' = 1.53 \sin^2 \alpha_i = \frac{dC_N}{d(\frac{x}{d})} \tag{3}$$

so that

$$\frac{dC_N}{dx} = \frac{1}{d} \frac{dC_N}{d(\frac{x}{d})} = \frac{C_N'}{d} \tag{4}$$

The restoring moment per unit length is

$$\frac{dM}{dx} = - x \frac{dN}{dx} = - x \frac{dC_N}{dx} qA = - 1.53 \frac{Aq}{d} x \sin^2 \alpha_i \doteq - 1.53 \frac{Aq}{d} x \alpha_i^2$$

$$\doteq - 1.53 \frac{Aq}{d} x \left[\frac{\dot{\theta}\, x \cos \theta}{V_{\infty}} \right]^2 \tag{5}$$

[redacted]

$$\doteq - 1.53 \frac{Aq}{d} \left[\frac{\dot{\theta} \cos \theta}{V_{\infty}} \right]^2 x^3$$

[redacted]

Hence, within the small angle approximation,

$$dM = -1.53 \frac{Aq}{d} \left[\frac{\dot{\theta} \cos \theta}{V_\infty} \right]^2 x^3 \, dx \quad \text{and}$$

$$M = -1.53 \frac{Aq}{d} \left[\frac{\dot{\theta} \cos \theta}{V_\infty} \right]^2 \int_{-\ell/2}^{\ell/2} x^3 \, dx \tag{6}$$

$$= -3.06 \frac{Aq}{d} \left[\frac{\dot{\theta} \cos \theta}{V_\infty} \right]^2 \frac{x^4}{4} \Bigg|_0^{\ell/2}$$

It must be noted that since x is an even function, integration between $-\ell/2$ and $\ell/2$ for the damping moment would inevitably lead to a zero value, which is not true physically. This is remedied by simply taking twice the integral over half the cylinder length.

Finally

$$M = -.0239 \left(\frac{\rho A \ell^4}{d}\right) \cos^2\theta \, \dot{\theta}^2 \tag{7}$$

So that by definition

$$C_{M_{\dot{\theta}}} = \frac{M}{qAd} \Big/ \frac{\dot{\theta} d}{2V_\infty} = .0956 \left(\frac{\ell^4}{d^3}\right) \frac{\cos^2\theta}{V_\infty} \dot{\theta} \tag{8}$$

The parameter $\dfrac{C_{M_{\dot{\theta}}}}{\dfrac{\ell^4}{d^3} \dfrac{\dot{\theta}}{V_\infty}}$ is plotted in Fig. 2 for a cylinder of finess ratio 20.3 versus normal α. [redacted]

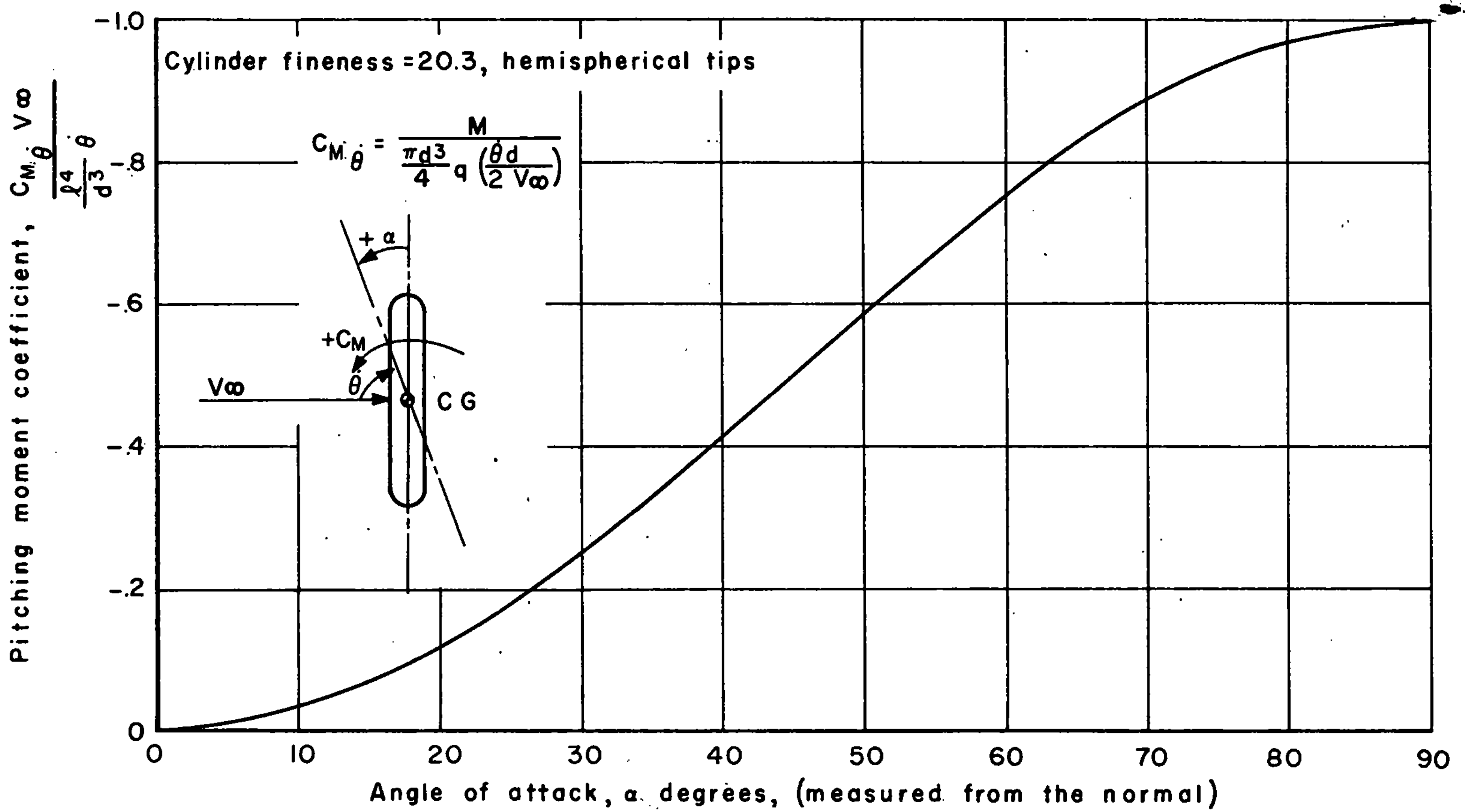

Fig. 2—Pitching moment due to pitch rate vs angle of attack

STATIC PITCHING MOMENT FOR IN-PLANE ROTATION

As a starting point, the pressure coefficient over a sphere was taken from Ref. 2, based on Newtonian flow corrected for centrifugal effects (Case 5, Fig. 18). The pressure distribution over the windward side of the spherical tips was integrated to get the forces and hence moments about the center of gravity of the right circular cylinder. The contribution to moments of the vacuum assumed over the lee side of the spherical tips was neglected and is considered negligible compared to the windward moments. The approximation made here to the variation of pressure coefficient with elevation angle ψ, and azimuth angle ϕ is

$$C_p = 2(1 - K \cos^2 \psi)(1 - K \cos^2 \phi), \text{ where } K = 1.1. \tag{9}$$

The spherical angles ψ and ϕ are illustrated in Fig. 3. The way in which the limits of integration vary for azimuth angle, ϕ, as a function of normal angle of attack is shown in Fig. 4. From Fig. 4 certain advantages can be taken due to the symmetry of the spherical tips. Areas A will have the same drag forces, F_x, and opposed yaw forces, F_y.

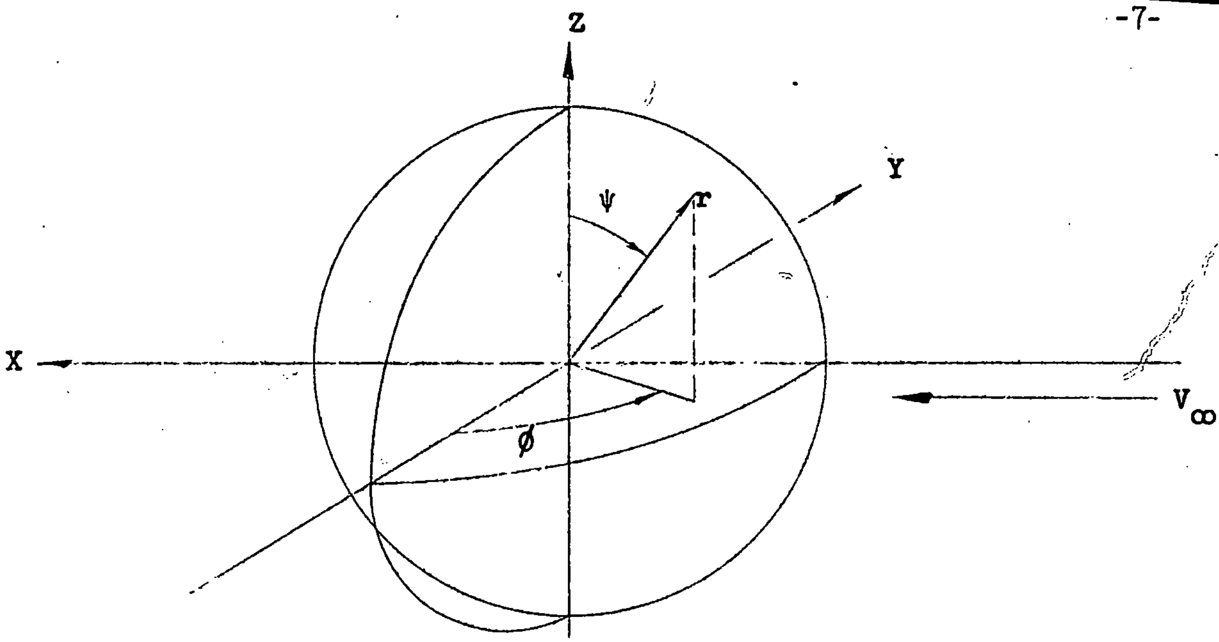

Figure 3

COORDINATE ORIENTATION

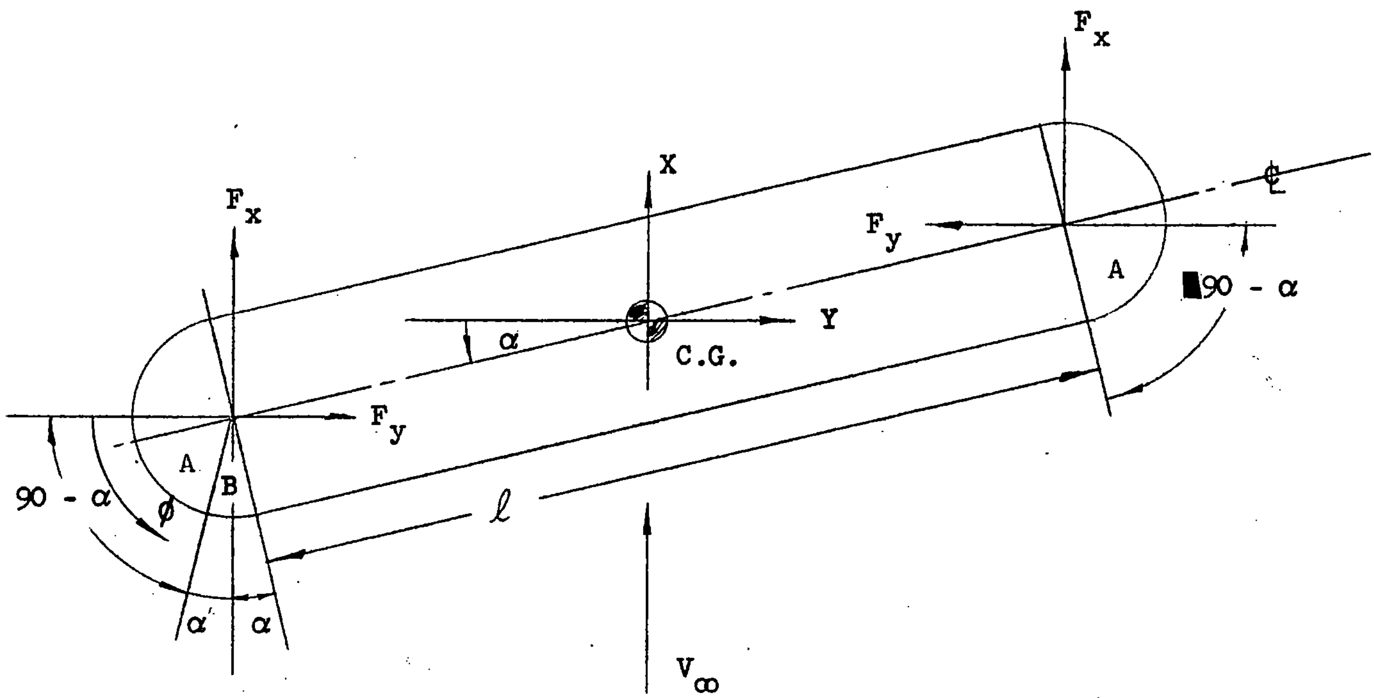

Figure 4

IN-PLANE ROTATION OF A CIRCULAR CYLINDER

From the geometry of the cylindrical rod, it is seen that the drag forces, F_x, will form equal and opposite moments about the center of gravity, while the yaw forces F_y will form the destabilizing couple of magnitude as follows:

$M_{(A)} = F_{y_{(A)}} \ell \sin\alpha$ where F_y is the yaw force on one end only.

Area B is symmetric with respect to the free stream direction, hence no resultant F_y force will appear, but only a drag force, F_x, acting so as to produce a stabilizing moment as given by

$$M_{(B)} = F_{x_{(B)}} \frac{\ell}{2} \cos\alpha$$

Hence

$$M = M_{(A)} + M_{(B)} = F_y \ell \sin\alpha \div \frac{1}{2} F_x \ell \cos\alpha \text{ (dropping A and B subscripts)} \qquad (10)$$

Furthermore, certain simplifications of the integration can be foreseen in that the elevation angle, ψ, will always be in the range $0 \leq \theta \leq 180^\circ$ which leaves only the azimuth angle, ϕ, as a variable, it being a function of α as follows:

Area	Range of ϕ
A	$0 \leq \phi \leq 90 - \alpha$
B	$90 - \alpha \leq \phi \leq 90 + \alpha$

From Figs. 3 and 4, it is seen that

$$F_x = \int_{\phi}\int_{\psi} C_p \, q \sin\psi \sin\phi \, dA \quad \text{and}$$

$$F_y = \int_{\phi}\int_{\psi} C_p \, q \sin\psi \cos\phi \, dA \tag{11}$$

Using Equation (9) and substituting into Eqs. 11, we get

$$F_x = \int_{90-\alpha}^{90^\circ+\alpha}\int_{0}^{180^\circ} 2(1 - K\cos^2\psi - K\cos^2\phi + K^2\cos^2\psi\cos^2\phi)\, q \sin\psi \sin\phi \, r^2 \, d\psi \, d\phi$$

nd $$F_y = \int_{0}^{90^\circ-\alpha}\int_{0}^{180^\circ} 2(1 - K\cos^2\psi - K\cos^2\phi + K^2\cos^2\psi\cos^2\phi)\, q \sin\psi \cos\phi \, r^2 \, d\psi \, d\phi \tag{12}$$

which when integrated between the limits $0 \leq \psi \leq 180^\circ$, become

$$\frac{F_x}{(1 - \frac{K}{3})\, q \, d^2} = \left[\frac{K}{3}\cos^3\phi - \cos\phi\right]_{90-\alpha}^{90^\circ+\alpha}$$

$$\frac{F_y}{(1 - \frac{K}{3})\, q \, d^2} = \left[\sin\phi - \frac{K}{3}\left\{\sin\phi\,(2 + \cos^2\phi)\right\}\right]^{90^\circ\mp\alpha} \tag{13}$$

[redacted]

The static moment of the cylinder was finally calculated by substituting Eqs. 13 into Eq. 10. The variation of C_M vs α is plotted in Fig. 5 for a finess ratio[redacted]of 20.3. The static moment is always stabilizing with respect to the rod position of zero angle of attack, and varies as $- \sin 2\alpha$, approximately. The zero moment slope coefficient is

$$\left(\frac{dC_M}{d\alpha}\right)_{\alpha=0} = -0.219 \tag{13}$$

It must be noted that the moment contribution of the cylindrical portion has been neglected.

III. DYNAMICS OF IN-PLANE ROTATION

For this mode, the equation of motion is

$$I \ddot{\theta} = M_\alpha \alpha + M_{\dot{\theta}} \dot{\theta} \tag{14}$$

It can be seen from Fig. 5, that the moment coefficient C_M varies as $- \sin 2\alpha$ so that the static moment can be expressed as

$$M_\alpha \alpha = C_M Aqd$$

$$= - 6 \sin 2\alpha\, qAd$$

With $\gamma \equiv 0$

$$M_\alpha \alpha = 6 \sin 2\theta qAd \tag{15}$$

It was shown in Eq. 7 that

$$M_{\dot{\theta}} \dot{\theta} = - .0239 \left(\frac{\rho A \ell^4}{d}\right) \cos^2\theta\, \dot{\theta}^2 \tag{16}$$

[redacted]

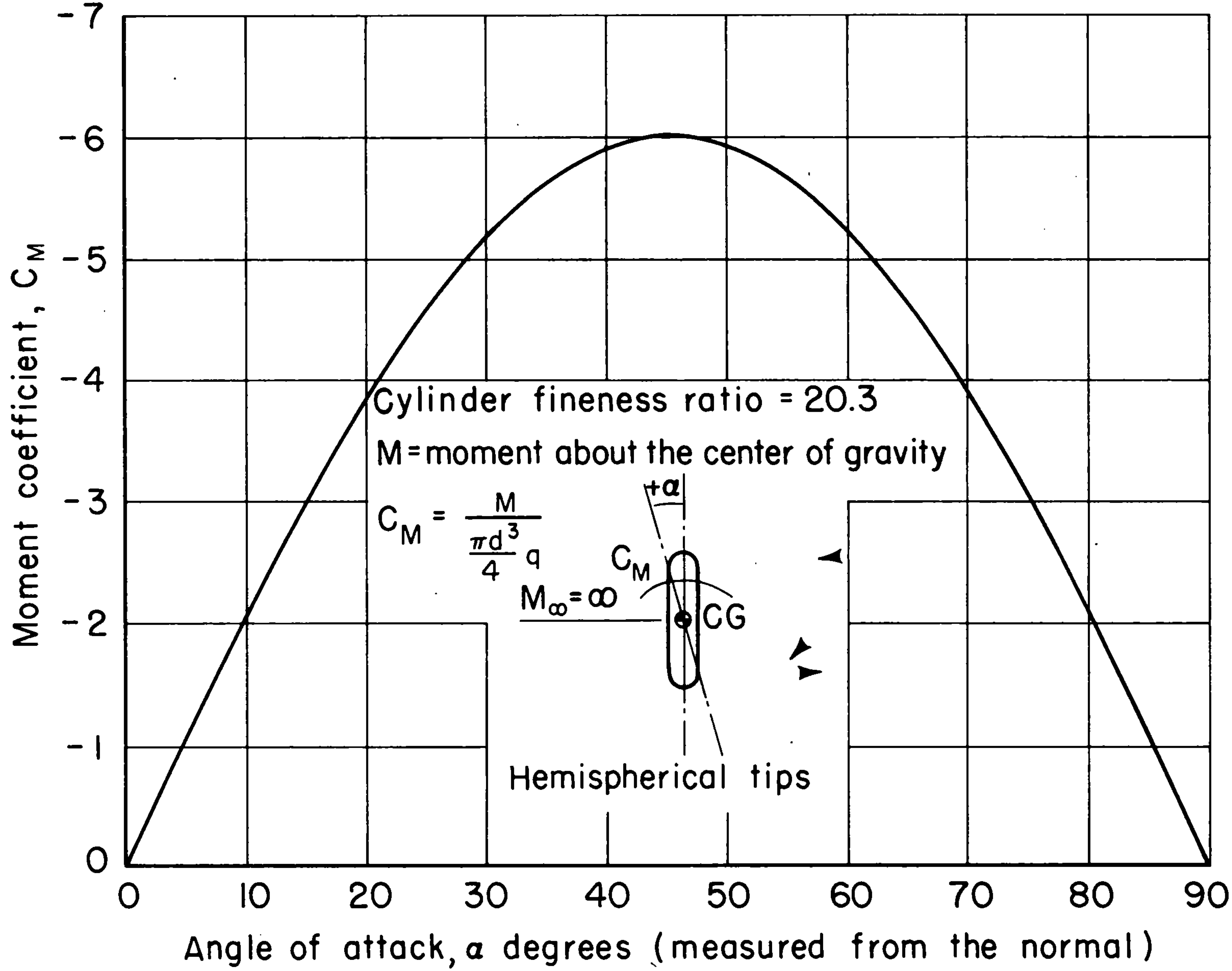

Fig. 5—Pitching moment vs angle of attack

At infinite Mach number

Hence

$$\ddot{\theta} = K_1 \sin 2\theta - K_2 \dot{\theta}^2 \cos^2 \theta \qquad (17)$$

where

$$K_1 = \frac{6qAd}{I}$$

and [redacted]

$$K_2 = .0239 \frac{\rho A \ell^4}{Id} \cdot \frac{\dot{\theta}}{|\dot{\theta}|}$$

The ratio $\dot{\theta}/|\dot{\theta}|$ is included in the definition of K_2 so that the damping moment will always oppose the motion of the rod. For positive values of $\dot{\theta}$, K_2 is a positive constant.

If it is assumed that the velocity vector, V_∞, has no angular [redacted] rate with respect to a fixed reference, then, considering only one rotational degree of freedom (i.e. end-over-end in-plane motion of the rod), Eq. 17 may be written in the following manner by changing the independent variable from t to θ

$$\dot{\theta} \frac{d\dot{\theta}}{d\theta} - K_1 \sin 2\theta + K_2 \dot{\theta}^2 \cos^2 \theta = 0 \qquad (18)$$

Equation 18 may be linearized by the following transformation

$$\dot{\theta}^2 = u;$$

Thus

$$\frac{1}{2} \frac{du}{d\theta} + \left[K_2 \cos^2\theta \right] u = K_1 \sin 2\theta \qquad (19)$$

The solution for $\dot{\theta}^2$ can now be written

$$\dot{\theta}^2 = 2K_1 \, e^{-K_2\left(\theta + \frac{\sin 2\theta}{2}\right)} \int \sin (2\theta) \, e^{+K_2\left(\theta + \frac{\sin 2\theta}{2}\right)} d\theta + C_1 \, e^{-K_2\left(\theta + \frac{\sin 2\theta}{2}\right)}$$

(20)

Let us first consider the undamped case, with K_2 equal to zero.

$$\dot{\theta}^2 = C_1 - K_1 \cos 2\theta \tag{21}$$

For initial conditions at t = 0, let $\theta = \pi/2$

$$\dot{\theta} = \dot{\theta}_o$$

then

$$C_1 = \dot{\theta}_o^{\,2} - K_1 \quad .$$

and

$$\dot{\theta}^2 = \dot{\theta}_o^{\,2} - K_1(1 + \cos 2\theta) \tag{22}$$

Equation 22 points out a salient feature of the rotational motion, that is, it defines the conditions under which the cylinder will rotate rather than oscillate. Since $\dot{\theta}^2$ cannot, from physical considerations, be negative, a minimum must occur with respect to θ. This minimum occurs when θ is equal to 0, π, 2π, etc. Thus the minimum value of $\dot{\theta}_o$ for which rotary motion will occur, for the undamped case, is

$$\dot{\theta}_o \geqq \sqrt{2K_1} \tag{23}$$

The angle θ can be found as a function of time for the undamped case in terms of an elliptic integral:

$$\int \frac{d\theta}{\sqrt{\dot{\theta}_o^{\,2} - K_1(1 + \cos 2\theta)}} = t + C_2 \tag{24}$$

The solution to Eq. 20 is more difficult to obtain when damping is included. However, if the $\frac{\sin 2\theta}{2}$ term which occurs in the exponential is neglected, the integral may be evaluated. This possibility is now examined.

A comparison of the magnitudes of θ and $\frac{\sin 2\theta}{2}$ indicates that the maximum influence of the $\frac{\sin 2\theta}{2}$ term occurs when θ is 45°, since its magnitude alternates between + 1/2 and - 1/2. If K_2, the damping constant, is relatively small, the rod will rotate through large values of θ, and the $\frac{\sin 2\theta}{2}$ term, becomes insignificant. On the basis of this approximation the solution for $\dot{\theta}^2$, including damping, is

$$\dot{\theta}^2 \doteq \frac{2K_1}{(K_2)^2 + 4} (K_2 \sin 2\theta - 2 \cos 2\theta) + C_1 e^{-K_2\theta} \tag{25}$$

with $\theta = \pi/2$, $\dot{\theta} = \dot{\theta}_o$ at $t = 0$

$$\dot{\theta}^2 = \left\{ \dot{\theta}_o^{\,2} - \frac{4K_1}{(K_2)^2 + 4} \right\} e^{K_2(\frac{\pi}{2} - \theta)} + \frac{2K_1\left[K_2 \sin 2\theta - 2 \cos 2\theta\right]}{(K_2)^2 + 4} \tag{26}$$

As in the undamped case, $\dot{\theta}^2$ has a series of minimum values as a function of θ. However, with damping, the value of $\dot{\theta}_o$ which will result in a specified number of revolutions is not constant.

Thus

for $\theta = \pi$

$$(\dot{\theta}_o)_{min} \geqq \sqrt{2K_1} \left\{ \frac{2 + 2 e^{K_2 \frac{\pi}{2}}}{(K_2)^2 + 4} \right\}^{1/2}$$

[redacted]

for $\theta = 2\pi$

$$(\dot{\theta}_o)_{min} \geqq \sqrt{2K_1}\left\{\frac{2 + 2\,e^{\frac{3K_2\pi}{2}}}{(K_2)^2 + 4}\right\}^{1/2}$$

It is to be noted that the position of static equilibrium of the cylinder is at $\alpha = 0$ (or $\theta = \frac{\pi}{2}$). During the course of rotation, the rod must be forced past $\alpha = 90^\circ$ or else it will return to the $\alpha = 0$ position.

In general

$$(\dot{\theta}_o)_{min} = \sqrt{2K_1}\left\{\frac{2 + 2\,e^{\frac{nK_2\pi}{2}}}{(K_2)^2 + 4}\right\}^{1/2} \qquad (27)$$

where n is an odd, positive integer.

IV. ROTATION IN THE NORMAL TO THE FLIGHT PATH PLANE (NORMAL ROTATION)

A brief development of the damping characteristics is presented followed by the dynamic properties of the cylinder for normal rotation.

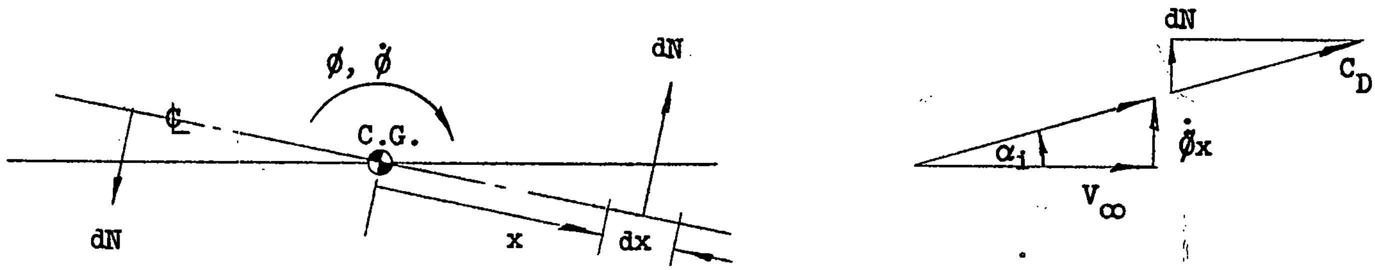

Figure 6

THE VELOCITY VECTOR IS NORMAL TO THE PLANE OF ROTATION

The tangent of the induced angle of attack for this mode is

$$\tan \alpha_i = \frac{\dot{\phi}\, x}{V_\infty} \doteq \alpha_i \doteq \sin \alpha_i \tag{28}$$

then

$$\frac{dN}{dx} = C_D\, qA \frac{d}{dx} (\sin \alpha_i) \tag{29}$$

therefore

$$\frac{dM}{dx} = -x \frac{dN}{dx} = -x\, C_D\, qA \frac{d}{dx} (\sin \alpha_i)$$

$$\doteq -x\, C_D\, qA \frac{d\alpha_i}{dx} \tag{30}$$

Hence

$$M = 2 \int_0^{\ell/2} -x\, C_D\, qA \frac{d}{dx} \left(\frac{\dot{\phi}\, x}{V_\infty}\right) dx$$

$$= -C_D \frac{qA\, \dot{\phi}}{V_\infty}\, x^2 \Big|_0^{\ell/2} = \frac{-C_D\, qA\, \dot{\phi}\, \ell^2}{4\, V_\infty} \tag{31}$$

$C_D = 1.2$ for a cylinder in cross flow at hypersonic speeds

Hence

$$M = -0.30 \frac{qA\, \ell^2\, \dot{\phi}}{V_\infty} \tag{32}$$

The motion for the in-plane mode is then

$$I\ddot{\phi} + 0.30 \frac{qA\,\ell^2}{V_\infty}\dot{\phi} = 0$$

$$\ddot{\phi} + K_3\,\dot{\phi} = 0 \qquad \text{where } K_3 = 0.30 \frac{qA\,\ell^2}{V_\infty} \qquad (33)$$

This is readily solvable for $\dot{\phi}$ as a function of t

$$\dot{\phi} = \dot{\phi}_0\, e^{-K_3 t} \qquad (34)$$

Equation 34 represents a logarithmically damped motion such that the final roll position is arbitrary.

V. REFERENCES

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2. Grimminger, G., E. P. Williams, G. B. W. Young, ''Lift on Inclined Bodies of Revolution in Hypersonic Flow,'' Journal of Aeronautical Sciences, Vol. 17, November 1950, pp. 675-690.