AN AFFINE FIELD DESCRIPTION OF GRAVITATION ELECTROMAGNETISM AND MATTER

Dominic G. B. Edelen
Mathematics Division
The RAND Corporation

P-2100

September 14, 1960

Reproduced by

The RAND Corporation • Santa Monica • California

The views expressed in this paper are not necessarily those of the Corporation

SUMMARY

This paper examins the adequacy of the Affine Field Theory as a descriptive geometric structure for the representation of physical events. It is shown that the theory leads to a derivation of elemental matter from purely geometric considerations in an abstract affinely connected space, and yields equations which describe mesonic phenomena in a "physical" space of four dimensions of the Riemann variety. The meson character arises solely from the interactions of gravitational and electromagnetic fields, where only geometric arguments enter into the Lagrangean function and where the field mass is an integration constant. In addition, it is shown that the theory also describes macroscopic matter, Maxwell-Lorentz electrodynamics, and gravitational fields interacting with uncharged matter in the absence of electromagnetic fields.

CONTENTS

SUMMARY iii

Section

1. INTRODUCTION 1
2. THE BASIC EQUATIONS 3
3. MACROSCOPIC MATTER 7
4. THE QUESTION OF GENERAL COVARIANCE..... 15
5. EXPLICIT FORM OF THE FIELD EQUATIONS 19
6. MAXWELL-LORENTZ ELECTRODYNAMICS 27
7. GRAVITATION IN THE ABSENCE OF ELECTROMAGNETISM 29
8. INTERACTING ELECTROMAGNETIC AND GRAVITATIONAL FIELDS IN THE PRESENCE OF MATTER 33
9. FIELD-FIELD INTERACTION AND THE RESULTING PICTURE OF ELEMENTARY MATTER 39

1. INTRODUCTION

We have presented, in previous communications, the basic equations of the Affine Field Theory and the general form which the components of affine connection may take for fields which are members of the non-null class.* It was shown in I that

*Edelen, D. G. B.: The Affine Theory of Electricity and Gravitation, Archive for Rat. Mech. Anal. 6 (1960).
———————: On the Form of the Affine Connection in Affine Field Theory, to be published.
We shall refer to these communications as I and II and their corresponding equations by I-

affine field theory described vacuum electrodynamics and vacuum gravitation theory by appropriate choices of the Lagrangean function. These considerations are extended in this communication to show that affine field theory also describes macroscopic matter, Maxwell-Lorentz electrodynamics, and gravitational fields interacting with uncharged matter in the absence of electromagnetic fields. In addition, it is shown that when gravitational and electromagnetic fields interact, equations of the vector meson type always result, in which the field mass is an integration constant and the Lagrangean function contains only geometric quantities analogus to the quantities which describe pure fields in previous theories. The resulting vector meson type equations are of a form which may be interpreted as holding a "physical" 4-dimensional space of the Riemann type. Thus, Affine field theory leads to a derivation of elemental matter from purely geometric considerations in a generally

affinely connected space, and yields equations which describe physical phenomena in a physical space of 4-dimensions of the Riemann type. Einstein's principle of general covariance is examined in the light of its epistemological basis and is found to be unacceptable. The principle of Observational Covariance is then formulated and it is shown that the Affine Field Theory identically satisfies this principle for the Lagrangean functions considered.

2. THE BASIC EQUATIONS

The space $\mathcal{E}$ used in I for the development of the affine field equations was a 4-dimensional Hausdorff space into which a structure was introduced by requiring that at every point there by defined a symmetric affine connection $L^{\alpha}_{\beta\gamma}$. The curvature tensor associated with $L^{\alpha}_{\beta\gamma}$ was defined in the usual manner

$$K^{\kappa}_{\lambda\nu\mu} \overset{\text{def}}{=} 2L^{\kappa}_{\lambda[\mu,\nu]} + 2L^{\kappa}_{\rho[\nu}L^{\rho}_{\mu]\lambda}, \tag{2.1}$$

from which the fundamental field tensors

$$A_{\alpha\beta} = -2C_{[\alpha\beta]} = 2L^{\epsilon}_{\epsilon[\beta,\alpha]} , \tag{2.2}$$

$$\begin{aligned} R_{\alpha\beta} = C_{(\alpha\beta)} &= L^{\epsilon}_{\epsilon(\alpha,\beta)} - L^{\epsilon}_{\alpha\beta,\epsilon} \\ &\quad + 2L^{\epsilon}_{\rho[\alpha}L^{\rho}_{\epsilon]\beta} ; \end{aligned} \tag{2.3}$$

were obtained ($C_{\alpha\beta} \overset{\text{def}}{=} K^{\epsilon}_{\alpha\beta\epsilon}$).

Let $\mathcal{L}_F(A_{\alpha\beta}, R_{\alpha\beta}, L^{\alpha}_{\beta\gamma}, Q_{(\rho)}; x)$, the field Lagrangean, be an arbitrary function of its arguments. Extremizing the field action $\mathcal{I}_f = \int_{D_4^*} \mathcal{L}_F \, dv$ with respect to the choice of $L^{\alpha}_{\beta\gamma}$ and $Q_{(\rho)}$, whose variations vanished on the boundary of D_4^* , yielded the following field equations:

$$H^{\alpha\epsilon}_{\ \ ,\epsilon} = S^{\alpha} \tag{2.4}$$

$$I^{\alpha\beta}_{:\gamma} = \frac{2}{5}\,\delta^{(\alpha}_{\gamma} I^{\beta)\epsilon}_{:\epsilon} + T^{\alpha\beta}_{..\gamma} \,, \tag{2.5}$$

$$\left(\mathcal{L}_F, {}_{Q_{(\rho)},\xi}\right), \xi - \mathcal{L}_{F;Q_{(\rho)}} = 0 \,; \tag{2.6}$$

where

$$H^{\alpha\beta} = \mathcal{L}_{F,A_{\alpha\beta}} \,, \tag{2.7}$$

$$I^{\alpha\beta} = \mathcal{L}_{F,R_{\alpha\beta}} \,, \tag{2.8}$$

$$S^{\alpha} = \frac{1}{10}\left(3I^{\alpha\epsilon}_{:\epsilon} - 2\partial_{L^{\epsilon}_{\alpha\epsilon}} \mathcal{L}_F\right) \,, \tag{2.9}$$

$$T^{\alpha\beta}_{..\gamma} = \frac{2}{5}\,\delta^{(\alpha}_{\gamma}\partial_{L^{\epsilon}_{\beta)\epsilon}} \mathcal{L}_F - \partial_{L^{\gamma}_{\alpha\beta}} \mathcal{L}_F \,, \tag{2.10}$$

and

$$I^{\alpha\beta}_{:\gamma} \overset{\text{def}}{=} I^{\alpha\beta}_{,\gamma} + L^{\alpha}_{\rho\gamma} I^{\rho\beta} + L^{\beta}_{\rho\gamma} I^{\alpha\rho} - L^{\rho}_{\rho\gamma} I^{\alpha\beta} \,. \tag{2.11}$$

From the definitions of the various terms given above we have the additional relations

$$L^{\alpha}_{[\beta\gamma]} \equiv 0 \,, \tag{2.12}$$

$$A_{(\alpha\beta)} \equiv 0 \,, \qquad A_{[\alpha\beta,\gamma]} = 0 \tag{2.13}$$

$$R_{[\alpha\beta]} \equiv 0 \,, \tag{2.14}$$

$$H^{(\alpha\beta)} \equiv 0 \,, \tag{2.15}$$

$$I^{[\alpha\beta]} = 0 \ , \tag{2.16}$$

$$T^{[\alpha\beta]}_{\ \ \ \ \ \ \gamma} = 0 \ , \tag{2.17}$$

$$T^{\alpha\beta}_{\ \ \ \ \beta} = T^{\beta\alpha}_{\ \ \ \ \beta} \equiv 0 \ , \tag{2.18}$$

and

$$S^{\alpha}_{\ ,\alpha} = 0 \ . \tag{2.19}$$

The system of equations (2.4 – 19) constituted the affine field theory as presented in I.

In order to discuss matter in a simple and direct manner, we find it convenient to enlarge the variational principle considered in I in the following way. In the next section, we develop the action of matter $\mathcal{I}_M = \int_{D_4^*} \mathcal{L}_M \, dv$, where the matter Lagrangean $\mathcal{L}_M$ is a definite function of $L^{\alpha}_{\beta\gamma}$, the symmetric coefficient tensor $h_{\alpha\beta}$ of the fundamental quadratic form $h_{\alpha\beta} \, dx^{\alpha} \, dx^{\beta}$, and additional quantities particular to the description of matter. Identify $Q_{(\rho)}$ with the tensor $h_{\alpha\beta}$ and consider the combination of matter and field actions as the total action

$$\mathcal{I}_T = \mathcal{I}_M + \mathcal{I}_F = \int_{D_4^*} (\mathcal{L}_M + \mathcal{L}_F) dv = \int_{D_4^*} \mathcal{L}_T \, dv \ . \tag{2.20}$$

Extremizing the total action with respect to $L^{\alpha}_{\beta\gamma}$ and $Q_{(\rho)} = h_{\alpha\beta}$, we obtain the affine field equations (2.4) through (2.6) and the subsidiary equations (2.7) through (2.19) on replacing $\mathscr{L}_F$ by $\mathscr{L}_T$ in equations (2.6, 9, 10). $\mathscr{L}_F$ remains in equations (2.7, 8) since $\mathscr{L}_M$ does not depend on $A_{\alpha\beta}$ and $R_{\alpha\beta}$. In order for $\mathscr{I}_T$ to be extremal we must add to the affine field equations the equations which result from extremizing the total action with respect to the choice of the quantities particular to the description of matter, where the variations of these quantities vanish on the boundary of D_4^* in conformity with the boundary requirements on the variations of the $L^{\alpha}_{\beta\gamma}$ and the $Q_{(\rho)} = h_{\alpha\beta}$. As is evident from the assumed functional form of $\mathscr{L}_F$, this latter extremization of $\mathscr{I}_T$ is equivalent to the extremization of $\mathscr{I}_M$.

It must be noted that the above extension of the affine field theory is introduced here solely for purposes of simplicity and directness, in that it yields Einstein gravitational theory, Maxwell–Lorentz electrodynamics, and Macroscopic matter representations in an easily recognizable and familiar form. We shall see, however, that considerations of just the field action alone offers a possible representation of (microscopic) matter, gravitation, and electrodynamics.

3. MACROSCOPIC MATTER

We envision macroscopic matter as a substance which forms a 3-dimensional continuum $\mathcal{E}_m$ of the 4-dimensional ground space $\mathcal{E}$ for any particular value of the "time" coordinate. Hence it may be referred to a system of coordinate values ξ, ζ, ν in a continuous manner. The intrinsic matter properties of such a continuum are described by two invariable distributions $\underline{dm} > 0$ and $\underline{de}$ defined over $\mathcal{E}_m$. These distributions will be referred to as the mass and charge distributions respectively. The "history" of matter is expressed by the fiber bundle of "world tubes" generated by a sufficiently fine neighborhood topology of $\mathcal{E}_m$.

We take for the definition of an infinitesimal length of a world tube (infinitesimal proper time) the expression*

$$ds = \sqrt{h_{\alpha\beta}dx^{\alpha}dx^{\beta}} \quad , \tag{3.1}$$

---

* $h_{\alpha\beta}dx^{2}dx^{\beta}$ is assumed positive in a timelike direction.

---

where $h_{\alpha\beta}$ is the symmetric coefficient tensor of the fundamental quadratic form of $\mathcal{E}$ and where x^{α} are the coordinates of a point in $\mathcal{E}$ which happens to coincide with the matter point ξ, ζ, ν at the moment $\underline{s}$ in its history. The "time" $\underline{s}$ is calculated for each world tube from the point in $\mathcal{E}$ corresponding to ξ, ζ, ν (i.e., the particular value of the time used in labeling $\mathcal{E}_m$ with the coordinates ξ, ζ, ν is $s = 0$).

Let Ω_m be the world tube which is generated from an infinitesimal element of $\mathcal{E}_m$ with mass distribution dm and charge distribution de. We take as the action of mass the expression

$$+ \int_{\mathcal{E}_m} dm \int_{\Omega_m} ds$$

and as the action of electricity the expression*

$$+ \int_{\mathcal{E}_m} de \int_{\Omega_m} L^{\mu}_{\mu\alpha} dx^{\alpha} \quad .$$

*The reason for $L^{\mu}_{\mu\alpha}$ instead of the vector potential of the electromagnetic field appearing in this expression is twofold: first, there is no vector potential defined at this point; and second, we have shown in I–Section 4 that $L^{\mu}_{\mu\alpha}$ possesses the operational properties of the vector potential in the affine field theory without the associated transformation properties, so that one is able to choose an arbitrary reference point for the null point of the potential.

The action of matter is thus given by

$$\mathcal{I}_M = \int_{\mathcal{E}_m} \left\{ dm \int_{\Omega_m} ds + de \int_{\Omega_m} L^{\mu}_{\mu\alpha} dx^{\alpha} \right\} \qquad (3.2)$$

Equation (3.2) may be expressed as a volumn integral over $\mathcal{E}$ in the following manner. Let μ_0 be the mass density function of $\mathcal{E}_m$, then

$$dm\ ds = \mu_0\ d\xi\ d\zeta\ d\nu\ ds = \mu_0\ d\xi\ d\zeta\ d\nu (ds)^2/ds$$

$$= \mu_0\ d\xi\ d\zeta\ d\nu\ h_{\alpha\beta}\ dx^\alpha\ dx^\beta/ds$$

$$= \mu_0\ h_{\alpha\beta}\ u^\alpha\ u^\beta\ d\xi\ d\zeta\ d\nu\ ds,$$

where $u^\alpha = dx^\alpha/ds$ is the instantaneous tangent vector to the world tube Ω_m. Since ξ, ζ, ν, s forms a continuous, and we shall also assume, a differentiable coordinate covering of $\mathcal{E}$, we have

$$\mu_0\ d\xi\ d\zeta\ d\nu\ ds = \mu_0 \left| \det \left(\frac{\partial(\xi,\zeta,\nu,s)}{\partial(x^1,x^2,x^3,x^4)} \right) \right| dv = \mu dv$$

where μ is the world mass density function. Thus

$$\int_{\mathcal{E}_m} dm \int_{\Omega_m} ds = \int_{D_4^*} \mu h_{\alpha\beta}\ u^\alpha\ u^\beta\ dv\ .$$

Similarly

$$\int_{\mathcal{E}_m} de \int_{\Omega_m} L^\mu_{\mu\alpha}\ dx^\alpha = \int_{\mathcal{E}_m} de \int_{\Omega_m} L^\mu_{\mu\alpha}\ u^\alpha\ ds =$$

$$= \int_{\mathcal{E}_m \times \Omega_m} \rho_0\ L^\mu_{\mu\alpha}\ u^\alpha\ d\xi\ d\zeta\ d\nu\ ds$$

$$= \int_{D_4^*} \rho_0 u^\alpha L^\mu_{\mu\alpha} \left| \det \left(\frac{\partial(\xi,\zeta,\nu,s)}{\partial(x^1,x^2,x^3,x^4)} \right) \right| dv =$$

$$= \int_{D_4^*} \rho u^\alpha\ L^\mu_{\mu\alpha}\ dv\ ,$$

where ρ is the world charge density function. We thus obtain the following restatement of the action of matter which will be required in later sections

$$\mathcal{I}_M = \int_{D_4^*} \left\{ \mu h_{\alpha\beta} u^\alpha u^\beta + \rho u^\alpha L^\mu_{\mu\alpha} \right\} dv = \int_{D_4^*} \mathcal{L}_M \, dv \, . \qquad (3.3)$$

In order to obtain the equations of motion of the matter continuum (equations of the world tubes) and to extremize the total action, we must extremize the functional $\mathcal{I}_M$ with respect to the choice of the world tubes. Denote the operation of varying the world tubes by δ_w, then

$$\delta_w \int de \int_{\Omega_m} L^\mu_{\mu\alpha} \, dx^\alpha = \int_{\mathcal{E}_m} de \, \delta_w \int_{\Omega_m} L^\mu_{\mu\alpha} \, dx^\alpha$$

since only the world tubes are to be varied, not the intrinsic properties of matter or space. Now

$$\begin{aligned} \delta_w \int_{\Omega_m} L^\mu_{\mu\alpha} \, dx^\alpha &= \int_{\Omega_m} (dx^\alpha \, \delta_w L^\mu_{\mu\alpha} + L^\mu_{\mu\alpha} \, d \, \delta_w x^\alpha) \\ &= \int_{\Omega_m} (dx^\alpha \, \delta_w L^\mu_{\mu\alpha} - dL^\mu_{\mu\alpha} \, \delta_w x^\alpha) \\ &\quad + \int_{\Omega_m} d(L^\mu_{\mu\alpha} \, \delta_w x^\alpha). \end{aligned}$$

Thus, since $\delta_w x^\alpha$ vanishes at the extremities of the world tube (the initial and final states of the continuum are not to be varied in accordance with the vanishing of variations on the

boundary of D_4^*), we have

$$\overset{\delta}{w}\int_{\Omega_m} L^{\mu}_{\mu\alpha}\, dx^{\alpha} = \int_{\Omega_m} (L^{\mu}_{\mu\alpha,\beta} - L^{\mu}_{\mu\beta,\alpha})\ \overset{\delta}{w}x^{\beta}\, dx^{\alpha} \qquad (3.4)$$

upon noting that

$$dL^{\mu}_{\mu\alpha} = L^{\mu}_{\mu\alpha,\beta}\, dx^{\beta}$$

$$\overset{\delta}{w}L^{\mu}_{\mu\beta} = L^{\mu}_{\mu B,\alpha}\ \overset{\delta}{w}x^{\alpha} \quad .$$

The last equation above results from the fact that $L^{\mu}_{\mu\alpha}$ only changes under $\overset{\delta}{w}$ by the change induced in evaluating $L^{\mu}_{\mu\alpha}$ at the varied world point. Similarly

$$\overset{\delta}{w}\int_{\mathcal{E}_m} dm \int_{\Omega_m} ds = \int_{\mathcal{E}_m} dm\ \overset{\delta}{w}\int_{\Omega_m} ds \ .$$

Setting

$$Q = h_{\alpha\beta}\, u^{\alpha}\, u^{\beta} \ ,$$

we have

$$\overset{\delta}{w}\int_{\Omega_m} ds = \overset{\delta}{w}\int_{\Omega_m} \sqrt{Q}\, ds = \frac{1}{2}\int_{\Omega_m} \overset{\delta}{w} Q\, ds \qquad (3.5)$$

$$= \frac{1}{2}\int_{\Omega_m} \left\{ + h_{\alpha\beta,\gamma} u^{\alpha} u^{\beta}\ \overset{\delta}{w}\, x^{\gamma} + 2h_{\alpha\beta} u^{\alpha}\ \overset{\delta}{w}\, u^{\beta} \right\} ds$$

$$= \int_{\Omega_m} -\left\{ h_{\alpha\beta}\, \frac{du^{\alpha}}{ds} + h_{\lambda\beta}\, \Gamma^{\lambda}_{\mu\nu}(h_{\xi\zeta})\ u^{\mu} u^{\nu} \right\} \overset{\delta}{w}x^{\beta}\, ds,$$

where $h_{\beta[\alpha,\gamma]}u^{\alpha}\,u^{\gamma} \equiv 0$ and the vanishing of $\overset{\delta}{w}x^{\beta}$ on the extremities of the world tube have been used.* Combining equations (3.4)

*$\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}(h_{\xi\zeta})$ stands for the Christoffel symbols formed from the $h_{\xi\zeta}$ field.

and (3.5) and converting the integration over the world tubes and over $\mathcal{E}_m$ to a volume integration over $\mathcal{E}$, we obtain

$$\overset{\delta}{w}\mathcal{J}_M = \int_{D_4^*} \left\{ -\mu \left(h_{\alpha\beta}\frac{du^{\alpha}}{ds} + h_{\lambda\beta}\,\Gamma^{\lambda}_{\mu\nu}(h_{\xi\zeta})\,u^{\mu}\,u^{\nu} \right) + 2\rho\, L^{\mu}_{\mu[\alpha,\beta]}\,u^{\alpha} \right\} \overset{\delta}{w}x^{\beta}\,dv \quad .$$

Thus, since $2L^{\mu}_{\mu[\alpha,\beta]} = A_{\beta\alpha}$ by equation (2.2), we have upon annuling $\overset{\delta}{w}\mathcal{J}_M$, multiplying the resulting equations through by $h^{\beta\sigma}$, and relabeling indices

$$\mu\frac{du^{\alpha}}{ds} + \mu\,\Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})\;\; u^{\beta}\,u^{\gamma} - \rho h^{\alpha\sigma}\,A_{\sigma\nu}\,u^{\nu} = 0. \quad (3.6)$$

Equations (3.6) are the "equations of motion" for the matter continuum. By this we mean each world tube of the fiber bundle' generated by $\mathcal{E}_m$ satisfies equations (3.6).

There are several significant conclusions which may be drawn from equations (3.6). First, if the matter continuum is made up of sufficiently small and discreet "lumps," we may interpret equations (3.6) as the equations of motion for a lump

where $\mu = m_0$, is identified with the mass of the lump and $\rho = e_0$ identified with the charge of the lump. In this case equations (3.6) are just those equations which describe the motion of a classic particle of mass m_0 and charge e_0 in the presence of an electromagnetic field with field tensor $-A_{\alpha\beta}$ and a gravitational field with potential tensor $h_{\alpha\beta}$.

Second, we tentatively identified the tensor field $A_{\alpha\beta}$ with the electromagnetic field tensor in Section 4 of I, on noting that $A_{\alpha\beta}$ has the operational properties of an electromagnetic field tensor (was of Maxwellian form), and corroborated this interpretation in Section 8 of I by showing that the solution manifold of the affine field equations in a proper semi-empty space was isomorphic with the solution manifold of vacuum electordynamics with $A_{\alpha\beta}$ as the field tensor. Since we did not have an expression for the pondromotive force which the electromagnetic field exerts on a charged particle in I, we could actually state no more than that $A_{\alpha\beta}$ was proportional to the electromagnetic field tensor, where the factor of proportionality was a constant. Equation (3.6) now shows that the $A_{\alpha\beta}$ field is just the negative of the electromagnetic field tensor, since the last term in equation (3.6) is an expression for the pondromotive force generated by the $A_{\alpha\beta}$ and differs from the accepted expression for the pondromotive force by the factor of -1. The factor of proportionality being -1 is not a surprising result since equation (2.2) states that $A_{\alpha\beta}$ is the negative of the curl of $L^{\mu}_{\mu\alpha}$, the latter quantity having been identified with a potential form for the electromagnetic field in I—Section 4.

The most important and final conclusion is obtained as follows: Let the Lagrangean function for the fields be an arbitrary function of class C^2 whose transformation properties under coordinate transformations of ξ are unspecified. Under this assumption the affine field equations will not be tensorial equations and hence will not be covariant with respect to coordinate transformations. The world tube equations, however, are tensorial equations and thus covariant, as is evident upon writing equations (3.6) in the equivalent form

$$\mu u^{\beta} \nabla_{\beta} u^{\alpha} - \rho h^{\alpha\sigma} A_{\sigma\nu} u^{\nu} = 0 \qquad (3.7)$$

(∇_{β} stands for the covariant derivative formed from the affinity $\Gamma^{\alpha}_{\beta\gamma}$ $(h_{\xi\zeta})$) and noting that $A_{\alpha\beta}$ is a tensor regardless of whether the affine field equations are tensorial or not. Thus, even though the affine field equations be non-tensorial, the interactions of the resulting fields with the matter continuum assumes the covariant form (3.7).

4. THE QUESTION OF GENERAL COVARIANCE

Based on epistemological arguments in which acceptance or rejection of hypotheses rested on "observable facts of experience," Einstein stated the following principle of general covariance: "The general laws of Nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any (coordinate) substitutions whatever (generally covariant)."* This principle was interpreted by Einstein and his associated to imply that the equations

*Einstein, A.: Die Grundlage der allgemeinen Relativitätstheorie, Annalen der Physik 49 (1916), Section 3.

which describe fields as well as those which describe matter must be tensorial equations.

The statement that both the equations which describe fields and the equations which describe matter must be tensorial equations appears to be in error if one goes back to the basic epistemological criteria upon which the principle of general covariance is founded, namely: "observable facts of experience." The most which is required by this epistemological criteria is that observable quantities are to be expressed by equations which hold good in all systems of coordinates. We shall refer to this statement as The Principle of Observational Covariance. To require anything more than this must rest ultimately on ad hoc assumptions or on an expansion of the epistemological criteria. The requirement that both the field equations and

the matter equations be tensorial equations is easily seen to be a sufficient condition for the satisfaction of the principle of observational covariance. It becomes, however, a necessary condition only if both the fields and matter are observable quantities.

Matter, per sè, is a fortiori observable on a macroscopic level, in that it forms the basis of our known facts of experience, and we know how to make observations of matter in the laboratory. Such observations are represented mathematically by describing the matter world tubes. When we say that we observe fields, however, what we mean is that we observe matter which is being acted upon by something which we term a field (i.e., we observe the motion of test particles). We do not observe the fields themselves, but only their effects on test particles. In actuality, what we basically mean by fields, and the way in which we fundamentally define fields is in terms of their interactions with test particles. When we write down systems of equations to "describe" fields, what is meant is that these equations describe quantities which determine the magnitude, direction, etc. of the interactions of a something termed fields, whose presence results in certain characteristic changes in the matter world tubes. From the "observable facts of experience" criteria, fields thus have no a fortiori existential properties in themselves, but rather are abstractions arrived at by cata-gorization of the types and natures of interactions between matter and the something which we term fields. This becomes

even more evident if we anticipate the results of future sections and note that the fields themselves are sourced by and hence result from the aggregate of matter existent in space, at least with respect to Einsteinean gravitation theory and Maxwell-Lorentz electrodynamics. To satisfy the principle of observational covariance, all that need be required is that the equations which describe the interactions of fields with matter be covariant, but not that the field equations themselves necessarily have such properties. In other words, general covariance of field equations is not a necessary condition for conformity with the principle of observational covariance. The necessary condition is that only the field-matter interactions must be covariantly formulated. We therefore must reject on the basis of the observable facts of experience the principle of general covariance as interpreted by Einstein in terms of requiring both matter and fields to be covariantly formulated, and replace it by the principle of observational covariance with its weaker requirement of covariant descriptions of matter and matter-field interactions.

We showed, in the close of the last section, that the equations which describe macroscopic matter interacting with the fields described by the affine field equations were always covariant for $\mathcal{I}_m$ given by equation (3.2) even though the affine field equations be not necessarily covariant. We may thus conclude that affine field theory with $\mathcal{I}_m$ given by equation (3.2) satisfies the principle of observational covariance automatically. If the affine field equations happen to be tensorial it is a

fortuitous situation and can greatly simplify the analysis, as will be seen below. It must be noted, however, this need not be the case in order for the affine field theory to be in conformity with the principle of observational covariance, since this conformity is automatically satisfied under equation (3.2).

It has often been stated that a field theory must be a covariant theory, for if not, one is unable to obtain the field identities which play such a central role in general relativity and the recent work in unified field theory. We have shown, however, that under the Principle of Extremal Invariance* one

---

*Edelen, Dominic G. B.: The Invariance Group for Hamiltonian Systems of Partial Differential Equations, I. Analysis, Archive for Rat. Mech. Anal. 5 (1960) Section 7.

---

is able to define systems of coordinate transformations which are naturally associated with the field equations in such a way that the field equations remain invariant under such transformations. The geometry of the ground space defined by such transformation groups plays an analogous role to the field identities, with the added property that the conservation laws have real information content rather than simply state identities.*

---

*Fletcher, J. G.: Local Conservation Laws in Generally Covariant Theories, Rev. Mod. Phys. 32, No. 1 (January, 1960).

---

5. EXPLICIT FORM OF THE FIELD EQUATIONS

The field Lagrangean considered in the remainder of this communication is the scalar density

$$\mathcal{L}_F = \sqrt{-h}\left\{\Lambda + a h^{\alpha\beta} R_{\alpha\beta} - \frac{1}{4} b h^{\alpha\delta} h^{\beta\gamma} A_{\delta\gamma}\right\} . \tag{5.1}$$

The quantities a, b, and Λ are constants, $h_{\alpha\beta}$ is the coefficient tensor of the fundamental quadratic form, and

$$h = \det (h_{\alpha\beta}) < 0 \tag{5.2}$$

$$h h^{\alpha\beta} = h_{,h_{\alpha\beta}} \quad h_{[\alpha\beta]} \equiv 0 \tag{5.3}$$

are assumed to hold. Combining equations (3.3) and (5.1), as prescribed by equation (2.20), we obtain the following form for the total Lagrangean

$$\begin{aligned}\mathcal{L}_T = \sqrt{-h}\left\{\Lambda + a h^{\alpha\beta} R_{\alpha\beta} - \frac{1}{4} b h^{\alpha\phi} h^{\beta\gamma} A_{\alpha\beta} A_{\delta\gamma}\right\} \\ + \mu h_{\alpha\beta} u^\alpha u^\beta + \rho u^\alpha L^\epsilon_{\epsilon\alpha} .\end{aligned} \tag{5.4}$$

It is evident, upon inspection, that every term in $\mathcal{L}_T$ with the exception of the last is a scalar density. The lack of invariance of the last term of $\mathcal{L}_T$ gives no trouble, however, and will be eliminated at the close of this section by subtracting from $\mathcal{L}_T$ a term of the form $(W^\alpha)_{,\alpha}$ provided the Lorentz gauge condition is satisfied. The subtraction of such a form from $\mathcal{L}_T$ will not effect the field equations since the variation of such an expression vanishes identically under the condition that all variations vanish on the boundary of D_4^*.*

*Edelen, D. G. B.: The Invariance Group for Hamiltonian Systems of Partial Differential Equations, I. Analysis, Archive for Rat. Mech. Anal. 5, No. 2 (1960).

With the total Lagrangean, we may explicitly evaluate the subsidiary quantities $H^{\alpha\beta}$, $I^{\alpha\beta}$, S^{α}, and $T^{\alpha\beta}_{\cdot\cdot\gamma}$ in terms of the fundamental variables $h_{\alpha\beta}$, $A_{\alpha\beta}$, $R_{\alpha\beta}$, and u^{α}. Substituting equation (5.4) into equations (2.7) and (2.8) yields

$$I^{\alpha\beta} = a\sqrt{-h}\, h^{\alpha\beta} \tag{5.5}$$

and

$$H^{\alpha\beta} = -\frac{1}{2}\, b\sqrt{-h}\, A_{\delta\gamma}\, h^{\alpha\delta}\, h^{\beta\gamma} \,. \tag{5.6}$$

Since $h_{\alpha\beta}$ and $A_{\alpha\beta}$ are tensors, we see that $I^{\alpha\beta}$ and $H^{\alpha\beta}$ are contravariant tensor densities of rank 2 and weight one. Thus, the colon derivative defined by equation (2.11) becomes the covariant derivative formed from the affinity $L^{\alpha}_{\beta\gamma}$.

The tensors $h_{\alpha\beta}$ and $h^{\alpha\beta}$ will be used to raise and lower indices in the usual manner. Equation (5.6) may then be written as

$$H^{\alpha\beta} = -\frac{1}{2}\, b\, A^{\alpha\beta}_{\cdot\cdot}\sqrt{-h} \quad , \tag{5.7}$$

where

$$A^{\alpha\beta}_{\cdot\cdot} = A_{\delta\gamma}\, h^{\alpha\delta} h^{\beta\gamma} \,.$$

It must be noted that the operations of raising and lowering indices commutes with the operation of covariant differentiation only if $h_{\alpha\beta;\gamma} = 0$
(i.e.: $W^{\cdot}_{\beta;\gamma} = (W^{\alpha}\, h_{\alpha\beta})_{;\gamma} = h_{\alpha\beta}\, W^{\alpha}_{;\gamma} + W^{\alpha}\, h_{\alpha\beta;\gamma}$) .

Using equation (5.4), we have

$$\mathcal{L}_{T,L^{\alpha}_{\beta\gamma}} = \rho u^{(\beta}\delta^{\gamma)}_{\alpha} \tag{5.8}$$

so that

$$\mathcal{L}_{T,L^{\epsilon}_{\alpha\epsilon}} = \frac{5}{2}\,\rho u^{\alpha}\ . \blacksquare \tag{5.9}$$

Substituting equations (5.8,9) into equations (2.9,10) gives

$$S^{\alpha} = \frac{3}{10}\, I^{\alpha\epsilon}_{\ ;\epsilon} - \frac{\rho}{2}\, u^{\alpha} = \frac{3}{10}\,(a\sqrt{-h}\, h^{\alpha\beta})_{;\beta} - \frac{\rho}{2}\, u^{\alpha} \tag{5.10}$$

and

$$T^{\alpha\beta}_{\ \ .\gamma} \equiv 0\ . \tag{5.11}$$

The affine field equations (2.4,5) thus read

$$\left(-\frac{1}{2}\, b\sqrt{-h}\, A^{\alpha\beta}_{\ \ .}\right)_{,\beta} = \frac{3}{10}\left(a\sqrt{-h}\, h^{\alpha\beta}\right)_{;\beta} - \frac{\rho}{2}\, u^{\alpha} \tag{5.12}$$

$$\left(a\sqrt{-h}\, h^{\alpha\beta}\right)_{;\gamma} = \frac{2}{5}\,\delta^{(\alpha}_{\gamma}\left(a\sqrt{-h}\, h^{\beta)\epsilon}\right)_{;\epsilon} \tag{5.13}$$

The continuation from this point requires the results established in II. In order to apply these results we must determine whether the field $I^{\alpha\beta}$ is a member of the non-null class, the proper null class, or the improper null class. These classes were defined, respectively, as the classes of $I^{\alpha\beta}$ fields for which the determinant of $I^{\alpha\beta}$ does not vanish, vanishes because $I^{\alpha\beta} = 0$, or vanishes with $I^{\alpha\beta} \neq 0$ for some (α, β).* From equation (5.5) we have

---

*cf. II Section 3

$$\det (I^{\alpha\beta}) = a^4 h \ .$$

Thus, since we have assumed h to be non-zero, $I^{\alpha\beta}$ is a member of the non-null class if $a \neq 0$ and a member of the proper null class if $a = 0$. Hence, we must distinguish between these two cases.

Case 1: $a \neq 0$. For this case $I^{\alpha\beta}$ is a member of the non-null class. In addition, we have seen that $I^{\alpha\beta}$ is a tensor density of rank two and weight one. The results of II-Section 6 are thereby directly applicable. In fact, comparing equations (5.5) and (II-6.1) we see that they are identical. The field equations (5.12) are thus equivalent to the system of equations*

---

*cf. II-6.7

$$h_{\alpha\beta;\gamma} = M_\gamma \, h_{\alpha\beta} - 2h_{\gamma(\alpha} M_{\beta)} \ , \qquad (5.14)$$

where M_γ is an arbitrary vector. Equation (5.14) is readily solvable for $L^\alpha_{\beta\gamma}$ yielding (cf. II-6.5)

$$L^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}(h_{\xi\zeta}) - \frac{1}{2}\left[M_\gamma \, \delta^\alpha_\beta + M_\beta \, \delta^\alpha_\gamma - 3M^\alpha_{\cdot} \, h_{\beta\gamma}\right] \qquad (5.15)$$

as the general form of the affinity in this case. We also have by equations (II-6.9,10)

$$R_{\mu\lambda} = J_{\mu\lambda} + 3h_{\mu\lambda} \nabla_\rho M^\rho_{\cdot} - \frac{3}{2} M_\mu \, M_\lambda \qquad (5.16)$$

and

$$A_{\mu\lambda} = 2M_{[\mu,\lambda]} \ , \tag{5.17}$$

where $J_{\mu\lambda}$ is the contraction of the curvature tensor formed from the affinity $\Gamma^{\alpha}_{\beta\gamma}\ (h_{\xi\zeta})$ and ∇_{μ} stands for the covariant derivative formed from $\Gamma^{\alpha}_{\beta\gamma}\ (h_{\xi\zeta})$. Thus, by equations (I-4.5) and (5.17), M_{α} is related to the vector potential f_{α} of the Maxwellian field by the equation

$$M_{\alpha} = \frac{1}{2} f_{\alpha} \ . \tag{5.18}$$

We may use equation (5.18) to rewrite the above equations in terms of the vector potential f_{α} which will be required later:

$$h_{\alpha\beta;\gamma} = \frac{1}{2} f_{\gamma} h_{\alpha\beta} - h_{\gamma(\alpha} f_{\beta)} \ , \tag{5.19}$$

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}\ (h_{\xi\zeta}) - \frac{1}{4}\left[f_{\gamma}\,\delta^{\alpha}_{\beta} + f_{\beta}\,\delta^{\alpha}_{\gamma} - 3f^{\alpha}_{\cdot}\,h_{\beta\gamma}\right], \tag{5.20}$$

$$R_{\mu\lambda} = J_{\mu\lambda} + \frac{3}{2} h_{\mu\lambda}\,\nabla_{\rho}\,f^{\rho}_{\cdot} - \frac{3}{8} f_{\mu}\,f_{\lambda} \ , \tag{5.21}$$

$$A_{\mu\lambda} = f_{[\mu,\lambda]} \ , \tag{5.22}$$

With $L^{\alpha}_{\beta\gamma}$ given by equation (5.20), the field equations (5.12) are <u>algebraically</u> satisfied in terms of the 14 unknown functions $h_{\alpha\beta}$ and f_{α} . There are, however, exactly 14 remaining field equations, the 4 equations (5.12) and the 10 equations (2.6)* for the determination of the 14 unknown functions. To

*There are only 10 independent equations in the set (2.6) since we have identified $Q_{(\rho)}$ with the symmetric tensor $h_{\alpha\beta}$.

these 14 equations must be added the 4 equations (3.6) for the determination of the u^β, so that there are a total of 18 equations for the determination of 18 unknowns in this case.

Case 2: $a = 0$. For this case $I^{\alpha\beta}$ is a member of the proper-null class and in addition, we loose the field equations (2.5) since they are identically satisfied. We assume that the field $A_{\alpha\beta}$ is non-zero for some (α, β). If this is to be true, $L^\epsilon_{\epsilon[\beta,\alpha]}$ must be non-zero for some (α, β) by equation (2.2). This condition is easily satisfied if we take (5.19) as the definition of the $L^\alpha_{\beta\gamma}$ provided f_α is not a gradient vector. Thus, equations (5.18) through (5.21) carry over directly from case 1.

This case differs from the previous case, however, in a fundamental manner. We no longer have equations (2.5) and hence the expression (5.19) for the affinity is not unique. In fact, the affinity may be chosen arbitrarily from the class of affinities which satisfy the condition $L^\epsilon_{\epsilon[\beta,\alpha]} \neq 0$ for some (α, β). For example, another possible choice for the affinity is given by Theorem I-8. We have chosen the form given by equation (5.19) simply to obtain the same basic formulas in both cases considered and thereby simplify the succeeding considerations. If we are to obtain a determanistic system, however, in the light of the indeterminism in the form which the affinity may assume, it can be shown that the tensor field $h_{\alpha\beta}$ must be assumed to be a given field and thus not subject to variations. Hence the field equations (2.6) no longer hold, since $h_{\alpha\beta}$ is no longer subject to variation, and we obtain the 4 field equations (5.12) for

the determination of the 4 functions f_γ. To these equations must be added the 4 equations (3.6) for the determination of the four u^β, so that there are a total of 8 equations for the determination of 8 unknowns in this case.

We now have the results required to eliminate the lack of invariance arising from the last term of $\mathcal{L}_T$. From equations (5.20) we have

$$L^\epsilon_{\epsilon\alpha} = \Gamma^\epsilon_{\epsilon\alpha}\ (h_{\xi\zeta}) - \frac{1}{2}\ f_\alpha = (\ell n\sqrt{-h})_{,\alpha} - \frac{1}{2}\ f_\alpha$$

since $\Gamma^\epsilon_{\epsilon\alpha}\ (h_{\xi\zeta}) = (\ell n\sqrt{-h})_{,\alpha}$. Thus

$$\rho u^\alpha\ L^\epsilon_{\epsilon\alpha} = -\frac{1}{2}\ \rho u^\alpha\ f_\alpha + \rho u^\alpha\ (\ell n\sqrt{-h})_{,\alpha}\ .$$

Now, substituting equation (5.10) into equation (2.19) we have

$$(\rho u^\alpha)_{,\alpha} = \frac{3}{10}\ (I^{\alpha\epsilon}_{;\epsilon})_{,\alpha}\ .$$

The term $I^{\alpha\epsilon}_{;\epsilon}$ may be directly evaluated using equations (5.5) and (5.19) giving

$$I^{\alpha\epsilon}_{;\epsilon} = \frac{5}{2}\ a\sqrt{-h}\ f^\alpha_. \qquad (5.23)$$

so that

$$(\rho u^\alpha)_{,\alpha} = \frac{3}{4}\ a(\sqrt{-h}\ f^\alpha_.)_{,\alpha}\ .$$

Thus since

$$(\ell n\sqrt{-h})_{,\alpha}\ \rho u^\alpha = (\rho u^\alpha\ \ell n\sqrt{-h})_{,\alpha} - \ell n\sqrt{-h}\ (\rho u^\alpha)_{,\alpha}$$

we have

$$2\rho u^{\alpha} L^{\epsilon}_{\epsilon\alpha} = -\frac{1}{2}\rho u^{\alpha} f_{\alpha} - \frac{5}{2} a(\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} \ell n\sqrt{-h}$$

$$+ (\rho u^{\alpha} \ell n\sqrt{-h})_{,\alpha} \quad .$$

Hence, if we subtract the divergence term $(\rho u^{\alpha} \ell n\sqrt{-h})_{,\alpha}$ from $\mathcal{L}_T$, the last term of $\mathcal{L}_T$ becomes

$$-\frac{1}{2}\rho u^{\alpha} f_{\alpha} - \frac{5}{2} a(\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} \ell n\sqrt{-h} \quad .$$

We thus see that if either

(i) $a = 0$

(ii) $f_{\alpha} = 0$

or

(iii) the Lorentz gauge condition

$$(\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} = 0$$

is satisfied, the last term of $\mathcal{L}_T$ becomes the scalar density $-\frac{1}{2}\rho u^{\alpha} f_{\alpha}$ which is usually encountered in variational representations of electrodynamics.

6. MAXWELL-LORENTZ ELECTRODYNAMICS

We shall show that classic Maxwell-Lorentz electrodynamics is obtained as a special case of the affine field theory. Set $a = 0$ and $b = 1$ in equation (5.4). The $I^{\alpha\beta}$ field is then a member of the proper null class so that the results of Case 2 of the previous section are applicable. Hence, we obtain the system of equations

$$(\sqrt{-h}\ A^{\alpha\beta}_{\cdot\cdot})_{,\beta} = +\rho u^{\alpha} \tag{6.1}$$

and

$$\frac{du^{\alpha}}{ds} + \mu\ \Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})\ u^{\beta}\ u^{\gamma} - \rho h^{\alpha\sigma}\ A_{\sigma\nu}\ u^{\nu} = 0 \tag{6.2}$$

for the determination of the 8 unknown functions u^{α} and f_{α}, where

$$A_{\alpha\beta} = f_{[\alpha,\beta]}\ , \tag{6.3}$$

$$S^{\alpha} = -\frac{1}{2}\rho u^{\alpha} \qquad S^{\alpha}_{,\alpha} = 0\ , \tag{6.4}$$

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta}) - \frac{1}{4}\left[f_{\gamma}\ \delta^{\alpha}_{\beta} + f_{\beta}\ \delta^{\alpha}_{\gamma} - 3f^{\alpha}_{\cdot}\ h_{\beta\gamma}\right]\ , \tag{6.5}$$

and

$$h_{\alpha\beta;\gamma} = \frac{1}{2}\ f_{\gamma}\ h_{\alpha\beta} - h_{\gamma(\alpha}f_{\beta)}\ . \tag{6.6}$$

As noted in the previous section, the tensor $h_{\alpha\beta}$ is assumed given. Since $h_{\alpha\beta}$ is the coefficient tensor of the fundamental quadratic form, the ability to choose $h_{\alpha\beta}$ is equivalent to the ability to choose the coordinate system in which the above

equations are to be solved.* The Lorentz gauge condition

*Equation (6.6) does not place any restriction on the tensor $h_{\alpha\beta}$ other than that of differentiability since (6.6) is identically satisfied for arbitrary $h_{\alpha\beta}$ under equation (6.5).

$(\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} = 0$ obviously can be satisfied with no loss of generality, since the vector f_{β} is determined by equations (6.1) and (6.3) only to within an arbitrary gradient vector. Since the above system of equations constitutes Maxwell–Lorentz electrodynamics we have demonstrated the result stated at the beginning of this section.

There is a point of particular importance which we note here in connection with the Lorentz gauge condition. We may not effect the Lorentz gauge condition by the simultaneous conformal, gauge, and projective transformations developed in II–Section 5 which leave the affinity (6.5) invariant, since the world tube equations (6.2) are not transformed into world tube equations by such transformations. The only time this may be done is when $\mu = \rho = 0$ (i.e.: a pure radiation field), in which case the analysis is equivalent to that of I–Section 9. This situation is not too surprising since the affinity (6.5) was not a unique choice. Any affinity would have done which was such that the $A_{\alpha\beta}$ field would not be a null field.

7. GRAVITATION IN THE ABSENCE OF ELECTROMAGNITISM

We shall show that gravitation in the absence of electromagnitism is a special case of the extended affine field theory. Set $b = \rho = 0$ in equation (5.4) and assume that $a \neq 0$. The $I^{\alpha\beta}$ field is then a member of the non-null class and hence the results of Case 1 Section 5 are applicable. Since b and ρ are assumed to be zero, equation (5.12) implies that

$$\frac{3}{10}\left(a\sqrt{-h}\, h^{\alpha\beta}\right)_{;\beta} = \frac{3}{10} I^{\alpha\epsilon}_{\ ;\epsilon} = 0.$$

Substituting this result into equation (2.5) and noting that $T^{\alpha\beta}_{\ \ \cdot\cdot\gamma}$ vanishes for the $\mathcal{L}_T$ considered* we have

---

*Cf. equation 5.11.

$$I^{\alpha\beta}_{\ ;\gamma} = 0 \tag{7.1}$$

Equation (7.1) together with equations (5.5) and (5.14) then imply

$$M_\beta = \frac{1}{2} f_\beta = 0 \tag{7.2}$$

and

$$h_{\alpha\beta;\gamma} = 0 \ . \tag{7.3}$$

Hence the electromagnetic field tensor $A_{\alpha\beta}$ vanishes, as is evident from equation (5.21). The conditions $b = \rho = 0$ thus imply the absence of an electromagnetic field. In addition, we have by equation (7.3) that the space $\mathcal{E}$ reduces to a metric space with metric tensor $h_{\alpha\beta}$ (cf. I-Section 2), and by equations

(7.2) and (5.20) that

$$R_{\mu\lambda} = J_{\mu\lambda} \cdot \tag{7.4}$$

The field equations (5.12) are identically satisfied since all these equations vanish separately, and the field equations (5.13) are algebraically satisfied by

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}\ (h_{\xi\zeta}) \tag{7.5}$$

obtained from equations (7.2) and (7.3). Thus, the only remaining equations are the 4 equations

$$\frac{du^{\alpha}}{ds} + \Gamma^{\alpha}_{\beta\gamma}\ (h_{\xi\zeta})\ u^{\alpha}\ u^{\gamma} = 0 \tag{7.6}$$

obtained by variation of the world tubes of matter, and the 10 equations

$$J^{\alpha\beta}_{\cdot\cdot} - \frac{1}{2}\ h^{\alpha\beta}\ J_{\mu\nu}\ h^{\mu\nu} - \frac{\Lambda}{2a}\ h^{\alpha\beta} = \frac{\mu}{a\sqrt{-h}}\ u^{\alpha}\ u^{\beta} \tag{7.7}$$

obtained by variation of the $h_{\alpha\beta}$. Equations (7.6,7) are, however, just those equations which describe Einsteinean gravitation theory in the absence of electromagnitism upon identifying $\frac{\Lambda}{2a}$ with the cosmological constant and $\frac{1}{a}$ with the matter–field interaction coefficient.

Equations (7.5) and (7.6) allow us to construct a conservation law for the mass density μ similar to equation (6.4) for the charge density. Set

$$G^{\alpha\beta} = J^{\alpha\beta} - \frac{1}{2}\ h^{\alpha\beta}\ J_{\mu\nu}\ h^{\mu\nu} \cdot \tag{7.8}$$

From the Bianchi identity* we have

---

*Schouten, J. A.: Ricci Calculus (Second Edition), Springer-Verlag, Berlin Gottingen Heidelberg (1954), p. 148, equ.'s 5.28,29.

$$G^{\alpha\beta}_{\ ;\beta} = 0 \ . \tag{7.9}$$

Substituting from equation (7.7) we then have

$$\left(\frac{\mu}{\sqrt{-n}}\, u^{\alpha}\, u^{\beta} + \frac{\Lambda}{2}\, h^{\alpha\beta}\right)_{;\beta} = 0 \tag{7.10}$$

Now, equation (7.6) is equivalent to

$$u^{\alpha}_{\ ;\beta}\, u^{\beta} = 0$$

and the covariant derivative of $h^{\alpha\beta}$ vanishes by equation (7.3) so that the raising and lowering of indices commutes with covariant differentiation. Thus, (7.10) states the desired conservation law

$$\left(\frac{\mu}{\sqrt{-n}}\, u^{\beta}\right)_{;\beta} = 0$$

upon noting that (7.10) must hold for all vectors u^{α} under equations (7.6,7).

8. INTERACTING ELECTROMAGNETIC AND GRAVITATIONAL FIELDS IN THE PRESENCE OF MACROSCOPIC MATTER

We now turn to the description of interacting electromagnetic and gravitational fields in the presence of matter. Using the results stated in Section 5 for the Lagrangean function $\mathcal{L}_T$, we obtain the following system of equations

$$\frac{b}{2}\left(\sqrt{-h}\ A^{\alpha\beta}_{\cdot\cdot}\right)_{,\beta} = -\frac{3a}{4}\sqrt{-h}\ f^{\alpha}_{\cdot} + \frac{\rho}{2}u^{\alpha}\ , \tag{8.1}$$

$$A_{\mu\lambda} = f_{[\mu,\lambda]}\ , \tag{8.2}$$

$$\mu\frac{du^{\alpha}}{ds} + \mu\,\Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})\,u^{\beta}u - \rho h^{\alpha\nu}A_{\nu\tau}\,u^{\tau} = 0, \tag{8.3}$$

$$a\left(R^{\alpha\beta}_{\cdot\cdot} - \frac{1}{2}h^{\alpha\beta}R_{\mu\nu}h^{\mu\nu}\right) - \frac{\Lambda}{2}h^{\alpha\beta} \tag{8.4}$$

$$= \frac{\mu}{\sqrt{-h}}u^{\alpha}u^{\beta} - b\left(A^{\alpha\nu}_{\cdot\cdot}A^{\beta}_{\cdot\nu} - \frac{1}{4}h^{\alpha\beta}A_{\mu\nu}A^{\mu\nu}_{\cdot\cdot}\right)\ ,$$

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta}) - \frac{1}{4}\left[f_{\gamma}\,\delta^{\alpha}_{\beta} + f_{\beta}\,\delta^{\alpha}_{\gamma} - 3f^{\alpha}_{\cdot}\,h_{\beta\gamma}\right]\ , \tag{8.5}$$

where equation (8.4) is obtained by computing the variation of $\mathcal{L}_T$ with respect to $h_{\alpha\beta}$ (i.e.: using equation (2.6) under the identification of $Q_{(\rho)}$ with $h_{\alpha\beta}$). If we set $a = 0$ and $b = 1$ in the above system of equations we obtain the equations of Section 6 which describe Maxwell–Lorentz electrodynamics, and if we set $b = \rho = 0$ we obtain the equations of Section 7 which describe gravitation in the absence of electromagnetic effects. It is thus natural to make the tentative assumption that a description of interacting electromagnetic and gravitational

fields in the presence of matter is provided by the above system. Under this assumption, equations (8.1,2) describe the electromagnetic field, equation (8.3) describes the world tubes of matter, and equation (8.4) describes the gravitational field, for the form of the affinity given by equation (8.5).

Examining equation (8.4) we see that the right side is just the momentum-energy tensor of uncorrelated mass and the electromagnetic field, on noting that $A_{\alpha\beta}$ has been identified with the negative of the electromagnetic field tensor. We thus obtain equations formally similar to the equations of Einstein's general theory of relativity. There is a significant difference, however. Rewriting equation (8.4) by using (5.50) or (II-6.13) gives, on lowering the (α,β) indices,

$$a(J_{\alpha\beta} - \tfrac{1}{2} J_{\mu\nu} h^{\mu\nu} h_{\alpha\beta}) = \frac{\mu}{\sqrt{-h}} u^{\cdot}_{\alpha} u^{\cdot}_{\beta} - b A^{\ \ \nu}_{\alpha\cdot} A_{\beta\nu}$$
$$+ \frac{\Lambda}{2} h_{\alpha\beta} + \frac{b}{4} h_{\alpha\beta} A_{\mu\nu} A^{\mu\nu}_{\cdot\cdot} + a(\tfrac{3}{2} h_{\alpha\beta} \nabla_{\rho} f^{\rho}_{\cdot} \qquad (8.6)$$
$$+ \tfrac{3}{8} f_{\alpha} f_{\beta} - \tfrac{3}{16} f_{\rho} f^{\rho}_{\cdot} h_{\alpha\beta}) ,$$

where $J_{\alpha\beta} - \frac{1}{2} J_{\mu\nu} h^{\mu\nu} h_{\alpha\beta} = G_{\alpha\beta}$ is the Einstein tensor and ∇_{ρ} is the covariant derivative formed from the affinity $\Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})$. Thus, although equation (8.4) is of the same formal form as Einstein's equations, when written out explicitly in terms of the Einstein tensor they assume the significantly different form (8.6).

The system of equations (8.1,2,6) exhibit certain new properties not encountered in previous sections. From equations (5.10) and (5.22) we have

$$S^{\alpha} = \frac{3a}{4} \sqrt{-h}\, f^{\alpha}_{\cdot} - \frac{\rho}{2} u^{\alpha} \ . \tag{8.7}$$

Hence, by equation (2.19) or directly from equation (8.1) the following result is obtained

$$(\rho u^{\alpha})_{,\alpha} = \frac{3a}{4} (\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} \ . \tag{8.8}$$

From this equation we see that instead of obtaining a statement as to the conservation of macroscopic charge, we have the statement that at every point of space macroscopic charge is generated where the source strength is given by the expression $\frac{3a}{4} (\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha}$. In Section 6 we saw that we could apply the Lorentz gauge condition $(\sqrt{-h}\, f^{\alpha}_{\cdot})_{,\alpha} = 0$ since in that case f_{α} was only determined to within an arbitrary gradient vector. This is not the case here as is obvious from inspection of equation (8.1). Hence, under the above description of interacting gravitational and electromagnetic fields in the presence of matter, it may be concluded that the macroscopic charge density is not necessarily conserved and that the Lorentz gauge condition is not necessarily satisfied.

We know, at least on the laboratory scale, that macroscopic charge density is conserved and that Maxwell–Lorentz electrodynamics holds so that the Lorentz gauge condition is applicable.

Although these conditions can not be satisfied by the above description, they may be approximated if we take the constant a sufficiently small that the first term on the right side of equation (8.1) is only significant on a cosmological scale. Under this choice of a we obtain the normal laboratory predictions, but on a cosmological scale we would predict that macroscopic charge density is constantly being created or destroyed depending on the sign of the term $(\sqrt{-h}\, f^{\alpha}_{\bullet})_{,\alpha}$. The physical model thus resulting predicts constant creation similar, though not identical, to Hoyl's hypothesis. The predictions are sufficiently close to the hypothesis of Hoyl, however, that we may tentatively use Hoyl's results to evaluate the constant a.

With a chosen by the above considerations as to the laboratory conservation of macroscopic charge density, the question arises as to whether equation (8.6) is an admissible representation of the gravitational field. We saw, in the last section that a was to be identified with the reciprocal of the matter-field interaction coefficient and hence was a significantly large number. There thus would appear to be a conflict between this choice of a and the one obtained by the requirement of laboratory conservation of macroscopic charge density, notwithstanding the additional terms appearing on the right side of equation (8.6). We shall defer the demonstration that there is no actual conflict to the next section. In fact, it will be shown that the extra terms which occur in equations (8.1) and (8.6) are essential to the existence of a pure field description

of matter and occur in just the required manner to yield the unquantized form of the basic messonic field representations in a natural and straightforward manner.

9. FIELD-FIELD INTERACTION AND THE RESULTING PICTURE OF ELEMENTARY MATTER

In order to resolve the apparent conflict pointed out in the last section and to develop certain fundamental consequences of Affine Field Theory, we consider the case of interacting gravitational and electromagnetic fields in the absence of macroscopic matter. Set $\rho = \mu = \mathbf{A} = 0$ in equation (5.4) so that the Lagrangean function to be studied is given by

$$\mathcal{L} = \sqrt{-h}\,(aR_{\alpha\beta}\,h^{\alpha\beta} - \frac{b}{4}\,A_{\alpha\beta}\,A_{\gamma\delta}\,h^{\alpha\gamma}\,h^{\beta\delta}) \ . \tag{9.1}$$

Applying the results of Section 5, we obtain the following system of field equations

$$b(\sqrt{-h}\,A^{\alpha\beta}_{\cdot\cdot})_{,\beta} + \frac{3a}{2}\sqrt{-h}\,f^{\alpha}_{\cdot} = 0 \ , \tag{9.2}$$

$$A_{\mu\nu} = f_{[\mu,\nu]}, \tag{9.3}$$

$$a(J_{\alpha\beta} - \frac{1}{2}\,J_{\mu\nu}\,h^{\mu\nu}\,h_{\alpha\beta}) \tag{9.4}$$

$$= -b\,(A^{\blacksquare\nu}_{\alpha\cdot}\,A_{\beta\nu} - \frac{1}{4}\,h_{\alpha\beta}\,A_{\mu\nu}\,A^{\mu\nu}_{\cdot\cdot})$$

$$+ a(\frac{3}{2}\,h_{\alpha\beta}\,\nabla_{\rho}\,f^{\rho}_{\cdot} + \frac{3}{8}\,f_{\alpha}\,f_{\beta} - \frac{3}{16}\,f_{\rho}\,f^{\rho}_{\cdot}\,h_{\alpha\beta}),$$

$$S^{\alpha} = \frac{3a}{4}\,\sqrt{-h}\,f^{\alpha}_{\cdot} \ , \tag{9.5}$$

for $L^{\alpha}_{\beta\gamma}$ given by

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}\,(h_{\xi\zeta}) - \frac{1}{4}\Big[f_{\gamma}\,\delta^{\alpha}_{\beta} + f_{\beta}\,\delta^{\alpha}_{\gamma} - 3f^{\alpha}_{\cdot}\,h_{\beta\gamma}\Big]; \tag{9.6}$$

where $J_{\alpha\beta}$ is the contracted curvature tensor formed from $\Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})$ and ∇_ρ is the covariant differential operator formed from $\Gamma^{\alpha}_{\beta\gamma}(h_{\xi\zeta})$. Applying equation (2.19), we have, from equation (9.5)

$$0 = \frac{3a}{4}\left(\sqrt{-h}\, f^{\alpha}_{\cdot}\right)_{,\alpha} = \frac{3a}{4}\nabla_\alpha\left(\sqrt{-h}\, f^{\alpha}_{\cdot}\right) \qquad (9.7)$$

$$= \frac{3a}{4}\sqrt{-h}\,\nabla_\alpha f^{\alpha}_{\cdot} \quad ,$$

since ∇_ρ annuls $h_{\alpha\beta}$ and hence $\sqrt{-h}$. Thus, using equation (9.7), equation (9.4) becomes

$$a\left(J_{\alpha\beta} - \frac{1}{2} J_{\mu\nu}\, h^{\mu\nu}\, h_{\alpha\beta}\right) \qquad (9.8)$$

$$= -b\left(A^{\nu}_{\alpha\cdot}\, A_{\beta\nu} - \frac{1}{4} h_{\alpha\beta}\, A_{\mu\nu}\, A^{\mu\nu}_{\cdot\cdot}\right)$$

$$+ a\left(\frac{3}{8} f_\alpha\, f_\beta - \frac{3}{16} f_\rho\, f^{\rho}_{\cdot}\, h_{\alpha\beta}\right) \quad .$$

The above system of field equations are not directly recognizable in their present form, but may be transformed into a familiar system as follows. Set

$$h'_{\alpha\beta} = \lambda' h_{\alpha\beta} \qquad (9.9)$$

where λ is a constant to be determined. Since equation (9.9) is a conformal transformation with constant generating function, the affinity (9.6) is invariant as is evident by inspection. Thus λ may be looked upon as an integration constant resulting in the solution of $I^{\alpha\beta}_{;\gamma} = \frac{2}{5}\delta^{(\alpha}_{\gamma} I^{\beta)\epsilon}_{;\epsilon}$ for $L^{\alpha}_{\beta\gamma}$, as given by

equation (9.6). Substituting equation (9.9) into the above system of field equations and dropping the primes we have

$$\frac{\sqrt{-h}}{b}\,(\sqrt{-h}\,A^{\alpha\beta}_{\cdot\cdot})_{,\beta} + \frac{3a\lambda}{2}\,f^{\alpha}_{\cdot} = 0 \tag{9.10}$$

$$a(J_{\alpha\beta} - \tfrac{1}{2}\,J_{\mu\nu}\,h^{\mu\nu}\,h_{\alpha\beta}) \tag{9.11}$$

$$= -\frac{b}{\lambda}\,(A^{\blacksquare\nu}_{\alpha\cdot}\,A_{\beta\nu} - \tfrac{1}{4}\,h_{\alpha\beta}\,A_{\mu\nu}\,A^{\mu\nu}_{\cdot\cdot})$$

$$+ a(\tfrac{3}{8}\,f_{\alpha}\,f_{\beta} - \tfrac{3}{16}\,f_{\rho}\,f^{\rho}_{\cdot}\,h_{\alpha\beta})$$

$$S^{\alpha} = \frac{3a\lambda}{4}\sqrt{-h}\,f^{\alpha}_{\cdot} \tag{9.12}$$

in place of equations (9.1,3,4). Setting b = 1,

$$\kappa^2 = \frac{3a\lambda}{2}\ , \tag{9.13}$$

and adjusting the scale of f_α by the equation

$$f_{\alpha} = \sqrt{\frac{2}{3}}\,\kappa\,\phi_{\alpha} \tag{9.14}$$

we finally obtain the following system of field equations

$$\frac{1}{\sqrt{-h}}\,(\sqrt{-h}\,{}'A^{\alpha\beta}_{\cdot\cdot})_{,\beta} + \kappa^2\,\phi^{\alpha}_{\cdot} = 0\ , \tag{9.15}$$

$${}'A_{\mu\nu} = \phi_{[\mu,\nu]}\ , \tag{9.16}$$

$$a(J_{\alpha\beta} - \tfrac{1}{2}\,J_{\mu\nu}\,h^{\mu\nu}\,h_{\alpha\beta}) \tag{9.17}$$

$$= -a\,({}'A^{\blacksquare\nu}_{\alpha\cdot}\,{}'A_{\beta\nu} - \tfrac{1}{4}\,h_{\alpha\beta}\,{}'A_{\mu\nu}\,{}'A^{\mu\nu}_{\cdot\cdot})$$

$$+ \frac{a\kappa^2}{4}\,(\phi_{\alpha}\,\phi_{\beta} - \tfrac{1}{2}\,\phi_{\rho}\,\phi^{\rho}_{\cdot}\,h_{\alpha\beta})\ ,$$

$$S^{\alpha} = \sqrt{\frac{2}{3}}\,\kappa^{3}\,\sqrt{-h}\,\phi^{\alpha}_{\cdot}\quad , \tag{9.18}$$

for $L^{\alpha}_{\beta\gamma}$ given by

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}\,(h_{\xi\zeta}) - \sqrt{\frac{1}{24}}\;\kappa\left[\phi_{\gamma}\,\delta^{\alpha}_{\beta} + \phi_{\beta}\,\delta^{\alpha}_{\gamma} - 3\phi^{\alpha}_{\cdot}\,h_{\beta\gamma}\right] \tag{9.19}$$

This final system of field equations admits simple and direct interpretations and conclusions.

(1) Equations (9.15), 16) are easily recognized as the equations which describe a vector meson field with the mass of the particles equal to κ and with unity spin.*

---

*Schweber, S. S., Bethe, H. A., Hoffmann, F., de.: Mesons and Fields, Vol. I, Fields Row, Peterson and Co., White Plains, N.Y. (1955), p. 107.
Wentzel, G.: Quantum Theory of Fields, Interscience Publications, N.Y. (1949), Ch. II.

---

(2) From equation (9.13) we see that if there is no interaction included in the Lagrangean (i.e., a = 0) then the field mass vanishes and we obtain just Maxwell-Lorentz electrodynamics as in Section 7. We may thus conclude that the vector meson character of the fields is a direct consequence of the interaction between the $A_{\alpha\beta}$ field and the $R_{\alpha\beta}$ field.

(3) The field mass κ occurs as an integration constant and does not appear in the Lagrangean function. In fact the Lagrangean function contains only geometric quantities and has the same form as encountered in studies of gravitation and electrodynamics in the absence of macroscopic matter,* in

---

*Fletcher: op. cit.

contrast to the usual case in which the Lagrangean function has to be specially rigged to give the vector mesonic field by inclusion of terms of the form

$$\sqrt{-h}\,\kappa^2\,\phi_\alpha\,\phi^\alpha_{\cdot} \quad .$$

in the Lagrangean function.

(4) The constant a cancels out of equation (9.17) and does not appear in equation (9.15) so that the apparent conflict pointed out in the last section does not occur. Hence a may be taken equal to unity with no loss of generality.

(5) Equation (9.17) is just the form taken by the equations which describe the gravitational field generated by a vector meson field of mass κ and spin one. In addition, they show that in those highly localized regions where ϕ_α is not small there is a large spatial curvature and hence a large gravitational field. This is not surprising since the very existence of a vector meson field has been shown to depend on the interaction between the gravitational field and the $A_{\alpha\beta}$ field, and presents the possibility of constructing stable static charged particles.

(6) The field equations (9.15, 16, 17) may be viewed as describing a situation in a Riemann space with metric tensor $h_{\alpha\beta}$ since only quantities which are well defined in such a space appear. We may thus view the general affinely connected space $\mathcal{E}_4$ as a subsidiary space used for geometric description purposes under the affine field equations, where the Lagrangean functions involved contain only geometric arguments (i.e.: we have seen that we may take a = b = 1 with no loss of generality

and hence the Lagrangean function given by equation (9.1) contains only geometric arguments). Once the potential form of the general affine connection $L^{\alpha}_{\beta\gamma}$ has been obtained and the Lagrangean function chosen appropriately, the resulting field equations can be considered as describing situations in a "physical" 4-dimensional space of the Riemann variety with metric tensor $h_{\alpha\beta}$. Under this interpretation, we see that the affine field theory provides a method of <u>deriving</u>, rather than assuming, vector meson type equations. In addition, the meson nature and hence the elemental partical nature arises solely from the interaction between the geometric curvature fields $R_{\alpha\beta}$ and $A_{\alpha\beta}$. We may conclude that the elementary particalistic nature of matter results directly from interactions between the curvature fields $R_{\alpha\beta}$ and $A_{\alpha\beta}$ in an abstract geometric space in which only purely geometric Lagrangean functions are required if affine field theory is applied. The final field equations which result are then well defined in physical Riemann space in which they are to be solved to yield physical predictions.

One may show equally well, by a method directly similar to the one used above, that the conflict noted at the end of the previous section when macroscopic matter was considered, disappears with similar conclusions and with the preservation of the constant creation property. This latter property arises from the generation of vector mesonic particles by the electromagnetic field of the macroscopic matter in the presence of its gravitational field.