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Mathematics of Strategic Indirect Bomb Damage Assessment for Point Targets

H. G. Massey and R. D. Shaver

A Report prepared for

UNITED STATES AIR FORCE PROJECT RAND

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SANTA MONICA, CA. 90406

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PREFACE

This Report presents some mathematical formulas and models related to the problem of evaluating the utility of bomb damage assessment in nuclear weapon attacks on hardened point targets. It includes some new mathematical results for calibrating the probability of target damage with bomb damage assessment information and for the analysis of shoot-look-shoot campaigns. This methodological work was carried out as a part of a broader force posture study that deals with capabilities for deterring nuclear attacks on the U.S. strategic forces. A companion study, applying the formulas and models developed in this report to specific force engagements, is in progress and will be reported at a later date.

The models presented here should be of use to Air Force officers in Hq USAF and Hq SAC concerned with strategic force posture analysis.

SUMMARY

This Report derives mathematical formulations related to the problem of evaluating the utility of bomb damage assessment (BDA) in nuclear weapon attacks on hardened point targets. The mathematical models apply specifically to *indirect* BDA: inferring the status of the target from observation of the miss distance -- that is, the distance between the detonation and the target.

It is shown that for a noisy observation of miss distance, with observational inaccuracy described by a circular normal distribution with standard deviation ω, the probability that the target has been damaged can be expressed by the target damage function used for perfect observations, with the weapon/target parameters suitably transformed. Based on the Defense Intelligence Agency damage function D, which we have used here, the probability of target damage P_d can be estimated from the observed miss distance θ by

$$P_d = D\left(\frac{WR}{E}, \frac{\beta^2\theta}{E}\right),$$

where

$$E = \sqrt{(\sigma_d \cdot WR)^2 + \beta^2\omega^2} \text{ and } \beta^2 = \frac{\alpha^2}{\alpha^2+\omega^2},$$

and WR is the weapon radius, σ_d the target sigma number, and α the standard deviation of the delivery distribution. The derivation applies for other commonly used target damage functions as well, and the results are extended herein to derive P_d as a function of the number of weapons shot at a target and the number of BDA observations for cases when more than one weapon is employed against the target.

Finally, the results are applied to constructing distributions of miss distance observations. Models are proposed for analyzing single shoot-look-shoot campaigns with noisy observations and for finding the miss distance distribution for multiple shoot-look-shoot campaigns with exact observations. The miss distance distribution is then combined with the P_d formulas derived earlier to compute the expected fraction of targets damaged after any number of shoot-look sequences with a specified minimum-miss-distance stopping rule.

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CONTENTS

I. INTRODUCTION

Strategic bomb damage assessment (BDA)* is a necessary part of the *look* in a *shoot-look-shoot* sequence for attacking hardened targets (e.g., silos) with nuclear weapons. Traditionally, BDA has been categorized as *direct* and *indirect*, according to whether or not the damage done to the target is actually seen. In many cases, and particularly in the case of hardened silos, the targets of interest -- e.g., the missiles enclosed in the silos -- are not visible. In such circumstances, only indirect BDA -- i.e., BDA using *other* observables to *infer* target status -- is possible.

This Report develops the mathematical foundation for evaluating the utility of strategic indirect BDA against point targets. Target status is inferred by observing the distance between the target location and the weapon detonation point. We refer to this observed distance as the *miss* distance. This measurement can be obtained in numerous ways, e.g., by locating the weapon's crater with respect to the target. This report will not concern itself with how miss distance estimates are obtained, but with converting them into target status assessments.

Section II derives the mathematical formulation for estimating P_d, the probability that the target has been damaged, where there has been a noisy observation of the miss distance. This derivation applies to a broad class of commonly used target damage functions; it is applied

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*Strategic bomb damage is defined in this report as the assessment of damage done by a nuclear weapon, usually against a point target.

here specifically to the Defense Intelligence Agency (DIA) circular coverage damage function.

The DIA methodology does not apply in the case of certain types of targets (including hardened ICBM silos), if more than one weapon is employed. Section III specifies a target damage function and derives P_d as a function of the number of shots at the target and the number of BDA *looks*. The derived expressions are related back to the DIA damage function of Section II. The results are also extended to account for mixed weapon types and imperfectly reliable weapons.

Section IV applies the results derived in Sections II and III to construct restrike algorithms and models useful in analyzing single shoot-look-shoot campaigns with noisy observations and multiple shoot-look-shoot campaigns with exact observations.

II. CALCULATING P_d WITH AND WITHOUT BDA INFORMATION

P_d WITH A KNOWN IMPACT POINT

The estimated probability that the target has been damaged P_d depends on: (1) the distance between the actual impact point (x,y) and the target, henceforth chosen as the origin $(0,0)$, and (2) the target-damage function, which expresses the conditional probability that a target is damaged, given that the miss distance is $\sqrt{x^2+y^2}$.*

As is standard, let the target-damage function be defined by the DIA damage function D.** If $s = \sigma_d \cdot WR$, where WR is the weapon radius and σ_d is the target's sigma number, then

$$P_d = D\left(\frac{WR}{s}, \frac{\sqrt{x^2+y^2}}{s}\right) \tag{2.1}$$

The weapon radius is an expression of the mean hardness of the target set (for a weapon of given yield); the target sigma number expresses the uncertainty in the precise hardness of any particular target within the set. The damage function D can be interpreted as the integral of a circular Gaussian distribution, with standard deviation s, over a circle of radius WR, offset a distance $\sqrt{x^2+y^2}$ from the origin. The vulnerability number system used to determine the weapon radius for a given weapon yield and target hardness is described in [1] and [2].

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*For convenience we shall pose the problem as if the weapon detonated upon impact with the ground. Should weapons be air burst, then an equivalent miss distance could be obtained by adjusting the miss distance to produce an equivalent target overpressure if a ground burst were assumed.

**The DIA damage function is tabulated in [2]. A brief statement of the mathematical expressions defining the function D and the parameters σ_d and WR is presented in the Appendix.

P_d WITH NO IMPACT POINT OBSERVATION

When there is no information about the actual impact point, the probability that the target is damaged depends on (1) the best estimate of the impact point, (2) the delivery distribution, and (3) the target-damage function. If we assume an unbiased delivery distribution, then the best estimate $(\bar{x},\bar{y})$ of the actual impact point is the aim point (0,0). We further assume that the delivery density $f(x,y)$ of the actual impact point (x,y) around the aim point is circular normal,* i.e.,

$$f(x,y) = \frac{1}{2\pi\alpha^2} \exp\left(-\frac{x^2+y^2}{2\alpha^2}\right) \tag{2.2}$$

where $\alpha = \mathrm{CEP}/\sqrt{\log_e(4)}$ is the delivery standard deviation. We term $f(x,y)$ the *a priori delivery density*.

The P_d of a weapon aimed at the target is given by the product of the damage function expressed in Eq. (2.1) and $f(x,y)$, integrated over all possible values of x and y. Thus, the expected value of P_d with respect to the distribution of impact points is

$$P_d = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D\left(\frac{WR}{s}, \frac{\sqrt{x^2+y^2}}{s}\right) \cdot f(x,y)\,dx\,dy \tag{2.3}$$

Now, Snow [4] shows that combining a circular Gaussian distribution with the DIA function produces another DIA function, with the actual miss distance replaced by its best estimate and the variable s by the root-mean-square of s and α. Thus,

$$P_d = D\left(\frac{WR}{\psi}, \frac{\sqrt{\bar{x}^2+\bar{y}^2}}{\psi}\right) \tag{2.4}$$

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*This assumption is for convenience only; the results of this section can be readily generalized to elliptical normal distributions [3,4].

where $\widehat{\sqrt{\bar{x}^2+\bar{y}^2}}$ is the best estimate of the miss distance (which is zero here, since $\bar{x} = 0$ and $\bar{y} = 0$) and $\psi = \sqrt{s^2+\alpha^2}$.

P_d WITH IMPACT POINT OBSERVATION

Suppose that we have a noisy observation of an impact point -- by a satellite, reconnaissance aircraft, etc. -- and that the closeness of the observed impact point to the actual impact point depends upon the accuracy of the observational measurement. Let (θ_x, θ_y) be the location of the observed impact point, a distance $\theta = \sqrt{\theta_x^2+\theta_y^2}$ from the target. We ask: How may this information on the actual impact point be reflected in our estimate of the single-shot probability of damage?

Assume that the *observed* impact point is distributed about the *actual* impact point (x,y) with a circular normal density, $g(\theta_x, \theta_y | x,y)$, given by

$$g(\theta_x, \theta_y | x,y) = \frac{1}{2\pi\omega^2} \exp\left[-\frac{(\theta_x - x)^2 + (\theta_y - y)^2}{2\omega^2}\right] \tag{2.5}$$

where the *observational inaccuracy* ω -- the standard deviation of this observational distribution -- reflects the uncertainty in the determination of the actual impact point, i.e., the inaccuracy of the measurement. When $\omega = 0$, the actual impact point is known exactly; when $\omega \to \infty$, observational knowledge (except, possibly, burst information) is essentially absent.

Now, using θ, we can replace our a priori delivery distribution, f(x,y), with an *a posteriori delivery distribution*, which will give a better prediction of the actual impact point. Let $\hat{f}(x,y|\theta_x,\theta_y)$ represent the *a posteriori delivery density** of the actual impact point. Then, by application of Bayes's rule, we obtain

$$\hat{f}(x,y|\theta_x,\theta_y) = \frac{g\left(\theta_x,\theta_y|x,y\right)\cdot f(x,y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\theta_x,\theta_y|x,y\right)\cdot f(x,y)\,dxdy} \tag{2.6}$$

which reduces to

$$\hat{f}(x,y|\theta_x,\theta_y) = \frac{1}{2\pi\gamma^2}\exp\left[-\frac{\left(x-\beta^2\theta_x\right)^2+\left(y-\beta^2\theta_y\right)^2}{2\gamma^2}\right] \tag{2.7}$$

where

$$\beta^2 = \frac{\alpha^2}{\alpha^2+\omega^2} \qquad \text{and} \qquad \gamma^2 = \frac{\alpha^2\omega^2}{\alpha^2+\omega^2}$$

Again, this is a circular Gaussian distribution, whose standard deviation γ combines the standard deviations of the a priori delivery distribution and the observational distribution. Note that $\left(\beta^2\theta_x,\beta^2\theta_y\right)$ is the mean of the a posteriori distribution; thus, it serves as the best estimate of the impact point.

Now, replacing f in Eq. (2.3) with $\hat{f}$, defined by (2.7), is tantamount again to combining a circular Gaussian delivery

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*We will use circumflexes to denote a posteriori densities.

distribution with the DIA damage function; so, the expression for P_d with an observed miss distance θ is

$$P_d(\theta,\omega) = D\left(\frac{WR}{E}, \frac{\beta^2\theta}{E}\right) \tag{2.8}$$

with $\beta^2\theta$ being the best estimate of the miss distance between the actual impact point and the target, and

$$E = \sqrt{s^2+\gamma^2} = \sqrt{(\sigma_d \cdot WR)^2 + \frac{\alpha^2\omega^2}{\alpha^2+\omega^2}} \tag{2.9}$$

Here again, WR is the weapon radius, σ_d the target's sigma number, α the a priori delivery standard deviation, and ω the observational inaccuracy.*

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*In passing we note that ω has little influence on E and, thus, on our estimate of P_d, if (1) $\alpha^2 \ll \omega^2$ or (2) $\omega^2 \ll (\sigma_d \cdot WR)^2$. In the first case, the observational data adds essentially no useful information above our knowledge of the delivery CEP. The second case, however, has an important implication on the selection of relevant BDA collection systems, namely, that they need only be able to determine the impact point with sufficient accuracy to insure condition (2) in order to provide what is effectively perfect indirect BDA information.

III. CALCULATING P_d FOR MULTIPLE ATTACKS ON NONUNIFORM TARGETS, USING BDA INFORMATION

In 1968 Latter and Thomas [5]* showed that the standard DIA method for calculating the expected damage from attacks involving multiple warheads per target** was not applicable to the important cases where there is a variation in target-to-target hardness for a set of targets, or where there is uncertainty about the exact value of hardness for a single target. By considering variations in target-to-target hardness, one can arrive at substantially greater weapon requirements to achieve a desired damage probability against a target than with the DIA method. This section derives some mathematical expressions for calculating P_d which include these factors. It then extends them to include single and multiple BDA observations.

ASSUMPTIONS AND DEFINITIONS

In this section, for simplicity, we shall assume a target-damage function that is linear in form. We shall interpret this function as the combination of two components: a target hardness or lethal radius distribution and a target response (damage) distribution.

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*Snow [6] extends that work considerably.

**As used by DIA [1], the single-shot damage probability P_d is found from the function D and then applied to multiple weapons. The damage probability from n shots is $1-\left(1-P_d\right)^n$.

Figure 1 shows the target-damage function and its components. The target lethal radius distribution $H(\ell)$ is linear Gaussian,* with mean L_R and standard deviation $s = \sigma_d \cdot L_R$:

$$H(\ell) = \frac{1}{\sqrt{2\pi s^2}} \exp\left[-\frac{(\ell - L_R)^2}{2s^2}\right] \tag{3.1}$$

The target response distribution gives the probability that a target is damaged, given the lethal radius ℓ and the miss distance d. Here we assume that the target-response distribution is a cookie cutter: If the weapon's miss distance d lies within a lethal radius ℓ from the target, then the target is damaged; otherwise no damage is done. We symbolize this by $p(\text{damage}|d,\ell)$:

$$p(\text{damage}|d,\ell) = \begin{cases} 1 & \text{if } d \leq \ell \\ 0 & \text{otherwise} \end{cases}$$

The mean target lethal radius L_R is a measure of the mean hardness of a target class against weapons of a specified yield. It differs slightly from WR used in the preceding section: $WR = L_R \cdot \sqrt{1+\sigma_d^2}$, approximately, as noted in [6]. For values of $\sigma_d \leq .3$, the linear formulation used here is a good approximation of the DIA damage function used in Section II.

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*We limit ℓ to positive values. (The normalization factor for $\sigma_d \leq .3$ is sufficiently near 1.0 to be ignored.) The results derived are negligibly different if negative values of ℓ are permitted or if the DIA function, used in the previous section, is substituted for this formulation.

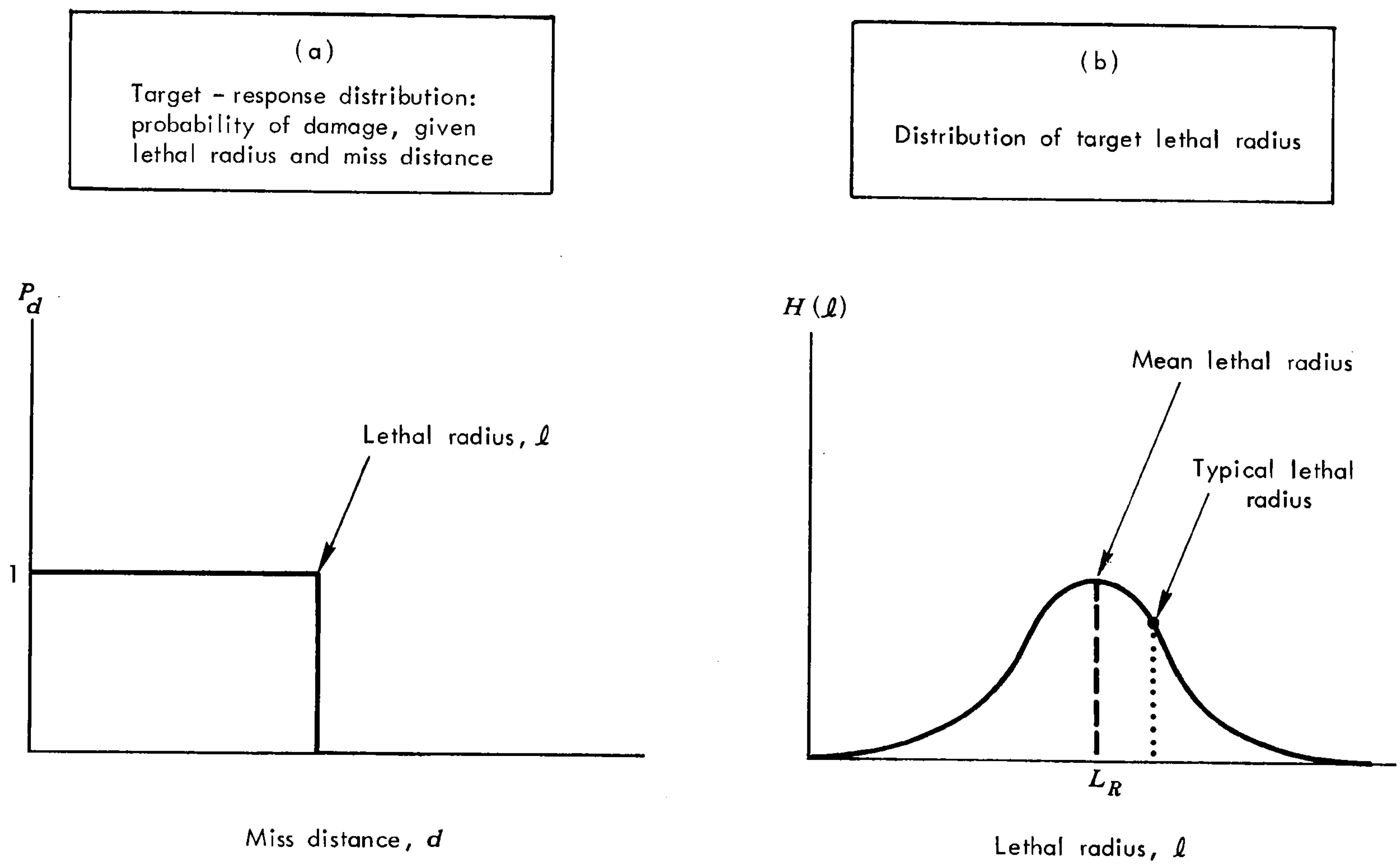

Fig. 1 — The target damage function

P_d FOR ONE WEAPON, NO BDA OBSERVATION

Since the weapon's actual impact point (x,y) is taken to be normally distributed around the intended target with standard deviation α, the miss distance d where

$$d = \sqrt{x^2+y^2}$$

is distributed according to the Rayleigh distribution and the probability that $d \leq \ell$ is given by

$$p(d \leq \ell) = \int_0^{\ell} \frac{z}{\alpha^2} \exp\left(-\frac{z^2}{2\alpha^2}\right) dz = 1-\exp\left(-\frac{\ell^2}{2\alpha^2}\right) \qquad (3.2)$$

Thus,

$$p(d > \ell) = 1-p(d \leq \ell) = \exp\left(-\frac{\ell^2}{2\alpha^2}\right) \qquad (3.3)$$

Therefore, the probability that the target has been damaged, given no information about the weapon's impact point, is

$$P_d(1 \text{ weapon}) = \int_0^{\infty} H(\ell) \cdot p(d \leq \ell) \cdot p(\text{kill} \mid d, \ell)\, d\ell \qquad (3.4)$$

$$= \int_0^{\infty} H(\ell) \cdot p(d \leq \ell)\, d\ell \qquad (3.4)$$

After some algebra, this simplifies to

$$P_d(1 \text{ weapon}) = 1-\Omega_1 \exp\left(-\frac{\Omega_1^2 L_R^2}{2\alpha^2}\right) \qquad (3.5)$$

where

$$\Omega_1^2 = \frac{\alpha^2}{\alpha^2+s^2} \tag{3.6}$$

Equation (3.5) is the equivalent, for the linear formulation of the target-damage function used in this section, of Eq. (2.4) with $(\bar{x},\bar{y}) = (0,0)$.

P_d FOR MULTIPLE SHOTS, NO BDA OBSERVATIONS

Now assume that there are n identical weapons independently aimed at a single target, each with its respective (unobserved) miss distance d_i. The target is assumed damaged if at least one impact lies within a lethal radius ℓ about the target. By analogy with Eq. (3.4) we see that

$$P_d(\text{n weapons}) = \int_0^\infty H(\ell)\cdot\left[1-\prod_{i=1}^{n} p(d_i>\ell)\right]d\ell \tag{3.7}$$

where

$$p(d_i>\ell) = \exp\left(-\frac{\ell^2}{2\alpha^2}\right), \text{ for all } i\leq n.$$

But,

$$\prod_{i=1}^{n}\exp\left(-\frac{\ell^2}{2\alpha^2}\right) = \exp\left(-\frac{n\ell^2}{2\alpha^2}\right)$$

so,

$$P_d(\text{n weapons}) = \int_0^\infty H(\ell) \cdot \left[1-\exp\left(-\frac{n\ell^2}{2\alpha^2}\right)\right] d\ell \qquad (3.8)$$

By direct substitution of $\bar{\alpha}_n = \alpha/\sqrt{n}$, we again have a formula resembling the damage probability for a single weapon, but with $\bar{\alpha}_n$ replacing α (see Eqs. [3.2] and [3.4]). Thus,

$$P_d(\text{n weapons}) = 1-\Omega_n \exp\left(-\frac{\Omega_n^2 L_R^2}{2\bar{\alpha}_n^2}\right) \qquad (3.9)$$

where

$$\Omega_n^2 = \frac{\bar{\alpha}_n^2}{\bar{\alpha}_n^2+s^2}$$

Note that $\bar{\alpha}_n = \alpha/\sqrt{n}$ reflects the well-known property of sampling theory that if the variation among all possible samples of size one (the population) is α, then the variation among all possible samples of size n is $\alpha/\sqrt{n}$.

The result that the probability of damage for n shots against a given target can be obtained by dividing the CEP by the square root of n and then using the single-shot formula with the adjusted CEP is quite general. Had we used the DIA damage response formula, P_d for n shots would have been expressed as

$$P_d = D\left(\frac{WR}{\psi_n}, 0\right) = 1-\exp\left(-\frac{WR^2}{2\psi_n^2}\right) \qquad (3.10)$$

where $\psi_n = \sqrt{s^2+\alpha^2/n}$.

P_d FOR ONE SHOT, ONE BDA OBSERVATION, MULTIPLE RESTRIKES

Now suppose that one weapon is fired at the target, and we obtain a BDA observation (θ_x,θ_y), as in the previous section. The actual impact point (x,y) is described by the a posteriori delivery density $\hat{f}(x,y|\theta_x,\theta_y)$ described in Eq. (2.7).

Suppose that after the initial shot, which has an observed miss distance of d_0, we fire n restrike weapons, each with its (unobserved) respective miss distance d_i. As above, the target is assumed damaged if at least one impact lies within a lethal radius ℓ of the target. By analogy with Eq. (3.7) we see that

$$P_d(\theta,\text{n restrikes}) = \int_0^\infty H(\ell)\cdot\left\{1-\left[p(d_0>\ell|\theta)\cdot\prod_{i=1}^{n} p(d_i>\ell)\right]\right\}d\ell \qquad (3.11)$$

As before,

$$\prod_{i=1}^{n} p(d_i>\ell) = \exp\left(-\frac{n\ell^2}{2\alpha^2}\right)$$

while for $d_0^2 = x^2+y^2$,

$$p(d_0>\ell|\theta) = 1-\iint_{x^2+y^2\leq\ell^2} \hat{f}(x,y|\theta_x,\theta_y)\,dxdy \qquad (3.12)$$

Thus,

$$P_d(\theta,\text{n restrikes})$$

$$=\int_0^{\infty} H(\ell)\cdot\left\{1-\exp\left(-\frac{n\ell^2}{2\alpha^2}\right)\cdot\left[1-\iint_{x^2+y^2\leq\ell^2}\hat{f}(x,y|\theta_x,\theta_y)\,dxdy\right]\right\}d\ell$$

If we expand and regroup terms, this can be rewritten as

$$P_d(\theta,\text{n restrikes})$$

$$=1-\Omega_n\exp\left(-\frac{\Omega_n^2L_R^2}{2\bar{\alpha}_n^2}\right)\cdot\left[1-\int_0^{\infty}\bar{H}_n(\ell)\iint_{x^2+y^2\leq\ell^2}\hat{f}(x,y|\theta_x,\theta_y)\,dxdyd\ell\right] \quad (3.13)$$

where $\bar{\alpha}_n$ and Ω_n are defined as above, and

$$\bar{H}_n(\ell) = \frac{1}{\sqrt{2\pi s^2\Omega_n^2}}\exp\left[-\frac{(\ell-\Omega_n^2L_R)^2}{2\Omega_n^2s^2}\right]$$

$\bar{H}_n(\ell)$ is again a linear Gaussian distribution. For a useful range of values for the parameters, the term in square brackets in Eq. (3.13) can be approximated by a linear Gaussian distribution which combines the variances of $\bar{H}_n(\ell)$ and $\hat{f}(x,y|\theta_x,\theta_y)$. Specifically, for parameter values such that $\Omega_n^2L_R^2\geq3\cdot\Lambda$, where

$$\Lambda=\sqrt{\Omega_n^2s^2+\frac{\alpha^2\omega^2}{\alpha^2+\omega^2}}\ ,$$

we have the approximation

$$P_d(\theta, n \text{ restrikes}) = 1-\Omega_n \exp\left(-\frac{\Omega_n^2 L_R^2}{2\bar{\alpha}_n^2}\right) \cdot \int_0^{\beta^2\theta} G_n(\ell)\, d\ell \tag{3.14}$$

with

$$G_n(\ell) = \frac{1}{\sqrt{2\pi\Lambda^2}} \exp\left[-\frac{\left(\ell-\Omega_n^2 L_R\right)^2}{2\Lambda^2}\right]$$

For the case of an exact observation we have

$$\theta_x \equiv x, \theta_y \equiv y$$

In this case,

$$\hat{f}(x,y|\theta_x,\theta_y) = \begin{cases} 1 \text{ if } x = \theta_x, y = \theta_y \\ 0 \text{ otherwise} \end{cases} \tag{3.15}$$

Thus,

$$p(d_0 > \ell|\theta) = \begin{cases} 1 \text{ if } \ell > \theta \\ 0 \text{ otherwise} \end{cases} \tag{3.16}$$

Substituting Eq. (3.16) into Eq. (3.11) and carrying out the appropriate algebra as before, we obtain the expression:

$$P_d(\theta, n \text{ restrikes}) = 1-\Omega_n \exp\left(-\frac{\Omega_n^2 L_R{}^2}{2\bar{\alpha}_n^2}\right) \int_0^{\theta} \bar{H}_n(\ell)\, d\ell \tag{3.17}$$

Note that $\bar{H}_n(\ell)$ is linear Gaussian with mean $\Omega_n^2 L_R$ and standard deviation $\Omega_n \cdot s$.

P_d FOR MULTIPLE WEAPONS, MULTIPLE BDA OBSERVATIONS

The above can be extended immediately to include multiple observations. If j is the number of observations, one after each weapon, and n is the number of weapons salvoed after the final observation, then

P_d[j observations (θ_i), and n restrikes after j^{th} observation]

$$= \int_0^\infty H(\ell)\left\{1-\left[\prod_{i=1}^{j} p(d_i>\ell\,|\,\theta_i)\right]\left[\prod_{k=1}^{n} p(d_k>\ell)\right]\right\}d\ell \qquad (3.18)$$

where

$$p(d_i>\ell\,|\,\theta_i) = 1-\iint_{x_i^2+y_i^2\le\ell^2} \hat{f}\left(x,y\,|\,\theta_{x_i},\theta_{y_i}\right)dxdy$$

and $p(d_k>\ell)$ is defined as before. At present, no analytical solution to this last expression exists, but numerical schemes for solving this integral are readily available.

For the case of exact observations, only the closest impact point need be considered, and thus Eq. (3.17) applies, with θ equal to the minimum of the θ_i values.

P_d FOR IMPERFECTLY RELIABLE WEAPONS

In the preceding discussion weapons were assumed to be perfectly reliable or 100 percent reprogrammable. The formulas involving multiple weapon shots can be extended easily to allow for less-than-perfect reliability, as follows. We assume that shots are independent and that, on any shot, the probability that a weapon fired at a target will arrive and detonate in the vicinity of the target is equal to R where $0 \leq R \leq 1$. Let $P_k(z)$ represent the probability of damage for z perfectly reliable weapons (with or without BDA observations). Let $P_d(R,n)$ represent the probability of damage for n weapons of reliability $R<1$. Then,

$$P_d(R,n) = \sum_{z=0}^{n} \binom{n}{z} R^z (1-R)^{n-z} P_d(z) \tag{3.19}$$

where

$$\binom{n}{z} = \frac{n!}{z!(n-z)!}$$

<u>P_d FOR MIXED WEAPON TYPES</u>

The multiple-weapon formulas derived in this section can be generalized to allow for weapons differing in either yield (i.e., mean lethal radius) or CEP by analogy to Latter and Thomas [5]. Equation (3.9) is applicable to the general case if we let

$$a_i = \frac{\alpha_i}{L_{R_i}} \text{ and } \frac{1}{\bar{a}_n^2} = \sum_{i=1}^{n} \frac{1}{a_i^2} \tag{3.20}$$

Then,

$$P_d(\text{n weapons, no BDA}) = 1-\Omega_n \exp\left\{-\frac{\Omega_n^2}{2\bar{a}_n^2}\right\} \tag{3.21}$$

$$\Omega_n^2 = \frac{\bar{a}_n^2}{\bar{a}_n^2+\sigma_d^2} \tag{3.22}$$

Generalization of Eq. (3.17) requires that the observation θ be replaced by $t = \theta/L_{R_0}$, where L_{R_0} is the mean lethal radius of the weapon used for the pre-BDA strike. The general form is:

$$P_d(\theta,\text{n restrikes}) = 1-\Omega_n \exp\left(-\frac{\Omega_n^2}{2\bar{a}_n^2}\right)\int_0^t \bar{H}_n(\ell)\,d\ell \qquad (3.23)$$

where Ω_n and $\bar{a}_n$ are as defined in Eq. (3.22) and (3.20), respectively, and $\bar{H}_n(\ell)$ is Gaussian with mean Ω_n^2 and standard deviation $\Omega_n \cdot \sigma_d$.

IV. THE MATHEMATICS OF SHOOT-LOOK-SHOOT CAMPAIGNS

The previous sections have concentrated on deriving P_d formulas for a single target. To evaluate the utility of BDA in a campaign, one must consider not only estimates of individual target P_d's, but also the distribution of observed miss distances and a restrike allocation algorithm. In this section we will discuss methods of calculating shoot-look-shoot (BDA) campaign outcomes for single and multiple BDA looks.

SHOOT-LOOK-SALVO CAMPAIGNS WITH NOISY OBSERVATIONS

Assume a target set of m identical targets. We fire one missile at each target, and for each target j have an observed miss distance θ_j; we then select a salvo of n_j restrike weapons to be fired at the j^{th} target. The total expected number of targets damaged by an attack of N weapons (N>m) is

$$T(\text{damage}) = \sum_{j=1}^{m} P_d(\theta_j, n_j \text{ restrikes}) \qquad (4.1)$$

where

$$N = m + \sum_{j=1}^{m} n_j \qquad (4.2)$$

and P_d is as derived in Section III. If the distribution of θ_j were known, then the allocation of n_j subject to the constraint (4.2) could be determined such that T(damage) is maximized.*

*Other reasonable criteria for weapon allocation algorithms exist: we have chosen the one most commonly used.

As in Section III, assume that θ_j, the observed miss distances, are distributed about the target normally with standard deviation $\sqrt{\alpha^2+\omega^2}$. Thus, the distribution function for θ is the Rayleigh distribution

$$F(\theta) = 1-\exp\left[-\frac{\theta^2}{2(\alpha^2+\omega^2)}\right] \tag{4.3}$$

where α and ω are defined as before.

Knowing how θ is distributed does not determine how θ_j are to be selected in the analysis. Two methods are most common: (1) by Monte Carlo, or random selection with θ appropriately weighted by $\frac{dF(\theta)}{d\theta}$, or (2) by assuming that θ_j are evenly distributed, again with appropriate weighting. In the latter case, solving (4.3) for θ

$$\theta^2 = -\,2(\alpha^2+\omega^2)\log_e[1-F(\theta)]$$

One can pick m uniform values for $F(\theta)$ where $0<F(\theta)<1$, such that

$$F_j(\theta) = \frac{2j-1}{2m} \text{ for } 1\leq j\leq m$$

Then,

$$\theta_j^2 = -2(\alpha^2+\omega^2)\log_e\left(\frac{1+2m-2j}{2m}\right), \text{for } 1\leq j\leq m \tag{4.4}$$

These values for θ_j can be used directly in Eq. (4.1).

We can maximize T(damage) on the target set by allocating one weapon at a time to the target which produces the maximum *marginal* return on each shot. For each target, calculate

$$\Delta T_j = P_d(\theta_j, n) - P_d(\theta_j, n-1)$$

By selecting that target which maximizes ΔT_j, and repeating this process until all the weapons are employed, the optimal allocation is obtained.

MULTIPLE SHOOT-LOOK-SHOOT CAMPAIGNS WITH EXACT OBSERVATIONS

When multiple looks are possible, no ready method for determining a best allocation for N exists, if noisy observations are present.* For many cases of military interest, the observational noise may be negligible; hence we present here a mathematical formulation of the problem for cases when $\omega = 0$.

Now consider a multiple shoot-look-shoot campaign in which weapons are salvoed at a target set with BDA observations between salvos. The observations are used to limit the number of targets to be fired at on subsequent salvos. On the first salvo, one weapon is fired at each target, and an *exact* observation of miss distance d is obtained for each target in the set. On the second salvo, one weapon is fired at each target whose observed miss distance is greater than a *specified* minimum miss distance τ; and another set of exact miss distance observations is obtained. Third salvo weapons

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*In theory the problem could be solved by means of dynamic programming [7], but the computational problems would be large.

are assigned to targets whose minimum miss distance after the first two salvos is greater than τ. This sequence continues indefinitely (until all targets have a miss distance less than τ^*), or for a specified number of salvos n.

Using the delivery distribution functions from Section III, the miss distances are distributed by the Rayleigh distribution (Eqs. [3.2], [3.3]). Therefore, the miss distance distribution after the first salvo is:

$$F_1(t) = p(d \leq t) = 1-\exp\left(-\frac{t^2}{2\alpha^2}\right) \tag{4.5}$$

The fraction of targets receiving a second weapon is

$$[1-F_1(\tau)] = \exp\left(-\frac{\tau^2}{2\alpha^2}\right)$$

and thus the distribution of *minimum* miss distances after two salvos is

$$F_2(t) = p(d_{min} \leq t) = \begin{cases} F_1(t)+[1-F_1(\tau)]p(d_2 \leq t), & \text{for } t \leq \tau \\ 1-p(d_1>t)p(d_2>t), & \text{for } t>\tau \end{cases}$$

or, after substituting for $p(d_1>t)$, $p(d_2>t)$, and $F_1(t)$,

$$F_2(t) = \begin{cases} \left[1-\exp\left(-\frac{t^2}{2\alpha^2}\right)\right]\left[1+\exp\left(-\frac{\tau^2}{2\alpha^2}\right)\right], & \text{for } t \leq \tau \\ 1-\exp\left(-\frac{t^2}{\alpha^2}\right), & \text{for } t>\tau \end{cases}$$

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*By proper selection of τ, this scheme can be shown to maximize T(damage) for any given weapon stockpile.

By induction,

$$F_n(t) = \begin{cases} F_{n-1}(t)+[1-F_{n-1}(\tau)]p(d<t), \text{ for } t\leq\tau \\ 1-\prod_{i=1}^{n} p(d_i>t), \text{ for } t>\tau \end{cases}$$

and therefore,

$$F_n(t) = \begin{cases} \left[1-\exp\left(-\frac{t^2}{2\alpha^2}\right)\right] \cdot \sum_{k=0}^{n-1} \exp\left(-\frac{k\tau^2}{2\alpha^2}\right), \text{ for } t\leq\tau \\ 1-\exp\left(-\frac{nt^2}{2\alpha^2}\right), \text{ for } t>\tau \end{cases}$$

Using the above minimum miss distance distribution and the target-damage function defined in Section III, we can compute the expected fraction of targets damaged Q after the n^{th} salvo, for a specified value of τ, i.e.,

$$Q(\tau,n \text{ salvos}) = \sum_{k=0}^{n-1} \exp\left(-\frac{k\tau^2}{2\alpha^2}\right) \cdot \int_0^{\tau} H(\ell)\cdot\left[1-\exp\left(-\frac{\ell^2}{2\alpha^2}\right)\right] d\ell$$

$$+\int_{\tau}^{\infty} H(\ell)\cdot\left[1-\exp\left(-\frac{n\ell^2}{2\alpha^2}\right)\right] d\ell \qquad (4.8)$$

This can be rewritten in terms of Ω_n, $\overline{\alpha}_n$, and $\overline{H}_n(\ell)$ -- used in Eq. (3.13) -- as:

$$Q(\tau, n \text{ salvos}) =$$

$$\left[\sum_{k=0}^{n-1} \exp\left(-\frac{k\tau^2}{2\alpha^2}\right)\right] \cdot \left[\int_0^\tau H(\ell)\,d\ell - \Omega_1 \exp\left(-\frac{\Omega_1^2 L_R^2}{2\alpha^2}\right) \cdot \int_0^\tau \overline{H}_1(\ell)\,d\ell\right]$$

$$+ \int_\tau^\infty H(\ell)\,d\ell - \Omega_n \exp\left(-\frac{\Omega_n^2 L_R^2}{2\overline{\alpha}_n^2}\right) \cdot \int_\tau^\infty \overline{H}(\ell)\,d\ell \qquad (4.9)$$

The expected number of weapons fired, $N(\tau, n \text{ salvos})$ expressed as a multiple of the number of first salvo targets, is:

$$N(\tau, n \text{ salvos}) = \sum_{k=0}^{n} \exp\left(-\frac{k\tau^2}{2\alpha^2}\right) \qquad (4.10)$$

For unlimited salvos (unlimited looks), this becomes

$$N(\tau, n \to \infty) = \frac{1}{1 - \exp\left(-\frac{\tau^2}{2\alpha^2}\right)} \qquad (4.11)$$

and the fraction of targets killed becomes $Q(\tau, n \to \infty)$

$$= \frac{1}{1 - \exp(-\tau^2/2\alpha^2)} \left[\int_0^\tau H(\ell)\,d\ell - \Omega_1 \exp\left(-\frac{\Omega_1^2 L_R^2}{2\alpha^2}\right) \int_0^\tau \overline{H}_1(\ell)\,d\ell\right]$$

$$+ \int_\tau^\infty H(\ell)\,d\ell \qquad (4.12)$$

Both (4.11) and (4.12) are readily calculated.

Appendix
THE DIA DAMAGE FUNCTION

DIA [2] gives the probability of damage P(r) to a point target at a distance r from ground zero of a weapon detonation by the formula

$$P(r) = D\left(\frac{WR}{\sigma_d \cdot WR}, \frac{r}{\sigma_d \cdot WR}\right) \tag{A-1}$$

where D is the circular coverage function defined below in Eq. (A-5), σ_d is the target sigma number, and WR is the weapon radius.

The parameter σ_d is numerically equal to the ratio of the difference between ground ranges at .31 and .69 probability to the weapon radius:

$$\sigma_d = \frac{r_{.31} - r_{.69}}{WR} \tag{A-2}$$

where r_x is defined as the ground range at which $P(r_x) = x$. The weapon radius is best understood by imagining an infinite array of identifical targets, evenly distributed in such a way that there is one target in each unit of area. Then WR is the radius of a circle, centered at ground zero, within which there are as many targets undamaged, on the average, as there are targets damaged (to the specified degree) outside the circle. Hence,

$$\pi(WR)^2 = \int_0^\infty P(r) \cdot 2\pi r\,dr \tag{A-3}$$

and thus,

$$WR = \sqrt{2 \cdot \int_0^{\infty} P(r) \cdot r dr} \qquad \text{(A-4)}$$

The circular coverage function used in Eq. A-1 above is defined as the integral of the symmetric bivariate Gaussian distribution with unit standard deviation, centered at the origin, over a circle of radius R with center a distance r from the origin:

$$D(R,r) = \frac{1}{2\pi} \iint_{x^2+y^2 \leq R^2} \exp\left[-\frac{1}{2}(x-r)^2 - \frac{1}{2}y^2\right] dxdy \qquad \text{(A-5)}$$

For a Gaussian distribution with standard deviation σ, this integral reduces to $D(R/\sigma, r/\sigma)$.

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