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Cardinal Utility from Intensity Comparisons

Lloyd S. Shapley

A Report prepared for

UNITED STATES AIR FORCE PROJECT RAND

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PREFACE

Utility theory is concerned with the possibility of assigning numerical indices to represent the values of the possible outcomes or decisions in an optimization problem. Project RAND has frequently drawn upon (and occasionally contributed to) this body of methodological research. An example is the recent development of a numerical measure of technological progress in the field of aircraft turbine engine design[(1)]; an important question for that study was whether the index that was constructed was "cardinal" or "ordinal"--or something in between. A cardinal utility index would enable one to use concepts that involve _magnitudes_, e.g., the rate of progress per dollar spent, or the relative importance of two "breakthroughs" in technology. An ordinal index, on the other hand, would provide only a _ranking_ of the different states of the art, with no quantitative implications.

Several approaches can be adopted in an attempt to transform an ordinal index into a cardinal index.[(2)] The classical method is to introduce probabilities and inquire into the gambling odds that would make different outcomes equally desirable, e.g., "Would state A be as good as a 30-70 chance of state B or state C?" The method of _intensity comparisons_, discussed here, requires answers to questions like "Would an improvement from A to B be bigger or smaller than an improvement from C to D?" In many applications the latter approach is preferable; as one of the authors of an important early study in the field[(3)] remarked: "We found that Naval officers were quite

willing to compare improvements, whereas probability comparisons seemed too abstract for them."[4]

The present report provides a new set of underlying theoretical conditions, or axioms, under which intensity comparisons enable one to "cardinalize" a given ordinal utility scale. It was prepared under a Rand project entitled "Analytical Methodology Research."

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(1) See R-1017-ARPA/PR: Measuring Technological Change: Aircraft Turbine Engines (A. J. Alexander and J. R. Nelson), May 1972; R-1061-PR: Technological Change Through Product Improvement in Aircraft Turbine Engines (R. Shishko), May 1973; R-1288-PR: Relating Technology to Acquisition Costs: Aircraft Turbine Engines (J. R. Nelson and F. S. Timson), March 1974. Other recent Rand studies having at least the potential for nontrivial applications of utility theory include R-1422-PR: Attacking Hardened Air Bases (AHAB): A Decision Analysis Aid for the Tactical Commander (C. R. Neu), August 1974; R-1435-PR: Improving SAC Aircrew and Aircraft Scheduling to Increase Resource Effectiveness (M. B. Berman), July 1974; and R-1536-RC: A Framework for Exploring Escalation Control (W. M. Jones), June 1974.

(2) For a survey, see R-904/4-NSF: Game Theory in Economics - Chapter 4: Preferences and Utility (L. S. Shapley and M. Shubik), December 1974.

(3) R. J. Aumann and J. B. Kruskal: "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," Naval Research Logistics Quarterly 6 (1959), pp. 1-16.

(4) R. J. Aumann, correspondence.

SUMMARY

A proof that a cardinal utility function is determined if changes between outcomes, as well as the outcomes themselves, can be ranked in order of desirability. It is assumed that a numerical ordinal utility function with convex range is already given, and that the ranking of changes is continuous and enjoys a certain "crossover" property that is characteristic of difference comparisons.

CONTENTS

1. INTRODUCTION

We present in this report a proof of the following result: if a domain of outcomes can be preference-ordered by a numerical utility function with convex range, and if an "intensity" ordering also exists, satisfying certain axioms, that compares the relative desirability of different changes from one outcome to another, then there is an essentially unique numerical utility function that simultaneously describes both the preferences among the outcomes and their intensities. A precise statement of the theorem will be found at the end of the next section.

This work was prompted by a desire to compare and interrelate the various types of considerations that enable one to pass from an ordinal utility scale (i.e., determined up to a continuous order-preserving transformation) to a cardinal utility scale (i.e., determined up to a linear order-preserving transformation). A number of conceptually very different means to this end are described in [6], to which the reader is referred for an extended discussion. They depend on the ideas of money, risk, bargaining, and decentralized decision-making, in addition to the "intensity of desire" concept that we axiomatize here.

Our axioms require, in effect, that the intensity ordering be continuous (Axiom A3, below), that it be consistent with the given preference ordering (Axioms A1 and A1'), and that it enjoy a certain "crossover" property that is characteristic of difference comparisons (Axiom A2). These conditions all appear to be plausible in the abstract, and arguably testable in practice. Our assumption that a

numerical utility function exists is a familiar and well-understood idea; we have no reason here to derive it once again from more basic properties of preference relations (see Debreu [1]). Finally, our convex-range assumption, i.e., that there are no gaps in the set of possible utility values, is surely satisfied in many applications--for example, when the outcomes are described with the aid of continuous parameters, with respect to which the preference ordering is continuous. It is in a sense an opposite extreme to the assumption that the domain of alternatives is finite, a case that has been elegantly treated by Scott [5].

The field of utility axiomatization has been very thoroughly cultivated in the past two decades, and a number of derivations of cardinal utility from preference-intensity considerations have been published since the original 1955 paper by Suppes and Winet [7]. Most treatments, however, start "further back" with axioms about an abstract binary preference relation and derive both the numerical scale and the cardinality property together; our object is to isolate the latter problem. Our perusal of several surveys of the literature* leads us to believe that the present result is not a duplication or immediate corollary of previous work, but fills a small yet significant gap.

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*See especially [2], [3], [8]. It should be noted that there is a formal correspondence between "intensity of desire" models and "probabilistic utility" models, whereby results proved in either field may be re-interpreted and applied to the other.

2. THE THEOREM

We shall assume that the domain of alternative outcomes or states has been identified with a nonempty, convex subset $\mathcal{D}$ of the real line, preference-ordered in accordance with the natural ordering $\geqq$ of the real numbers. Let $\succsim$ be a weak complete order on $\mathcal{D} \times \mathcal{D}$ satisfying the following axioms:

A1: $(x, z) \succsim (y, z) \Longleftrightarrow x \geqq y$.

A1': $(z, x) \succsim (z, y) \Longleftrightarrow x \leqq y$.

A2: $(x, y) \sim (z, w) \Longleftrightarrow (x, z) \sim (y, w)$.

A3: $\{x, y, z, w: \ (x, y) \succsim (z, w)\}$ is a closed set in $\mathcal{D} \times \mathcal{D} \times \mathcal{D} \times \mathcal{D}$.

The ordered pair $(x, y) \in \mathcal{D} \times \mathcal{D}$ is intended to represent the prospect of replacing state y by state x; one could read it "x in lieu of y" or "x over y". The statement "$(x, y) \succsim (z, w)$" is intended to mean that getting x over y gives at least as much added satisfaction as getting z over w--or (if $x \leqq y$) at most as much added dissatisfaction.

Axioms A1 and A1' ensure <u>consistency</u> between the two orderings. Actually A1' can be proved from A1, A2, A3; the reader interested in this kind of "logical economy" is referred to the proof in the Appendix. From the standpoint of "economic logic," however, A1' is exactly as plausible as A1. The <u>crossover</u> axiom, A2, expresses the idea that differences are being compared; similar axioms have been used by Suppes and Zinnes [8], Pfanzagl [4], and probably others. Axiom A3 is a convenient form of the <u>continuity</u> or <u>Archimedean</u> axiom; note that it does not require that $\mathcal{D}$ itself be a closed set.

THEOREM. *There exists a function* u *from* $\mathcal{B}$ *to the reals such that*

(1) $$x \geqq y \Longleftrightarrow u(x) \geqq u(y)$$

and

(2) $$(x, y) \succsim (z, w) \Longleftrightarrow u(x) - u(y) \geqq u(z) - u(w)$$

for all x, y, z, w, *in* $\mathcal{B}$; *moreover, this function is unique up to an order-preserving linear transformation*.

3. TWO LEMMAS

We begin with two lemmas; similar propositions have sometimes been taken as axioms in treatments that lack our strong assumptions on $\mathscr{D}$. (See [8].)

LEMMA 1. *Let* Z *be any element of* $\mathscr{D} \times \mathscr{D}$. *If* x', x'', *and* y *in* $\mathscr{D}$ *are such that*

$$(x', y) \succsim Z \succsim (x'', y),$$

then there exists a unique x^* *in* $\mathscr{D}$ *such that* $(x^*, y) \sim Z$, *and we have* $x' \geqq x^* \geqq x''$.

Proof. Define

$$x^* = \inf\{x \in \mathscr{D}: \ (x, y) \succsim Z\}.$$

To see that this is an element of $\mathscr{D}$, we note *first* that the set whose "inf" is in question includes x', and so is nonempty; *second*, that the set is bounded below by x'', since $(x, y) \succsim Z \succsim (x'', y)$ implies $x \geqq x''$ by A1, and so has its "inf" between x'' and x'; and *third* that $\mathscr{D}$ is convex, and so contains everything between x'' and x'. Similarly, define

$$x^{**} = \sup\{x \in \mathscr{D}: \ Z \succsim (x, y)\};$$

this likewise* is in $\mathcal{D}$. By A3 we have

$$(x^*, y) \gtrsim Z \gtrsim (x^{**}, y),$$

so $x^* \geqq x^{**}$ by A1. If $x^* > x^{**}$ then we get an immediate contradiction by comparing $(\frac{x^* + x^{**}}{2}, y)$ with Z (again using the convexity of $\mathcal{D}$). So $x^* = x^{**}$, and the required properties are easily verified. Q.E.D.

<u>LEMMA 2</u>. <u>Let</u> x <u>and</u> z <u>be elements of</u> $\mathcal{D}$ <u>such that</u> $x > z$. <u>Then there is a unique</u> $y^* \in \mathcal{D}$ <u>such that</u> $(x, y^*) \sim (y^*, z)$, <u>and we have</u> $x > y^* > z$.

<u>Proof</u>. Define

$$y^* = \sup\{y \in \mathcal{D}: \ (x, y) \gtrsim (y, z)\};$$

this is an element of $\mathcal{D}$ because the set in question is nonempty and bounded above by x, and $\mathcal{D}$ is convex. Similarly define

$$y^{**} = \inf\{y \in \mathcal{D}: \ (y, z) \gtrsim (x, y)\} \in \mathcal{D}.$$

If $y^* < y^{**}$, then comparing $(x, \frac{y^* + y^{**}}{2})$ with $(\frac{y^* + y^{**}}{2}, z)$ yields an immediate contradiction; hence we must have $y^* \geqq y^{**}$. By A1 and A1' we have

*For the sake of the proof in the Appendix, we note that A1' is <u>not</u> involved in these arguments, only A1.

$$(y^*, z) \succsim (y^{**}, z) \qquad \text{and} \qquad (x, y^{**}) \succsim (x, y^*),$$

and by A3 we have

$$(x, y^*) \succsim (y^*, z) \qquad \text{and} \qquad (y^{**}, z) \succsim (x, y^{**}).$$

As these four comparisons form a loop, we actually have "$\sim$" throughout. Hence $y^* = y^{**}$, and the required properties are easily verified. Q.E.D.

4. PROOF OF THE THEOREM

Let a_0 and a_1 be two fixed elements of $\mathcal{B}$ with $a_0 < a_1$. (The theorem is trivial if $\mathcal{B}$ consists of a single point.) After assigning the values $u(a_0) = 0$ and $u(a_1) = 1$, we shall show that the function u has a unique extension to the rest of $\mathcal{B}$ satisfying (1) and (2).

For convenience, we shall denote the element (a_1, a_0) of $\mathcal{B} \times \mathcal{B}$ by the symbol "1", and (a_0, a_0) by "0". Clearly $1 \succ 0$, and by A1 and A2 we have that $(x, x) \sim 0$ for all x in $\mathcal{B}$. Moreover, for any y in $\mathcal{B}$, either

(i) there is a unique element of $\mathcal{B}$, which we shall denote by $T_1(y)$, such that $(T_1(y), y) \sim 1$; or

(ii) we have $1 \succ (x, y)$ for all x in $\mathcal{B}$.

To see this, note that if (ii) fails, then there is an $x' \in \mathcal{B}$ such that $(x', y) \succsim 1$; applying Lemma 1 with $Z = 1$ and $x'' = y$ then yields (i).

We may now define a_2 to be $T_1(a_1)$ if (i) holds for $y = a_1$, and similarly $a_3 = T_1(a_2)$, etc., continuing until we are stopped (if ever) by an occurrence of (ii) for some $y = a_p$. Analogously, we may characterize T_{-1} by $(T_{-1}(x), x) \sim (a_0, a_1)$, and define $a_{-1} = T_{-1}(a_0)$, $a_{-2} = T_{-1}(a_{-1})$, etc., until we are stopped (if ever) by an occurrence of the analog of (ii), i.e., $(x, a_p) \succ (a_0, a_1)$ for all x in $\mathcal{B}$. Denote by $\mathcal{A}$ the sequence

$$\mathcal{A} = \{\ldots a_{-1}, a_0, a_1, a_2, \ldots\};$$

this sequence may be finite or infinite in either direction.

We shall now begin the process of extending the utility function u. Define

$$(3) \qquad u(a_p) = p$$

for all $a_p \in \mathcal{A}$. It is easy to show that (2) holds whenever x, y, z, w are all members of $\mathcal{A}$. Indeed, we have

$$(a_n, a_{n-1}) \sim 1 \sim (a_m, a_{m-1}),$$

so A2 gives us

$$(a_n, a_m) \sim (a_{n-1}, a_{m-1}),$$

which, by a finite iteration, yields

$$(a_n, a_m) \sim (a_{n-d}, a_{m-d}).$$

This equation is valid for all integers m, n, d such that the four points in question are all in $\mathcal{A}$. This proves the equality relation in (2). The inequality follows with the aid of A1 and the easily established fact that $n > m$ implies $a_n \succ a_m$.

We shall now use Lemma 2 to extend u to points of $\mathcal{B}$ that lie between the points of $\mathcal{A}$. Set $b_0 = a_0$ and let b_1 be the element of $\mathcal{B}$ provided by Lemma 2 such that

$$(a_1, b_1) \sim (b_1, a_0).$$

Thus, b_1 is "half way" between a_0 and a_1. We can now construct a sequence

$$\mathcal{B} = \{\ldots\ b_{-1},\ b_0,\ b_1,\ b_2,\ \ldots\}$$

based on b_0 and b_1 just as we constructed the sequence $\mathcal{A}$ from a_0 and a_1. Since $(b_2, b_1) \sim (b_1, b_0)$ we have $b_2 = a_1$, and it is routine to verify that

$$b_{2p} = a_p \tag{4}$$

holds for every p such that a_p is in $\mathcal{A}$. In exactly the same way, we may construct a sequence

$$\mathcal{C} = \{\ldots\ c_{-1},\ c_0,\ c_1,\ c_2,\ \ldots\}$$

so that $c_{2p} = b_p$ holds whenever $b_p \in \mathcal{B}$, and so on indefinitely.

Let us now introduce a more flexible notation. Define $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_1 = \mathcal{B}$, $\mathcal{A}_2 = \mathcal{C}$, ..., and set

$$a(0, p) = a_p \in \mathcal{A}_0$$
$$a(1, p) = b_p \in \mathcal{A}_1$$
$$a(2, p) = c_p \in \mathcal{A}_2$$

and so on. In general, $a(n, p)$ for $p \neq 1$ is obtained from the construction based on (i), (ii), applied to the points a_0 and $a(n, 1)$, while $a(n, 1)$ for $n > 0$ is the "half-way" point between $a(n - 1, 1)$ and a_0, given by Lemma 2. By iterating (4), we have that $a(n, p) = a(m, q)$ whenever $p/2^n = q/2^m$. We may therefore extend u to all points in $\mathcal{A}_\infty = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ by

$$u(a(n, p)) = p/2^n.$$

We see easily that (2) still holds in the extended domain: merely choose n so large that x, y, z, w are all members of $\mathcal{A}_n$ and proceed as at (3), with $\mathcal{A}_n$ in place of $\mathcal{A}$. To complete the construction of u, it remains only to show that $\mathcal{A}_\infty$ is <u>dense</u> in $\mathcal{D}$; this will enable us to extend u by continuity to all of $\mathcal{D}$. Moreover, (1) and (2) will remain satisfied as a result of A3. For this we shall need the following lemma.

<u>LEMMA 3</u>. <u>None of the sets $\mathcal{A}_n$ has a point of accumulation in $\mathcal{D}$</u>.

<u>Proof</u>. Suppose $\lim_{k\to\infty} a(n, p_k) = a^* \in \mathcal{D}$, where (without loss of generality) $\{p_k\}$ is an increasing sequence of integers. Denote the ordered pair $(a(n, 1), a_0)$ by "1_n". Then we have

$$(a(n, p_{k+1}), a(n, p_k)) \underset{\approx}{>} 1_n > 0.$$

Passing to the limit with the aid of A3, we find

$$(a^*, a^*) \gtrsim 1_n > 0,$$

which is a contradiction. Q.E.D.

To prove that $\mathcal{a}_\infty$ is dense in $\mathcal{B}$ we shall show that any $x \in \mathcal{B}$ is in the closure of $\mathcal{a}_\infty$. Without loss of generality, $x \geqq a_0$. For n = 1, 2, ..., define

(5) $$y_n = \sup\{y \in \mathcal{a}_n\colon\ y \leqq x\}.$$

Since a_0 belongs to the set in question, this is a number between a_0 and x, and so an element of $\mathcal{B}$ by convexity. By Lemma 3, y_n is not an accumulation point of $\mathcal{a}_n$, so it must actually be a member of $\mathcal{a}_n$, say $y_n = a(n, p_n)$, and we have

(6) $$1_n > (x, y_n) \gtrsim 0,$$

since otherwise $a(n, p_n + 1)$ would exist in $\mathcal{a}_n$ and would be $\leqq x$, contradicting (5). We wish to consider what happens in (6) as $n \to \infty$.

First consider the numbers y_n. Since each $\mathcal{a}_n$ includes $\mathcal{a}_{n-1}$, they form a nondecreasing sequence, bounded above by x; call the limit of this sequence y^*. Since y^* lies between a_0 and x it belongs to $\mathcal{B}$, by convexity. Now consider the behavior of the 1_n. We have

$$1_n \approx (a(n, 2), a(n, 1)) = (a(n - 1, 1), a(n, 1)).$$

The numbers a(n, 1) form a decreasing sequence, bounded below by a_0; call the limit a^*. Being between a_1 and a_0, a^* belongs to $\mathcal{D}$. (Actually, $a^* = a_0$, but we don't need this fact.) So if we pass to the limit in (6), we can assert

$$(a^*, a^*) \succsim (x, y^*) \succsim 0,$$

by A3. This shows that $x = y^*$. Since y^* is in the closure of $\mathcal{A}_\infty$ and x was arbitrary in $\mathcal{D}$, we have proved that $\mathcal{A}_\infty$ is dense in $\mathcal{D}$ and hence that a utility function u satisfying (1) and (2) can be constructed.

Uniqueness follows easily from the constructive nature of our proof. Indeed, any other utility function u' satisfying (1) and (2) can be "normalized" by a linear order-preserving transformation so that $u'(a_0) = 0$, $u'(a_1) = 1$. But then, at each step of our extension, u' will be forced to agree with u. This completes the proof of the theorem.

APPENDIX: PROOF THAT A1' FOLLOWS FROM A1, A2, A3

We first observe that Lemma 1 is available, since its proof did not make use of A1'. Proceeding by contradiction, suppose that A1' fails. Then there exist elements a, b, c of $\mathcal{D}$ such that

(7) $$(a, c) \succsim (a, b)$$

and

(8) $$c > b.$$

We distinguish two cases: $c > a$ and $c \leqq a$.

If $c > a$ then we have

$$(b, b) \sim (c, c) \succ (a, c) \succsim (a, b),$$

by A2, A1 and (7), respectively. We may apply Lemma 1 with Z = (a, c) to find $x \in \mathcal{D}$ such that

$$(x, b) \sim (a, c) \qquad \text{and} \qquad b \geqq y \geqq a.$$

Hence we have

$$(a, a) \sim (c, c) \succ (b, c) \sim (y, a) \succsim (a, a),$$

by A2, (8) and A1, A2, and A1 respectively, a contradiction.

For the other case, if $c \leqq a$ then $b < a$, by (8). The argument of the previous case can now be repeated, interchanging b and c and reversing all inequalities. Q.E.D.

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