R-675-HUD
APRIL 1971

RESPONSE OF EMERGENCY UNITS: THE EFFECTS OF BARRIERS, DISCRETE STREETS, AND ONE-WAY STREETS

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This study was sponsored by the U. S. Department of Housing and Urban Development. Its contents, however, do not purport to represent the official views or policy of its sponsor.

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Published by The Rand Corporation

PREFACE

This report is part of a New York City-Rand Institute study of the deployment of emergency services. It presents the mathematical details of calculations which are required to determine realistic estimates of travel distances of vehicles travelling city streets. Such formulas for estimated travel distances are frequently required in models designed to improve the locations, patrol areas, or dispatching policy for emergency units such as fire engines and police patrol cars.

SUMMARY

In realistic urban environments, emergency response vehicles (units) may encounter barriers (e.g., rivers) and one-way streets which impede rapid response. In this report we derive formulas for the increase in mean travel distance associated with such impediments. We consider vehicles whose locations at time of dispatch may vary over some region. Such a model is most appropriate for police patrol cars. Calls for service are assumed to be uniformly and independently distributed over the region. The techniques illustrated are applicable to systems in which the response unit is dispatched from fixed locations and to systems with more complicated spatial descriptions.

Examining the formulas derived here, one finds several important system insensitivities. First, barriers of even moderate size typically increase mean response distance by less than 10 percent. Second, restricting the response unit and the incidents to a discrete rectangular grid of two-way streets adds at most 1/3 of a block length to the mean travel distance computed from the corresponding continuous distribution model. Third, restricting the response unit and the incidents to a large discrete grid of one-way streets adds two block lengths to the mean travel distance computed from the corresponding two-way street model. However, for about 6.2 percent of the responses on a one-way street grid, the response unit will have to travel an average of six extra block lengths. Employing reasonable parameter values, the one-way constraints would cause mean travel time to be increased 3 minutes or more in these cases.

ACKNOWLEDGMENTS

The author gratefully acknowledges the support of the U.S. Department of Housing and Urban Development for this work. He also wishes to thank Jan Chaiken for several useful suggestions.

CONTENTS

FIGURES

I. INTRODUCTION

Recent work on emergency urban service systems has shed light on certain aspects of design of response areas* and location of response vehicles (units).** That work has indicated how the precise nature of the travel distance influences area design and location decisions. It has also illustrated that several system characteristics are relatively insensitive to certain design and location decisions.

In this report we examine how travel distance of emergency urban services is increased by certain impediments to travel. In turn, we shall consider (1) barriers, (2) discrete (two-way) street grids, and (3) one-way street grids. By examining simple models that include these complications, we can get some idea of how each impedes response. In each case, we assume that the response unit and incidents are uniformly, independently distributed over the unit's "response area," in a way that is described separately in each section. The model is appropriate for police patrol cars and other services whose units do not have fixed locations. Several insensitivities are found which provide further insight into system operation. A summary and discussion of results are given in Section V.

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*Carter, G. M., J. M. Chaiken, and E. Ignall, Response Areas for Two Emergency Units, The New York City-Rand Institute, R-532, March 1971.

**Larson, R. C. and K. Stevenson, On Insensitivities in Urban Redistricting and Facility Location, The New York City-Rand Institute, R-533, March 1971.

II. BARRIERS

Consider the rectangular n by m response area illustrated in Fig. 1a. Assume that the response distance between two points (x_1, y_1) and (x_2, y_2) is given by the "right-angle" distance metric,

$$d = |x_1 - x_2| + |y_1 - y_2|.$$

If the coordinates are random variables, (X_1, Y_1) and (X_2, Y_2), then the expected response distance, E[D], is given by

$$E[D] = E\left(|X_1 - X_2|\right) + E\left(|Y_1 - Y_2|\right).$$

We will examine the case in which the coordinates are uniformly distributed over the response area and are mutually independent. For this case it is known that*

$$E[D] = \frac{1}{3}[n+m]. \qquad (1)$$

Now assume we erect a barrier of height y = a at x = b (Fig. 1b). The response distance for this case can be written as the sum of "old" response distance and a "perturbation" distance,

$$D' = D + D_e, \qquad (2)$$

where

D_e = "extra" distance travelled due to the barrier.

*See, for example, Larson, R. C., "Models for the Allocation of Urban Police Patrol Forces," MIT Operations Research Center Technical Report No. 44, 1969, p. 100.

FIG. 1a: RECTANGULAR RESPONSE AREA WITH NO BARRIER

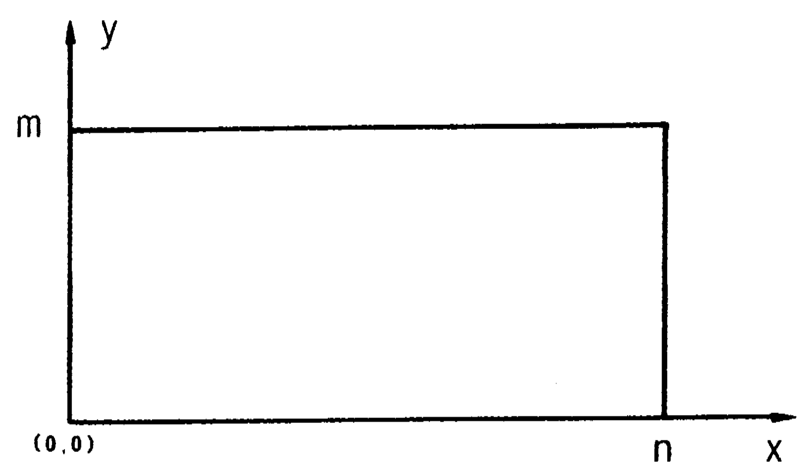
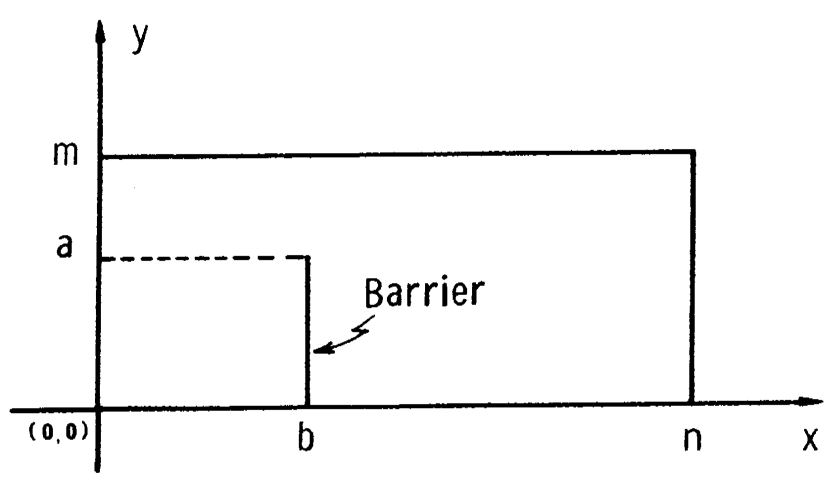


FIG. 1b: RECTANGULAR RESPONSE AREA WITH ONE SIMPLE BARRIER



The expected response distance is

$$E[D'] = E[D] + E[D_e]. \tag{3}$$

Since $E[D]$ is known we need only compute $E[D_e]$. It is convenient to write

$$E[D_e] = \sum_i E[D_e|A_i]\ P[A_i], \tag{4}$$

where the "events" A_i are mutually exclusive and collectively exhaustive. In particular, consider the events

$A_1 : D_e > 0$, and

$A_2 : D_e = 0$.

Event A_1 requires that the following inequalities be satisfied:*

$Y_1 < a$,

$Y_2 < a$,

$MIN[X_1, X_2] \leq b$, and

$MAX[X_1, X_2] > b$.

When coordinate positions are uniformly distributed and independent, we easily compute

$$P[A_1] = 2\left(\frac{b}{n} \cdot \frac{a}{m}\right) \left(\frac{n-b}{n} \cdot \frac{a}{m}\right). \tag{5}$$

*We arbitrarily group points $(x = b,\ y < a)$ with all points to the left of the barrier.

Conditioned on event A_1, we can write

$$(D_e|A_1) = 2\ \mathrm{MIN}[V_1, V_2],$$

where

$$V_1 = a - (Y_1|A_1) \text{ and}$$
$$V_2 = a - (Y_2|A_1).$$

Thus, V_1 and V_2 are uniformly independently (u.i.) distributed on $[0, a]$.

Then, by standard methods,

$$F_{D_e}(s|A_1) = P[D_e \leq s|A_1]$$
$$= \begin{cases} 0 & \text{if } s \leq 0 \\ 1 - \frac{1}{a^2}\left(a - \frac{s}{2}\right)^2 & \text{if } 0 < s \leq 2a \\ 1 & \text{if } s > 2a. \end{cases}$$

Using this result, we get

$$E[D_e|A_1] = \frac{2}{3}\,a. \tag{6}$$

Thus, using (1), (3), (4), (5), and (6), we have

$$E[D'] = \frac{1}{3}(n+m) + \frac{2a^2}{n^2m^2}\,b(n-b)\,\frac{2}{3}\,a. \tag{7}$$

As an example, let $n = m = 1$ and $b = \frac{1}{2}$.

Then,

$$E[D'] = \frac{2}{3}\left(1 + \frac{a^3}{2}\right). \tag{8}$$

This function is plotted for $0 \leq a \leq 1$ in Fig. 2. Note that a barrier of height $a = \frac{1}{2}$, say, yields a mean response distance of 0.708, which is only about 6.3 percent greater than the mean response distance with no barrier (0.677).

We can use (7) to find the mean response distance for the case in which a barrier extends completely from $y = 0$ to $y = m$ with one "crossing point" at $y = a$ (e.g., a river at $x = b$ with a bridge at $y = a$). The response distance for this case can be written

$$D'' = D' + D_e',$$

where

D_e' = extra distance travelled compared to the distance D' of (2).

By symmetry with the case of one simple barrier,

$$P[D_e' > 0] = 2\left(\frac{b}{n} \cdot \frac{m-a}{m}\right)\left(\frac{n-b}{n} \cdot \frac{m-a}{m}\right)$$

$$E[D_e' | D_e' > 0] = \frac{2}{3}(m-a).$$

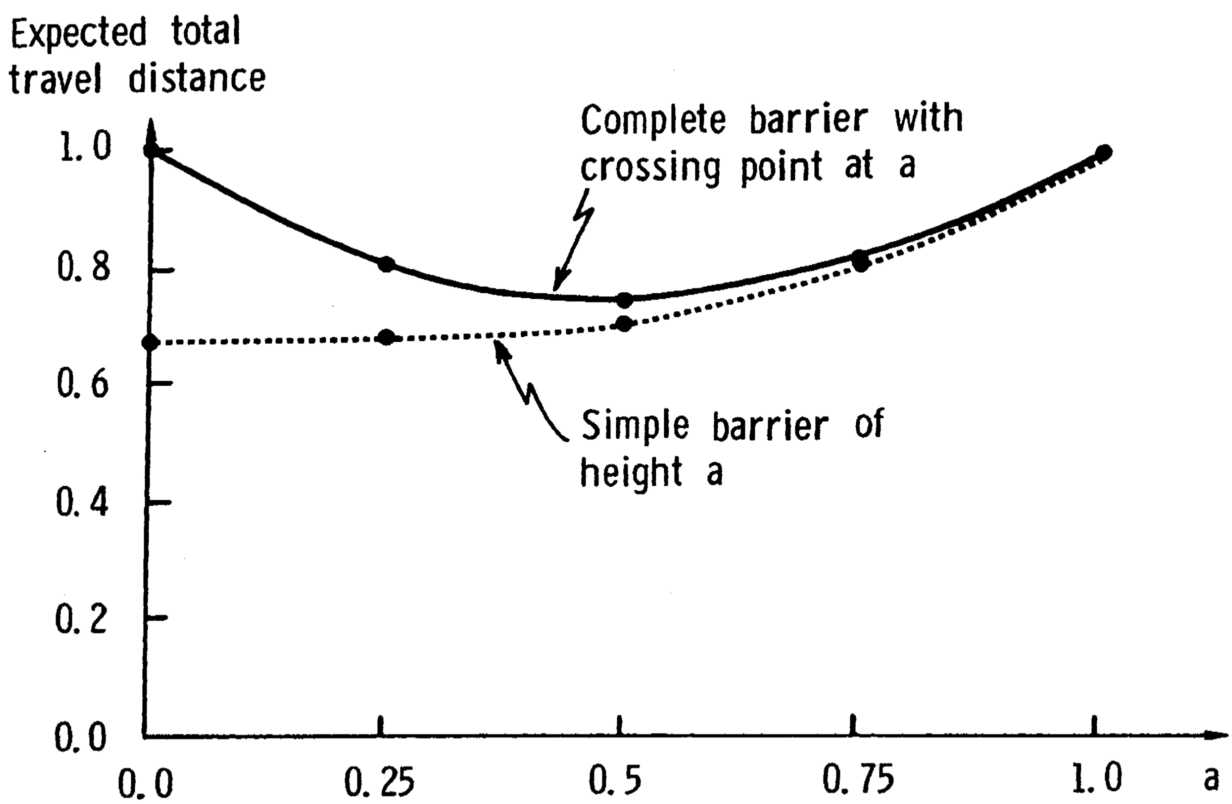
Thus,

$$E[D''] = \frac{1}{3}(n+m) + \frac{4b(n-b)}{3n^2m^2}\left(a^3 + (m-a)^3\right). \qquad (9)$$

This function is also plotted in Fig. 2 for the example $n = m = 1$, $b = \frac{1}{2}$.

An interesting and important property of these results is the relative insensitivity of mean travel distance to moderately large barriers.

FIG. 2: EXPECTED TOTAL TRAVEL DISTANCE FOR A SQUARE RESPONSE AREA OF UNIT AREA: TWO TYPES OF BARRIERS



III. DISCRETE GRID OF STREETS

We now examine the case in which response units and incidents are confined to an n by m discrete grid of equally spaced two-way streets forming square blocks of unit area, as illustrated in Fig. 3. Assume that the positions of the response unit (X_1, Y_1) and the incident (X_2, Y_2) are independent and uniformly distributed over the streets of the grid. In addition, assume that the response distance D between (X_1, Y_1) and (X_2, Y_2) is a shortest path that remains on the streets of the grid. We wish to obtain an expression for E[D].

In the development we allow n and m to assume any non-negative integer values, excluding the trivial case $n = m = 0$. Although we call n and m the "sector dimensions," note that since blocks are defined to have unit area, n and m are dimensionless integers.

We write

$$E[D] = \sum_{i=1}^{4} E[D|A_i]\, P[A_i], \qquad (10)$$

where the A_i's and their probabilities are:

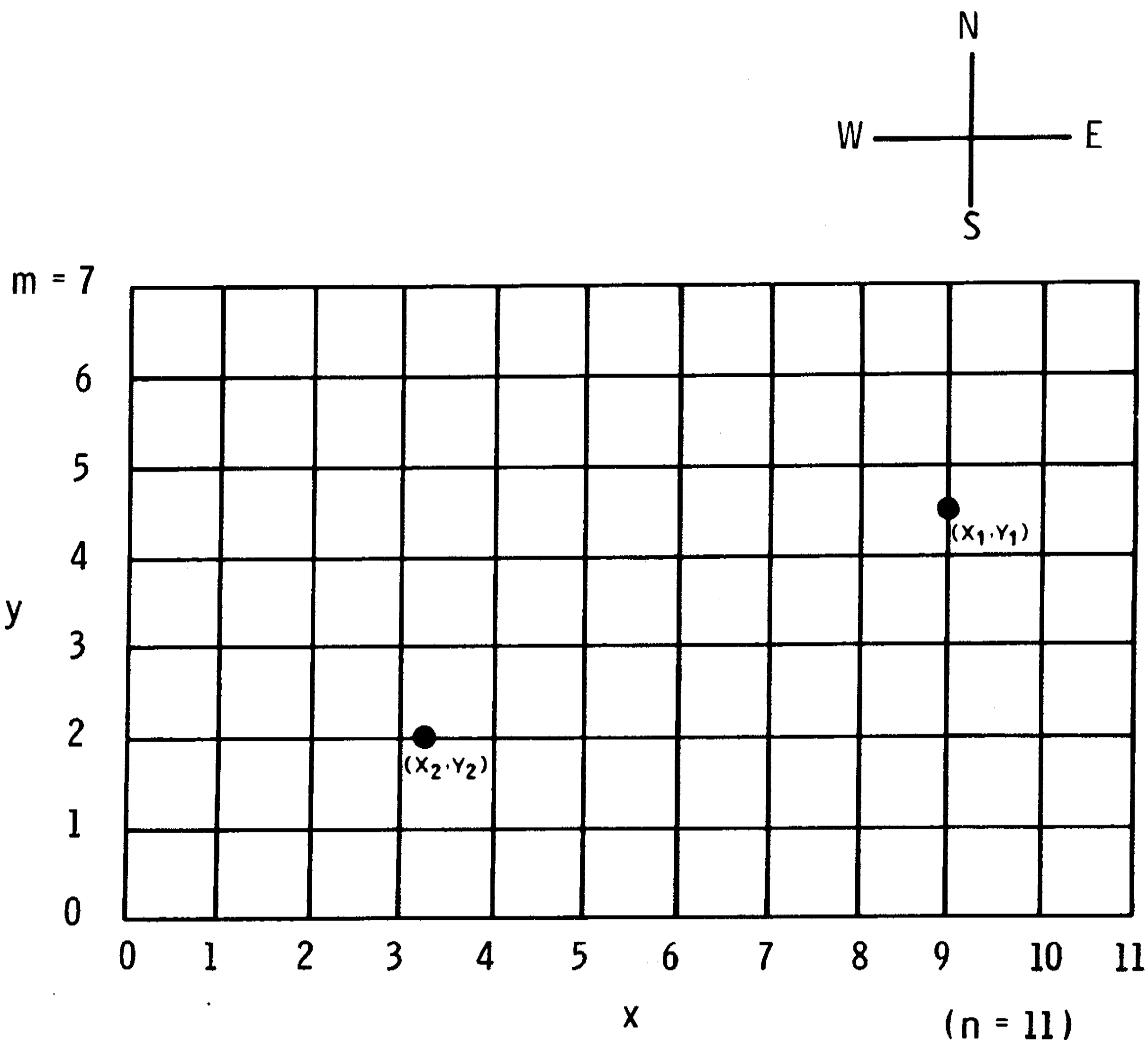
A_1: Response unit and incident both on East-West (E-W) streets $\qquad P[A_1] = \frac{n^2(m+1)^2}{[(n+1)m + (m+1)n]^2}$

A_2: Response unit on E-W street, incident on North-South (N-S) street $\qquad P[A_2] = \frac{n(m+1)\, m(n+1)}{[(n+1)m + (m+1)n]^2}$

A_3: Response unit and incident both on N-S streets $\qquad P[A_3] = \frac{m^2(n+1)^2}{[(n+1)m + (m+1)n]^2}$

A_4: Response unit on N-S street, incident on E-W street $\qquad P[A_4] = \frac{n(m+1)\, m(n+1)}{[(n+1)m + (m+1)n]^2}$

FIG. 3: DISCRETE GRID REPRESENTING RECTANGULAR RESPONSE AREA



Consider event A_1. We can write

$$E[D|A_1] = E[D_x|A_1] + E[D_y|A_1], \tag{11}$$

where

D_x = total E-W response distance, and

D_y = total N-S response distance.

If the response unit did not have to follow N-S streets but could travel N-S anywhere, then since X_1 and X_2 are u.i. distributed over [0, n],

$$E[D_x|A_1] = E\left((|X_1 - X_2|)|A_1\right) = \frac{1}{3} n. \tag{12}$$

But if $Y_1 \neq Y_2$ and if X_1 and X_2 both fall in the same set of N-S blocks, (i.e., if $[X_1] = [X_2]$),* then

$$D_x = |X_1 - X_2| + 2 \text{ MIN}[V, 1 - W],$$

where

$$V = \text{MIN}[U_1, U_2]$$

$$W = \text{MAX}[U_1, U_2],$$

and U_1 and U_2 are u.i. distributed over [0, 1]. The probability of this occurring is $\frac{1}{n} \cdot \frac{m}{m+1}$. Thus we can write

$$E[D_x|A_1] = E\left((|X_1 - X_2|)|A_1\right) + E[2 \text{ MIN}(V, 1 - W)] \frac{1}{n} \cdot \frac{m}{m+1} . \tag{13}$$

By standard methods, we compute the expected value of the perturbation term,

$$E[2 \text{ MIN}(V, 1 - W)] = \frac{1}{3} . \tag{14}$$

---

*We define $[w] \equiv$ largest integer less than or equal to w.

Using (12), (13), and (14),*

$$E[D_x|A_1] = \frac{1}{3}\left(n + \frac{m}{n(m+1)}\right). \tag{15}$$

We must now find $E[D_y|A_1]$. Note that

$$(D_y|A_1) = \left((|Y_1 - Y_2|)|A_1\right). \tag{16}$$

For any pair (i,j) (i=0,1,...,m; j=0,1,...,m),

$$P[Y_1 = i,\ Y_2 = j] = \frac{1}{(m+1)^2}\ . \tag{17}$$

Using (16) and (17), it is not difficult to show that

$$P[D_y = 0|A_1] = \frac{1}{m+1}$$

$$P[D_y = k|A_1] = \frac{2(m-k+1)}{(m+1)^2} \qquad k=1,2,\ldots,m. \tag{18}$$

We can now compute

$$E[D_y|A_1] = \sum_{k=0}^{m} k\ P[D_y = k|A_1].$$

Using (18) and the facts that

$$\sum_{i=0}^{m} i = \frac{m(m+1)}{2}$$

and

$$\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}\ ,$$

---

*Eq. (15) corrects Eq. (B-1) given in Larson, R. C., "Operational Study of the Police Response System," MIT Operations Research Center Technical Report No. 26, 1967, p. 180.

we have

$$E[D_y|A_1] = \frac{m(m+2)}{3(m+1)} . \tag{19}$$

Combining (15) and (19), we have

$$E[D|A_1] = \frac{1}{3}\left(n+m+ \frac{m(n+1)}{n(m+1)}\right). \tag{20}$$

Consider event A_3. By symmetry we see this is the same as event A_1 with x and y interchanged. Thus, $E[D|A_3]$ is found by interchanging n and m in (20).

Consider event A_2. We have

$$(D|A_2) = \left((|X_1 - X_2|)|A_2\right) + \left((|Y_1 - Y_2|)|A_2\right).$$

For the E-W distance we can write

$$\begin{aligned} E\left((|X_1 - X_2|)|A_2\right) &= \sum_{i=0}^{n} E\left((|X_1 - X_2|)|X_2 = i, A_2\right) P[X_2 = i|A_2] \\ &= 2\sum_{i=0}^{n} E[X_1 - X_2|X_2 = i, X_1 \geq X_2, A_2] \, P[X_1 \geq X_2|X_2 = i, A_2] \, P[X_2 = i|A_2] \end{aligned} \tag{21}$$

Since X_1 is uniformly distributed on [0, n],

$$E[X_1 - X_2|X_2 = i, X_1 \geq X_2, A_2] = \frac{n-i}{2} \tag{22}$$

$$P[X_1 \geq X_2|X_2 = i, A_2] = \frac{n-i}{n} . \tag{23}$$

Since X_2 is uniformly distributed on the integers $0,1,\ldots,n$,

$$P[X_2 = i|A_2] = \frac{1}{n+1} \qquad i=0,1,\ldots,n. \tag{24}$$

Using (22), (23), and (24) in (21),

$$E\left((|X_1 - X_2|)|A_2\right) = 2\sum_{i=0}^{n} \frac{(n-i)^2}{2n(n+1)}$$

or,

$$E\left((|X_1 - X_2|)|A_2\right) = \frac{2n+1}{6} . \tag{25}$$

Similarly, we obtain

$$E\left((|Y_1 - Y_2|)|A_2\right) = \frac{2m+1}{6} . \tag{26}$$

Combining these results, we have

$$E[D|A_2] = \frac{1}{3}(n+m+1). \tag{27}$$

Consider event A_4. By symmetry we see this is the same as event A_2 with x and y interchanged. Thus

$$E[D|A_4] = E[D|A_2].$$

Finally, the desired result, E[D], is found by substituting the above results in (10), yielding

$$E[D] = \frac{1}{3}(n+m) + \frac{4n(m+1)m(n+1)}{3[(n+1)m+(m+1)n]^2} \tag{28}$$

For practical problems one wishes to know how closely the continuous formulation approximates the more exact discrete grid formulation. In fact, one can easily show that

$$\frac{1}{3}(n+m) \leq E[D] \leq \frac{1}{3}(n+m+1), \tag{29}$$

where the left-hand inequality becomes an equality when n or m is zero and the right-hand inequality becomes an equality when $n = m$. Thus, <u>the continuous approximation is never in error by more than 1/3 of a block length</u>.

IV. ONE-WAY STREETS

We now modify the model of the previous section and assume that response units and incidents are confined to a discrete grid of equidistant one-way streets, the direction of travel alternating from street to street (Fig. 4). As in Section III, we assume that the positions of the response unit (X_1, Y_1) and the incident (X_2, Y_2) are independent and uniformly distributed over the grid. We also assume that the response distance D' from (X_1, Y_1) to (X_2, Y_2) is a shortest path that remains on the streets of the grid and that obeys the one-way constraints. We wish to discover how the complication of one-way streets adds to the mean response distance found in Section III.

We write

$$D' = D + D_e,$$

where

D = discrete grid travel distance of Section III

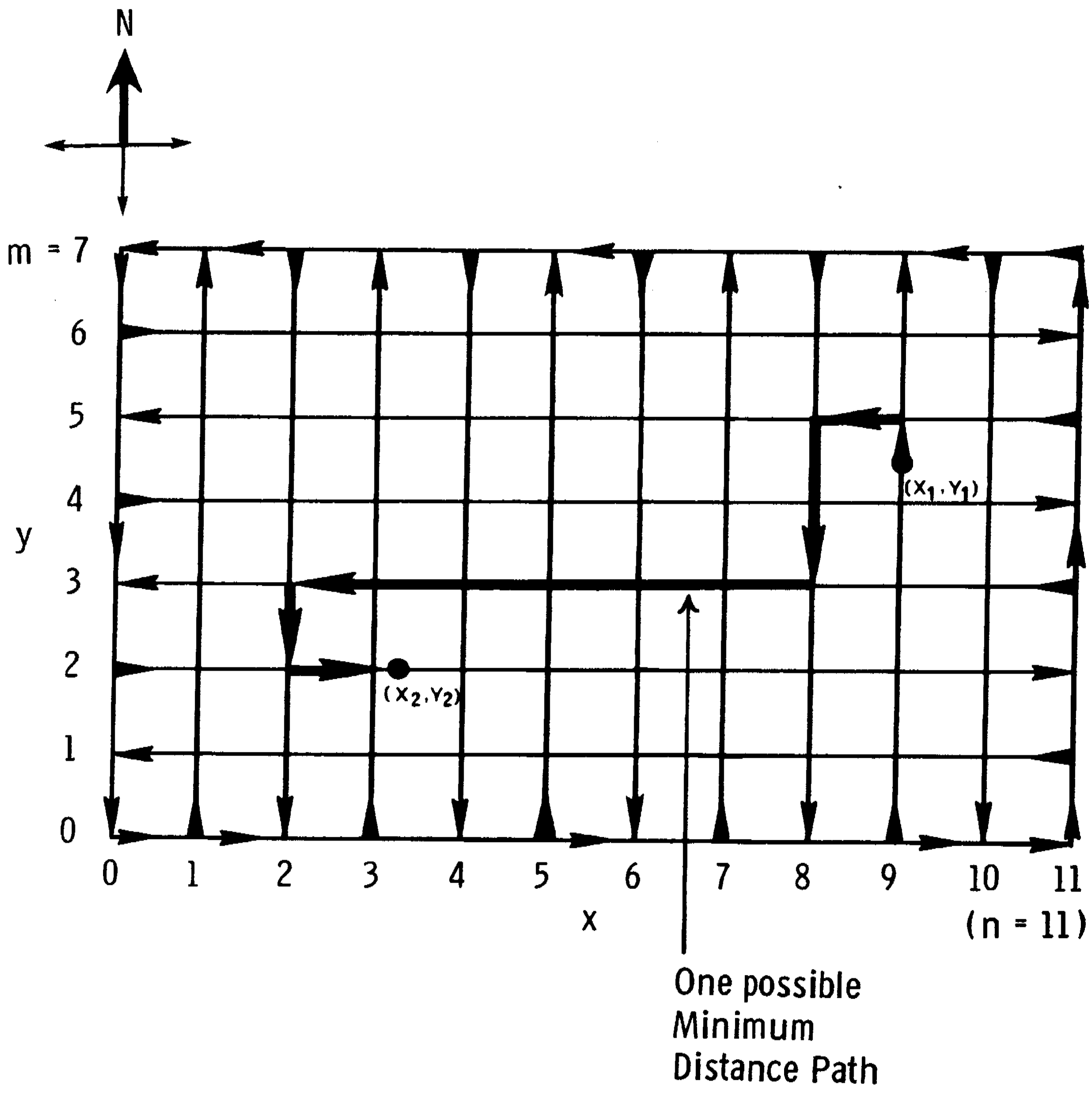
D_e = extra distance travelled from (X_1, Y_1) to (X_2, Y_2) due to the one-way constraints.

On a grid without one-way constraints, the response distance D can be written

$$D = \begin{cases} |U_1 - U_2| & \text{if incident and response unit are on the same block and the same street,} \\ K + \mathrm{MIN}[U_1 + U_2,\ 2 - U_1 - U_2] & \text{if incident and response unit are on parallel blocks and parallel streets,}^{*} \\ K + U_1 + U_2 & \text{otherwise,} \end{cases}$$

*Two blocks are parallel if one is directly East of the other or if one is directly North of the other. Otherwise, they are non-parallel. Two streets are parallel if they both run N-S or if they both run E-W.

FIG. 4: DISCRETE GRID REPRESENTING RECTANGULAR RESPONSE AREA WITH ONE-WAY STREETS



where

U_1, U_2 are u.i. distributed on [0,1];

K = number of street segments (block lengths) to traverse between the segment originally containing the response unit and the segment containing the incident.

The random variable U_1 is the distance the response unit has to travel to the first street intersection encountered on the response path. U_2 is the distance the response unit has to travel from the last street intersection to the incident.

We will be concerned with the asymptotic behavior of $E[D_e]$ for large regions and thus we can assume that the probability is negligibly small that the incident and response unit fall on both parallel blocks and parallel streets or on the same block and same street. Now, if we impose one-way constraints (assuming incident and response unit are not on both parallel blocks and parallel streets or on the same block and same street), the new response distance can be written

$$D' = U_1' + U_2' + K'$$

$$= U_1' + U_2' + K + K_e$$

where

U_1', U_2' are u.i. distributed on [0,1];

K' = total number of segments to traverse between the segment originally containing the response unit and the segment containing the incident;

K_e = extra number of segments to traverse due to the one-way constraints.

Note that since

$$D_e = U_1' + U_2' + K + K_e - (K + U_1 + U_2),$$

we have

$$E[D_e] = E[K_e].$$

We will thus focus our attention on the asymptotic probability behavior of K_e.

In fact, we wish to show that as the area of the response area becomes increasingly large, the asymptotic probability distribution of K_e is given as follows:

$$P[K_e = i] = \begin{cases} \frac{1}{4} & i=0,1,3 \\ \frac{1}{8} & i=4 \\ \frac{1}{16} & i=2,6 \\ 0 & \text{otherwise.} \end{cases} \qquad (30)$$

Thus, the mean extra distance travelled (in block lengths) due to one-way constraints is

$$E[D_e] = E[K_e] = \sum_i i\ P[K_e = i] = 2. \qquad (31)$$

The remainder of this section is devoted to proving (31). The nonmathematical reader may wish to skip to the summary and discussion in Section V.

To obtain this result (31) we can always assume the situation illustrated in Fig. 5, where the origin of the grid ($x = 0$, $y = 0$) is the base (lower vertex) of the street segment originally containing the response unit.* For convenience we again label directions of travel East (E), West (W), North (N), and South (S). A street segment is denoted by the coordinates of its end-points (vertices), ordered according to the direction of travel. For instance, the segment originally containing the response unit is segment [(0,0), (0,1)].

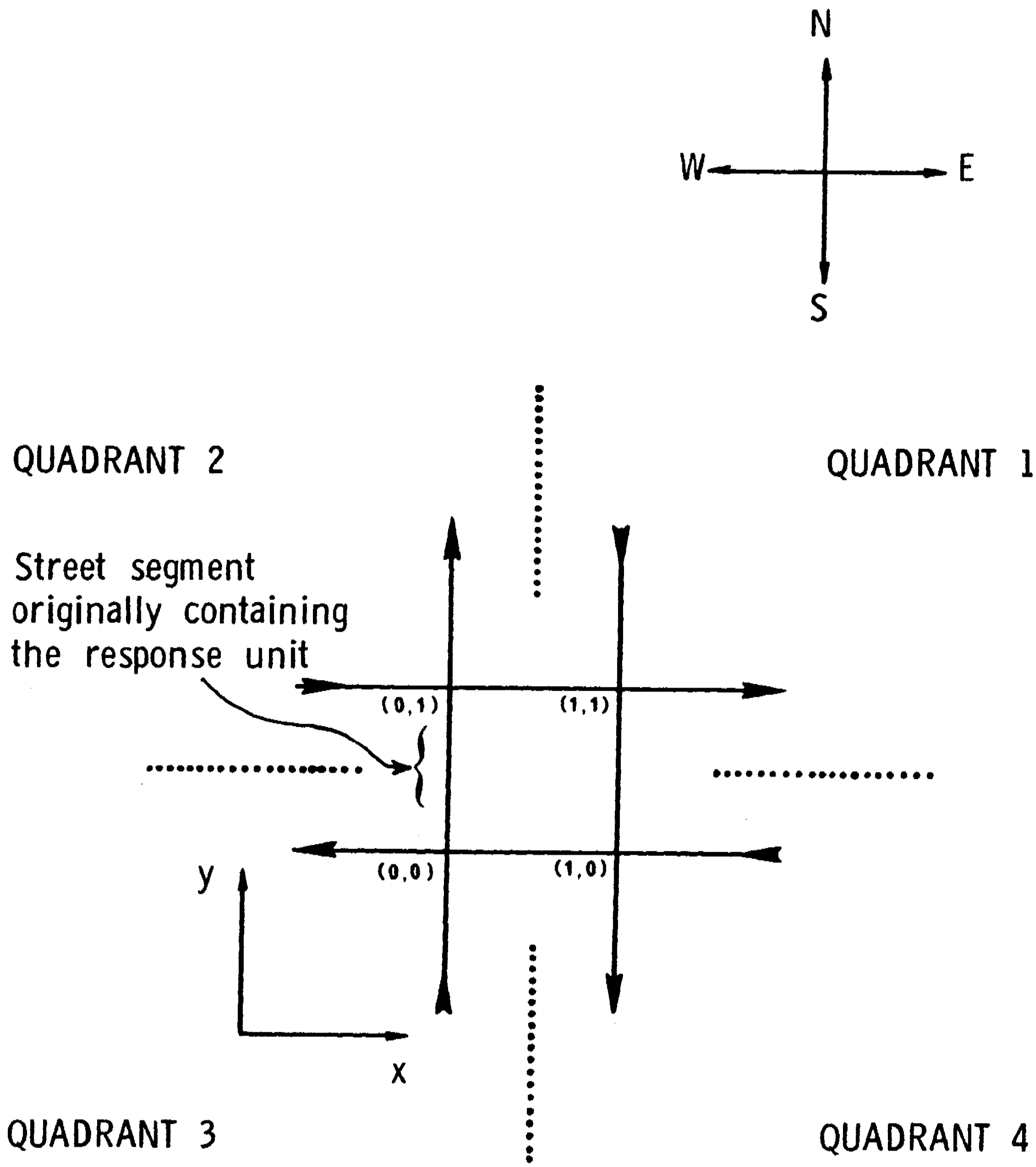
In order to study the case of one-way constraints, it is useful first to collect some results for the case of no one-way constraints. Usually, the latter case is denoted by unprimed variables (e.g., D, K, U_1, U_2) and the former case is denoted by primed variables (e.g., D', K', U_1', U_2').**

---

*The situation in Fig. 5 is obtained by translating the origin of the coordinate system to the base of the segment containing the response unit, rotating it so that the other vertex is point ($x = 0$, $y = 1$), and perhaps "flipping" the x-axis.

**The obvious exception is the variable K_e.

FIG. 5: REDEFINED COORDINATE SYSTEM FOR A LARGE GRID OF ONE-WAY STREETS



For the case of no one-way constraints, define the random variable

$$D\Big((k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2);\ (i,j),\ (i+\delta_1,\ j+\delta_2)\Big) \equiv$$

total travel distance from the initial position of the response unit on segment $[(k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2)]$ to the incident on segment $[(i,j),\ (i+\delta_1,\ j+\delta_2)]$, assuming no one-way constraints.*

The pairs $\{\Delta_1,\ \Delta_2\}$ and $\{\delta_1,\ \delta_2\}$ can assume values $\{1,0\}$, $\{0,1\}$, $\{-1,0\}$, and $\{0,-1\}$. The number of segments to traverse between segment $[(k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2)]$ and segment $[(i,j),\ (i+\delta_1,\ j+\delta_2)]$, given no one-way constraints, is

$$K\Big((k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2);\ (i,j),\ (i+\delta_1,\ j+\delta_2)\Big) =$$

$$\mathrm{MIN}\Big(|i-k|,\ |i-k-\Delta_1|,\ |i+\delta_1-k|,\ |i+\delta_1-k-\Delta_1|\Big) \tag{32}$$

$$+\ \mathrm{MIN}\Big(|j-\ell|,\ |j-\ell-\Delta_2|,\ |j+\delta_2-\ell|,\ |j+\delta_2-\ell-\Delta_2|\Big).$$

Thus, the total travel distance, given no one-way constraints, can be written

$$D\Big((k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2);\ (i,j),\ (i+\delta_1,\ j+\delta_2)\Big) =$$

$$K\Big((k,\ell),\ (k+\Delta_1,\ \ell+\Delta_2);\ (i,j),\ (i+\delta_1,\ j+\delta_2)\Big) + U_1 + U_2. \tag{33}$$

Since we are concerned with responses from segment $[(0,0),\ (0,1)]$, for notational compactness we define

$$K\Big((0,0),\ (0,1);\ (i,j),\ (i+\delta_1,\ j+\delta_2)\Big) \equiv K\Big((i,j),\ (i+\delta_1,\ j+\delta_2)\Big). \tag{34}$$

---

*With no one-way constraints, the two notations $[(i,j),\ (i+\delta_1,\ j+\delta_2)]$ and $[(i+\delta_1,\ j+\delta_2),\ (i,j)]$ both refer to the same (two-way) segment.

From (32) we have

$$K\Big((i,j),\ (i+\delta_1,\ j+\delta_2)\Big) = \begin{cases} \mathrm{MIN}\Big(|i|,\ |i+\delta_1|\Big) + \mathrm{MIN}\Big(|j-1|,\ |j-1+\delta_2|\Big), & j \geq 1 \\ \mathrm{MIN}\Big(|i|,\ |i+\delta_1|\Big) + \mathrm{MIN}\Big(|j|,\ |j+\delta_2|\Big), & j \leq 0. \end{cases} \tag{35}$$

The shortest distance between <u>vertex</u> (i,j) and <u>vertex</u> (k,ℓ), given no one-way constraints, is

$$S[(i,j),\ (k,\ell)] \equiv |i-k| + |j-\ell|. \tag{36}$$

Now we also define

$K(i,j) \equiv$ number of segments to traverse between <u>segment</u> $[(0,0),\ (0,1)]$ and <u>vertex</u> (i,j), given no one-way constraints.

Clearly,

$$K(i,j) = \begin{cases} |i| + |j-1| & j \geq 1 \\ |i| + |j| & j \leq 0 \end{cases} \tag{37}$$

This completes our summary of grids with no one-way constraints.

The primed variables (D', K', U_1', U_2') have definitions identical to the corresponding unprimed variables, except one-way constraints are assumed.

Given one-way constraints, we can write

$$K'\Big((i,j),\ (i+\delta_1,\ j+\delta_2)\Big) = K'(i,j), \tag{38}$$

since the number of segments required to get to segment $[(i,j),\ (i+\delta_1,\ j+\delta_2)]$ is equal to the number of segments required to get to vertex (i,j).

Now we consider response to incidents in each of the four quadrants, where we define*

quadrant 1: All segments $[(k,\ell), (k+\delta_1, \ell+\delta_2)]$, where $k \geq 0$, $\ell \geq 1$, $k + \delta_1 \geq 0$, $\ell + \delta_2 \geq 1$;

quadrant 2: All segments $[(k,\ell), (k+\delta_1, \ell+\delta_2)]$, where $k \leq 0$, $\ell \geq 1$, $k + \delta_1 \leq 0$, $\ell + \delta_2 \geq 1$ (excluding segments $[(0,\ell), (0, \ell+1)]$);

quadrant 3: All segments $[(k,\ell), (k+\delta_1, \ell+\delta_2)]$, where $k \leq 0$, $\ell \leq 0$, $k + \delta_1 \leq 1$, $\ell + \delta_2 \leq 0$;

quadrant 4: All segments $[(k,\ell), (k+\delta_1, \ell+\delta_2)]$, where $k \geq 1$, $\ell \leq 0$, $k + \delta_1 \geq 0$, $\ell + \delta_2 \leq 0$.

Consider quadrant 1. The quadrant comprises segments of the following types with corresponding relative frequencies of occurrence:

Segment type $(i,j=1,2,\ldots)$		Relative Frequency
East:	$[(i-1, 2j-1), (i, 2j-1)]$	$\frac{1}{4}$
North:	$[(2(i-1), j), (2(i-1), j+1)]$	$\frac{1}{4}$
West 1:	$[(2i, 2j), (2i-1, 2j)]$	$\frac{1}{8}$
South 1:	$[(2i-1, 2j+1), (2i-1, 2j)]$	$\frac{1}{8}$
West 2:	$[(2i-1, 2j), (2i-2, 2j)]$	$\frac{1}{8}$
South 2:	$[(2i-1, 2j), (2i-1, 2j-1)]$	$\frac{1}{8}$

First consider East segments and North segments. We wish to show that

*Note that our four quadrants encompass all points for which the incident and response unit do not fall on both parallel blocks and parallel streets.

$$K'[(i-1,2j-1),\ (i,2j-1)] = K'[i-1,2j-1] = K[(i-1,2j-1),\ (i,2j-1)] \tag{39}$$

$$K'[(2(i-1),j),\ (2(i-1),j+1)] = K'[2(i-1),j] = K[(2(i-1),j),\ (2(i-1),j+1)]. \tag{40}$$

Since from (35)

$$K[(i-1,2j-1),\ (i,2j-1)] = i - 1 + 2j - 2,$$

to prove (39) we need only demonstrate a feasible path from vertex $(0,1)$ to vertex $(i-1,2j-1)$ of length $i + 2j - 3 = S[(0,1),\ (i-1,2j-1)]$. To prove (40), the path must be from vertex $(0,1)$ to vertex $(2(i-1),j)$ and of length $S[(0,1),(2(i-1),j)]$. Such paths, if demonstrated, must be minimal distance paths since there exist no shorter paths with no one-way constraints.

Let

$$K_n^*[k,\ell] \equiv \begin{cases} K'[k,\ell] & \text{if } n \geq K'[k,\ell], \\ \\ +\infty & \text{if } n < K'[k,\ell], \end{cases}$$

for all $n=0,1,2,\ldots$. We have the boundary condition

$$K_0^*[0,1] = 0 = K'[0,1]. \tag{41}$$

We can now compute $K'[k,\ell]$ for all $[k,\ell]$ by recursively using the relation

$$K_{n+1}^*[k,\ell] = 1 + \underset{\{\delta_1,\delta_2\}}{\mathrm{MIN}} \left\{K_n^*[k+\delta_1,\ \ell+\delta_2]\right\},\ n=0,1,2,\ldots . \tag{42}$$

For the interesting values of n, (42) states that the minimal distance $(n+1)$ to vertex (k,ℓ) is one plus the minimal distance (n) to an adjacent vertex.

For instance, applying (42) (with (41)) once, clearly

$$K_1^*[0,2] = K_1^*[1,1] = 1.$$

Using (42) repeatedly with increasing ℓ, we obtain

$$K_{\ell-1}^*[0,\ell] = \ell-1, \qquad \ell=1,2,\dots\ .$$

In particular,

$$K_{2j-2}^*[0,2j-1] = 2j-2.$$

Now, by successively increasing k, we obtain

$$K_{(k+2j-2)}^*[k,2j-1] = k+2j-2, \qquad k=0,1,2,\dots\ .$$

In particular,

$$K_{(i-1+2j-2)}^*[i-1,2j-1] = i+2j-3 = K'[i-1,2j-1],$$

as was to be shown. Using this path-tracing technique, we have demonstrated a feasible response path that goes North to $y = 2j-1$, then East to $x = i-1$. A similar argument, with the path going East then North proves (40). Thus,

$$K_e[(i-1,2j-1),\ (i,2j-1)] = K_e[(2(i-1),j),\ (2(i-1),j+1)] = 0. \qquad (43)$$

Now consider a West 1 segment. From (38),

$$K'[(2i,2j),\ (2i-1,2j)] = K'[2i,2j].$$

From (40),

$$K'[2i,2j] = 2j-1 + 2i.$$

But from (35),

$$K[(2i,2j),\ (2i-1,2j)] = 2j-1 + 2i-1.$$

Thus,

$$K_e[(2i,2j),\ (2i-1,2j)] = 1. \tag{44}$$

A similar argument shows that for South 1 segments

$$K_e[(2i-1,2j+1),\ (2i-1,2j)] = 1. \tag{45}$$

Now consider a West 2 segment. From (42) it is clear that

$$\begin{aligned} K'[(2i-1,2j),\ (2i-2,2j)] &= \mathrm{MIN}\begin{bmatrix} 1 + K'[(2i,2j),\ (2i-1,2j)] \\ 1 + K'[(2i-1,2j+1),\ (2i-1,2j)] \end{bmatrix} \\ &= \mathrm{MIN}\begin{bmatrix} 1 + K'[2i,2j] \\ 1 + K'[2i-1,2j+1] \end{bmatrix}. \end{aligned} \tag{46}$$

But, from (39), (40), and (35),

$$K'[2i,2j] = K'[2i-1,2j+1] = 2i + 2j-1.$$

Thus,

$$K'[(2i-1,2j),\ (2i-2,2j)] = 2i + 2j. \tag{47}$$

But from (35),

$$K[(2i-1,2j),\ (2i-2,2j)] = 2i + 2j-3. \tag{48}$$

Subtracting (48) from (47),

$$K_e[(2i-1,2j),\ (2i-2,2j)] = 3. \tag{49}$$

A similar argument shows that for South 2 segments

$$K_e[(2i-1,2j),\ (2i-1,2j-1)] = 3. \tag{50}$$

Now if we define

$$P_{K_e}[n|Q_i] = \text{Prob}[K_e = n|\text{incident in quadrant } i],$$

from (43), (44), (45), (49), and (50), we have

$$\begin{aligned} P_{K_e}[0|Q_1] &= \frac{1}{2} \\ P_{K_e}[1|Q_1] &= \frac{1}{4} \\ P_{K_e}[3|Q_1] &= \frac{1}{4}\ . \end{aligned} \tag{51}$$

A similar argument applies to quadrant 2 to show that

$$P_{K_e}[n|Q_2] = P_{K_e}[n|Q_1]. \tag{52}$$

Now consider quadrant 4. Consider a [(2i-1,j), (2i-1,j-1)] segment. By tracing a path from (0,1) East to (2i-1,1), then South to (2i-1,j), we demonstrate that

$$K'[(2i-1,j),\ (2i-1,j-1)] = K'[(2i-1,j)] = |j| + 2i.$$

This must be a shortest path since

$$S[(0,1),\ (2i-1,j)] = |j| + 2i.$$

But, from (35),

$$K[2i-1,j] = 2i + |j| - 1.$$

Thus,

$$K_e[(2i-1,j),\ (2i-1,j-1)] = 1.$$

In a similar manner we can show that corresponding to each segment type in quadrant 1 there is a segment type in quadrant 4 for which the travel distance is one segment greater than that in quadrant 1. Thus,

$$P_{K_e}[n|Q_4] = P_{K_e}[n-1|Q_1]. \tag{53}$$

Finally, consider quadrant 3. Consider a $[(2i-1,j),\ (2i-1,j-1)]$ segment. Clearly, we can write

$$K'[(2i-1,j),\ (2i-1,j-1)] = \mathrm{MIN}\begin{bmatrix} 1 + K'[(0,1),(1,1);\ (2i-1,j),(2i-1,j-1)] \\ \\ 1 + K'[(0,1),(0,2);\ (2i-1,j),(2i-1,j-1)] \end{bmatrix}. \tag{54}$$

But path tracing indicates that

$$\begin{aligned} K'[(0,1),(1,1);\ (2i-1,j),(2i-1,j-1)] &= K'[(0,1),(0,2);\ (2i-1,j),(2i-1,j-1)] \\ &= S[(1,1),(2i-1,j)] \\ &= |2i| + |j| + 3. \end{aligned} \tag{55}$$

Thus, substituting (55) in (54),

$$K'[(2i-1,j),(2i-1,j-1)] = 4 + |2i| + |j|. \qquad (56)$$

But, from (35),

$$K[(2i-1,j),(2i-1,j-1)] = |2i-1| + |j|. \qquad (57)$$

Subtracting (57) from (56), we have

$$K_e[(2i-1,j),(2i-1,j-1)] = 3. \qquad (58)$$

In a similar manner we can show that corresponding to each segment type in quadrant 1 there is a segment type in quadrant 4 for which the travel distance is 3 segments greater than that in quadrant 1. Thus,

$$P_{K_e}[n|Q_3] = P_{K_e}[n-3|Q_1]. \qquad (59)$$

Finally, the desired probability distribution is obtained by writing

$$P_{K_e}[n] = \sum_{i=1}^{4} P_{K_e}[n|Q_i]\ P[Q_i]. \qquad (60)$$

Since on a large grid an incident is equally likely to be in each of the four quadrants,

$$P[Q_i] = \frac{1}{4}, \quad i=1,2,3,4. \qquad (61)$$

Substituting (51), (52), (53), (59), and (61) into (60), we obtain the probability distribution given in (30).*

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*A little thought should convince one that (51), (52), (53), and (59) also hold for a system in which units are dispatched from fixed locations (all other assumptions remaining unchanged). For such a system, the probability distribution of K_e is obtained from (60), where in general $P[Q_i] \neq \frac{1}{4}$.

V. SUMMARY AND DISCUSSION

We have considered various models of response to incidents in urban environments that are complicated by (1) barriers, (2) confinement to two-way streets, and (3) confinement to one-way streets.

In Section II, Eq. (7) gives the mean right-angle response distance for a unit which is randomly positioned to an incident whose position is random and independent of the unit's position, given that the response area is rectangular and that a simple barrier in the response area impedes travel. The interesting feature of (7) is the relative insensitivity of the mean response distance to small and moderate-sized barriers (see Fig. 2). Equation (9) gives the mean response distance for the case in which the barrier completely partitions the response area, with one crossing point allowing travel between the partitioned regions (again see Fig. 2).

In Section III, we examined the case in which response units and incidents are confined to a discrete rectangular grid of equally spaced two-way streets forming square blocks of unit area (Fig. 3). It was assumed that the positions of the response unit and the incident are independent and uniformly distributed over the grid. The response distance between the response unit's initial position and the incident position was assumed to be a shortest path that remains on the streets of the grid. An exact expression for the mean response distance was obtained (Eq. (28)). Examination of this equation revealed that the continuous approximation is never in error by more than 1/3 of a block length. Thus, except for response areas of very few blocks, the continuum formulation is a very good approximation to the more exact discrete formulation.

In Section IV, we modified the model of Section III and assumed that response units and incidents are confined to a discrete grid of equidistant one-way streets, the direction of travel alternating from street to street (Fig. 4). As in Section III, we assumed that the positions of the response unit and the incident are independent and uniformly distributed over the grid. We also assumed that the response distance from the response unit's initial position to the incident position is a shortest path that remains on the streets of the grid and that obeys the one-way constraints. We found in (31) that for large grids the average extra distance travelled due to one-way streets is two block lengths. We also found that in (30) 25 percent of all responses require no additional travel distance because of one-way constraints. On the other hand, slightly over 6 percent of all responses require an additional mean travel distance of six block lengths. If response speed is 12 mph and a block length is 0.1 mile, the one-way constraints cause mean travel time to be increased 3 minutes in these cases. Heuristically, the intermediate results of Section IV state the following:

(1) If the incident is "in front" of the response unit (i.e., if the response unit does not effectively have to make a "U-turn"), the average extra distance travelled due to one-way constraints is one block length;

(2) If the incident is "behind" the response unit and if the next cross street allows the unit to turn in the direction of the incident, the average extra distance travelled due to one-way constraints is two block lengths;

(3) If the incident is "behind" the response unit and if the next cross street does not allow the unit to turn in the direction of the incident, the average extra distance travelled due to one-way constraints is four block lengths.

Since all models assumed a "roving" response unit,* they are particularly applicable to, say, police department operations where the response units correspond to police cars on patrol. The same techniques (e.g., "path tracing," adding perturbation variables) are applicable to systems (e.g., fire departments, emergency ambulance services) in which the response unit is dispatched from fixed locations and to systems with more complicated spatial descriptions.

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*An exception is discussed in the footnote on page 28.