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Game Theory in Economics-Chapter 3: The "Rules of the Game"
L. S. Shapley and M. Shubik

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PREFACE

This is the third chapter of a manuscript dealing with the applications of game theory to economic analysis. Work on this manuscript is now supported by the National Science Foundation (Grant GS-31253), in continuation of earlier support by United States Air Force Project RAND, as well as by The Rand Corporation itself, Yale University, and The Cowles Foundation for Research in Economics.

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SUMMARY

This chapter contains an overview of the concepts, methods, and formal models that are used in game theory to describe the possible courses of action in a multiperson competitive situation. Among the topics considered are the extensive and strategic forms of a game, Kuhn trees, information sets, pure, mixed, and behavior strategies, recurrent positions, stochastic games, duels, differential games, matrix representations, games on the square, and nonnumerical payoffs. There are many simple examples, and an extensive bibliography is provided.
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CHAPTER 3

THE "RULES OF THE GAME"

Oh, what a tangled web we weave... Scott

1. INTRODUCTION

The theoretical literature on bilateral monopoly, duopoly, and other situations that involve substantial bargaining rightly devotes much attention to the dynamics of individual moves. The details of process, be they price declarations, offers, counteroffers, threats, ploys, or bluffs, are often decisive in the resolution of the conflict. The information pattern is no less decisive as it unfolds over time, and the exact description of "who knows, what, when" can be extremely complex in a multilateral, multimove model. Game theory provides a framework and a formal language for dealing with these and all other details of the "rules of the game". This chapter will be devoted to an exposition and survey of this important area of descriptive game theory.

Nevertheless, our later interests will lead us away from details of process. The question of tactical skill will not usually be an issue in the kind of economic models that we shall study. Our point of view is that in the most important economic situations, the existing procedural rules are only of secondary significance; not only do they result from a long evolution, but they remain forever
subject to modification or repeal when they stand in the way of underlying economic or political forces. This applies particularly to the procedures and traditions surrounding the workings of the marketplace.*

Thus, in most of our illustrative examples and larger models, the procedural rules of the game will be stated verbally and rather informally. Unless there is a particular point to be made, we shall forgo any direct use of the exact descriptive tools discussed in this chapter. To the reader uninterested in formalism for its own sake, this may come as a relief. The reader more inclined to mathematical rigor should in every case be able to construct at least one set of formal procedural rules that will satisfy the informal description given.

*For example a commodity exchange might, by rule or custom, quote prices only in units of 1¢. Nevertheless a mathematical model that ignored that rule and permitted a continuum of prices might actually be more realistic in the long run, since it is a fair assumption that the 1¢ rule would soon be revised if it were found that substantial economic values turned on the possibility of smaller increments.

This point will be developed in more detail in Sec. 4, below.
2. THE CONCEPT OF A STRATEGY

Two descriptive forms in game theory are often contrasted: the extensive form and the strategic form.* In the former, we set forth each possible move and information state in detail, throughout the course of the play. In the latter, we content ourselves with a tabulation of overall strategies, together with the outcomes or payoffs that they generate.

A strategy, in this technical sense, means a complete description of how a player intends to play a game, from beginning to end. The test of completeness of a strategy is whether it provides for all contingencies that can arise, so that a secretary or agent or programmed computer could play the game on behalf of the original player, without ever having to return for further instructions.**

For a source of simple illustrations we shall turn from the rich field of balloon-popping to the still richer field of art dealing.***

*Called "normalized form" or "normal form" by other authors; the present term, however, seems more explanatory.

**Borel (1921). If negotiations or agreements are envisaged outside the formal rules, they must be assumed to take place before the selection of strategies, not during the actual course of play. Otherwise, the strategic form loses some validity as a descriptive tool.

NB. An incomplete citation (usually author and date) means that the item in question is listed in the Bibliography.

AN ART AUCTION. A painting is to be sold to the highest bidder. After each accepted bid, any player may make a higher bid. If several bids are entered at once, the auctioneer accepts the one nearest the rostrum.

Examples of strategies:

Player $P_1$ (nearest to the rostrum): "I will start bidding at $15 and increase my bid to $2 above anyone else's, provided that this is not higher than $24; otherwise I bid $24 if I can and then stop."

Player $P_2$: "I will bid $17 initially unless someone bids more before I get in. I will enter only this one bid, or no bid at all."

Player $P_3$: "I will wait until someone else has bid, then raise in units of $1 up to a top of $19, unless I am bidding against player 2, in which case my top is $22."

If there are no other players, the following sequence of bids would result from this trio of strategies (parenthesis indicate a simultaneous bid, not accepted):

<table>
<thead>
<tr>
<th>1st round:</th>
<th>$P_1$ bids $15$ ($P_2$ bids $17$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd round:</td>
<td>$P_2$ bids $17$ ($P_3$ bids $16$)</td>
</tr>
<tr>
<td>3rd round:</td>
<td>$P_1$ bids $19$ ($P_3$ bids $18$)</td>
</tr>
</tbody>
</table>

Table 1
If chance factors are present, a richer variety of lines of play can result, without necessarily increasing the complexity of the strategies themselves. To illustrate, we might modify the rules to require the auctioneer to choose at random among simultaneous bids. In order to describe the variety of plays that could result, if the players adopt the same three strategies as above, a "probability tree" diagram like Fig. 1 is useful. The notation "17_2" means an accepted bid of $17 by player P_2, etc., and the heavy dots separate equiprobable events. The probability of reaching a particular "end" is therefore \((1/2)^k\), where \(k\) is the number of heavy dots on the path. Thus, with probability 1/8 the left-most path is followed, and \(P_1\) gets the painting for $21.

2.1. Are Strategies Practical?

There is, in principle, no sacrifice of flexibility in adhering to a single, fixed strategy. Strategies that can be described in a few words, however, tend to produce a mechanical style of play. Highly adaptive behavior, if codified in strategic form, usually requires a lengthy and complex description. The reader will note that even the simple strategies illustrated above contain provision for many contingencies that did not arise in Table 1, or even in Fig. 1. In more realistic examples, a dismaying large part of the planning that goes into formulating a strategy
Fig. 1—A probability tree
is wasted when the game is played. Since planning and computing are not costless, the strategic form is almost never used in practice in multimove games. Its value is primarily conceptual and theoretical.*

Indeed, in multimove games the sheer number of strategies is usually such as to preclude any hope of a manageable enumeration, and other techniques must be found for practical work. Some calculations based on the game of Chess will help to bring home the astonishing scale of the problem.

Consider first a trivial game, which we might call "One-Move-Chess". Play is halted after one move on each side, and some arbitrary payoff is designated for each of the 400 possible end positions. In this game, White has just 20 strategies, corresponding to his 20 possible opening moves. But Black already has \(20^{20}\) strategies (about \(10^{26}\), or one hundred septillion!). We emphasize that these strategies are all operationally different, since any two of them will make different responses to at least one of White's moves.

Proceeding to "Two-Move-Chess", we have calculated that White has \(7.886 \times 10^{29}\) strategies, while for Black the number is in excess of \(10^{52}\). In "Forty-Move-Chess"

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*Rabin (1957) has shown using the theory of recursive functions that there are certain two-person win-lose games in which one player has a winning strategy but no "effectively computable" winning strategy, so that it would be impossible, e.g., to write a computer program for him that would defeat any opponent. Rabin's game has infinitely many positions, but the rules, unlike the winning strategies, are effectively computable, and each play is finite in length.
we estimate more than $10^{1000}$ on each side. For the real
game of Chess an estimate is more difficult to make, but
hyperastronomical numbers like $10^{10000}$ or $10^{20000}$ seem to
be indicated, since most strategies must include contingency-laden detailed instructions for playing an almost
endless variety of meandering games, continuing sometimes
for hundreds of moves until a fortuitous checkmate or
stalemate, a thrice-repeated position, a lack of mating
strength on both sides, or the "forty moves without pro-
gress" rule makes an end.

Note that the number of strategies is far greater
than the number of possible courses of play; this is typ-
ical in multimove games.

2.2. Perfect Information and Mixed Strategies

If at every point of time during the play of a game,
each player with a decision to make is competely apprised
of the state of affairs, then we have a game of perfect
information.* Of our examples thus far, the three-person
"Shooting Match" was of this type, but the "Dart Duel" and
"Art Auction" fell just short, since simultaneous actions
were possible. (They are examples of "almost-perfect" in-
formation; see Secs. 3.4 and 3.8 below.) It is easy to
think of examples where significant information is longer
deferred (e.g., the contents of an opponent's hand in a
card game, the effects of an advertising campaign, etc.),
or even withheld completely from some of the players.

*Von Neumann and Morgenstern (1944).
When perfect information is lacking in a game, a new strategic element becomes of great importance in the analysis of optimum play: the deliberate randomization of decisions. In mathematical terms, one sets up a probability distribution, called a mixed strategy, over the original set of "pure" strategies, and lets it control the actual play of the game. It is perhaps intuitively clear that doing this is of no help in a game of perfect information like Chess; one has no secrets to dissemble.* On the other hand, in the Chess players' little preliminary game of "Choosing Colors", in which one player conceals two pawns and the other picks a hand, it is clear that the only foolproof plan for either player is to randomize between right and left, with equal probabilities. Any other plan could be exploited by a knowledgeable opponent.**

2.3. Perfect Recall and Behavior Strategies

A further possibility is that a player at some point during the game may not be apprised of his own previous moves, or of the information on which they were based. We are thinking not of schizophrenics or amnesiacs, but of teams. North, at Bridge, must bid without knowing the contents of South's hand, yet North and South can usefully be regarded in other respects as agents of a single player.***

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*The first mathematical proof is due to Zermelo (1913).

**Note that the concealing player has perfect information, but nevertheless should randomize.

***G. L. Thompson (1953ab); E. Keeler (1971).
The technical term for this phenomenon is imperfect recall.*

In a game of imperfect recall it may be necessary for a player using a mixed strategy to make his random choice of strategy at the beginning of play, so that the actions of his agents will be properly correlated. In a game of perfect recall, on the other hand, the randomizations can be deferred until the actual decision points are reached, since there is no scope for profitable correlations.** Mixed strategies of this kind, with their on-the-spot randomizations, are called behavior strategies in the literature; they are far easier to work with, both in theory and practice, than general mixed strategies.***

2.4. Conclusion

When we reduce a game description from extensive to strategic form, we lose sight of the actual pattern of information. The strategic form has, of course, its own peculiar information pattern: each player makes just one move (a strategy choice), in ignorance of what the others are doing. There is perfect recall, trivially, but not perfect information. Mixed strategies may therefore be

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*Kuhn (1950b).

**The laws of Bridge appear to prohibit general mixed strategies, which would entail private understandings between partners. It seems likely, for this reason, that Bridge has no optimal strategies. See G. L. Thompson (1953b) and Keeler, op. cit.

***Nash and Shapley (1950), Kuhn (1950ab, 1953), Dalkey (1953).
worthwhile, and we cannot afford to dismiss them out of hand. But if we happen to know that the original model in extensive form had perfect information, then we can safely proceed to analyze the model in strategic form on the basis of pure strategies alone.

These informational concepts play only a minor role in the static economic models that will be our first concern, but their significance for economic analysis in general is far from negligible. To mention just two points: imperfect information provides a rational basis for explaining deliberately random behavior; imperfect recall pinpoints the fundamental informational obstacle to achieving optimality through decentralized decisionmaking.

So far we have discussed strategies and information without reference to any exact descriptive model. In the next section we shall sketch a series of formal models that can be used to provide a rigorous basis for the study of the move-and-information structures of various classes of games in extensive form.

EXERCISES. 1. Show that the number of strategies for the first player in Tic-Tac-Toe, disregarding the symmetry, lies between $9^{78} \cdot 5^6 = 810,518,890,625$ and $9^8 \cdot 5^6 \cdot 3^4 = 65,652,030,140,625$.

2. In a two-player game, $P_1$ is an individual and $P_2$ is a team consisting of two people $P_{21}$ and $P_{22}$. Everybody simultaneously calls out "Heads" or "Tails". If the
three calls are the same, then $P_2$ wins; otherwise, $P_1$ wins. Show that $P_2$ has a mixed strategy that wins with probability $1/2$, but no behavior strategy that wins with probability greater than $1/4$. 
3. MODELING THE EXTENSIVE FORM

The earliest general-purpose formal systems for describing games in extensive form were those of von Neumann and Morgenstern (1944) and Kuhn (1950b, 1953).** The one uses an approach through sets and partitions while the other leans more heavily on graphical or diagrammatic ideas; but the two approaches are nevertheless almost equivalent. Both are directly concerned only with finite games--i.e., games in which the number of players, the number of choices at each decision point, the number of attainable intermediate and terminal positions, and the number of moves in each possible play-through of the game, are all finite. Games such as Chess and Go (with suitable stop rules), Poker and Bridge (considering each deal as a separate game), and many other parlor games fall into this category. But most competitive situations found in the worlds of business or politics are only imperfectly modeled as finite games, since they usually have continuous strategic parameters, a continuous time frame, and an indefinite continuation into the future.

We shall first sketch the Kuhn model, which with minor variations has become the generally accepted general-purpose model for the extensive form. Then we shall note more briefly a number of other models that are better adapted to accommodate certain classes of infinite games.

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*Some readers may prefer to go directly to Sec. 4, reserving the present rather long and detailed section for reference or for later study.

**For others, see McKinsey (1952), Otter and Dunne (1953), Isbell (1957), and the references given below in Sec. 3.4.
3.1. The Kuhn Game Tree

Consider the following simple example of a finite game.

FINGERS. The first player holds up one or two fingers, and the second player holds up one, two or three fingers. If the total displayed is even, then $P_1$ pays $5 to $P_2$; if it is odd, then $P_2$ pays $5 to $P_1$.

Suppose the rules further state that $P_1$ moves first. Then a graphical representation of the game can be drawn as shown in Fig. 2. This is an example of the special kind of connected graph known as a "rooted tree". The nodes (vertices) are labelled with symbols $P_i$ or $O_i$, for players or outcomes respectively. The starting node, or root of the tree, has a distinguishing mark, "o".

Each node in the tree represents a position or state in which the game might be found by an observer. A node labelled $P_1$ is a decision point for that player: he is called upon to select one of the branches of the tree leading "out" of that node, i.e., away from the root. In our example, $P_1$ has two alternatives, one finger or two fingers; accordingly we have labelled "1" and "2" the two edges leading away from the initial node. After $P_1$'s move, the play

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*A graph is a finite collection of nodes and edges, with a node attached to each end of each edge. The graph is connected if it is possible to go from any node to any other node along some path of connecting edges. In general there may be several paths between two given nodes, so that closed loops or cycles exist. But if the connecting paths are all unique, the graph is said to be acyclic and is called a tree. A rooted tree is a tree with one node distinguished, called the root.
Outcomes:  \( O_1 \): \((-5, 5)\)
\( O_2 \): \((5, -5)\)

Fig. 2
progresses to one of the two nodes marked "P_2"; at either of these, P_2 has three alternatives, which we have labelled "1", "2", "3". Finally a terminal position is reached, and an outcome O_1 is designated. Thus, any path through the tree, from the initial node to one of the terminals, corresponds to a possible play of the game. For example, if P_1 shows one finger and P_2 three, then the play shown by the heavy lines in Fig. 2 results, and, since 1 + 3 is even, P_2 wins $5 and P_1 loses $5.

If random elements are present, another type of node will occur. For notational purposes, we may pretend that chance, or "Dame Fortune", is a player, and use the label "P_0" for those nodes at which the tree branches due to a chance event. The edges leading out from such a node are labelled with the probabilities of the corresponding alternatives. Figure 3 illustrates this: the move structure is similar to that of Fig. 2, but a chance move has been inserted, between the two personal moves, that doubles the stakes with probability 1/4.

This notational apparatus equips us to handle any finite game of perfect information. The diagrams are virtually self-explanatory. But the apparent ease and concreteness of the representation is illusory, since in games of any complexity the tree will be unmanageably large and can be fully drawn only in the imagination.
Outcomes:

$O_1$: (-$5$, $5$)
$O_2$: ($5$, -$5$)
$O_3$: (-$10$, $10$)
$O_4$: ($10$, -$10$)

Fig. 3
EXERCISE. Estimate the number of nodes in the Kuhn tree for Tic-Tac-Toe.

3.2. Information Sets

Let us now modify the "Fingers" example to require \( P_2 \) to move first, thus reversing the order of play. The tree (a) of Fig. 4 then results. This is of course quite a different game—indeed, the advantage is now entirely with the first player, who can win five dollars with ease. The essence of the difference, however, is not in the timing, as such, but in the information pattern. It is not the fact of the opponent's prior move, but the knowledge of that fact, that conveys the advantage.*

To modify the example again, suppose that \( P_1 \) (still making the second move) is forbidden to look at \( P_2 \)'s fingers until his own are up. A method for indicating this rule is shown in Fig. 4(b): a "balloon" is drawn, enclosing the three game positions among which \( P_1 \) is not allowed to distinguish. The set of points so enclosed is called an information set.** Of course, a more complicated example

*As von Neumann and Morgenstern put it: "preliminarity implies anteriority, but need not be implied by it" (1944; p. 51). In discussing information lag in Sec. 3.7 we shall show how the timing of moves can sometimes be modified drastically without affecting the essential information pattern.

**It might better be called a "lack-of-information" set, as it portrays a reduction of the information available to the player. The larger an information set, the less the player knows about the situation.
Fig. 4
could have many information sets per player. Naturally, no two information sets can have a node in common; nor can a single information set intersect any given path in more than one node. If an information set consists of a single node, the "balloon" is usually omitted.

Note that the player-label is attached to the information set, rather than to the nodes that comprise it. Note also that every node in the same information set must have the same number of edges issuing from it, and the way in which these edges correspond to each other must be made clear--either by their labels or by some definite convention, like always reading counter-clockwise from the incoming edge.

For another, more substantial example, let us describe the beginning of the Kuhn tree for a hand of Bridge. The root is a chance move (the deal), having about $5 \times 10^{28}$ equiprobable branches. Each leads to a different node at which the Dealer must decide on a bid. The Dealer's nodes, however, are grouped into information sets (about $6 \times 10^{11}$ of them)--one set for each possible hand he may hold. The multitude of points within each set represent the unknown-to-him contents of the other hands.

After the Dealer's selection of one of his 36 possible calls,* the next rank of nodes, belonging to player $P_2$ on the Dealer's left, is reached. These nodes are grouped in

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*I.e., 35 bids or a pass.
a different way, to reflect the fact that \( P_2 \) is informed only of his own hand and the Dealer's call. (Any inferences he may make from the Dealer's bid are not "information" in our present, technical sense, no matter how "informative" the bid may have been in the practical sense.) There will be from 2 to 36 edges issuing from each of these nodes, depending on the Dealer's call,* and each one leads to a node belonging to \( P_3 \), the Dealer's partner. The continuation of the tree should now be apparent.

The information set provides a basis for mathematical definitions of a number of significant informational concepts. **Perfect information**, for example, is simply characterized by the statement that every information set is a singleton. Since a precise definition of **perfect recall** (Sec. 2.3) is somewhat fussy, we defer it to Part II.

Another important idea is that of a **position of complete information**: this is a node with the property that the branch of the tree that it defines—consisting

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*For example, "Seven No Trump" by the Dealer leaves only two alternatives: "Pass" and "Double".

**The term "complete information" (not to be confused with "perfect information") was used by von Neumann and Morgenstern (1944) to express an underlying assumption of their theory (as well as most subsequent theories), to wit, that the players at the beginning of the game are completely informed as to the precise state of affairs, and can make all necessary calculations. The study of "games of incomplete information", which arise in many practical contexts, presents formidable conceptual and technical difficulties; for references to some recent work in this area, see Sec. 3.2 of Ch. I and Sec. 3.7 of this chapter.
of that node and all following nodes and edges—is "informationally independent" of the rest of the tree. This means that there is no information set that ties any node of that branch to any node elsewhere in the tree. A position of complete information can be regarded as the starting point of a new game—a "subgame" of the original game—and it is often important to be able to recognize such positions.

EXERCISE. Diagram the "Fingers" game if there is a probability $p$ of an "information leak", revealing $P_2$'s choice to $P_1$. Consider two cases: (a) $P_2$ is not told in advance that the leak will occur; (b) he is told, with probability $q$, but $P_1$ is not told whether $P_2$ has been told.

3.3. Strategies in the Kuhn Tree

As we have already noted, a strategy is a complete plan of action, covering all contingencies. For a game in extensive form, the simplest definition of a strategy is a function that associates with each of a player's information sets one of the alternatives issuing from that set. When all players have fixed their strategies, it is as though in a toy railroad lay-out all the switches had been set, by the various players in control or by chance, with those belonging to a common information set set the same way. A train, starting at the root of the tree, runs through the network and arrives at a terminal, thereby tracing out the play of the game and disclosing the outcome.
A little reflection will reveal that this simple definition of strategy, though perfectly valid, may be very redundant. The decision points that can actually occur on a player's later moves are sharply reduced by his own earlier decisions. Yet a strategy, in the "toy train" sense, sets all the switches—even those that cannot be reached by the train. It would tell a Chess player, for example, what to do on his second move for every one of his possible first moves, even though the same strategy will have prescribed some particular first move.* To avoid the unnecessary tabulation of strategies that differ only in this vacuous way, a notationally more complicated but practically more efficient definition of strategy can be formulated; we shall not pursue the matter here.**

The formal description of randomized strategies presents no special difficulties in the finite Kuhn tree. A mixed strategy is any probability distribution over strategies (Sec. 2.2). We may think of a deck of cards being shuffled, with each card bearing a complete contingency plan for playing the game. Geometrically, the set of all mixed strategies can be represented in the form of

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*For another example we may look ahead to Fig. 6. Any strategy that instructs $P_1$ to play "1" at the beginning automatically makes the choice at his other node irrelevant. Of his six strategies, only four are operationally distinct.

**See e.g. Otter and Dunne (1953). We were in fact already using such a definition implicitly in Sec. 2.1 in our counts of Chess and Tic-Tac-Toe strategies.
a simplex, with one vertex for each pure strategy and with dimension equal to the number of such vertices minus one. Figure 5 illustrates.

As previously remarked (Sec. 2.3), it is often more convenient and more intuitive to introduce strategic randomization by means of behavior strategies, wherein each player pre-selects, at each information set that belongs to him, a probability distribution over the alternatives available there. Geometrically, the set of all behavior strategies is also a convex polyhedron, with one vertex for each pure strategy. It is not ordinarily a simplex, however, but a Cartesian product of simplices. Its dimension may be substantially less than the dimension of the corresponding mixed-strategy simplex. Figure 6 illustrates.*

Behavior strategies are more special than mixed strategies. That is, the effect of any behavior strategy can always be duplicated by some mixed strategy, but not conversely.** As already noted, however, in the important class of "games of perfect recall" (Sec. 2.3) the added possibilities for randomization afforded by mixed strategies are superfluous.

*For an instructive comparison of mixed and behavior strategies in certain Poker models, see Kuhn (1950a) and Nash and Shapley (1950).

Six pure strategies (See game tree in Fig. 6)  
(5-dimensional, projected)  

Fig. 5 — Examples of mixed strategy simplices
Behavior strategy set for $P_1$:
(compare Fig. 5)

= cartesian product of

and

Fig. 6 — Behavior strategies
EXERCISES. 1. There are three firms in an industry. First they make production plans and announce them simultaneously, then they select their prices and announce them simultaneously. Assuming two possible production levels and two possible price levels, show that each firm has 512 strategies, of which only 32 are essentially different.

2. In the above example, what is the dimension of the space of behavior strategies?

3.4. Simultaneous Moves

In many game models, the rules require the players to move simultaneously. The Kuhn tree does not provide a way to represent this; instead it gives us a choice among two or more equivalent representations, in which the moves are depicted in different temporal order, but with no information revealed.* Thus, Fig. 4(b) showed moves for \( P_1 \) and \( P_2 \) that are effectively simultaneous; this could equally well have been indicated as in Fig. 7(a). Since this is a common sort of situation, a more concise and symmetrical notation, as illustrated in Fig. 7(b), is often useful. The double labels indicate that two simultaneous decisions are to be made.

In order to use this notation, the players must be equally informed; usually, in fact, it is used only at a

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*The question of what makes two patterns of information equivalent has an extensive literature; see e.g. Shapley (1950), Krentnel, McKinsey and Quine (1951), F. B. Thompson (1952a), Dalkey (1953).
Fig. 7—Simultaneous moves
position where information is perfect except for the simultaneity. Indeed, in an important class of games, called simultaneous games or games of almost-perfect information, the players are always fully informed of everything that has happened in the past, and lack only information about the immediate present--i.e., what moves are being made concurrently with their own. With the aid of the notation illustrated, such games can be diagrammed without the use of "balloons", and each node that actually appears in the diagram represents a position of complete information.

A game in strategic form is, trivially, a simultaneous game. Presently we shall consider other less trivial examples of this informational form.

EXERCISE. Make a diagram for the example of the three firms (see the preceding exercises), using the simultaneous move notation. Compare the number of nodes, edges, and paths with the number of nodes, edges, and paths in an equivalent Kuhn tree.

3.5. Positional Games

The game tree is an extremely useful device for didactic purposes, but a high price is often paid for its use, in terms of redundancy. The rules of many games permit the same physical "position" to be reached through various different sequences of moves. Yet in a tree each sequence of moves must lead to a different node. The tree convention
forces us to remember the history of the position, whether we want to or not.

Thus, in Chess, different sequences often "transpose" into the same board position, after a lapse of two or more moves by each player. In such a case, it would evidently be wasteful in practice to conduct separate analyses of the continuation.*

The obvious answer is to abandon the tree in favor of a directed graph,** thereby allowing different lines of play to merge. This leads to what we call a game in positional form.*** This resembles in some respects a flow diagram for a computer program. The reader with a taste for such things may enjoy tracing through the diagram in Fig. 8, which depicts the 3-person "Shooting Match" of Chapter 2, laid out in positional form. (The light lines refer to misses.) Although a natural and useful form for many applications, the positional form raises a series of new modeling problems that merit some discussion here:

*Actually, in Chess the history of the board position is sometimes relevant, because of the rules for castling, for en passant captures, and for draws by repeated position. In fact, a rigorous definition of "position" in Chess is not so easy to formulate.

**See the footnote on page 14. In a directed graph, each edge has a preassigned orientation, usually denoted by an arrow. In the case of a rooted tree the arrow is not needed, since the rule: "travel away from the root" gives an unambiguous orientation to each edge.

***See Milnor (1953), Shapley (1953b), Berge (1957ab), Holladay (1957), Isbell (1957), Hanner (1959), Vorobyev (1963, 1970a), and several papers in the Russian collection Positional Games edited by Vorobyev and Vrublevskaya (1967).
Fig. 8—The shooting match
Loops

If there are loops, i.e., cycles in the directed graph, then the same position can occur more than once, and a play of infinite length is possible. Several ways of treating this kind of situation have been developed; we shall not pursue the general case further, but a number of special structures with loops will be discussed in Sec. 3.6.

If there are no loops, then the positional representation can be considered merely as an abbreviation for a tree --namely the tree that results from "pulling apart" the merging lines of play and replacing each positional node by a "stack" of tree nodes, as many of them as that position had possible histories. Even when there are loops, it is sometimes useful to think in terms of the infinite tree that could be created by "pulling apart" the positional nodes in this way.

Information

Chance moves and simultaneous moves can be treated as before, but the use and interpretation of information sets become a good deal more complex. Usually it is assumed that all nodes having two or more incoming edges (more generally, positions having two or more possible histories) are positions of complete information. Indeed, most positional games in the literature are games of perfect or almost-perfect information.
Strategies

The proper way to define a strategy also becomes less clear. For one thing, the simple notion of a choice at each information set (the toy train model) would presuppose that a player makes a fixed choice in each position, regardless of how he got there. True, we may be able to argue (for certain classes of games and certain concepts of solution) that there is no good reason for a player to make different choices in the same position. But that is hardly sufficient grounds for making such erratic behavior impossible or illegal.

For example, the identical dispositions of goods and money in an economic model might arise either through a chance move, over which the players have no control, or through some action by one of the players which the other players might choose to regard as a "double cross" or "breach of faith", although not a violation of the rules. In such a case, some apparently "irrational" punitive measure might well be a reasonable response.*

In a game in positional form, strategies which do prescribe the same action no matter how (or how often) a position is reached are called stationary strategies.** A

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*The full implication of this point must await the consideration of the role of "threats" in the various solution concepts to be defined.

**Shapley (1953ab). Other terms sometimes used are Markov strategies, positional strategies, and (in the case of a loop) steady-state strategies.
stationary pure strategy makes a fixed choice for each position belonging to the player in question. A stationary mixed strategy, on the other hand, is a special kind of behavior strategy, in which the same probabilities are used at a position whenever it arises.

**Spot Payments**

An additional feature that is often found in games in positional form is that of "spot" payments, made during the course of play, which must be totaled to determine the final payoffs. More generally, the outcome may be some function of the cumulated "scores", as in a game of ruin or survival (see Sec. 3.6). Sometimes, when a significant period of real time is covered by the model, the later payments are discounted in relation to the earlier, before forming the sum.

The advantage of having a way of indicating spot payments in the diagram is that positions which differ only in the amounts that have accumulated can be represented by a single node. This is especially useful when we are representing the repeated play of a single basic game. For example, Fig. 9 represents the "fingers" game played three times. If spot payments were not used, we would need two nodes at the second level, corresponding to the two money states (-5, 5) and (5, -5) possible at that time, and three nodes at the third level, corresponding to (-10, 10), (0, 0), and (10, -10). (If there were
Fig. 9—Spot payments
more variety in the payoffs of the basic game, the saving would of course be much greater.)

Repeated play of the same game is a favorite tool in experimental gaming, and there is a considerable literature on the subject, both theoretical and empirical, with special emphasis on certain two-person nonzero-sum cases.*

EXERCISES. 1. Two players alternately name numbers from one to ten, and the player who makes the total 100 wins.** Describe the positional form of this game, and then solve it by determining who should win in each position, working backwards from the end.

2. An unusual game of solitaire is diagrammed in Fig. 10. Show that any mixed stationary strategy is better than any pure stationary strategy. What definition of "strategy" for games like this, in which a path may have several nodes in the same information set, will ensure that the mixed strategies will include the behavior strategies?***

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*For a bibliography, see Shubik (1970) or Guyer and Perkel (1972).

**Game attributed to Bachet de Mèziriac (1612); see Vorobyev's account (1970b).

***Shapley (1953b); Isbell (1957).
3. SPELL-A-NUMBER.* $P_1$ and $P_2$ alternately write down the digits of a real number between 0 and 1 in decimal form, with $P_1$ moving first. $P_1$ wins if the number thus produced lies strictly between $9/22$ and $13/22$; otherwise $P_2$ wins. Does either player have a winning strategy, and if so what is it?

*Games of this type (where the winning set may of course be much more complicated than a simple interval) have been studied by Gale and Stewart (1953), Wolfe (1955), Davis (1964), and Mycielski (1964b). They are related to the Banach-Mazur game of "pick an interval" and similar games; see Mycielski and Zieba (1955), Mycielski, Swierczkowski, and Zieba (1956), Oxtoby (1957), and Hanani (1960). Also see Mycielski and Steinhaus (1962), Mycielski (1964a, 1966) and Mycielski and Swierczkowski (1964), where such infinite games of perfect information are employed in investigating the axiomatic foundations of mathematics (the "Axiom of Determinateness").
3.6. Recursive, Stochastic, and Survival Games

While it is not difficult to imagine the Kuhn tree model being extended, with great generality, to the case of games of infinite duration,* the infinite models that receive the greatest attention usually have some kind of special repetitive structure, based on an essentially finite foundation. In this section we shall briefly describe a few of these models.

Recursive Games

Consider first a game consisting of a finite set of component games, tied together by a "supergame" that controls the transitions among the components. After playing one of the components, and depending on its outcome, the players are either sent back to play another component--possibly the same one over again--or they are paid off and the whole thing ends. A structure of this kind is called a recursive game.**

Figure 11 shows two examples. The first has just one component, which is repeated if both players choose "2". The second begins in component $B$, but may shift to $A$ or $C$, or even revisit $B$, before the end is reached.

It is plain that a recursive game may go on forever. This can happen even when the players adopt entirely

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*See for example, Aumann's paper (1964), in which infinite sets of alternatives at a node are also considered. Another kind of infinite generalization comes from the introduction of a continuous time scale (see Sec. 3.8).

**Everett (1957).
Fig. 11 — Recursive games
reasonable strategies. Thus, in the first example, \( P_2 \) would lose $5 if he should ever choose his first alternative. On the other hand, player \( P_1 \) would very likely lose $10 if he should choose his first alternative. Both players would therefore be well advised to persist in choosing their second alternatives.

In a recursive game, the payoffs in the case of non-terminating play are defined to be zero. Thus a notion of "status quo" is implicit. In particular, we cannot add or subtract a constant to the terminal payoffs of a recursive game and expect its strategic properties to remain unchanged.

**Stochastic Games**

The recursive game might be a good format for the trials of Sisyphus or the occupant of some circle of Hell, but it lacks several features that are needed for an adequate model of recurrent human affairs in general and economic affairs in particular. Spot payments and chance moves, which are not provided for, are important in many applications. Income may accrue during the game, and an individual's income as well as the options he faces may depend not only on previous decisions, but also on "acts of God", e.g., storms and drought, fires, disease or accidents. The probabilities of these events may themselves depend on previous decisions. A driver may have an accident when completely sober, but he has the option of increasing
the chance by drinking heavily prior to driving. Thus we
are led to the idea of controlled stochastic process—a
chance-dependent sequence of events in which the actors can
influence both the odds and the available future options,
as well as the gains or losses incurred.

Expressing this idea, a stochastic game*, like a recur-
sive game, consists of a "supergame" and a set of one or
more component games, the latter sometimes called states
as in the theory of stochastic processes. The players
start at whatever component has been designated as the
initial state, and choose their first-round "strategies"
for that component. Their choices determine both the spot
payments (positive or negative) that are awarded at that
time, and the probabilities that are used to decide
whether the game ends (without further payoff) or continues
for another round in some specified state.

More formally, suppose that there are two players and
n states, and that in the k-th state the first player has
r_k alternatives and the second player s_k alternatives.
Further suppose that upon reaching the k-th state, the

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*Shapley (1953a). For related work and extensions, see Sakaguchi (1956), Gillette (1957), Takahashi (1962),
Beniest (1963), Zachrisson (1964), Hoffman and Karp (1966),
Charnes and Schroeder (1967), Blackwell and Ferguson (1968),
Pollatschek and Avi-Itzhak (1969), Rogers (1969), Smith
(1969), Liggett and Lippman (1970), Maitra and Parthasarathy
players select alternatives i and j, where \( 1 \leq i \leq r_k \) and \( 1 \leq j \leq s_k \). Then the rules would specify a payment say \( a^k_{ij} \), and a set of \( n + 1 \) transition probabilities, say \( p^k_{ij} \), \( t = 0, 1, \ldots, n \), which govern the passage from state \( k \) to state \( t \). Here \( t = 0 \) means termination.

It can be seen that a recursive game is essentially a special kind of stochastic game, in which all probabilities are 0 or 1, and all payments are zero except at a set of "terminal" states, associated with the different outcomes of the recursive game. The greater generality of the stochastic model has its price, however, in technical and conceptual difficulties and in weaker results. For one thing, the sum of the spot payments may become large without bound, or may oscillate and fail to converge to a limit even though bounded. We sketch some of the devices that may enable one to cope with this problem.

The simplest device is to stipulate that all the numbers \( p^k_{ij} \) be positive, so that on every move there is a finite probability that the game will end.\(^*\) This has the effect of assuring that never-ending games have probability zero, no matter how the players behave, and that the total payments converge in expected value to a finite limit.

A second expedient is to apply a time discount to the payments. That is, a payment of \( a^k_{ij} \) received on the \( t \)-th

\(^*\)Shapley, op. cit.
round is considered to be worth only \((1 - \beta)^t a_{ij}^k\), where \(\beta\) is a constant between 0 and 1.* This too has the effect of making the sum of the spot payments converge, and sets a uniform upper bound on the total (discounted) payoffs.

A third approach is to set all \(p_{ij}^{k0} = 0\) and allow infinite total payments, but to assume that it is the rate of income, not total income, that motivates the players. One sums the first \(t\) spot payments and divides by \(t\). As \(t\) becomes large this expression, if it converges, approaches the limiting average payment, which is regarded as the true objective of the game.**

Generally speaking, the basic technique in analyzing both stochastic and recursive games is to postulate a value for each player in each state. Then each inner game can be studied by itself, with its payoff being the sum of the spot payment, if any, and the mathematical expectation of the postulated value (perhaps discounted) of the new state to which the action moves. If these inner games can now be solved, and their values expressed as functions of the postulated state-values, then a set of simultaneous equations is obtained, from which in principle the true

*Equivalently, we may suppose that the spot payments are banked and draw interest at the rate \(\beta/(1 - \beta)\) per round.

**This procedure turns out to be more or less equivalent to the limiting form of the discounted case as \(\beta \to 0\); see Gillette (1957), Hoffman and Karp (1966), Liggett and Lippman (1970).
state values can be determined. The value of the game itself is the value of the initial state game. We have already seen this method applied in the three-person "Shooting Match" in Chapter 2 (equation (2-2)); it has something in common with functional equation methods in dynamic programming, the theory of harmonic functions, and the calculus of variations.* We shall discuss it further in Sec. 3.9.

**Survival Games**

A distinction was just made between the spot payments, or their sum, and the players' "true" objectives. This distinction is carried further in the games known as survival or attrition games. In the prototypical game of survival, two players, each with some specified number of "chips", agree to sit down and play a game of say, two-handed poker (or any other finite zero-sum game), and to continue to replay it until one has lost his entire stake. There are only two terminal states, but there is also the possibility that the game never terminates. The payoff then need not be zero, as in a recursive game, or even a constant. For example, it might depend on the limiting average size of the players' fortunes (in chips). As might be expected, best play in each round will generally depend

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*Bellman (1957); Scarf (1957); Blackwell (1962, 1965). It may be noted that almost nothing is known about stochastic or recursive games with more than two players, and very little about the two-person nonconstant-sum case or the two-person constant-sum case with imperfect information in the supergame; see however Mertens and Zamir (1971), Kirman and Sobel (1972).
on the current fortunes, sometimes in a rather complicated way.* These games are closely related to the classical problems of "gambler's ruin" and "random walk", and their analysis leans heavily on the theory of semimartingales.**

A somewhat more general form of survival game is the game of attrition, sometimes studied in a military context. Each player has quantities of different types of weapons or other items; each "engagement" causes certain numbers of these to be lost or captured, depending on the strategic choices and perhaps on chance; there may or may not be a replacement cycle; and play continues until one of the players is wiped out. The outcome may be win-lose, or the payoff may depend on the numbers of surviving items.†

Another extension of the simple survival model is the game of economic survival,‡ in which the players are thought of as corporations, or owners of corporations. The players do not merely survive or go bankrupt; the stream of dividend payments also figures in the final evaluation. Moreover, an external interest rate is postulated, so that a firm may go out of business voluntarily,

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*Hausner (1952ab), Bellman (1954), Sakaguchi (1956), Milnor and Shapley (1957).

**See e.g. Snell (1952). For another kind of game based on martingale considerations see Dynkin (1969).


‡Shubik (1959), Shubik and Thompson (1959), Gerber (1972). Unlike most of the other models discussed in this section, the main interest in games of economic survival lies in the nonzero-sum case.
not because it fears bankruptcy but because its assets if liquidated will bring a higher rate of return in some other enterprise.

A game of economic survival is a species of stochastic game; however, it shares with the other survival and attrition models the property of playing virtually the same "inner" game over and over again. The only difference is that the asset levels of the participants may change from period to period. To regard these games as stochastic or recursive games would entail introducing a very large number of states representing all the possible distributions of assets, and would not do full justice to their essentially repetitive nature.

**EXERCISE.** 1. The following table describes a one-position recursive game (two-person, zero-sum):

<table>
<thead>
<tr>
<th>P₁'s choices</th>
<th>$10, -$10</th>
<th>-$10, $10</th>
</tr>
</thead>
<tbody>
<tr>
<td>play again</td>
<td>$10, -$10</td>
<td></td>
</tr>
</tbody>
</table>

What must P₁ do to win?

2. Suppose P₁ and P₂ have ten pennies between them and are committed to play "Matching Pennies" until one player wins them all. Model this survival game as a recursive game, and determine the value of each position by the method of simultaneous equations.
3.7. Information Lag

One other type of repetitive structure warrants mention here.* The essential idea can be illustrated by the following diagram:

\[ P_1: \begin{array}{c}
a_1 \ \ a_2 \ \ a_3 \ \ a_4 \\
\end{array} \begin{array}{c}
\ldots \ \ \ldots \\
\end{array} \\

\[ P_2: \begin{array}{c}
b_1 \ \ b_2 \ \ b_3 \ \ b_4 \\
\ldots \ \ \ldots \\
\end{array} \]

Fig. 12

Here time flows from left to right, and \( P_1 \) and \( P_2 \) choose the a's and b's respectively. The payoff is some function of the "string" \( a_1 b_1 a_2 b_2 a_3 \ldots \) that is produced. The information pattern is described by saying that \( P_1 \) knows all of the b's up to and including \( b_{t-r} \) when he selects \( a_t \) and that \( P_2 \) knows all the a's up to and including \( a_{t-s} \) when he selects \( b_t \). The numbers \( r \) and \( s \) have an obvious interpretation as information delay times. In the diagram above, \( r = 1 \) and \( s = 2 \), and \( P_1 \) has perfect information but not \( P_2 \).

It is interesting to note that only the sum, \( r + s \), is really significant. Thus, by renumbering moves as follows: \( a'_i = a_{i-1}, b'_i = b_{i+1} \) (and adjusting the beginning), the above pattern can be shifted into one which has \( r = 3 \) and \( s = 0 \):

*The following discussion is based upon Scarf and Shapley (1957).
Here \( P_2 \) rather than \( P_1 \) has the perfect information; yet the two patterns that we have diagrammed are clearly equivalent. This illustrates our previous point that the essential ordering of moves in a game is more a matter of information than simple chronology.

The number \( \lambda = r + s - 1 \) has been called the \textit{time lag} for this type of information pattern. If \( \lambda = 0 \) there is perfect information on both sides, and no shift as between Figs. 12 and 13 is possible. If \( \lambda = 1 \) there is almost-perfect information—i.e., the moves are effectively simultaneous. (The only shift possible in this case is illustrated by the difference between Figs. 4b and 7a.) In both these cases an analysis by recursive methods is possible, since there are many positions of complete information. But when \( \lambda > 1 \) the analysis becomes substantially more difficult, as the players may never be able to shake off the effects of their past uncertainties and make a fresh start.*

*Nevertheless, using the flexibility afforded by shifting the pattern to give one or the other player perfect information, a certain kind of "generalized subgame" can
There is a close connection between these games with information lag and "games of incomplete information", mentioned earlier, in which the players do not know all the rules when they start to play.*

3.8. Games of Timing

The extensive-form models we have described thus far treat the game as a sequence of discrete events, and they are not especially well adapted to a continuous time frame. Although the principal ideas we have been discussing can be carried over in some form to continuous time, there are severe technical as well as conceptual difficulties. Satisfactory analysis in this area has usually depended on exploiting special properties of the model in question.

The "Dart Duel" of Ch. 2, Sec. 2.1 was an example of a game with continuous time. Each player, watching the other, had to decide when to throw his dart. We recall

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*In addition to the references in Ch. 1, Sec. 2.3, see Krinskii and Ponomarev (1964), Rosenfeld (1964), Vorobyev (1966), Stearns (1967), Sweat (1968), Zamir (1969ab, 1970), and Harsanyi and Selten (1972).
that the rule for optimal play was to throw one's dart as soon as the accuracies (hit probabilities) of the two players added up to 1. The game was analyzed in strategic form; this was easy to do despite the continuous infinity of decision points, because there were not really very many contingencies to consider.* One can even begin to sketch a game tree as in Fig. 14; but to be complete this diagram would require an infinity of branches, one issuing from every point of every vertical line.

Continuous-time games in which there are only a finite number of possible events, like this one, are known as games of timing, or (when the context permits) duels. The early literature stressed two diametrically opposed information conditions. In the silent duel, the players learn nothing of what their opponents have done until the game is over; in the noisy duel, as above, they learn everything the instant it happens.**

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*A strategy for player $P_i$ is an ordered pair $(t_i, f_i)$; the number $t_i$ tells him when to act if the other does not act, while the function $f_i(t)$ tells him when to act if the other acts at $t < t_i$. In Chapter 2 we started by eliminating all functions $f_i$ but one, by a domination argument, and so were left with just the single strategic parameter $t_i$.

**The study of duels was initiated in 1949 by Blackwell and Girshick and others in a series of Rand papers, whose contents are summarized in Dresher (1961). Further references are Danskin and Gillman (1953), Shiffman (1953), Karlin (1953ac, 1959), Blackwell and Girshick (1954), Caywood and Thomas (1955), Vogel (1956), Restrepo (1957),
Key:
- T = throw
- W = wait

(a and b are functions of t)

Outcomes:
- $\odot_1$ = target B hit ($P_1$ wins)
- $\odot_2$ = target A hit ($P_2$ wins)
- $\odot_3$ = both hit (draw)
- $\odot_4$ = neither hit (draw)

Fig. 14 — The dart duel
In the silent case, the extensive form quickly gives way to the strategic form. Time becomes like any other continuous strategic parameter and poses no unusual technical problems. Given the lack of information, solutions in mixed strategies are to be expected; typically these will consist of probability densities over portions of the time interval, combined perhaps with some "atoms" of positive probability. Figure 15 illustrates the solution to the symmetric silent duel in which each player has one "bullet" and the accuracy at time $t$ is equal simply to $t$, for $0 \leq t \leq 1$.*

Returning to the "noisy" case, we remark that the players have not perfect information, but almost-perfect information. It is the limiting case of a simultaneous game (Sec. 3.4). Such games are usually best dealt with in positional form, working backwards from the end to determine the value of each position. But the instantaneous information lag may give trouble. At critical times, e.g., at the beginning of the game, mixing over several different actions may be essential, just as in a


The term "noisy" is unfortunately in conflict with engineering and information-theory usage, in which "noise" is the antithesis of information.

*The optimal probability of firing in the infinitesimal interval $[t, t + dt]$ is equal to $dt/(4t^3)$ for $1/3 \leq t \leq 1 - dt$. The mean of this distribution happens to be $1/2$, the optimal firing time of the associated noisy duel.
Fig. 15 — Probability densities for optimal firing
discrete game, to keep oneself from being too predictable. Mixing over the time of some action, continuously in a small interval, may also be resorted to in order to cope with situations where exact simultaneity must be avoided.*

Another source of trouble lies in the nature of the continuous time variable itself. Consider the following example:

MUSICAL CHAIRS. The first player to sit down after the music stops wins.

In this game, the best positional strategy is obvious: whenever you are standing and the music is not playing, sit down! But this positional strategy does not "solve" the game, or even define a playable strategy; it does not give sufficient instructions to the user. In fact, this particular game has no strategies, pure or mixed, that are even approximately optimal.

EXERCISE. Show that no pure strategy will suffice for the silent duel illustrated in Fig. 15. Verify the optimality of the mixed strategy given.

3.9. Differential Games

Games of timing involve only a discrete set of actions, occurring within a continuous time framework. When actions or other positive decisions are being taken continuously, as in the steering of a car or a ship, then we enter the

*See Dresher (1961), Shapley (1964).
realm of differential games. It would take us too far afield to attempt a full survey of this important and distinctive branch of game theory, which has a very extensive literature of its own.* Instead, we shall make a few general observations on how some of the strategic and informational principles discussed in the preceding sections relate to differential games.

Mathematically, the theory of differential games draws upon the fields of differential equations, dynamical systems, the calculus of variations, and—most especially—control theory (which may be regarded as the theory of one-person differential games). This mix of technical disciplines has tended to attract a separate class of mathematical specialists from those attracted to the general body of game theory, and their work often reveals an unawareness of basic game-theoretic issues and ideas. Some, for example, have yet to discover how unsuited terms like "optimal" and "rational" are, in connection with noncooperative equilibrium strategies in non-zero-sum games (see Chapter 8). Even in the zero-sum two-person case, where differential games have been applied with notable success, the control-theoretic approach, by its nature,

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*We do not attempt a full bibliography, either. Isaacs (1965) gives a highly readable account of the researches through which he almost single-handedly opened the field. We also recommend Avner Friedman's more recent book (1971), the volume of contributions edited by Kuhn and Szegö (1971), and, for current research, almost any issue of the IEEE Transactions on Automatic Control.
has tended to saddle the investigators with a narrow and sometimes inadequate view of the concept of strategy.

The cornerstone of control theory is a general proposition known as Pontryagin's Maximum Principle,* which can be interpreted as asserting the equivalence of two forms of optimization. In order to optimize "globally" (i.e., over an entire multistage or continuous process), one must optimize "locally" (i.e., at each stage or instant of the process). The idea of local optimization, however, presupposes the ability to set up a function that evaluates all possible intermediate stages or positions through which the process might pass, thereby enabling the decisionmaker to operate without memory or foresight. The values of the immediately following positions tell him what he needs to know about the future, while the positional parameters tell him what he needs to know about the past.

This intuitively appealing and analytically powerful concept retains at least some of its force when we pass to systems with two or more decisionmakers. We have seen it applied in several preceding sections. An indispensable key to its successful use is the existence of a strong value concept, like the minimax value for two-person constant-sum games. Nevertheless, the underlying logic of Pontryagin's principle begins to crumble when there

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*Pontryagin (1957), Rozonoer (1959), etc.; it is more or less equivalent to the "principle of optimality" of dynamic programming; see Bellman (1957) or Dreyfus (1965).
is more than one player.* Local optimality, though usually sufficient, is no longer inherently necessary for global optimality. This is because a strategy can be optimal in the game as a whole even if it does not provide for exploiting or "punishing" every error that the other player might commit.

One can also have local without global optimality, in games (or even in one-person control problems) with no comprehensive stop rules, like games of survival,** because the "solving backwards from the end" technique does not work. It is quite possible in such games to have passive "waiting" strategies which are locally optimal, because they maintain the value of the position, but which postpone forever the winnings that more aggressive action could achieve.***

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*Of course, a "principle" is not meant to be a rigorous theorem or a universal truth. It is rather (to the specialist) like a signpost telling where to look for theorems, or (to the nonspecialist) a vivid summing-up of the sense of a whole family of theorems. In any particular theorem expressing an instance of Pontryagin's principle there will be restrictive hypotheses or conditions of a more or less technical nature. Our present critique, however, does not turn on technicalities.

**Milnor and Shapley (1957) give specific examples.

***There is a simple king-and-pawn end game in Kriegspiel (double blind Chess) in which none of White's locally optimal strategies wins, although the value of the initial position is a win for White. (White has a center pawn on the sixth rank, blocked by the Black king.) The point is that White cannot force a win without running some risk of a draw, though he can make that risk arbitrarily small.
The following differential game—a sort of two-dimensional Achilles and the Tortoise—will serve to illustrate several of these points.

TAG. A "pursuer" P chases his "quarry" Q across an open field. The former wishes to minimize, the latter to maximize, the time to capture.*

Suppose that P is twice as fast as Q, and that there are no obstructions of any kind. Represent the players as moving points in the euclidean plane, and avoid the question of infinitesimal "dodging" by defining capture to be an approach to within some preassigned small distance ε. Then, referring to Fig. 16, if both players play optimally they will both head for the point D, and "ε-capture" will occur just to the left of that point. These optimal trajectories are in fact unique; nevertheless there are many optimal strategies, i.e., contingency plans. Consider, for example, the following strategies for P:

1. Run directly towards Q at all times.
2. When on the closed segment AB, run towards C; otherwise run towards Q.
3. First run to C; then observe Q's position and run to that point; then observe his position again and run to that point, and so on.**

*Isaacs (1965).

**The "ε-capture" rule fends off Zeno's paradox!
Fig. 16 — Tag
Strategy (1) is the obvious "Pontryagin" strategy; it is both optimal and locally optimal. It exploits every situation to the full, except for anticipating future "dodges" by Q. Wherever the two contestants find themselves, this strategy gives P the earliest assured time of capture. To many differential game theorists this would be the only "optimal" or "rational" strategy for P.

Strategy (2) is optimal but not locally optimal, even though it is, like (1), a positional strategy.* If Q should run "north", for instance, then P, when he reaches B, would be further from his quarry at E than he would have been had he followed the pursuit curve AB', as dictated by (1). He would nevertheless effect capture at worst at F, bettering the optimal capture time.** But Q's mistake in heading "north" would not have been exploited as fully by (2) as by (1).

Strategy (3) is also optimal, since the distance between P and Q is at least halved at each observation. It is not locally optimal, however. Indeed, it is not a positional strategy, even if we include time among the positional variables. P relies on his memory, rather than on continuous observation of his target.

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*To see that (2) is optimal, note that when P reaches B then Q will still be to the right of B. The only way that Q might outwit his opponent would be to lead him back to the segment AB, whereupon (2) would dictate a wrong turn. But Q will be caught before he can do this.

**If AC is the unit, then CD is one unit in length and CEF is $1/4 + \sqrt{5}/4 = 0.809$ units.
How does one choose among these (and many other) optimal strategies? A strategy like (3) may be easier to implement than (1), since so much less information must be acquired and processed. On the other hand, (1) and (2) require no memory capability, though (2) depends on a fixed landmark in the plane. The local optimality of (1) seems an attractive bonus. Yet if there is a real reason to believe that E will not or cannot play optimally, then our model of the situation is wrong, or at best incomplete, and probably none of (1), (2), (3) is optimal. For example, if the maneuver CE could be predicted, then the straight path AB'' would lead to an earlier capture than either AB or AB'. In other words, if we adduce evidence to support the contention that local optimality is a desirable criterion in itself (and not just a technically attractive way to assure optimality), then in doing so we change the rules, and therewith the class of optimal strategies and (probably) the value of the game.*

An indication of the intrinsic technical difficulty of differential game analysis is provided by the fact that if we put a circular pond in the field, across which the players can see but not travel ("Obstacle Tag"), then no general solution is yet known, locally optimal or otherwise.

*Locally optimal solutions yield another bonus: they solve a whole class of games at one blow, since they take no notice of the starting position. But how much would a real player pay for this aid to analysis, and in what coin?
This game was proposed many years ago by Isaacs.* An idea of the trouble may be obtained from Fig. 17. If P runs directly towards A and Q directly away from B, then a position P' is reached where P would rather be running towards the other bank of the pond.

Very serious conceptual difficulties arise when differential game theory is carried outside the domain of two-person zero-sum games (or other "inessential" games). The absence of a strong value concept means that the Pontryagin principle loses its grip on the situation. Functional equations can still sometimes be written down and solved, expressing a form of local optimality. But the "optimality" or "rationality" of the myopic and memoryless behavior they depict is seldom justifiable on heuristic grounds. The solution that is obtained is a form of noncooperative or "Nash" solution--an n-tuple of strategies with a weak equilibrium property (see Chapter 8). But although this solution may uniquely satisfy the conditions for local "optimality"; it is almost never the unique noncooperative equilibrium; moreover, among the class of noncooperative equilibria it is usually far from the best, or the most realistic. We shall return to this question in Chapter 8.

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EXERCISE. Suppose that the payoff in "Tag" is 1 for capture, 0 for escape. Show that every positional strategy for P is locally optimal, regardless of the relative speeds of P and Q. Is the same true for Q?

3.10. Models and Context

This section has been devoted to indicating how the modeler may use game theoretic methods to incorporate phenomena that extend over time. Sometimes, and for some
purposes, the relationship between the static and the
time-dependent model of an economic or social process is
reasonably good. It is not possible to make this con-
clusion _a priori_. It is our belief, however, that for
much economic analysis the relationship is sufficiently
good to make a study of static or steady-state models
extremely fruitful. At many places in the preceding dis-
cussion we have had to motivate our treatment by intuitive
or informal appeals to solution concepts.* When we turn
to the discussion in earnest of the various solution
concepts in the later chapters of this volume, and then
to actual economic models, it is desirable that the reader
should try to consider the degree to which each argument
might be made dynamic, even when no explicit dynamic
analysis is presented.

Game theoretic methods are a natural outgrowth of
attempts to extend models of individual rational choice.
Sometimes they are presented as though they are in com-
plete opposition to other ways of modeling human behavior.
This is by no means so. The model of a stochastic game,
for example, is not far removed from the type of flow
diagram called for in simulations and behavioral models
in modern psychology and social-psychology. For another
example, the controlled random walk represented by a game

*Indeed, most of the bibliographic references we have
given in this survey are concerned as much with solutions
as with problems of structure and modeling.
of survival is closely related to models used in learning theory. And the Kuhn tree itself is a generalization and elaboration of the familiar "decision tree" of modern decision theory.* These and many other contacts of game theory with related fields will be surveyed in Chapter 10.

*Raiffa, Decision Analysis (1968).
4. THE STRATEGIC FORM

In the strategic form of a game, the whole move-and-information structure—what we ordinarily think of as the "rules"—drops out of sight. Strategies are no longer regarded as complex sets of instructions, but as abstract objects to be manipulated formally, without regard for their meaning. The outcomes or payoffs can now be given in tabular form, or in mathematical formulas that may not reveal anything about the original extensive-form rules. It can be an interesting puzzle in cryptographic detection to try to infer the most likely information conditions, number of moves, etc., of a game that is given to us in its strategic form. (It is somewhat like trying to guess the withholding tax laws from a study of paychecks!)

The strategic form is easily realized "in vitro," i.e., in the gaming laboratory. Occasionally a real game "in vivo" presents itself in strategic form: the players must really make simultaneous, independent, all-encompassing, initial decisions that determine the whole course of action. But this is quite unusual. It is therefore worth asking what is lost or forgotten when a game is "reduced" from extensive to strategic form. To put the question more sharply, suppose that two games have different extensive-forms but come down to the same strategic form. To what degree may we regard them as analytically equivalent?
The answer, as so often in these matters, depends on the purpose and context of the analysis. If we are just considering the problem of optimal behavior in, say, a zero-sum two-person game, then we are fairly safe in working only with the strategic form.* Likewise, if we are only interested in solutions that depend on the "characteristic function" of a multiperson game (see Chapter 6), then again the strategic form is enough.

On the other hand, there are at least three important areas of investigation in which the reduction from extensive to strategic form destroys essential information, to wit:

1. descriptive modeling
2. study of nonoptimal behavior
3. study of negotiations.

The first requires no further comment. The strategic form is deliberately nondescriptive—that is its virtue. The second heading refers to the problem of making explanatory models for such things as errors, trickery, surprise, skill, habit, stupidity, limited computing or communicating ability, complexity for its own sake,** simplicity for its own sake, etc. Though seldom studied in a systematic, mathematical context, these are all proper topics for a general theory of games of strategy, and they all seem to require the apparatus of the extensive form.

*But see Aumann and Maschler (1972).

**Thus, a Chess or Go player who has given his weaker opponent a handicap will often seek to complicate the position.
The third area of investigation is the most treacherous, as the pitfalls are less apparent. Negotiations among the players are implicit, but only implicit, in most of the cooperative solution concepts we shall consider. As these solutions generally do not look behind the strategic form, there is an implicit presumption that negotiations during the play are not allowed, or are not relevant. It is a sad fact that we still lack a general theory of cooperative games in extensive form.* The standard solution theories tell us next to nothing about coalitional dynamics; they require us to assume, contrary to common experience, that all the "politicking"--the coalition-forming and deals and promises and threats etc.--takes place before the first move is actually played. Accordingly, it may be misleading, when writing a verbal account of the meaning of a cooperative solution, to interweave negotiatory events with the moves and acts of the formal game, even though to do so may make the "story" sound more plausible.

What then saves our present endeavors from unreality and futility? The best answer echoes the beginning of this chapter: we interest ourselves primarily in situations

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*The main obstacle seems to be that introducing explicit negotiations into the fabric of the extensive game makes the model too "ad hoc" and limits its generality. Perhaps the best hope for progress is with a recursive or stochastic model (see Sec. 3.7 above), where one might insert a suitably vague (and hence general) "renegotiation" session between each round.
where details of process, including details of negotiation, can be presumed irrelevant. To the extent that, say, the Chicago Board of Trade or the Republican National Convention are "games of skill," with their myriad rules and special procedures (written and unwritten), present cooperative game theory can say virtually nothing about them. But to the extent that these institutions may represent the battlegrounds for certain elemental economic or political forces, the present theory has at least some basis for making meaningful pronouncements.

Indeed, many economic, political, and diplomatic competitions (in contrast to parlor games and military applications) are modeled most realistically not in the extensive form, but rather in the strategic form or (more often) in the still more austere "characteristic function" form (Chapter 6). Inventing extensive-form "rules" for such games is often an exercise in artificiality. For example, political constitutions can be written—and can be game-theoretically analyzed—with little reference to the specific procedures for casting and counting ballots.* In fact these procedures often change from election to election, and they may even be declared "unconstitutional" if they are found to violate the intent of the founding

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* E.g., Shapley and Shubik (1954), Riker and Shapley (1968), Banzhaf (1968).
fathers. In short, there is an underlying game, buried in the constitution, that is independent of any particular extensive-form realization.

For a second example, in a later chapter we shall take a simple one-period duopoly situation and exhibit a very large number of different ways of sequencing the price and production decisions and the flow of information. Many of these models might be plausible renderings of different actual duopolies, observed at particular points in time. Tactically they are different. But the differences dwindle in significance when we reflect that the "rules" in reality are not rigidly fixed, and may even be partially under the control of the players. They can buy more information (or more secrecy), they can reschedule their decisions and announcements, and so on. To an operations researcher advising one of the actual firms, or to a Justice Department analyst evaluating a proposed merger, the absence of an adequate theory of negotiation in the extensive form might be regrettable. But a more detached observer, seeking only to understand the basic phenomenon of collusion in oligopoly, may be justified in hoping for some help from the existing theory of cooperative games.

To sum up, at the level of generality practiced in much of political science as well as mathematical economics, the goal of accuracy in modeling may not only permit us, but even require us to avoid most of the detail with which the extensive-form is concerned.
4.1 The Outcome Matrix

Thus far we have not taken much care to distinguish between outcomes and payoffs. Speaking generally, by an outcome we mean a possible state of affairs at the end of the game. But the statement of an outcome might have to include a description of the course of play that led to it, or even a probability distribution over several courses of play, as we depicted earlier in Fig. 1. Usually, however, the model-builder will summarily discard anything that he regards unimportant to the evaluation of the outcome by the players. In the examples of Chapter 2 we cared only about broken balloons, not how or when they were broken, while in the "Art Auction" earlier in this chapter we cared only about the last bid. Thus, from Table 1 (page 4) we may extract the "reduced" outcome

"P_1 gets the painting for $19,"

or from Fig. 1 (page 6)

"P_1 gets the painting for $19, $20, or $21
with probabilities \(\frac{7}{16}\), \(\frac{7}{16}\), and \(\frac{1}{8}\) respectively."

If there are just two players, and not too many strategies, the outcomes (suitably reduced) can be displayed in a matrix form, as suggested by Fig. 18. The outcome matrix is often too large to write out in full, and with more players the array becomes multidimensional.
Fig. 18 — An outcome matrix
Nevertheless it is a useful conceptual device. By its rectangular shape it reminds us that each strategy-choice of each player must "work" against all possible strategy-choices by the other players.

Figure 19a illustrates again with the auction example, assuming for simplicity that there are three players with two strategies apiece. Strategy No. 1 in each case is the one given on page 4, while No. 2 is the strategy of never bidding at all. Thus, in the box marked (*) only $P_1$ and $P_3$ are bidding, and the sequence of bids leading to the stated outcome is $15_1, 16_3, 18_1, 19_3, 21_1$.

4.2. Payoff Vectors and Matrices

Continuing with the example, suppose that the value of owning the painting is $27, $28, and $20, to $P_1$, $P_2$, and $P_3$, respectively. If all three adopt the specimen strategies on page 4, then $P_1$ will obtain the item at a price of $19, so his net gain is $8. If we take zero to represent the value to each player of the status quo, then the outcome in question yields the payoff vector $(8, 0, 0)$. Under another trio of strategies $P_3$ might win the painting at $22; the payoff vector would then be $(0, 0, -2)$. If the outcome is probabilistic, the easiest thing to do is to average the payoffs. Thus in Fig. 1, $P_1$'s average price is
(a) Outcomes

(b) Payoffs

Fig. 19
\[ \frac{7}{16}(19) + \frac{7}{16}(20) + \frac{1}{8}(21) = 19.6875. \]

The payoff vector for this outcome is therefore (7.3125, 0, 0) in "expected dollars".*

Like the outcomes, the payoffs can be represented in matrix form. The coordinates are as before, but the entries are payoff vectors. Thus, the number "12" starred in Fig. 19b refers to the gain of P_1 when he wins the auction with his first bid of $15, as might occur if the players were to form a "ring".**

It is common practice to display the payoffs to each player in a separate matrix; thus, a (finite) two-person game is often described as a "bimatrix" game. A "zero-sum" game is one in which the payoffs to all the players add up to zero; similarly "constant sum". In such games one player's payoff matrix can be omitted, as it can be determined from the others. In particular, a two-person zero-sum game in strategic form is almost always represented by just a single payoff matrix or payoff function, one player being designated the maximizer, the other the minimizer.

* It is clear that the problem of passing from outcome to payoffs is being grossly simplified. Indeed, the whole of utility theory lies between these two concepts, as well as some considerations concerning the nature of money. Chapters 4 and 5 will be devoted to these subjects.

** Duveen, op. cit.
4.3. More General Payoff Functions

Speaking generally, an n-person game in strategic form is nothing but a set of n real-valued functions in n variables:

\[
\begin{cases}
    P_1(\sigma_1, \sigma_2, \ldots, \sigma_n) \\
    P_2(\sigma_1, \sigma_2, \ldots, \sigma_n) \\
    \vdots \\
    P_n(\sigma_1, \sigma_2, \ldots, \sigma_n),
\end{cases}
\]

where the domain \( \Sigma_i \) of each player's strategies \( \sigma_i \) may be set up in any convenient way. The matrix form, with its standardized numbering of the strategies, may needlessly obscure the patterns of interstrategic relationships, and another functional notation is often more expressive and more amenable to analysis as well. For example, in a given instance it might be more natural to represent a strategy as a pair of numbers \((i_1, i_2)\) than a single number \(i\).

When there are infinitely many strategies, the matrix form is of course no longer available, but a matrix-like representation is sometimes still handy. Thus, in Chapter 2, we considered the following payoff function for player "A" in a two-person, constant-sum game:

\[
P_A(x, y) = \begin{cases} 
    a(x) & \text{if } x < y, \\
    1 - b(y) & \text{if } x > y, \\
    \frac{1}{2}[a(x) + 1 - b(y)] & \text{if } x = y.
\end{cases}
\]
Here, the functions $a$ and $b$ are assumed given, and the strategic variables $x$ and $y$ have the same domain, e.g., the closed interval $[t_1, t_2]$. This information is concisely displayed in Fig. 20a. (For another example, see Fig. 15.) Much of the early work on infinite games was focussed on such "games on the square," perhaps because of the strong visual analogy to matrix games. *

**EXERCISES.** 1. Write the payoff matrices associated with the games shown in Figs. 2, 4a, and 4b.

2. Analyze the two-person "game on the square" shown in Fig. 20b, where the strategies for both players either (a) range over the closed unit interval $0 \leq s_i \leq 1$; or (b) range over the open unit interval $0 < s_i < 1$.

4.4. Other Payoff Indicators

Sometimes the passage from "outcome" to "payoff" is blocked by imponderables of valuation or interpretation that relate directly to the imponderables of the "n-person problem" that we are trying to resolve. We may prefer not to dispose of these matters summarily, at the descriptive stage of the analysis, but rather to carry them over into the solution stage. For this purpose, more general forms of payoff indicator can be useful.

Fig. 20 — Games on the square
In simple cases, it may suffice to introduce an undetermined coefficient or parameter into the expression for the payoff, and then adjust it or let it vary over a range as the solutions take shape. For example, the trade-off ratio between two objectives (e.g., guns and butter) may be left undetermined.* More generally, it may be necessary to drop the idea that the payoff to each player is a number, whether determined or undetermined, or even an element of a completely ordered utility space.

In the next chapter, we shall discuss a variety of nonnumeric payoff indicators that have been used with some success in game-theoretic investigations. The price

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* Shapley (1959). Discussing the need for a theory of games with vector payoffs (i.e., several numerical components per player), F. D. Rigby (1959) describes the following experience in a military application:

"However, each pair of these strategies... generated both a time delay and losses to the moving forces.... Since the values of these delays and losses would be realized in a subsequent battle... it was clear that some sort of exchange ratio between attrition and delay must exist. Efforts to obtain estimates of such a ratio failed completely. In fact, several rather arbitrary weightings... were assumed..., but this was a rather unsatisfactory expedient.

"... the occurrence of vector payoff represents a failure to resolve some of the questions whose answers are needed in order to construct a game model. This being the case, one should not be disappointed that the theory does not produce a clear cut, well defined solution concept."
for so "loosening" the payoff/utility structure of the model must be paid, however, when the solution stage of the analysis is reached. Most solution concepts will require some adaptation or extension in order to cope with the new payoff scheme, and some may fall by the wayside.* The solutions that survive, for their part, are likely to be less determinate than usual, and less satisfactory in their interpretation.

In addition to the problem of setting up individual utility scales, there lies the possibility that the stakes of the different players in the outcome may not be cleanly separated. The end of the game, despite all our efforts at closing the model (see Chapter 1, Sec. 4.2), may fail to be "game-theoretically inert." The alliances and polarizations effected during the play of the game may have residual influences far into the future, and these influences, coalitional in nature, cannot be captured by merely adjusting the separate payoffs of individual players.

A suggestive approach to this modeling problem would be to use set-functions in place of payoff vectors, assigning thereby a "payoff" number (or other indicator) to every coalition. These set-functions would be tantamount to characteristic functions (see Chapter 6), representing future games. An additive set-function, in particular,

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*The "value" solution, for example, fails when only ordinal utilities are assumed; see the "Bargainers Paradox" in Chapter 4.
would correspond to the "inert" case, and would reduce to an ordinary payoff vector. Replacing the set-functions by their "value" solutions (see Chapter 7) would be one possible way of disposing of the problem summarily at the descriptive stage, but it would be interesting to try to carry the set-functional description through to the solution stage, after suitably generalizing the solution concepts to be employed.

In sum, weakening or generalizing the payoff concept does not really solve anything. It may serve to postpone a modeling problem, however, to a later point where it can be shown to be irrelevant, or where it is swallowed by a bigger problem. Failing that, it may at least defer the problem to a point where the analyst's "customer"--the ultimate user of the model--can fairly exercise his own intuition in judging the matter.
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