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Analytic Theory of the Effects of Atmospheric Scattering on the Current and Ionization Produced by the Compton Electrons from High-Altitude Nuclear Explosions

William Sollfrey

A Project AIR FORCE report
prepared for the
United States Air Force

Rand
SANTA MONICA, CA. 90406

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PREFACE

In recent years, considerable interest has developed within the Air Force in the electromagnetic pulse generated by a high-altitude nuclear explosion. Possible effects of this pulse on military equipment have been discussed, and there has been a rekindling of investigations in the basic physics and mathematics of the phenomenon. Large and expensive computer codes have been developed to determine the fields that may result. Many organizations have been involved in these operations.

The Rand Corporation has continuing efforts in this area, conducted under the Project RAND research project "System Vulnerabilities." The work has been communicated to the Air Force in several briefings at the Air Force Weapons Laboratory, Kirtland Air Force Base, and the findings are being documented in a series of reports.

This report presents an analytic treatment of the effects of atmospheric scattering on the current and ionization produced by the burst. Studies of the problem at other locations have involved either ad hoc physical assumptions or Monte Carlo techniques requiring very costly computer programs. The methods described here use analytical procedures as far as practicable, and then employ very rapid computer programs. Since this work is all of a fundamental character, it is presented in unclassified form. Applications to the electromagnetic pulses produced by actual nuclear explosions will be developed in a classified report.

The analysis and the computer program contained in this report should be useful to persons working with the theory or applications of electromagnetic pulses from high-altitude nuclear explosions. The report presents the most comprehensive investigation to date of the basic phenomena of current and conductivity.

SUMMARY

A high-altitude nuclear explosion generates gamma rays which are absorbed in the atmosphere. The Compton electrons produced by the gamma rays form a current, which then radiates an electromagnetic pulse, and also induce secondary ionization which determines the propagation of the electromagnetic field. The electrons move under the influence of their deflection by the earth's magnetic field, lose energy by the ionization process, and suffer Coulomb scattering by the atoms of the atmosphere. This report is devoted to an analytic investigation of the current and ionization produced by these combined effects.

Although there have been satisfactory previous treatments of magnetic deflection and of energy loss, the investigations of scattering have been fraught with difficulties. Methods to handle scattering have included (1) the "obliquity" theory, which contains ad hoc assumptions and uncertain physical meaning, and (2) the Monte Carlo technique. Although the latter method is theoretically sound, it involves lengthy and expensive computer programs. Accordingly, we have been impelled to develop a theory of scattering which is carried as far as possible analytically.

The theory is based on a distribution function which represents the number of electrons per unit volume in a unit velocity interval at a point in space and time. We then derive a transport equation which governs the evolution in time of this distribution under the indicated physical processes from the initial Compton production distribution. This differential equation is then solved analytically by an expansion to third order in the ratio of spatial spreading to velocity spreading, thus providing sufficient accuracy to cover the significant duration of the pulse (10 shakes), as is shown by detailed numerical investigation. The current and ionization are then calculated as integrals over the velocity distribution function.

These integrals are carried out analytically as far as we have found possible. A computer program has been written to evaluate the

integrals which remain. This program, included as an appendix, is rapid compared to the Monte Carlo technique.

Some typical results are presented. For a gamma-ray source taken to be a delta function in time, the current per gamma ray at high altitudes, which is essentially independent of scattering and energy loss, displays a peak at about 2 shakes for the selected choice of magnetic field and gamma-ray energy, and a slow decay thereafter. As the altitude decreases, the peak becomes lower, earlier, and narrower, all effects which would be expected by the interpretation that scattering reduces the coherence of the electron beam. The magnitude of reduction is considerable, thus at an altitude of 20 km and times later than 2 shakes, the current is reduced by more than a factor of four relative to the high-altitude current per gamma ray. The effects on ionization are much less striking, and the conductivity per gamma ray associated with the ionization is fairly insensitive to altitude.

The theoretical investigation and computer program of this report should prove valuable to all persons interested in the high-altitude electromagnetic pulse.

ACKNOWLEDGMENTS

It is a pleasure to thank C. M. Crain for numerous helpful suggestions and sage advice. The careful review of the manuscript by R. W. Huff is likewise much appreciated. The computer program could not have been written without the skilled cooperation of M. D. Lakatos.

Many of the ideas developed in this report have benefited from discussions with investigators at other institutions. The author wishes to thank all those who have cooperated unselfishly with him.

CONTENTS

PREFACE iii

SUMMARY v

ACKNOWLEDGMENTS vii

FIGURES xi

SYMBOLS xiii

Section
I. INTRODUCTION 1

II. DERIVATION OF THE TRANSPORT EQUATION 6
General Considerations 6
Coherent Acceleration 10
Atmospheric Scattering 14
Compton Electron Production 19
Connection with Current and Secondary Ionization ... 24

III. SOLUTION OF THE TRANSPORT EQUATION 28
The Green's Function 28
Elimination of the First Derivatives 32
Approximate Solution of the Differential Equation 35

IV. CALCULATION OF THE CURRENT AND IONIZATION 41

V. DEVELOPMENT OF THE COMPUTER PROGRAM 54

VI. RESULTS AND CONCLUSIONS 59

Appendix
THE COMPUTER PROGRAM 81

REFERENCES 97

FIGURES

1. Geometry and Coordinate System 7

2. Energy Loss Factor Versus Energy 13

3. Scattering Coefficient Versus Energy 18

4. Total Electron Production Versus Altitude 21

5. Electron Production per Unit Solid Angle 23

6. Electron Production per Unit Angle 23

7. Electron Velocity and Kinetic Energy 24

8a. Contributions to the Ionization--Altitude 20 km 60

8b. Contributions to the Ionization--Altitude 30 km 61

8c. Contributions to the Ionization--Altitude 40 km 62

9a. Contributions to the Current--Altitude 20 km 63

9b. Contributions to the Current--Altitude 30 km 64

9c. Contributions to the Current--Altitude 40 km 65

10a. Current Versus Initial Angle--Altitude 20 km 67

10b. Current Versus Initial Angle--Altitude 30 km 68

10c. Current Versus Initial Angle--Altitude 40 km 69

11a. Ionization Rate Versus Initial Angle--20 km 70

11b. Ionization Rate Versus Initial Angle--30 km 71

11c. Ionization Rate Versus Initial Angle--40 km 72

12. Current per Primary Gamma Ray 73

13. Ionization per Primary Gamma Ray 74

14. Relative Current 76

15. Equivalent Conductivity 77

SYMBOLS

The number following each definition refers to the equation in which each symbol first appears or to the appropriate location in the text.

$A(\zeta)$	function equivalent to $a(\beta)$ (4.22)
$\vec{B}$	magnetic field (2.7)
B	magnitude of earth's magnetic field (2.11)
$C(\beta)$	forward component of displacement (3.12)
$\overline{C}(\beta)$	forward component of displacement relative to position without scattering (3.13)
$\overline{C}(\beta')$	the function $\overline{C}$ regarded as a function of its lower limit β' with the upper limit set equal to $h(\cos\theta)$ (4.12)
$\underline{C}_1$	$\overline{C}(\beta') - \overline{C}(\beta_1)$ (4.12)
$D(\zeta',\theta,\varphi)$	ratio of retarded time to electron reference time (4.21)
$D_i^j(\beta)$	Legendre expansion coefficient (3.21)
E	electron energy (2.9)
E_γ	gamma-ray energy (before 2.29)
$\vec{E}$	electric field (2.37)
E_i	mean ionization energy (2.41)
E_i^j	Legendre expansion coefficient (3.21)
$F(\vec{r},\vec{v},t)$	electron velocity distribution function (Introduction)
$G(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi',t')$	Green's function of transport equation (3.5)
H	atmospheric scale height (before 2.1)

$H(r,\beta,\theta,\varphi,r',\beta',\theta',\varphi')$ reduced Green's function (3.8)

I ionization potential of air (2.12)

$I(r_o,\beta,\theta_o,\varphi_o,r',\beta',\theta',\varphi')$ reduced Green's function expressed in initial variables (after 3.13)

$\vec{J}$ total current (2.37)

$\vec{J}_C$ Compton current (after 2.37)

$\vec{J}_S$ conduction current (after 2.37)

K energy attenuation per unit angle of turn (4.16)

$K(\cos\theta)$ Klein-Nishina number distribution (2.29)

L zero-order differential operator (3.17)

$L(t-t')$ ionization delay function (2.44)

M first-order differential operator (after 3.16)

M_1 reduced first-order differential operator (after 3.26)

N number of atoms per unit volume (2.6)

$\dot{N}_e(\vec{r},\vec{v},t)$ Compton electron production rate (2.4)

$N_S(r,t)$ number of ionization electrons (2.38)

$P_i^j(\cos\theta)$ associated Legendre polynomial

Q second-order differential operator (after 3.16)

Q_1 reduced second-order differential operator (after 3.26)

$Q(\beta,\theta,\varphi,\theta',\varphi')$ differential cross section for elastic scattering (2.6)

$R(\beta,\theta_o,\varphi_o)$ inhomogeneous term in zero-order differential equation (3.20)

$S(\beta)$ sideward component of displacement (3.12)

$\overline{S}(\beta)$ sideward component of displacement relative to position without scattering (3.13)

$\overline{S}(\beta')$ the function $\overline{S}$ regarded as a function of its lower limit β' with the upper limit set equal to $h(\cos\theta)$ (4.12)

$\underline{S}_1$	$\bar{S}(\beta') - \bar{S}(\beta_1)$ (4.12)
S_{2m}	moments of powers of sin χ with respect to the cross section Q (2.23)
T_i^j	Legendre expansion coefficient (after 3.21)
$T(\zeta', \theta, \varphi)$	electron arrival time (before 4.20)
$T_m(\theta, \varphi)$	maximum electron arrival time (5.2)
$U(\zeta)$	scattering reduction factor (before 4.21)
U_i^j	Legendre expansion coefficient (after 3.21)
$V(\beta)$	bracketed factor in energy loss (after 2.14)
$W(\beta)$	energy loss function (2.7)
$W(r, \beta, \theta, \varphi, t)$	modified electron distribution function (3.1)
$X(\beta, \theta_o, \varphi_o, \beta_1, \theta_1, \varphi_1)$	Green's function of zero-order differential equation (3.25)
Y	yield of burst (after 2.30)
Z	number of electrons per atom (2.12)
Z_i	position change function (4.23)
$\vec{a}$	acceleration produced by coherent forces (2.1)
a_i	amplitude coefficients in cross section (2.24)
$a(\beta)$	coefficient of scattering term in cross section (3.11)
b_i	width coefficients in cross section (2.24)
c	velocity of light (before 2.1)
e	electron charge (2.7)
$f(t-r/c)$	gamma-source time dependence (2.29)
$g(r)$	total Compton electron production (2.29)
h_B	height of burst (Fig. 1)
$h(\cos \theta)$	Klein-Nishina velocity distribution (2.29)

i	summation index (3.21)
j	summation index (3.21)
k	gamma energy in units of mc^2 (before 2.29)
m	electron mass (2.7)
n	order of gaussian approximation to integrals (5.5)
q	momentum transfer (2.24)
$q(\beta)$	coefficient of position derivative in transport equation (3.11)
$\vec{r}$	position vector (Introduction)
r	distance from burst to electron location (Fig. 1)
r_o	initial position variable (3.12)
r_e	electron radius (e^2/mc^2) (2.12)
$s(k)$	dependence of total Compton cross section on gamma energy (2.31)
t	time (Introduction)
u	dependent variable in zero-order differential equation (3.20)
$\vec{v}$	velocity vector (Introduction)
v	magnitude of velocity vector (2.6)
v_x, v_y, v_z	rectangular components of velocity (2.1)
w	velocity direction change function (4.23)
x	rectangular coordinate perpendicular to line of sight and to magnetic field (Fig. 1)
y	rectangular coordinate along magnetic field (Fig. 1)
z	rectangular coordinate along line of sight away from burst (Fig. 1)

Λ_i	momentum widths in cross section (2.24)
$\underline{\Psi}(\vec{r},\vec{v},\Delta\vec{r},\Delta\vec{v})$	transition probability (2.3)
α	azimuthal angle of velocity after scattering with initial velocity as polar direction (2.20)
β	v/c (2.1)
γ	$(1-\beta^2)^{-\frac{1}{2}}$, electron total energy/$mc^2$ (2.13)
γ_o	initial electron total energy/mc^2 (4.15)
δ	Dirac delta function (2.3)
$\delta\vec{v}$	fluctuating velocity increment in Δt (2.2)
ε	permittivity of free space (2.40)
ε_i	1 if i = 0; 2 if i $\neq$ 0 (3.21)
$\varepsilon(\beta)$	coefficient of magnetic field term in transport equation (3.11)
$\zeta(\beta)$	equivalent turn angle (3.12)
η	obliquity factor (Introduction)
θ	angle of velocity from original direction (Introduction only)
θ	polar angle of velocity (2.1)
θ_o	initial polar angle variable (3.12)
λ	space azimuthal angle (before 2.1)
μ	gamma-ray attenuation (2.31)
μ	permeability of free space (2.37)
$\nu(r)$	collision frequency (2.38)
$\xi_i \rightarrow (\beta_i,\theta_i,\varphi_i)$	symbolic representation of velocity (after 3.25)
ρ	atmospheric density (after 2.14)
ρ_o	atmospheric density at sea level (after 2.14)
$\bar{\rho} = \rho/\rho_o$	relative atmospheric density (2.14)

σ	conductivity (2.38)
σ_c	total Compton scattering cross section (2.30)
$\tau = (t-r/c)$	retarded time (before 4.20)
τ_m	pulse duration (before 2.1)
φ	azimuthal angle of velocity (2.1)
φ_o	initial azimuthal angle variable (3.12)
χ	polar angle between initial and final directions of velocity after scattering (2.17)
Ψ	space polar angle (Fig. 1)
Ψ_B	angle between line of sight and vertical (Fig. 1)
$\Psi(\vec{r},\vec{v},\Delta\vec{v})$	reduced transition probability (2.3)
$\Delta\vec{r}$	position increment in Δt (2.2)
Δt	time increment including many scatterings (2.2)
$\Delta\vec{v}$	total velocity increment in Δt (2.2)
∂	partial derivative (2.6)
$\nabla_{\vec{v}}$	gradient operator in velocity space (2.6)
∇^2	Laplacian in angles of the velocity (2.22)
$d\omega$	element of solid angle (after 2.5)

I. INTRODUCTION

The theory of the generation of electromagnetic (EM) pulses by high-altitude nuclear explosions was first put on a firm footing in investigations by Karzas and Latter.[1] In their theory, the prompt gamma rays from the burst interact with the atmosphere via the Compton effect to produce high-velocity electrons. These Compton electrons are accelerated by the earth's magnetic field to produce a radiating current and lose energy by ionization to cause atmospheric conductivity. The propagation of the fields through the conducting medium is treated by the "high-frequency" approximation, which neglects certain spatial derivatives relative to time derivatives and is regarded as sufficiently accurate for the effective duration of the pulse.

Following this initial theoretical investigation, considerable effort was expended in developing computer codes to apply the theory to practical cases. Among these codes were LEMP I,[2] developed at the Mission Research Corporation, and HEMP,[3] HEMP-B,[4] and CHEMP,[5] developed at the Air Force Weapons Laboratory. The codes were generally of such character that additional theoretical developments could be incorporated without changing the basic structure of the code. As the physics of the calculations have been improved, the codes steadily have become more complex and costly to operate.

There have been numerous theoretical improvements to the basic studies. The dependence of atmospheric conductivity on field strength has been treated by Baum[6] and by Higgins et al.[7] The CHEMP code[5] was designed to include self-consistent electron motions, allowing the electromagnetic field to interact with the electrons which produce it. The lag in the secondary ionization process was investigated by Longmire and Longley.[8] There have been several studies of how the current is affected by atmospheric scattering.

When atmospheric scattering is not included in the theory, the motion of the electrons is determined by their deflection by the magnetic field (electromagnetic field for self-consistent calculations) and by the energy loss to ionization. Thus, there is a unique and

invertible relation between the position and velocity of an electron at a given time and its position and velocity at its birth. In the presence of scattering, an electron suffers many random deflections in the course of its motion. Consequently, neither its position nor its velocity can be explicitly deduced from initial conditions. At any space-time location there will be a distribution of electrons with respect to velocity, and this enormously complicates the situation.

The treatments of scattering to date have proceeded along two essentially independent lines. First, there is the "obliquity" theory, developed in Ref. 8 and later improved.* In this model, the electron velocity distribution is replaced by a moving mean position whose velocity depends on the angular speed of the motion through an "obliquity" factor $\eta = < 1/\cos \theta >$, where θ is the angle of the velocity from its initial direction. The assumptions involved in this theory have remained obscure, and it has generally been justified by showing its agreement, if any, with the Monte Carlo theory.

Since the scattering phenomenon is a sequence of random processes, it is amenable to the Monte Carlo technique. In this method, the motion of each electron is followed individually. The scattering is included by selecting random numbers from distributions which represent the number of scatterings per unit path length and the angular deflection suffered at each scattering. The investigation via computer code was developed by Knutson[9] and improved by Wittwer et al.[10] This procedure allows a fully satisfactory treatment of scattering effects. However, it is extremely costly in computer time, and the change of any parameter requires a complete Monte Carlo investigation.

The difficulties associated with these treatments of scattering have impelled us to develop a new theory. Atmospheric scattering is basically a statistical process. The obliquity model employs "lumped" statistics, in that the characteristics of the velocity distribution are lumped into essentially a single parameter, whereas the Monte Carlo

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*C. L. Longmire, description of new obliquity given at a meeting at Air Force Weapons Laboratory, November 1973.

technique uses individual statistics, treating the electrons as independent. We shall treat the problem as a general stochastic process, employing the methods reviewed and summarized many years ago by Chandrasekhar.[11]

The body of this report is devoted to an exposition of this method of treating atmospheric scattering effects. The theory is based on a distribution function $F(\vec{r},\vec{v},t)$ which represents the number of Compton electrons per unit volume per unit velocity interval at the position $\vec{r}$ at the time t with velocity vector $\vec{v}$. In Section II, we derive the partial differential equation which governs the evolution in time of the distribution function F as affected by the various physical effects. This equation is referred to variously as the Boltzmann, Fokker-Planck, or transport equation. We shall use the last term. Effects included are production of electrons by gamma rays, with the appropriate Klein-Nishina distributions of number and energy versus initial direction, electron transport, deflection by the earth's magnetic field, energy loss by ionization, and atmospheric scattering. Self-consistent field effects are not included, since they make the transport equation nonlinear and essentially intractable. The current and the density of secondary (ionization) electrons will be represented in terms of the distribution function F.

Our purpose is to carry out the solution of the transport equation by analytic procedures as far as possible. These procedures are developed in Section III. We introduce a Green's function, which represents the distribution of electrons from an instantaneous uni-velocity point source. The function F is then found as a suitable convolution of the Green's function and the Compton electron production function. The transport equation for the Green's function is transformed by introducing new variables which eliminate all first derivatives from the equation. The resulting equation is then treated by successive approximations in which the expansion parameter is essentially the ratio of the effect of positional derivatives to the effect of velocity derivatives. This procedure is carried to third order. Explicit post hoc calculations in Section V show that the expansion is sufficiently

accurate for times up to 10 shakes (1 shake = 10^{-8} sec), which covers the significant duration of the pulse.

In Section IV, the current and the ionization density are calculated from these representations of the distribution function. As many integrals as possible are calculated analytically. The remaining integrations, which only involve the direction of the initial velocity, are brought into a form suitable for computer programming.

A computer program written to evaluate these integrals proved to be very rapid compared with the Monte Carlo technique. The program is described in Section V and some typical results given and discussed. The program itself is included as an appendix.

In Section VI, results are shown for a typical magnetic field (0.6 gauss, perpendicular to the direction of the gamma rays) and a selected gamma-ray energy (1.6 MeV). At high altitudes, the effects of scattering and energy loss are small, and the current reduces to the Karzas-Latter result. It may be expected that the effect of scattering will be to reduce the coherence of the electron distribution. Accordingly, as the altitude decreases, the peak value of the current should decrease, and the peak should occur at earlier times, since the electron beam cannot hold together. The results show precisely such behavior. For the particular case shown in Fig. 12, with a delta function time dependance for the gamma-ray source, the current per gamma ray at high altitudes peaks at 2.2 shakes with a peak value of 1.27 in the units of the figure. At 30 km, the peak occurs at 1.6 shakes with a value of 1.06; at 24 km the current peaks at 1.2 shakes with a value of 0.69; and at 18 km it peaks at 0.6 shake and is down to 0.31. Since the main electron deposition by the gamma rays takes place at these lower altitudes, we see that the effective radiating source is very considerably reduced by scattering, which will cause an attendant reduction in the electromagnetic field produced by this source. Although the secondary electron density is also affected by scattering, the results are much less striking than those for the current. The steady reduction of ionization with increasing altitude is primarily because the energy loss is proportional to the atmospheric density. The conductivity per gamma ray, which is

proportional to the number of secondary electrons divided by the atmospheric density, is relatively insensitive to altitude.

The calculations of the radiated field from the current and ionization are considerably complicated by the nonlinear dependence of the effective electron collision frequency on the electric field. At low altitudes, the effect is manifested by making the conductivity field-dependent, whereas at high altitudes (> 50 km), the effective conduction current must be treated by swarm theory. Since our main purpose in this report is to present the effects of atmospheric scattering on the collective Compton electron distribution, we have avoided this thicket by not calculating the fields produced by the current. This makes it very difficult to compare our results directly with other investigations, which usually give only the radiated fields.

This report presents our unclassified contribution to the theory of the effects of atmospheric scattering on the current and ionization density associated with high-altitude nuclear explosions. We have investigated other phenomena and will report on them separately. Applications of the theory to the pulses generated by specific nuclear weapons will be presented in a classified report.

II. DERIVATION OF THE TRANSPORT EQUATION

GENERAL CONSIDERATIONS

The geometrical configuration and coordinate system are shown in Fig. 1. A nuclear explosion occurs at height h_B, which will be assumed low enough that the curvature of the earth can be safely neglected, but high enough that it is above the principal altitudes at which gamma rays are absorbed. Thus, burst altitudes of 100 to perhaps 500 km are considered. This neglect of the earth's curvature simplifies the mathematics without losing any of the essential physics. An observer is on the ground, with the line of sight from burst to observer making an angle Ψ_B from the vertical. This observer could also be in an airplane.

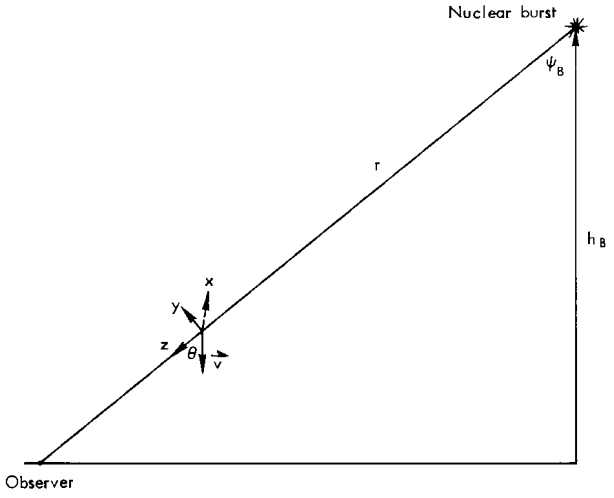


Fig. 1—Geometry and coordinate system

A rectangular coordinate system is shown, with the z-axis along the line of sight directed from the burst to the observer. This will also be the radial direction of a spherical coordinate system. Since the effect of scattering will be to reduce the electromagnetic pulse, we shall consider circumstances which tend to both simplify the analysis and maximize the pulse in the absence of scattering. For this purpose,

we shall assume that the earth's magnetic field is perpendicular to the line of sight and will define its direction as the y-direction. Since the gamma rays produce electrons which are predominantly directed forward, this assumption maximizes the magnetic deflection. Self-consistent field effects are neglected, and the earth's magnetic field is assumed uniform over the region in which significant interaction occurs between gamma rays and the atmosphere. This region has only a 30 to 40 km height range, so the variation of magnetic field is legitimately small. The methods of this report can be used for an arbitrary direction of the magnetic field, but the analysis becomes very much more complicated in detail. The coordinate system is completed by an x-coordinate.

Consider an electron which is going to contribute to the electromagnetic pulse within the first 10 shakes. This electron must be sufficiently close to the line of sight that the sum of the travel time of the gamma ray to the birth point of the electron, the age of the electron when it emits radiation, and the travel time of the radiation to the observer does not exceed by more than 10 shakes the travel time of light along the line of sight. As a typical example, for a burst at 100 km and an angle from the vertical of 60 deg, an electron born at a height of 30 km must be within 1.6 km of the radial axis. Introduce spherical coordinates r,Ψ,λ, with the origin at the burst and the radial direction along the line of sight. Then the polar angle Ψ will be less than 1 deg. The differential equation we shall derive later will involve derivatives with respect to the spherical angles. Since the solution is nonsingular with respect to angles and the total angular variation is small, we shall neglect these spherical angle derivatives of the electron number density compared to the radial derivative. We have worked out the extremely tedious calculations required by not making this approximation and find that the current and conductivity are only modified by a few percent over the time interval. Furthermore, the distance traveled by an electron in the 10 shake retarded time interval is small compared to the atmospheric scale height, so we shall neglect the variation of the atmospheric density over this distance. We thus neglect effects on the order of $c\tau_m/H$, where c is the velocity of light, τ_m the pulse duration, and H the scale height.

The gamma rays produce electrons in all directions. Hence, the velocity vector $\vec{v}$ shown in Fig. 1 can have all possible directions and magnitudes up to the maximum permitted by the Compton effect. The rectangular components of the velocity (v_x, v_y, v_z) are given in terms of its magnitude $v = \beta c$ and direction angles θ, φ, by

$$v_x = \beta c \sin\theta \cos\varphi \tag{2.1a}$$

$$v_y = \beta c \sin\theta \sin\varphi \tag{2.1b}$$

$$v_z = \beta c \cos\theta \tag{2.1c}$$

The polar angle θ is indicated in Fig. 1.

We shall now introduce stochastic procedures, following the method described on p. 42 of Ref. 11. An electron moves under the action of the coherent acceleration processes (i.e., energy loss and magnetic deflection) and the incoherent scattering process. We assume a time interval Δt exists such that the electron undergoes many infinitesimal scatterings, but that its position and velocity do not change appreciably. Then the increments in velocity and position in this time interval are given by:

$$\Delta\vec{v} = \vec{a}\Delta t + \delta\vec{v} \tag{2.2a}$$

$$\Delta\vec{r} = \vec{v}\Delta t \tag{2.2b}$$

where $\vec{a}$ is the acceleration produced by the coherent processes, and $\delta\vec{v}$ is a fluctuating quantity with a definite law of distribution. Thus, the transition probability that an electron with velocity $\vec{v}$ at position $\vec{r}$ will undergo a velocity change $\Delta\vec{v}$ and position change $\Delta\vec{r}$ is given by the function:

$$\underline{\Psi}(\vec{r},\vec{v},\Delta\vec{r},\Delta\vec{v}) = \Psi(\vec{r},\vec{v},\Delta\vec{v})\ \delta(\Delta\vec{r} - \vec{v}\Delta t) \tag{2.3}$$

Here, δ denotes the Dirac delta function and is not to be confused with the fluctuation velocity $\delta\vec{v}$.

Let us now define the distribution function F such that

$$F(\vec{r},\vec{v},t)\ d\vec{r}d\vec{v}$$

is the expected number of electrons at time t in an element of spatial volume $d\vec{r}$ in the vicinity of $\vec{r}$ and in an element of velocity volume $d\vec{v}$ in the vicinity of $\vec{v}$. We ask how many electrons are in this phase space volume at time $t + \Delta t$. Clearly, this is the sum of those produced by the gamma rays in the volume during the time Δt and the net change produced by inflow/outflow from the volume as a result of the acceleration forces. Thus (see Ref. 11, Eq. (3.24)), we have the integral equation:

$$F(\vec{r},\vec{v},t+\Delta t)\ d\vec{r}d\vec{v} = \dot{N}_e(\vec{r},\vec{v},t)\ \Delta t d\vec{r}d\vec{v}$$

$$+ \iint F(\vec{r}-\Delta\vec{r},\vec{v}-\Delta\vec{v},t)\ \underline{\Psi}(\vec{r}-\Delta\vec{r},\vec{v}-\Delta\vec{v},\Delta\vec{r},\Delta\vec{v})\ d\Delta\vec{r}d\Delta\vec{v}d\vec{r}d\vec{v} \qquad (2.4)$$

Here, $\dot{N}_e$ is the rate of production of electrons by gamma rays, and the integral term depicts transitions in and out of the phase space volume. The representation (2.3) permits integration over $d\Delta\vec{r}$. With a slight change of variables, Eq. (2.4) takes the form:

$$F(\vec{r}+\vec{v}\Delta t,\vec{v}+\vec{a}\Delta t,t+\Delta t)\ d\vec{r}d\vec{v} = \dot{N}_e(\vec{r},\vec{v},t)\ \Delta t d\vec{r}d\vec{v}$$

$$+ \int F(\vec{r},\vec{v}-\delta\vec{v},t)\ \Psi(\vec{r},\vec{v}-\delta\vec{v},\Delta\vec{v})\ d\delta\vec{v}d\vec{r}d\vec{v} \qquad (2.5)$$

Under the previous assumptions, the spatial dependence of F involves the radial coordinate only. The functions on the left and right sides of Eq. (2.5) can be expanded in Taylor series, and the limit $\Delta t \to 0$ taken. The transport equation results, which governs the evolution in time of the distribution function F:

$$\frac{\partial F}{\partial t} + v \cos\theta \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F + \nabla_{\vec{v}} \cdot \vec{a}F = \dot{N}_e(r,\vec{v},t)$$

$$+ Nv \int_o^{\pi}\int_o^{2\pi} \sin\theta' \, d\theta' d\varphi' Q(\beta,\theta,\varphi,\theta',\varphi')[F(r,\beta,\theta',\varphi',t)$$

$$- F(r,\beta,\theta,\varphi,t)] \qquad (2.6)$$

Here, $\nabla_{\vec{v}}$ is the gradient operator in velocity space, N is the number of atoms per unit volume, and Q is the differential cross section for elastic scattering from a velocity described by the direction angles θ',φ' to the angles θ,φ. The transition probability Ψ is the product NvQ. The two terms in brackets in the integral indicate that scattering takes place both into and out of the element of volume in velocity space. The space divergence operator on the left represents the net flow through the spatial element of volume, taking into account the change in the element of surface area $r^2 d\omega$, where $d\omega$ is an element of solid angle.

Equation (2.6) is the fundamental transport equation. We shall now consider the explicit representation of the coherent acceleration, scattering, and production terms.

COHERENT ACCELERATION

The coherent acceleration is given by the magnetic deflection and the energy loss expressions. The equation of motion of a relativistic electron under these forces is:

$$\frac{d}{dt}\frac{m\vec{v}}{(1-\beta^2)^{\frac{1}{2}}} = -e\vec{v} \times \vec{B} - W(\beta)\frac{\vec{v}}{v} \qquad (2.7)$$

Here, β is the velocity of the electron divided by c, the velocity of light, $\vec{B}$ is the earth's magnetic field, and $W(\beta)$ is related to the energy loss in a manner to be demonstrated. Self-consistent field effects have been omitted in Eq. (2.7). If we take the scalar product of $\vec{v}$ and Eq. (2.7) and simplify, there results:

$$\frac{d}{dt}\frac{mc^2}{(1-\beta^2)^{\frac{1}{2}}} = - vW(\beta) \tag{2.8}$$

The left side is the rate of change of the electron energy. Consequently, the function $W(\beta)$ is given by

$$W(\beta) = - \frac{dE}{dS} \tag{2.9}$$

Thus, $W(\beta)$ is the energy loss per unit path length dS. This is the quantity calculated and tabulated in the literature.

We have assumed here that all the inelastic scattering processes can be represented as an energy loss term which is parallel to the electron velocity. Actually, these inelastic processes also involve change in direction, but such effects are quite small. The dominant energy loss process at Compton energies is ionization, with small momentum change per collision. The change in the direction of the velocity of the primary for such ionizing collisions has been shown experimentally to be small. We neglect the very rare large-angle inelastic scattering events. With these assumptions, the use of the term "coherent" for energy loss seems satisfactory. This treatment is consistent with all other investigators.

When Eq. (2.8) is used in (2.7), we obtain for the coherent acceleration $\vec{a}$ the form:

$$\vec{a} = - \frac{e}{m}(1-\beta^2)^{\frac{1}{2}}\vec{v} \times \vec{B} - \frac{W(\beta)(1-\beta^2)^{3/2}\vec{v}}{mc^2\beta} \tag{2.10}$$

This expression is to be substituted into Eq. (2.6). The first term is perpendicular to the velocity, and the divergence will only involve derivatives with respect to the angles θ,φ. The second term is parallel to the velocity, and only derivatives with respect to the velocity magnitude will appear. We have assumed the magnetic field to be in the y-direction. When the derivatives are simplified, we have:

$$\nabla_{\vec{v}} \cdot \vec{a}F = -\frac{eB}{m}(1-\beta^2)^{\frac{1}{2}}\left[-\cos\varphi\,\frac{\partial F}{\partial\theta} + \frac{\cos\theta\,\sin\varphi}{\sin\theta}\,\frac{\partial F}{\partial\varphi}\right]$$

$$-\frac{1}{\beta^2}\,\frac{\partial}{\partial\beta}\left[\beta^2(1-\beta^2)^{3/2}\,\frac{W(\beta)}{mc}\,F\right] \qquad (2.11)$$

where B is the magnitude of the earth's field and we have replaced the derivative with respect to v by that with respect to β.

The function $W(\beta)$, the energy loss per unit path length, has been calculated by Bethe;[12] it is given by:

$$\frac{W(\beta)}{mc} = \frac{2\pi NZr_e^2 c}{\beta^2}\left[\log\left\{(mc^2)^2\,(\gamma-1)(\gamma^2-1)/2I^2\right\} - \left(\frac{2}{\gamma}-\frac{1}{\gamma^2}\right)\log 2 + \frac{1}{\gamma^2} + \frac{1}{8}\left(1-\frac{1}{\gamma}\right)\right] \qquad (2.12)$$

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} \qquad (2.13)$$

Here, N is the number of atoms per unit volume, Z is the number of electrons per atom (Z = 7.24 for air), and $r_e = e^2/mc^2$. The quantity I is an effective ionization potential which is 80.5 eV for air. The total energy of the electron is γmc^2 and its kinetic energy is $(\gamma-1)\,mc^2$.

The expression (2.12) has dimensions of inverse time. We shall measure time in shakes, so the multiplying factor becomes

$$2\pi NZr_e^2 c = 0.05522\,\bar{\rho}\quad [\text{shakes}^{-1}] \qquad (2.14)$$

The quantity $\bar{\rho} = \rho/\rho_o$ is the ratio of the air density at the altitude at which the electrons are located to the density at sea level. The factor in brackets in Eq. (2.12) will be designated $V(\beta)$ and is plotted against electron kinetic energy in Fig. 2, where we also plot $V(\beta)/\beta$. It may be seen from Fig. 2 that $V(\beta)/\beta$ only varies slightly over most of the range in energy. If it is replaced by the constant value 21, the error will not exceed 3.5 percent for energies between 250 KeV and 2 MeV, and will only be 20 percent at 100 MeV.

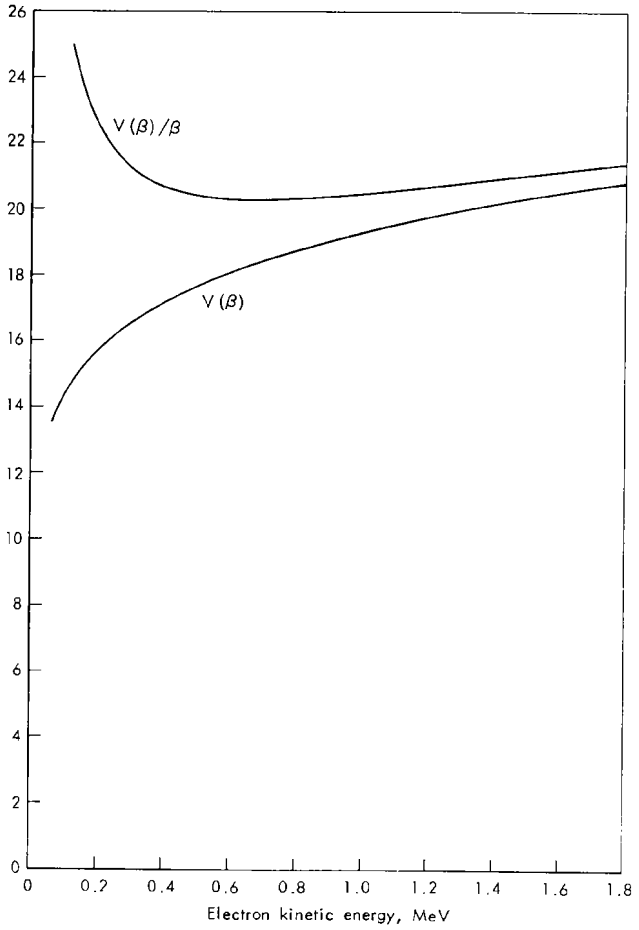


Fig. 2—Energy loss factor versus energy

Thus, we shall use the approximate form:

$$\frac{W(\beta)}{mc} = \frac{1.16\ \bar{\rho}}{\beta} \quad [\text{shakes}^{-1}] \tag{2.15}$$

The coefficient of the magnetic field term is

$$\frac{eB}{m} = 0.17588\ B\ (\text{gauss}) \quad [\text{shakes}^{-1}] \tag{2.16}$$

The form (2.15) for the energy loss per unit path length is equivalent to a constant loss per unit time, so the energy decreases linearly with time in the reference frame of the electron, down to relatively low

energies. At these low energies, the electron loses energy much faster and essentially stops. We shall use the form (2.15) for all energies and will not expect to affect the current or ionization significantly.

ATMOSPHERIC SCATTERING

The atmospheric scattering term on the right hand side of Eq. (2.6) involves the differential cross section for elastic scattering. This will generally be a function of the change in the momentum of the electron. As discussed above, we shall neglect all inelastic scattering processes, which will be regarded as represented by the energy loss. The change in momentum involves the invariant magnitude of the velocity, and the cosine of the angle between the initial and final direction of the velocity. We thus have:

$$Q(\beta,\theta,\varphi,\theta',\varphi') = Q[\beta,\ \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] \quad (2.17)$$

We shall call the second argument $\cos\chi$. The function Q is strongly peaked at small values of χ. We therefore expand the function $F(r,\beta,\theta',\varphi',t)$ in a Taylor series in $\theta'-\theta$, $\varphi'-\varphi$. To second order, we have:

$$F(r,\beta,\theta',\varphi',t) = F(r,\beta,\theta,\varphi,t) + \frac{\partial F}{\partial\theta}(\theta'-\theta) + \frac{\partial F}{\partial\varphi}(\varphi'-\varphi) + \frac{1}{2}\frac{\partial^2 F}{\partial\theta^2}(\theta'-\theta)^2 + \frac{\partial^2 F}{\partial\theta\partial\varphi}(\theta'-\theta)(\varphi'-\varphi) + \frac{1}{2}\frac{\partial^2 F}{\partial\varphi^2}(\varphi'-\varphi)^2 \quad (2.18)$$

Since the integration is over the direction cosines associated with θ',φ', the angle differences should be represented in terms of differences of direction cosines. Again to second order:

$$\theta'-\theta = \frac{(\cos\theta - \cos\theta')}{\sin\theta} - \frac{\cos\theta\,(\cos\theta' - \cos\theta)^2}{2\sin^3\theta} \quad (2.19a)$$

$$\varphi'-\varphi = \frac{\sin\theta'\sin(\varphi'-\varphi)}{\sin\theta}\left[1 + \frac{\sin\theta - \sin\theta'\cos(\varphi'-\varphi)}{\sin\theta}\right] \quad (2.19b)$$

These expressions are to be inserted into Eq. (2.18), then (2.18) itself is to be put into the integral of Eq. (2.6). With $\varphi'-\varphi$ as a new azimuthal angle, the terms in $\partial F/\partial\varphi$ and $\partial^2 F/\partial\theta\partial\varphi$ are odd functions and integrate to zero. The integration over the direction of $\vec{v}$ can be simplified by rotating the coordinate system so the direction θ,φ represents the polar direction. In this coordinate system, the angle χ, defined following Eq. (2.17), is the polar angle. We will call the corresponding azimuthal angle α. The relation between the quantities required in Eq. (2.19) for the θ',φ' and χ,α systems is:

$$\cos\theta - \cos\theta' = \cos\theta\,(1 - \cos\chi) + \sin\theta\sin\chi\cos\alpha \tag{2.20a}$$

$$\sin\theta - \sin\theta'\cos(\varphi'-\varphi) = \sin\theta\,(1 - \cos\chi) - \cos\theta\sin\chi\cos\alpha \tag{2.20b}$$

$$\sin\theta'\sin(\varphi'-\varphi) = \sin\chi\sin\alpha \tag{2.20c}$$

Since Q does not involve α, the integration over α is immediate. To this point, the scattering integral of Eq. (2.6) becomes:

$$2\pi Nv\int_o^\pi \sin\chi\, d\chi Q(\beta,\cos\chi)\left[\left\{1-\cos\chi-\frac{1}{4}\sin^2\chi\right\}\frac{\cos\theta}{\sin\theta}\frac{\partial F}{\partial\theta} + \frac{1}{4}\sin^2\chi\left(\frac{\partial^2 F}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}\right)\right] \tag{2.21}$$

Again, keeping only second-order terms, this reduces to:

$$2\pi Nv\cdot\frac{1}{4}S_2\cdot\nabla^2 F \tag{2.22a}$$

$$S_2 = \int_o^\pi \sin^3\chi\, Q(\beta,\cos\chi)\, d\chi \tag{2.22b}$$

$$\nabla^2 F = \frac{\partial^2 F}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial F}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2} \tag{2.22c}$$

We carried this procedure to sixth order; the result is:

$$2\pi Nv\left[\begin{array}{l}\left(\frac{1}{4}\,S_2+\frac{3}{32}\,S_4+\frac{5}{96}\,S_6\right)\nabla^2F\\ +\left(\frac{1}{64}\,S_4+\frac{13}{1152}\,S_6\right)\nabla^2\nabla^2F+\frac{1}{2304}\,S_6\,\nabla^2\nabla^2\nabla^2F\end{array}\right] \tag{2.23a}$$

$$S_{2m}=\int_o^{\pi}\sin^{2m+1}\chi\;Q(\beta,\cos\chi)\,d\chi \tag{2.23b}$$

The moments S_{2m} can be expected to fall rapidly with m. We now require an expression for the elastic scattering cross section Q. We are concerned with electron energies such that the momentum transfer is small and the Born approximation to scattering is valid. From the review article by Motz et al.,[13] we have selected the formula of Molière,[14] which represents an improved version of the Fermi-Thomas theory. This formula is:

$$Q(\beta,\cos\chi)=\frac{4Z^2r_e^2}{1-\beta^2}\left[\sum_1^3\frac{a_i}{\Lambda_i^2+q^2}\right]^2 \tag{2.24a}$$

$$q=2\beta\gamma\sin\chi/2 \tag{2.24b}$$

$$\Lambda_i=Z^{1/3}\,b_i/121 \tag{2.24c}$$

$$a_1=0.10\qquad a_2=0.55\qquad a_3=0.35 \tag{2.24d}$$

$$b_1=6\qquad b_2=1.2\qquad b_3=0.3 \tag{2.24e}$$

This expression may be substituted into Eq. (2.23b) for the scattering moments. All integrals may be performed analytically. The result is an expansion in $\Lambda_i/2\beta\gamma$, beginning with a logarithmic term (in S_2 only), then constants, then positive even powers. This expansion parameter will be small for all significant values of β, so only

the logarithm and constants will be kept. To this order:

$$S_2 = 4Z^2 r_e^2 \frac{(1-\beta^2)}{\beta^4} \left[\log \frac{242\ \beta\gamma}{Z^{1/3}} - 1.0793 \right] \tag{2.25a}$$

$$S_4 = 4Z^2 r_e^2 \frac{(1-\beta^2)}{\beta^4}\ 2/3 \tag{2.25b}$$

$$S_6 = 4Z^2 r_e^2 \frac{(1-\beta^2)}{\beta^4}\ 2/5 \tag{2.25c}$$

We shall modify these formulas by replacing Z^2 by $Z(Z+1)$. This is the "standard" procedure* for including scattering by the orbital electrons, whereas Eq. (2.24) corresponds to scattering by the screened nucleus. When Eqs. (2.25) are substituted into (2.23a), the coefficient of $\nabla^2 F$ becomes

$$2\pi Z(Z+1) r_e^2 Nc \frac{(1-\beta^2)}{\beta^3} \log\ (114.78\ \beta\gamma Z^{-1/3}) \tag{2.26}$$

If the higher order terms had not been included, the coefficient in the argument of the logarithm would have been 82.24. The logarithm is between 3 and 5 over the energy range. The coefficients of the higher derivatives in Eq. (2.23a), corresponding to the logarithm in Eq. (2.26), are respectively 0.05972 and 0.00069, so it is fully legitimate to neglect the fourth and sixth derivative terms. Substituting numbers into Eq. (2.26) yields:

$$0.455\ \bar{\rho} \frac{(1-\beta^2)}{\beta^3} \log\ (59.33\ \beta\gamma)\quad [\text{shakes}^{-1}] \tag{2.27}$$

*See Ref. 12, pp. 261 and 284 for many other references.

The logarithm and log/β are plotted against energy in Fig. 3. Again, log/β varies only slightly over the entire range. We shall replace it by the constant value 5.52, which is in error by only 3 percent between 250 KeV and 2 MeV, and only 20 percent at 100 KeV. The ratio between the scattering coefficient and the energy loss coefficient is more nearly constant than either of them. This was initially pointed out in Ref. 8. (The argument of the log was 65.36 βγ in Ref. 8, representing a slightly different choice of lower limit in the integration.) With this constant value, the scattering integral reduces to:

$$2.51\ \bar{\rho}\ \frac{(1-\beta^2)}{\beta^2}\ \nabla^2 F \quad [\text{shakes}^{-1}] \tag{2.28}$$

This will be used in Eq. (2.6) at all energies.

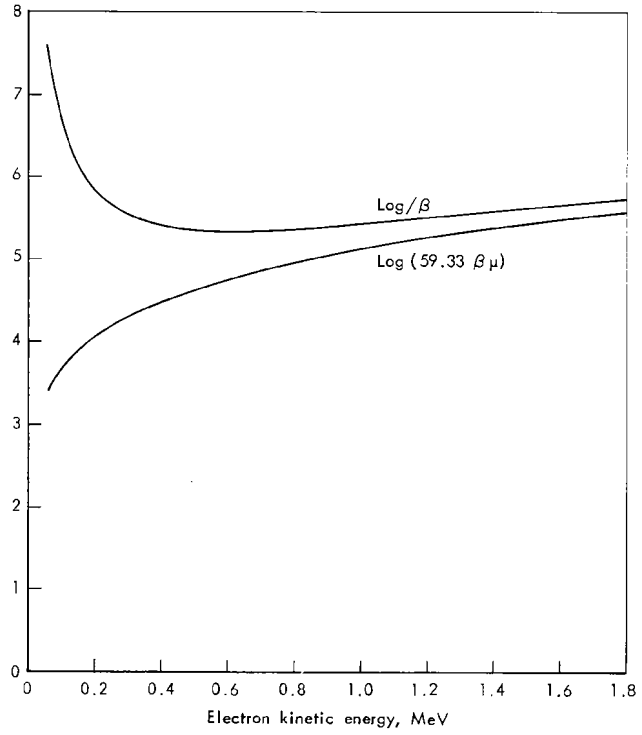


Fig. 3—Scattering coefficient versus energy

COMPTON ELECTRON PRODUCTION

The term $\dot{N}_e$ in Eq. (2.6) represents the production of Compton electrons by gamma rays and acts as a driving function for the differential equation. The gamma rays will interact with the atmosphere via the Compton effect, and the number of electrons produced per unit time will be proportional to the number of gamma rays which undergo interaction and to the differential cross section for production of an electron with a specified momentum vector. At each scattering of a primary gamma ray, a secondary gamma ray will also be produced, which will in turn be scattered at a later time, leading to a general cascade process. The retarded time until a second scattering takes place will usually exceed the 10-shake retarded time scale of this report, so we will limit our considerations here to the production of electrons by the first scatterings of gamma rays from the burst. We have investigated the cascade process on a longer time scale and will report the results in a separate publication.

We shall consider a source of gamma rays of a single energy, $E_\gamma = kmc^2$. If the burst is represented by an energy distribution, our results can be integrated over k to obtain the total current and ionization. The nature of the Compton scattering process is such that there is a unique relation between the angle at which an electron is emitted and its velocity. The function $\dot{N}_e$ can then be written as a product of four factors:

$$\dot{N}_e(r,\vec{v},t) = \frac{g(r)\ f(t - r/c)\ K(\cos\theta)\ \delta(\beta - h(\cos\theta))}{2\pi\beta^2} \qquad (2.29)$$

These functions have the following interpretations. First, g(r) is the total number of gamma rays which are absorbed per unit volume at the distance r from the burst. Let σ_c be the total Compton scattering cross section, given by:[15]

$$\sigma_c = 2\pi r_e^2 \left[\frac{2 + 8k + 9k^2 + k^3}{k^2(1 + 2k)^2} - \frac{(2 + 2k - k^2)\ \log\ (1 + 2k)}{2k^3} \right] \qquad (2.30)$$

Let the yield of the burst in MeV of gamma rays be Y. Then the function g(r) is given by

$$g(r) = \frac{Y\mu\bar{\rho}}{4\pi E_\gamma r^2} \exp\left(-\mu \int_o^r \bar{\rho}(r')\, dr'\right) \tag{2.31a}$$

$$\mu = 2\pi r_e^2 Z N_o s(k) = 18.42\ s(k)\ [km^{-1}] \tag{2.31b}$$

where s(k) is the function in brackets in Eq. (2.30), N_o is the atomic density at sea level, and $\bar{\rho}$ is the ratio of the density at r to the sea level density. Equation (2.31a) involves a mixture of units. With some further changes of units, we will write

$$g(r)\ [cm^{-3}] = \frac{3.66\ 10^{10}\ Y_{KT}\ s(k)\ \bar{\rho} \exp\left(-18.42\ s(k) \int_o^r \bar{\rho}(r')\, dr'_{KM}\right)}{E_\gamma\ [MeV]\ r_{KM}^2} \tag{2.32}$$

where now Y_{KT} is the gamma yield in kilotons, the distance r from the burst is measured in kilometers, the gamma energy is in MeV, and the total deposition g(r) is per cubic centimeter. The integral in the exponent is proportional to the total air mass traversed by the gamma rays to reach the distance r. We have plotted g(r) versus altitude in Fig. 4 for a burst at 100 km altitude, a yield of 1 kT, a gamma energy of 1.6 MeV (k = 3.1311), and an angle from the vertical of 60 deg. At this energy, s(k) = 0.3321. The production for other yields, energies, and geometries will have the same general character, a peak at some altitude in the 20 to 40 km range with a sharp drop-off at lower altitudes, and an exponential decline at higher altitudes essentially proportional to density. We have used the *Cospar International Reference Atmosphere 1961*.[16] The peak is actually quite broad, becoming half the value at the height of maximum (33 km) at 25.5 and at 44.5 km. The density varies by a factor of 37 over this altitude range.

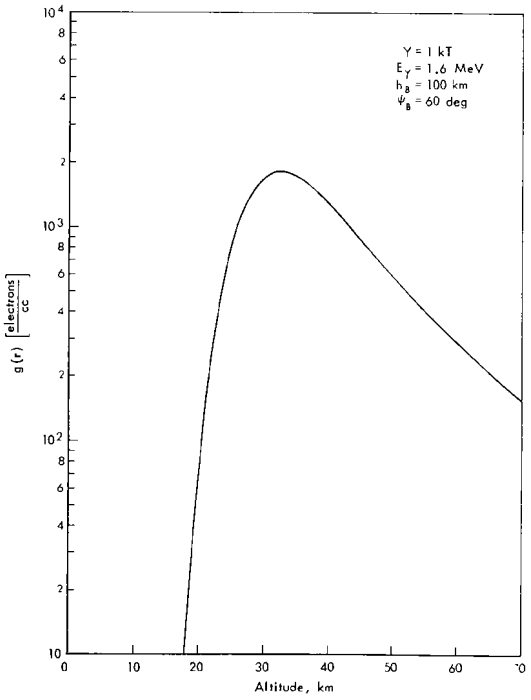


Fig. 4—Total electron production versus altitude

The function f(t - r/c) represents the time history of the gamma source. Rather than use specific functions, we shall calculate with a delta function for f. The actual time dependence of the source can then be brought in by convolution. The computer program in the appendix can be augmented by a simple integration process to obtain the current and ionization produced by an arbitrary gamma-ray source time dependence. No further differential equations need be solved. We anticipate that any actual time dependence of the gamma source will cause the current to display a slower initial rise rate than will be obtained with the delta function.

The function K(cos θ) corresponds to the angular distribution of the Compton production process. The literature generally gives this Klein-Nishina distribution in terms of the differential cross section for producing a gamma ray of momentum $\vec{k}'$ from a gamma ray of momentum $\vec{k}$. The cross section for electron production is obtained from this expression by replacing $\vec{k}'$ by $\vec{k} - \vec{p}$, where $\vec{p}$ is the electron momentum,

and then multiplying by the ratio of the unit solid angle for the photon $\vec{k}'$ to that of the electron $\vec{p}$. We normalize so the integral of K(cos θ) sin θ is unity. Omitting the detailed algebraic manipulations, there results:

$$K(x) = \frac{2(1+k)^2 x}{s(k)[(1+k)^2 - k^2x^2]^2}\left[\frac{(1+k)^2 - k(2+k)x^2}{(1+k)^2 - k^2x^2} + \frac{(1+k)^2 - k^2x^2}{(1+k)^2 - k(2+k)x^2} - \frac{4(1+k)^2 x^2(1-x^2)}{[(1+k)^2 - k(2+k)x^2]^2}\right] \tag{2.33}$$

where x = cos θ. We have plotted K(cos θ) and K(cos θ) sin θ in Figs. 5 and 6 for E_γ = 1.6 MeV (k = 3.1311). These are respectively the number per unit solid angle and the number per unit angle. Note the factor of 10 change in scale between Figs. 5 and 6. The sharp peak of K(cos θ) at small angles is evident in Fig. 5. However, the curve of Fig. 6, the number versus angle distribution, which will appear in the calculation of current and ionization, begins at zero, peaks at 10 deg, and falls off essentially linearly. This much broader distribution indicates that the often-made assumption that the Klein-Nishina distribution can be replaced by forward-directed electrons is fraught with considerable danger. Only half the number of electrons are produced at angles below 27 deg.

The final delta function factor in Eq. (2.29) relates the electron velocity to angle. The normalization is such that the integral over β will be unity. The electron velocity and kinetic energy are found from the Compton formulas, and are:

$$h(\cos\theta) = \frac{2k(1+k)\cos\theta}{(1+k)^2 + k^2\cos^2\theta} \tag{2.34}$$

$$E(\cos\theta) = \frac{mc^2\, 2k^2\cos^2\theta}{(1+k)^2 - k^2\cos^2\theta} \tag{2.35}$$

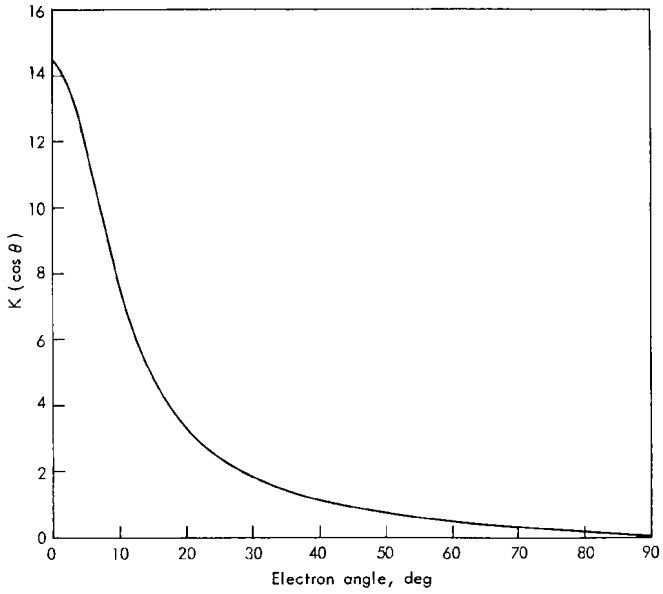


Fig. 5—Electron production per unit solid angle; $E_\gamma = 1.6$ MeV

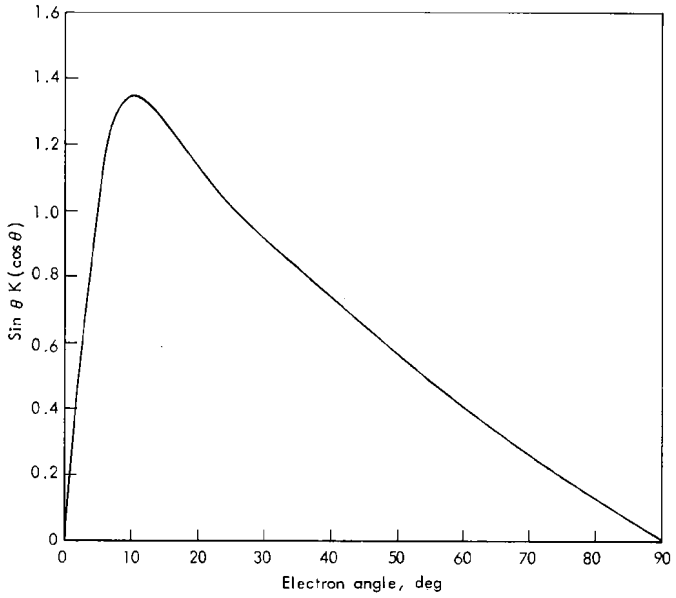


Fig. 6—Electron production per unit angle; $E_\gamma = 1.6$ MeV

These functions are plotted against angle in Fig. 7 for E_γ = 1.6 MeV. Although the velocity drops slowly with angle, the energy drops rather steeply. The average energy per electron can be calculated using Figs. 6 and 7; it is almost exactly 0.8 MeV. Thus, half the energy of the original gamma rays has gone into the production of these first-generation electrons and half remains in the photon cascade for later deposition. Half the total energy of the distribution is associated with electrons produced at angles less than 18 deg, so the effective energy distribution E(cos θ) K(cos θ), not shown, is narrower than the electron number distribution.

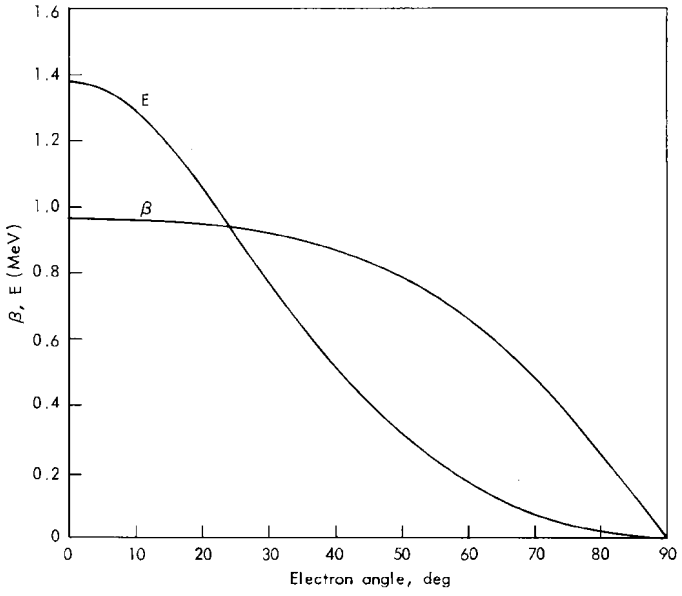


Fig. 7—Electron velocity and kinetic energy; E_γ = 1.6 MeV

CONNECTION WITH CURRENT AND SECONDARY IONIZATION

At this point, we can write the complete equation for the evolution of the distribution function F. Using the results of the previous subsections, we have:

$$\frac{\partial F}{\partial t} + \beta c \cos\theta \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F + \frac{eB}{m}(1-\beta^2)^{\frac{1}{2}}\left(\cos\varphi \frac{\partial F}{\partial \theta} - \frac{\cos\theta \sin\varphi}{\sin\theta}\frac{\partial F}{\partial \varphi}\right)$$

$$- \frac{1.16\,\bar{\rho}(r)}{\beta^2}\frac{\partial}{\partial \beta}\beta(1-\beta^2)^{3/2} F - 2.51\,\bar{\rho}(r)\frac{(1-\beta^2)}{\beta^2}\left\{\frac{\partial^2 F}{\partial \theta^2}\right.$$

$$\left. + \frac{\cos\theta}{\sin\theta}\frac{\partial F}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial \varphi^2}\right\}$$

$$= \frac{g(r)\, f(t - r/c)\, K(\cos\theta)\, \delta\left(\beta - h(\cos\theta)\right)}{2\pi\beta^2} \qquad (2.36)$$

This is the complete transport equation. The time is in shakes and, if Eq. (2.32) is to be used for the total production, the units of F are electrons per cubic centimeter. The reduced density $\bar{\rho}(r)$ is a function of altitude or, equivalently, of range to the burst.

The physical quantities to be determined from F are found from Maxwell's equations. In MKS units, these are:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2.37a)$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu\vec{J} \qquad (2.37b)$$

where $\vec{J}$ denotes the total current, including the Compton current $\vec{J}_C$ and the conduction current $\vec{J}_S$. At altitudes below 50 km, the secondary electrons experience so many collisions with the neutral molecules of the atmosphere in a time during which the electromagnetic field changes appreciably that the conduction current is a true conduction term $\sigma\vec{E}$. The conductivity is then given by:

$$\sigma = \frac{e^2 N_S(r,t)}{m\nu(r)} \qquad (2.38)$$

where $\nu(r)$ is the collision frequency which is proportional to the density and $N_S(r,t)$ is the total number of secondary electrons generated in the atmosphere by the ionization process.

The Maxwell equations are reduced and simplified by dropping the angular derivatives and employing the high-frequency approximation of Ref. 1. This enables us to write a single equation

$$\frac{\partial}{\partial r} rE + \frac{\mu c \sigma}{2} rE = \frac{\mu c}{2} rJ_c \tag{2.39}$$

where the arguments of E and J_c are r, t - r/c. The derivative is with respect to r, with t - r/c held constant. Only the transverse components of E or J_c appear in Eq. (2.39). Thus, the quantities we shall calculate from F are $\mu c J_c/2$ and N_S. We note that, although $\nu(r)$ has been written as a function of position only, it actually is strongly dependent on the field, so Eq. (2.39) is a nonlinear equation for the field which must be integrated step-wise. Since we shall not calculate fields in this report, we shall not treat Eq. (2.39) further.

The Compton current is obtained from the velocity distribution function by multiplying by $e\vec{v}$ and integrating over all values of the velocity. This yields:

$$\frac{\mu c}{2} J_c = \frac{e}{2\varepsilon} \int_o^1 \beta^3 d\beta \int_o^{\pi} \int_o^{2\pi} \sin^2 \theta \cos \varphi \, d\theta d\varphi \, F(r,\beta,\theta,\varphi,t) \tag{2.40}$$

The numerical factor $e/2\varepsilon = 0.90467 \times 10^{-8}$. The other component will integrate to zero.

For the secondary electrons, we have the rate relation

$$E_i \delta \dot{N}_S = \frac{dE}{dt} F \tag{2.41}$$

which asserts that the number of secondaries produced per unit time multiplied by the mean energy per secondary equals the number of primaries multiplied by the energy loss per unit time of the primaries. Using a mean energy per secondary of 34 eV and with the help of Eq. (2.15), we have

$$\delta\dot{N}_S = 17435\ \bar{\rho} F \tag{2.42}$$

The total rate of production of secondaries is thus:

$$\dot{N}_S = 17435\ \bar{\rho} \int_0^1 \beta^2 d\beta \int_0^{\pi} \int_0^{2\pi} \sin\theta\ d\theta d\varphi\ F(r,\beta,\theta,\varphi,t) \tag{2.43}$$

The integral on the right is also the total number of primaries N_c.

To obtain the actual number of secondaries from their rate of production, we shall use the model of Ref. 8, which takes into account the manner in which the electrons produced by Compton ionization in turn produce additional ionization. With this model, we have:

$$N_S = \int_{-\infty}^{t} dt' \dot{N}_S(t')\ L(t-t') \tag{2.44a}$$

$$L(t-t') = \frac{34}{86}\left[1 + 0.16\ \log\left(1 + 33100\ \bar{\rho}(t-t')\right)\right] \tag{2.44b}$$

$$= 1 \text{ if Eq. (2.44b) exceeds 1} \tag{2.44c}$$

We have modified the actual formula of Ref. 8 by introducing $\bar{\rho}$ instead of the atomic density N and have expressed time in shakes. Electron attachment has been neglected, which should be satisfactory at these altitudes on this time scale. We thus see that the quantities to be calculated from F are the integrals of Eqs. (2.40) and (2.43).

With Eqs. (2.36), (2.40), and (2.43), we have completed the derivation and shall now proceed to the solution. Lengthy and complicated mathematical treatment follows. We considered delegating this material to appendices, but feel the report would be impossible to follow if we leaped from the formulation of the transport equation to (1) its solution, which in abbreviated form in Eq. (3.27) occupies ten lines of solid mathematical symbols, or (2) the representations for current and ionization rate, which are of comparable length. We have attempted to make the presentation clear without becoming either too concise or hopelessly bogged down in detail.

III. SOLUTION OF THE TRANSPORT EQUATION

THE GREEN'S FUNCTION

Equation (2.36) represents the evolution in time of the electron distribution function in response to the Compton production function. We shall introduce a Green's function, which is the response to an instantaneous point source with a specified velocity vector. The true distribution function can then be determined by a convolution between the Green's function and the Compton production function. The equation for the Green's function is simpler to solve than is Eq. (2.36).

First, we shall consider the distance scale of the electron motion. In the absence of scattering, the fastest electrons of the initial distribution will spiral around the field lines with a helical radius of about 100 m. We do not expect scattering to change this scale significantly. This scale is small compared to the 6 km atmospheric scale height, so we shall neglect the variation of the density $\bar{\rho}(r)$ and the deposition $g(r)$ over the electron path. We extract a velocity-dependent factor, and write:

$$F = \frac{g(r)\ W(r,\beta,\theta,\varphi,t)}{\beta(1-\beta^2)^{3/2}} \tag{3.1}$$

$$\frac{\partial W}{\partial t} + \beta c \cos\varphi \frac{\partial W}{\partial r} + \frac{eB}{m}(1-\beta^2)^{\frac{1}{2}} \left(\cos\varphi \frac{\partial W}{\partial \theta} - \frac{\cos\theta \sin\varphi}{\sin\theta} \frac{\partial W}{\partial \varphi}\right)$$

$$- 1.16\ \bar{\rho} \frac{(1-\beta^2)^{3/2}}{\beta} \frac{\partial W}{\partial \beta} - 2.51\ \bar{\rho} \frac{(1-\beta^2)}{\beta^2} \nabla^2 W$$

$$= \frac{(1-\beta^2)^{3/2}}{2\pi\beta} f(t - r/c)\ K(\cos\theta)\ \delta\left(\beta - h(\cos\theta)\right) \tag{3.2}$$

$$\frac{\mu c}{2} J_c(r', t') = \frac{eg(r')}{2\varepsilon} \int_0^1 \frac{\beta'^2 d\beta'}{(1-\beta'^2)^{3/2}} \int_0^{\pi} \int_0^{2\pi} \sin^2\theta' \cos\varphi'\ d\theta' d\varphi'$$

$$W(r',\beta',\theta',\varphi',t') \tag{3.3}$$

$$\dot{N}_S(r',t') = 17435\ \bar{\rho}g(r') \int_o^1 \frac{\beta' d\beta'}{(1-\beta'^2)^{3/2}} \int_o^\pi \int_o^{2\pi} \sin\theta'\, d\theta' d\varphi'$$

$$W(r',\beta',\theta',\varphi',t') \tag{3.4}$$

where the primes have been introduced to facilitate later work.

The point of the transformation (3.1) is, of course, to eliminate the velocity factor under the β derivative. The initial condition on the distribution W is that W vanish for $t < r/c$.

The Green's function G, a function of the 10 variables, r,β,θ,φ,t, $r',\beta',\theta',\varphi',t'$, is taken to satisfy the differential equation

$$\frac{\partial G}{\partial t} + \beta c \cos\theta \frac{\partial G}{\partial r} + \frac{eB}{m}(1-\beta^2)^{\frac{1}{2}} \left(\cos\varphi \frac{\partial G}{\partial \theta} - \frac{\cos\theta \sin\varphi}{\sin\theta} \frac{\partial G}{\partial \varphi}\right)$$

$$- 1.16\, \bar{\rho} \frac{(1-\beta^2)^{3/2}}{\beta} \frac{\partial G}{\partial \beta} + 2.51\, \bar{\rho} \frac{(1-\beta^2)}{\beta^2} \nabla^2 G$$

$$= \frac{(1-\beta^2)^{3/2}\, \delta(r-r')\, \delta(\beta-\beta')\, \delta(\theta-\theta')\, \delta(\varphi-\varphi')\, \delta(t-t')}{\beta \sin\theta} \tag{3.5}$$

We choose the initial condition $G = 0$ for $t > t'$.

Multiply Eq. (3.2) by $\beta G \sin\theta/(1-\beta^2)^{3/2}$; Eq. (3.5) by $\beta W \sin\theta/(1-\beta^2)^{3/2}$; add; and integrate over all values of r,β,θ,φ,t. The several terms of the integration on the left lead, after integration by parts, to the following boundary terms:

$$GW\Big|_{t=-\infty}^{t=\infty},\quad GW\Big|_{r=0}^{r=\infty},\quad GW\Big|_{\beta=0}^{\beta=1},\quad \sin\theta\, GW\Big|_{\theta=0}^{\theta=\pi},$$

$$\sin\varphi\, GW\Big|_{\varphi=0}^{\varphi=2\pi},\quad \sin\theta\left(\frac{W\partial G}{\partial\theta} - \frac{G\partial W}{\partial\theta}\right)\Big|_{\theta=0}^{\theta=\pi},\quad \left(\frac{W\partial G}{\partial\varphi} - \frac{G\partial W}{\partial\varphi}\right)\Big|_{\varphi=0}^{\varphi=2\pi}$$

(3.6)

The first term vanishes at $t = \infty$ from the condition on G, and at $t = -\infty$ from the condition on W. If t is finite, W vanishes for $r \to \infty$. We take $G(r = 0) = 0$. Since the function $K(\cos\theta)$, Eq. (2.34), has an upper limit, there will be no electrons at $\beta = 1$, so the third boundary term vanishes at $\beta = 1$. To make the lower limit contribution vanish, we take $G = 0$ for $\beta < \beta'$. We assume G and W are bounded functions of θ, which makes the θ boundary terms vanish, and that they are periodic functions of the velocity azimuthal angle φ with period 2π, as is required by the physics. As a result the φ boundary terms vanish and the entire left side integration is zero. On the right side, the second term can be evaluated completely because of the delta functions, and is simply $-W(r',\beta',\theta',\varphi',t')$. Thus, W can be expressed in terms of the Green's function G by:

$$W(r',\beta',\theta',\varphi',t') =$$

$$\int_{-\infty}^{\infty} dt \int_{o}^{\infty} dr \int_{\beta'}^{1} d\beta \int_{o}^{\pi} \int_{o}^{2\pi} \sin\theta \, d\theta d\varphi \; f(t - r/c) \; K(\cos\theta) \; \delta\big(\beta - h(\cos\theta)\big)$$

$$\frac{G(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi',t')}{2\pi} \tag{3.7}$$

The time may be eliminated from Eq. (3.5). It has the following solution, which meets the initial and boundary conditions:

$$G(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi',t') = \delta\left(t-t' + \int_{\beta'}^{\beta} \frac{d\beta_1 \beta_1}{1.16\,\bar{\rho}} (1-\beta_1^2)^{3/2}\right)$$

$$\frac{H(r,\beta,\theta,\varphi,r',\beta',\theta',\varphi')}{1.16\,\bar{\rho}} \tag{3.8}$$

When this form is substituted into Eq. (3.5), the terms in δ' cancel, and the terms in δ factor out on both sides. The integral in Eq. (3.8) can be evaluated, and the argument of the delta function becomes $t-t' + (\gamma-\gamma')/1.16\,\bar{\rho}$. Since this must be zero, we see that the energy

of each electron decreases linearly with time, exactly as it must for the purely coherent energy loss process. Substitute Eq. (3.8) into Eq. (3.7) and evaluate the t integration, and we have:

$$W(r',\beta',\theta',\varphi',t') =$$

$$\int_o^\infty dr \int_{\beta'}^1 d\beta \int_o^\pi \int_o^{2\pi} \sin\theta \, d\theta d\varphi \, f\left[t' - \frac{r}{c} + \frac{(\gamma'-\gamma)}{1.16\,\bar{\rho}}\right]$$

$$\frac{K(\cos\theta)\,\delta(\beta - h(\cos\theta))\,H(r,\beta,\theta,\varphi,r',\beta',\theta',\varphi')}{(2\pi\ 1.16\,\bar{\rho})} \tag{3.9}$$

The differential equation satisfied by H will be:

$$\frac{\partial H}{\partial \beta} - \frac{\beta^2 c\cos\theta}{1.16\,\bar{\rho}(1-\beta^2)^{3/2}}\frac{\partial H}{\partial r} - \frac{eB}{m}\frac{\beta}{1.16\,\bar{\rho}(1-\beta^2)}\left(\cos\varphi\frac{\partial H}{\partial\theta} - \frac{\cos\theta\sin\varphi}{\sin\theta}\frac{\partial H}{\partial\varphi}\right)$$

$$- \frac{2.164}{\beta(1-\beta^2)^{\frac{1}{2}}}\nabla^2 H = \frac{\delta(r-r')\,\delta(\beta-\beta')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}\ ;\quad H = 0,\ \beta < \beta' \tag{3.10}$$

We shall define the three functions $q(\beta)$, $\varepsilon(\beta)$, and $a(\beta)$ as the coefficients in this equation:

$$q(\beta) = \frac{\beta^2}{1.16\,\bar{\rho}(1-\beta^2)^{3/2}} \tag{3.11a}$$

$$\varepsilon(\beta) = \frac{eB\beta}{m\ 1.16\,\bar{\rho}(1-\beta^2)} \tag{3.11b}$$

$$a(\beta) = \frac{2.164}{\beta(1-\beta^2)^{\frac{1}{2}}} \tag{3.11c}$$

We now tackle the solution of Eq. (3.10).

ELIMINATION OF THE FIRST DERIVATIVES

The procedure for simplification of Eq. (3.10) is indicated on p. 39 of Ref. 11. The procedure is to introduce as new independent variables a set of first integrals of the motion in the absence of scattering. When the differential equation is expressed in these variables, the first derivatives will disappear from Eq. (3.10), leaving only $\partial H/\partial\beta$ and second derivatives from the scattering term. The new variables are essentially the integration constants of the motion of an electron under magnetic deflection with energy loss, with the velocity magnitude β regarded as the independent variable.

In the absence of scattering, the y-direction cosine of velocity will remain constant, whereas the x- and z-direction cosines will correspond to a steady rotation. Expressing the direction cosines in terms of their initial values, we take as the trial representation for the new variables

$$\sin\theta\cos\varphi = \sin\theta_o\cos\varphi_o\cos\zeta(\beta) - \cos\theta_o\sin\zeta(\beta) \tag{3.12a}$$

$$\sin\theta\sin\varphi = \sin\theta_o\sin\varphi_o \tag{3.12b}$$

$$\cos\theta = \sin\theta_o\cos\varphi_o\sin\zeta(\beta) + \cos\theta_o\cos\zeta(\beta) \tag{3.12c}$$

$$r = r_o - c\left[\sin\theta_o\cos\varphi_o\,S(\beta) + \cos\theta_o\,C(\beta)\right] \tag{3.12d}$$

This expresses the old variables θ,φ,r in terms of the new variables θ_o,φ_o,r_o. The functions $\zeta(\beta)$, $C(\beta)$, $S(\beta)$ are to be determined to eliminate the first derivative terms in Eq. (3.10). Since the initial value of the magnitude of the velocity is β', we must choose $\zeta(\beta') = C(\beta') = S(\beta') = 0$. The inversion of Eq. (3.12) to express the new variables in terms of the old yields:

$$\sin\theta_o\cos\varphi_o = \sin\theta\cos\varphi\cos\zeta(\beta) + \cos\theta\sin\zeta(\beta) \tag{3.13a}$$

$$\sin\theta_o\sin\varphi_o = \sin\theta\sin\varphi \tag{3.13b}$$

$$\cos\theta_o = -\sin\theta\cos\varphi\sin\zeta(\beta) + \cos\theta\cos\zeta(\beta) \tag{3.13c}$$

$$r_o = r + c\left(-\sin\theta\cos\varphi\,\bar{S}(\beta) + \cos\theta\,\bar{C}(\beta)\right) \tag{3.13d}$$

$$\bar{C}(\beta) = C(\beta)\cos\zeta(\beta) + S(\beta)\sin\zeta(\beta) \tag{3.13e}$$

$$\bar{S}(\beta) = C(\beta)\sin\zeta(\beta) - S(\beta)\cos\zeta(\beta) \tag{3.13f}$$

The reduced Green's function H will be denoted by I when expressed in terms of the new variables. The derivatives of H may be expanded by the chain rule in terms of derivatives of I and derivatives of the new variables with respect to the old. The latter set is found by differentiating Eq. (3.13). The manipulation is tedious and will not be presented. To save writing, drop the arguments from ζ,C,S, and let d/dβ be denoted by a prime.

When the expressions for the derivatives of r_o,θ_o,φ_o are substituted into the chain rule relations to obtain the derivatives of H, and the resulting forms are substituted into Eq. (3.10), an enormous amount of algebraic simplification takes place. The following formulas are obtained for the coefficients of the first derivative terms:

$$\left(\frac{\partial I}{\partial r_o}\right) = c\left[-\sin\theta\cos\varphi(\bar{S}' + 2a\bar{S} - \varepsilon\bar{C}) + \cos\theta(\bar{C}' + 2a\bar{C} + \varepsilon\bar{S} - q)\right] \tag{3.14a}$$

$$\left(\frac{\partial I}{\partial\theta_o}\right) = \frac{(\zeta'-\varepsilon)\cos\varphi_o - a\cos\theta_o}{\sin\theta_o} \tag{3.14b}$$

$$\left(\frac{\partial I}{\partial\varphi_o}\right) = \frac{(\zeta'-\varepsilon)\cos\theta_o\sin\varphi_o}{\sin\theta_o} \tag{3.14c}$$

The coefficients of $\sin\theta\cos\varphi$ in Eq. (3.14a) may be set equal to zero, which gives a pair of differential equations for $\bar{C}$ and $\bar{S}$ which may be solved easily. Setting $\zeta' = \varepsilon$ eliminates Eq. (3.14c) and the first term of Eq. (3.14b). The second term is part of the transformed Laplacian.

The solutions for the functions $\zeta, \bar{C}, \bar{S}$ satisfying the proper initial conditions are:

$$\zeta = \int_{\beta'}^{\beta} d\beta_1 \, \varepsilon(\beta_1) \tag{3.15a}$$

$$\bar{C} = \int_{\beta'}^{\beta} d\beta_1 \, q(\beta_1) e^{-2\int_{\beta_1}^{\beta} d\beta_2 \, a(\beta_2)} \cos \int_{\beta_1}^{\beta} d\beta_3 \, \varepsilon(\beta_3) \tag{3.15b}$$

$$\bar{S} = \int_{\beta'}^{\beta} d\beta_1 \, q(\beta_1) e^{-2\int_{\beta_1}^{\beta} d\beta_2 \, a(\beta_2)} \sin \int_{\beta_1}^{\beta} d\beta_3 \, \varepsilon(\beta_3) \tag{3.15c}$$

$$C = \int_{\beta'}^{\beta} d\beta_1 \, q(\beta_1) e^{-2\int_{\beta_1}^{\beta} d\beta_2 \, a(\beta_2)} \cos \int_{\beta'}^{\beta_1} d\beta_3 \, \varepsilon(\beta_3) \tag{3.15d}$$

$$S = \int_{\beta'}^{\beta} d\beta_1 \, q(\beta_1) e^{-2\int_{\beta_1}^{\beta} d\beta_2 \, a(\beta_2)} \sin \int_{\beta'}^{\beta_1} d\beta_3 \, \varepsilon(\beta_3) \tag{3.15e}$$

It is clear from these equations that ζ represents the angle through which the velocity of the electron would turn in the magnetic field while changing velocity magnitude from β' to β in the absence of scattering. The functions C and S ($\bar{C}$ and $\bar{S}$) describe the forward and sideward motion of the electron during this turn, and the exponential is associated with the effects of scattering.

The coefficients of the second derivatives must now be computed. These require straightforward but lengthy manipulations. The result is the transformed differential equation:

$$\frac{\partial I}{\partial \beta} = a(\beta)\left[\begin{array}{l} \dfrac{\partial^2 I}{\partial \theta_o^2} + \dfrac{\cos\theta_o}{\sin\theta_o}\dfrac{\partial I}{\partial \theta_o} + \dfrac{1}{\sin^2\theta_o}\dfrac{\partial^2 I}{\partial \varphi_o^2} \\ \\ \quad - 2c\left\{(C\sin\theta_o - S\cos\theta_o\cos\varphi_o)\dfrac{\partial^2 I}{\partial r_o \partial\theta_o} + S\dfrac{\sin\varphi_o}{\sin\theta_o}\dfrac{\partial^2 I}{\partial r_o\partial\varphi_o}\right\} \\ \\ \quad + c^2\left\{C^2 + S^2 - (C\cos\theta_o + S\sin\theta_o\cos\varphi_o)^2\right\}\dfrac{\partial^2 I}{\partial r_o^2} \end{array}\right]$$

$$+ \frac{\delta(r_o - r')\,\delta(\beta-\beta')\,\delta(\theta_o-\theta')\,\delta(\varphi_o-\varphi')}{\sin\theta_o} \qquad (3.16a)$$

$$I = 0 \quad \beta < \beta' \qquad (3.16b)$$

APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATION

Define the differential operators on the second and third lines of Eq. (3.16a) as M and Q respectively. We see that the first line differential operator includes only derivatives with respect to the initial velocity angles θ_o, φ_o. The second line differential operator M involves mixed derivatives with respect to initial position r_o and initial velocity angles, and the third line differential operator Q contains only position derivatives.

We would expect that the primary effect of scattering for small changes in velocity magnitude (early times) would be to change the velocity direction. As the velocity change increases, the effect on position becomes significant. We would therefore anticipate that an expansion in terms of the "ratio" of the operators M and Q to the first line operator would in some sense be rapidly convergent for small velocity changes, and hope that this convergence persists for the significant duration of the current. We shall perform this expansion to third order and obtain after the fact that the convergence has remained satisfactory.

We see from Eqs. (3.15d,e) that, for small changes in velocity from the initial velocity β', C is proportional to $(\beta-\beta')$ and S to $(\beta-\beta')^2$. The operator M is thus a first-order and Q a second-order operator, both in the velocity change and in the order of the derivative with respect to r_o. Let us introduce the operator:

$$L = \frac{\partial}{\partial\beta} - a(\beta)\left[\frac{\partial^2}{\partial\theta_o^2} + \cot\theta_o \frac{\partial}{\partial\theta_o} + \csc^2\theta_o \frac{\partial^2}{\partial\varphi_o^2}\right] \tag{3.17}$$

There should be no confusion between the differential operator L and the ionization lag function L of Eq. (2.44b). We now make the symbolic expansion:

$$I = I_o + I_1 + I_2 + I_3 \tag{3.18}$$

substitute into Eq. (3.16a), and match terms of corresponding orders. There results:

$$LI_o = \frac{\delta(r_o-r')\ \delta(\beta-\beta')\ \delta(\theta_o-\theta')\ \delta(\varphi_o-\varphi')}{\sin\theta} \tag{3.19a}$$

$$LI_1 = MI_o \tag{3.19b}$$

$$LI_2 = MI_1 + QI_o \tag{3.19c}$$

$$LI_3 = MI_2 + QI_1 \tag{3.19d}$$

The expansion can be continued indefinitely, but the calculation of the higher terms becomes so complex that we have not attempted to proceed beyond third order, and have in some parts of the calculation stopped at second order.

From the structure of Eqs. (3.19), we see that we wish to solve the general equation:

$$Lu = \frac{\partial u}{\partial \beta} - a(\beta)\left[\frac{\partial^2 u}{\partial \theta_o^2} + \frac{\cos\theta_o}{\sin\theta_o}\frac{\partial u}{\partial \theta_o} + \frac{1}{\sin^2\theta_o}\frac{\partial^2 u}{\partial \varphi_o^2}\right] = R(\beta,\theta_o,\varphi_o) \qquad (3.20)$$

Since the operator L does not involve r_o, the dependence of the right side on r_o need not be indicated, and u is to be regular in θ_o,φ_o. This equation is obviously suitable for an expansion in Legendre functions in θ_o,φ_o. Thus, we assume the following solution form, in which the orthogonality properties of the Legendre functions yield the coefficient representation:

$$u = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{\varepsilon_j}{2\pi}(i+\tfrac{1}{2})\frac{(i-j)!}{(i+j)!}P_i^j(\cos\theta_o)\left[D_i^j(\beta)\cos j\varphi_o + E_i^j(\beta)\sin j\varphi_o\right] \qquad (3.21a)$$

$$\begin{matrix} D_i^j \\ E_i^j \end{matrix} = \int_o^{\pi}\int_o^{2\pi} \sin\theta_1\, d\theta_1 d\varphi_1\, P_i^j(\cos\theta_1)\, u(\beta,\theta_1,\varphi_1) \begin{matrix} \cos j\varphi_1 \qquad (3.21b) \\ \sin j\varphi_1 \qquad (3.21c) \end{matrix}$$

Here we have written $\varepsilon_j = 1$ if $j = 0$, 2 if $j \neq 0$, and have, in anticipation of later results, changed the letter for the dummy integration variable. We assume a corresponding expansion for R, with coefficients T_i^j, U_i^j. Substitute these expansions into Eq. (3.20), multiply by the factors of Eqs. (3.21b,c), and integrate. The differential operator in θ_o,φ_o in Eq. (3.20) reduces to $-i(i+1)$ when applied to the appropriate term in the expansion. The orthogonality of the Legendre functions then completely separates the terms of the expansion, leaving the differential equations

$$\frac{\partial}{\partial \beta} D_i^j + i(i+1)\, a(\beta)\, D_i^j = T_i^j \qquad (3.22a)$$

$$\frac{\partial}{\partial \beta} E_i^j + i(i+1)\, a(\beta)\, E_i^j = U_i^j \qquad (3.22b)$$

Assuming that everything vanishes at $\beta = \beta_-'$, the solution of these equations is:

$$D_i^j = \int_{\beta_-'}^{\beta} d\beta_1 T_i^j(\beta_1) e^{-i(i+1)\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \tag{3.23}$$

and correspondingly for E_i^j. Put these solutions into the expansion (3.21a), obtaining

$u(\beta,\theta_o,\varphi_o) =$

$$\int_{\beta_-'}^{\beta} d\beta_1 \int_o^{\pi}\int_o^{2\pi} \sin\theta_1\, d\theta_1 d\varphi_1 \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{\varepsilon_j}{2\pi}(i+\tfrac{1}{2})\frac{(i-j)!}{(i+j)!} e^{-i(i+1)\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$P_i^j(\cos\theta_o)\, P_i^j(\cos\theta_1)\cos j(\varphi_o-\varphi_1)\, R(\beta_1,\theta_1,\varphi_1) \tag{3.24}$$

Using the addition theorem for Legendre functions, this becomes symbolically:

$u(\beta,\theta_o,\varphi_o) =$

$$\int_{\beta_-'}^{\beta} d\beta_1 \int_o^{\pi}\int_o^{2\pi} \sin\theta_1\, d\theta_1 d\varphi_1\, R(\beta_1,\theta_1,\varphi_1)\, X(\beta,\theta_o,\varphi_o,\beta_1,\theta_1,\varphi_1) \tag{3.25a}$$

$X(\beta,\theta_o,\varphi_o,\beta_1,\theta_1,\varphi_1) =$

$$\frac{1}{2\pi}\sum_{i=0}^{\infty}(i+\tfrac{1}{2}) e^{-i(i+1)\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} P_i\left(\cos\theta_o\cos\theta_1 + \sin\theta_o\sin\theta_1\cos(\varphi_o-\varphi_1)\right) \tag{3.25b}$$

The function X is the Green's function for the differential equation (3.20). We shall write the arguments of X symbolically as (ξ_o,ξ_1), where the letter ξ_1 stands for the triad $\beta_1,\theta_1,\varphi_1$, and will write $d\omega_1$ for $\sin\theta_1\, d\theta_1 d\varphi_1$. In all these equations, β' stands for a value of β immediately below β'. We now apply this general solution to the sequence of Eqs. (3.19a-d). We see immediately that:

$$I_o = \delta(r_o-r')\, X(\xi_o,\xi') \tag{3.26}$$

The differential operators M,Q may be split into products of $\partial/\partial r_o$ and operators in β,θ_o,φ_o, which we denote M_1,Q_1. The $\partial/\partial r_o$ will act to produce derivatives of $\delta(r_o-r')$, whereas the other operators will appear in integrals of the type in Eq. (3.25a). We shall write down the results of the successive solution of Eqs. (3.19a-d) without the intermediate details:

$$\begin{aligned}
I = {} & \delta(r_o-r')\, X(\xi_o,\xi') - 2c\,\delta'(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, M_1(\xi_1)\, X(\xi_1,\xi') \\
& + 4c^2\,\delta''(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, M_1(\xi_1) \int_{\beta'}^{\beta_1} d\beta_2 \int d\omega_2 \\
& \quad X(\xi_1,\xi_2)\, M_1(\xi_2)\, X(\xi_2,\xi') \\
& + c^2\,\delta''(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, Q_1(\xi_1)\, X(\xi_1,\xi') \\
& - 8c^3\,\delta'''(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, M_1(\xi_1) \int_{\beta'}^{\beta_1} d\beta_2 \int d\omega_2 \\
& \quad X(\xi_1,\xi_2)\, M_1(\xi_2) \int_{\beta'}^{\beta_2} d\beta_3 \int d\omega_3\, X(\xi_2,\xi_3)\, M_1(\xi_3)\, X(\xi_3,\xi') \\
& - 2c^3\,\delta'''(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, M_1(\xi_1) \int_{\beta'}^{\beta_1} d\beta_2 \int d\omega_2 \\
& \quad X(\xi_1,\xi_2)\, Q_1(\xi_2)\, X(\xi_2,\xi') \\
& - 2c^3\,\delta'''(r_o-r') \int_{\beta'}^{\beta} d\beta_1 \int d\omega_1\, X(\xi_o,\xi_1)\, Q_1(\xi_1) \int_{\beta'}^{\beta_1} d\beta_2 \int d\omega_2 \\
& \quad X(\xi_1,\xi_2)\, M_1(\xi_2)\, X(\xi_2,\xi')
\end{aligned} \tag{3.27}$$

This representation for I holds for $\beta > \beta'$. For $\beta < \beta'$, I = 0. At $\beta = \beta'_+$, I reduces to the first term. If we let β approach β' from above, X tends to $\delta(\theta_o-\theta')\ \delta(\varphi_o-\varphi')/\sin\theta$, so I satisfies the proper discontinuity condition.

The reduced Green's function H is found from I by re-expressing the variables r_o,θ_o,φ_o in terms of r,θ,φ, using Eq. (3.13). Then H is to be substituted in Eq. (3.9) to obtain W, the response to the Compton production function, as represented in the variables r',ξ',t'. This representation is then to be inserted into Eqs. (3.3) and (3.4) and, after a change of lettering, integrated over ξ'. The result will be the current and the ionization production rate. We shall carry out these procedures in the next section.

IV. CALCULATION OF THE CURRENT AND IONIZATION

The highest order term in the expansion (3.27) involves integration over nine variables. There are four integrations in Eq. (3.9) and three in Eq. (3.3) or (3.4). Thus, we have to deal with 16-fold integrals. We shall now perform analytically as many as we have found possible. We shall interchange orders freely but carefully and will make great use of the orthogonality properties of the Legendre functions.

The first integration is the outermost, the integration over θ',φ, in Eqs. (3.3) and (3.4). We observe that in Eq. (3.27) these variables occur only in the extreme right factors of each term in the expansion. In $X(\xi,\xi')$, the angular variables appear only as arguments of Legendre functions. We thus have to evaluate the integrals:

$$\int d\omega' \, P_i \left[\cos\theta_k \cos\theta' + \sin\theta_k \sin\theta' \cos(\varphi_k - \varphi')\right] \tag{4.1a}$$

$$\int d\omega' \sin\theta' \cos\varphi' \, P_i \tag{4.1b}$$

where θ_k,φ_k refer to the first arguments of X in the appropriate term of Eq. (3.27). Rotate the coordinate system so that the pole is in the direction θ_k,φ_k, and Eq. (4.1) becomes:

$$\int d\omega' \, P_i(\cos\theta') \tag{4.2a}$$

$$\int d\omega' \left[\sin\theta_k \cos\varphi_k \cos\theta' + \cos\theta_k \cos\varphi_k \sin\theta' \cos\varphi' - \sin\varphi_k \sin\theta' \sin\varphi'\right] P_i(\cos\theta') \tag{4.2b}$$

The first integral is zero unless $i = 0$, in which case it equals 4π. The second and third terms in the bracket of Eq. (4.2b) vanish when integrated over φ'. The first integrates to zero unless $i = 1$, when

it becomes $4\pi \sin\theta_k \cos\varphi_k/3$. Using these results in the series representation then yields:

$$\int d\omega' \, X(\xi_k,\xi') = 1 \tag{4.3a}$$

$$\int d\omega' \sin\theta' \cos\varphi' \, X(\xi_k,\xi') = \sin\theta_k \cos\varphi_k \, e^{-2\int_{\beta'}^{\beta_k} d\beta'' a(\beta'')} \tag{4.3b}$$

Since M_1 is a differential operator, when it is applied to a constant the result is zero. Hence, there is no contribution to the ionization density from the second, third, fifth, and seventh terms of the expansion (3.27). We shall make a further simplification by dropping the third-order terms in the current. This will be justified by later numerical work.

Next are the angular integrations in Eq. (3.27). We shall demonstrate the procedure by calculating the first-order term in the current and then writing out the result for the complete expression. In this term, the integral to be evaluated is:

$$\int d\omega_1 \, X(\xi_o,\xi_1) \, M_1(\xi_1) \sin\theta_1 \cos\varphi_1 \tag{4.4}$$

The differential operator M_1 is given in Eq. (3.16a), with the variables to be given the subscript 1. There results:

$$M_1(\xi_1) \sin\theta_1 \cos\varphi_1 = C_1 \sin\theta_1 \cos\theta_1 \cos\varphi_1 - S_1(1 - \sin^2\theta_1 \cos^2\varphi_1) \tag{4.5}$$

Insert this into the integral in Eq. (4.4) corresponding to the *ith* term in the series for X, and rotate the coordinate system so that the pole is in the θ_o,φ_o direction. We then have, after simplification:

$$\int d\omega_1 \, P_i(\cos\theta_1) \Big[- S_1 + (\sin\theta_o \cos\varphi_o \cos\theta_1 + \cos\theta_o \cos\varphi_o \sin\theta_1 \cos\varphi_1 - \sin\varphi_o \sin\theta_1 \sin\varphi_1) \Big\{ C_1(\cos\theta_o \cos\theta_1 - \sin\theta_o \sin\theta_1 \cos\varphi_1) + S_1(\sin\theta_o \cos\varphi_o \cos\theta_1 + \cos\theta_o \cos\varphi_o \sin\theta_1 \cos\varphi_1 - \sin\varphi_o \sin\theta_1 \sin\varphi_1) \Big\} \Big] \tag{4.6}$$

Integrating over φ_1 yields:

$$2\pi \int_o^\pi \sin\theta_1 \, d\theta_1 P_i(\cos\theta_1) \left[\begin{array}{l} C_1 \cos\theta_o \sin\theta_o \cos\varphi_o \left(\frac{3}{2}\cos^2\theta_1 - \frac{1}{2} \right) \\ \quad + S_1 \left\{ -1 + \frac{1}{2}\sin^2\theta_1 \right. \\ \quad \left. + \sin^2\theta_o \cos^2\varphi_o \left(\frac{3}{2}\cos^2\theta_1 - \frac{1}{2} \right) \right\} \end{array} \right] \tag{4.7}$$

From the orthogonality properties of the Legendre functions, the integral is zero unless i = 0 or 2. Evaluating these cases and restoring the series representation gives the result:

$$\int d\omega_1 \, X(\xi_o, \xi_1) \, M_1(\xi_1) \sin\theta_1 \cos\varphi_1 =$$

$$C_1 \cos\theta_o \sin\theta_o \cos\varphi_o \, e^{-6\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$- S_1 \left[\frac{2}{3} + \left(\frac{1}{3} - \sin^2\theta_o \cos^2\varphi_o \right) e^{-6\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right] \tag{4.8}$$

In this expression we have:

$$C_1 = \int_{\beta'}^{\beta_1} d\beta_2 q(\beta_2) e^{-2\int_{\beta_2}^{\beta_1} d\beta_3 a(\beta_3)} \cos \int_{\beta'}^{\beta_2} d\beta_4 \varepsilon(\beta_4) \qquad (4.9)$$

and a corresponding expression for S_1. All the angular integrals of Eq. (3.27) can be evaluated in exactly the same manner, but the manipulations become very tedious for the higher order terms. We write out the result to third order in the integrand for the ionization and to second order in the current:

$$\begin{aligned}
\int_I d\omega' = {} & \delta(r_o - r') + c^2 \delta''(r_o - r') \int_{\beta'}^{\beta} d\beta_1 a(\beta_1) \left[\left(C_1^2 + S_1^2 \right) \left(\frac{2}{3} + \frac{1}{3} e^{-6\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right) \right. \\
& \left. - \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right)^2 e^{-6\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right] \\
& + \frac{4}{5} c^3 \delta'''(r_o - r') \int_{\beta'}^{\beta} d\beta_1 a(\beta_1) \int_{\beta'}^{\beta_1} d\beta_2 a(\beta_2) e^{-6\int_{\beta_2}^{\beta_1} d\beta_3 a(\beta_3)} \\
& \left[\left(C_1 C_2 + S_1 S_2 \right) \left(C_2 \cos\theta_o + S_2 \sin\theta_o \cos\varphi_o \right) \left(2e^{-12\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} + 3e^{-2\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \right) \right. \\
& + \left(C_2^2 + S_2^2 \right) \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) \left(e^{-12\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} - e^{-2\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \right) \\
& \left. - 5 \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) \left(C_2 \cos\theta_o + S_2 \sin\theta_o \cos\varphi_o \right)^2 e^{-12\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \right]
\end{aligned} \qquad (4.10)$$

$$\int I\, d\omega' \sin\theta' \cos\varphi' = e^{-2\int_{\beta'}^{\beta} d\beta_1 a(\beta_1)} \Bigg[\delta(r_o - r') \sin\theta_o \cos\varphi_o$$

$$+ 2c\, \delta'(r_o - r') \int_{\beta'}^{\beta} d\beta_1 a(\beta_1) \Bigg\{ S_1 \left(\frac{2}{3} e^{2\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} + \frac{1}{3} e^{-4\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right)$$

$$- \sin\theta_o \cos\varphi_o \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) e^{-4\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \Bigg\}$$

$$+ \frac{1}{5} c^2\, \delta''(r_o - r') \int_{\beta'}^{\beta} d\beta_1 a(\beta_1) \Bigg\{ \left(C_1^2 + S_1^2 \right) \sin\theta_o \cos\varphi_o \left(1 + 4e^{-10\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right)$$

$$- 2S_1 \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) \left(1 - e^{-10\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \right)$$

$$- 5 \sin\theta_o \cos\varphi_o \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right)^2 e^{-10\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \Bigg\}$$

$$+ \frac{4}{5} c^2\, \delta''(r_o - r') \int_{\beta'}^{\beta} d\beta_1 a(\beta_1) \int_{\beta'}^{\beta} d\beta_2 a(\beta_2) e^{-4\int_{\beta_2}^{\beta_1} d\beta_3 a(\beta_3)}$$

$$\Bigg\{ 2S_2 \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) \left(1 - e^{-10\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \right)$$

$$- \Big\{ \left(C_1 C_2 + S_1 S_2 \right) \sin\theta_o \cos\varphi_o + S_1 \left(C_2 \cos\theta_o + S_2 \sin\theta_o \cos\varphi_o \right) \Big\} \left(3 + 2e^{-10\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \right)$$

$$+ 10 \sin\theta_o \cos\varphi_o \left(C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o \right) \left(C_2 \cos\theta_o + S_2 \sin\theta_o \cos\varphi_o \right) e^{-10\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3)} \Bigg\} \Bigg] \tag{4.11}$$

The next step is to replace the variables r_o,θ_o,φ_o by r,θ,φ using Eq. (3.13). The integrations which remain to evaluate the current and the ionization density are over the variables $r,\beta,\theta,\varphi,\beta',\beta_1,\beta_2$. The variable r appears in the argument of a delta function, $r-r' + c(\bar{C}\cos\theta - \bar{S}\sin\theta\cos\varphi)$, its derivatives, and f. The derivative with respect to r of the delta function goes over into $\partial/\partial(ct')f$ after the r integration, where the argument of f is $t'-r'/c - (\gamma-\gamma')/1.16\,\bar{\rho} + \bar{C}\cos\theta - \bar{S}\sin\theta\cos\varphi$. Since $t'-r'/c$ is the retarded time at the electron location, we have now shown that, to the approximation of neglecting the variation of $\bar{\rho}$ and g over the electron motion, the current and ionization are functions of retarded time only, as is of course to be expected.

The variable β appears as the argument of a delta function $\beta - h(\cos\theta)$ and in integration limits. Since β must exceed β', the β' integration must be brought inside the θ integration, and then the limits on β' become 0,h. Referring back to Eqs. (3.15b,c), when β is replaced by h, we shall regard $\bar{C},\bar{S}$ as functions of the lower limit β'. We then define:

$$\underline{C}_1 = \bar{C}(\beta') - \bar{C}(\beta_1) = \int_{\beta'}^{\beta_1} d\beta_2 q(\beta_2) e^{-2\int_{\beta_2}^{h} d\beta_3 a(\beta_3)} \cos\int_{\beta_2}^{h} d\beta_4 \varepsilon(\beta_4) \tag{4.12}$$

and a corresponding equation for $\underline{S}_1$. The function $\bar{C}(\beta')$ differs from Eq. (4.12) only in that the outermost upper limit is h. We have the relations:

$$C_1 = e^{2\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} (\underline{C}_1\cos\zeta + \underline{S}_1\sin\zeta) \tag{4.13a}$$

$$S_1 = e^{2\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} (\underline{C}_1\sin\zeta - \underline{S}_1\cos\zeta) \tag{4.13b}$$

$$C_1 \cos\theta_o + S_1 \sin\theta_o \cos\varphi_o = e^{2\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$(\underline{C}_1 \cos\theta - \underline{S}_1 \sin\theta \cos\varphi) \tag{4.13c}$$

$$C_1^2 + S_1^2 = e^{4\int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} (\underline{C}_1^2 + \underline{S}_1^2) \tag{4.13d}$$

$$C_1C_2 + S_1S_2 = e^{2\int_{\beta_1}^{\beta} d\beta_3 a(\beta_3) + 2\int_{\beta_2}^{\beta} d\beta_3 a(\beta_3)} (\underline{C}_1\underline{C}_2 + \underline{S}_1\underline{S}_2) \tag{4.13e}$$

The point of introducing these functions is that $\overline{C}(\beta')$ and $\overline{S}(\beta')$, functions of only one variable, are easy to compute. The underscored functions are then obtained by subtraction, or the polynomials in $\overline{C}(\beta') - \overline{C}(\beta_1)$ may be expanded and the integrations thereby separated into products of integrals. This procedure greatly shortens the computing time.

These discussions show how to integrate over the variables r,β. We shall now assume that $f(t - r/c)$, the gamma-ray time history, is a delta function whose argument is:

$$t' - r'/c - \int_{\beta'}^{h} \frac{\beta d\beta}{1.16\ \overline{\rho}\ (1-\beta^2)^{3/2}} \left\{ 1 - \beta e^{-2\int_{\beta}^{h} a(\beta_1)d\beta_1} \left(\cos\theta \cos\int_{\beta}^{h} d\beta_2 \varepsilon(\beta_2) - \sin\theta \cos\varphi \sin\int_{\beta}^{h} d\beta_2 \varepsilon(\beta_2) \right) \right\} \tag{4.14}$$

The derivatives of the delta function with respect to $t-t'$ which arise from the r_o derivatives in I may be taken outside the integrations.

Thus, Eq. (4.14) indicates how we may integrate over β'. Choose values of $t'-r'/c$, θ, and φ. Find the value of β' such that Eq. (4.14) is zero. Set β' equal to that value in Eqs. (4.10) and (4.11). Divide by the derivative of Eq. (4.14) with respect to β', which is of course the integrand of Eq. (4.14). The result will be the integral over β'.

The structure of Eq. (4.14) is such that the integral increases monotonically as β' decreases from h. At $\beta' = h$, expression (4.14) is $t'-r'/c$ and is positive. When $\beta' = 0$, the integral has a maximum value which is a function of θ and φ. For sufficiently small values of $t'-r'/c$, there will be a zero for Eq. (4.14) as a function of β for all values of θ and φ. As $t'-r'/c$ increases, it reaches a value which exceeds the maximum value of the integral for some values of θ,φ, which then do not contribute to the integration over β'. Those values of θ,φ correspond to electrons which have slowed to zero velocity, so of course they should not contribute. Eventually, $t'-r'/c$ becomes so large that it exceeds the integral for all θ,φ and there is no further contribution to the current or ionization.

We have reached this point in the solution of the transport equation and the calculation of the current and ionization without utilizing the specific forms of the functions q, ε, and a. We shall now employ Eq. (3.11) to simplify the expressions. We first evaluate ζ and the integral of a. With Eqs. (3.11b) and (3.15a), and β set equal to h, we have:

$$\zeta' = \int_{\beta'}^{h} d\beta_1 \varepsilon(\beta_1) = \frac{eB}{m\ 1.16\ \bar{\rho}} \int_{\beta'}^{h} \frac{\beta_1 d\beta_1}{(1-\beta_1^2)} \tag{4.15a}$$

$$= \frac{eB}{m} \frac{1}{1.16\ \bar{\rho}} \log \sqrt{\frac{1-\beta'^2}{1-h^2}} \tag{4.15b}$$

$$= \frac{1}{K} \log \frac{\gamma_o}{\gamma'} \tag{4.15c}$$

where the constant K is equal to:

$$K = 6.595 \frac{\bar{\rho}}{B} \tag{4.16}$$

and γ_o is the initial electron total energy, including the rest energy in units of mc^2.

We can invert Eq. (4.15c) to express β' and γ' in terms of ζ', yielding:

$$\gamma' = \gamma_o \, e^{-K\zeta'} \tag{4.17a}$$

$$\beta' = \left(1 - \frac{e^{2K\zeta'}}{\gamma_o^2}\right)^{\frac{1}{2}} \tag{4.17b}$$

Thus, as the electron turns in the magnetic field through an angle ζ in the absence of scattering, the energy decreases exponentially with ζ. Since the smallest possible value of β is zero, we see from Eq. (4.15c) that the maximum angle of turn of the electron is directly proportional to the magnetic field, inversely proportional to the density, and proportional to the logarithm of the initial energy. For electrons with the energy distribution of Fig. 7, the logarithm has a maximum value 1.306 at $\theta = 0$ and is down to 0.7 at 40 deg. We shall consider $\log \gamma_o = 1$ as representative, corresponding to electrons slightly below the mean energy, 0.8 MeV. With a magnetic field of 0.6 gauss, the maximum angle of turn will be 72 deg at an altitude of 20 km, 347 deg at 30 km, and 1568 deg (4.36 revolutions) at 40 km. This cutoff of the angle of turn is caused entirely by the energy loss process.

The integral of a can be evaluated using Eq. (3.11c). We shall arrange the limits in the exponents of Eqs. (4.10) and (4.11) so that the upper limit is always h. After simplification, we obtain:

$$\int_{\beta}^{h} d\beta_1 a(\beta_1) = 1.082 \log \left[\frac{\gamma_o - 1}{\gamma_o + 1} \frac{\gamma + 1}{\gamma - 1}\right] \tag{4.18a}$$

$$e^{-2\int_{\beta}^{h} d\beta_1 a(\beta_1)} = \left[\frac{(\gamma - 1)}{(\gamma + 1)} \frac{(\gamma_o + 1)}{(\gamma_o - 1)}\right]^{2.164} \tag{4.18b}$$

We shall introduce ζ as a new independent variable. We have, for the function $\bar{C}$ (see Eq. (4.12)), the expressions:

$$\bar{C}(\zeta') = \frac{m}{eB}\int_o^{\zeta'} d\zeta \left(\gamma_o^2 e^{-2K\zeta} - 1\right)^{\frac{1}{2}} \left[\frac{\gamma_o e^{-K\zeta} - 1}{\gamma_o e^{-K\zeta} + 1} \frac{\gamma_o + 1}{\gamma_o - 1}\right]^{2.164} \cos \zeta \tag{4.19}$$

For $\bar{S}$, cos ζ is replaced by sin ζ. This representation depends directly on ζ', related to β' through Eq. (4.17b), and indirectly on θ, which appears in γ_o. Equation (4.19) is in precise form for step-by-step numerical integration.

If there were no scattering, we would set a = 0. The functions $\bar{C},\bar{S}$ then represent the forward and sideward components of position of an electron with initial energy γ_o which has turned through an angle ζ' from its initial direction. The first effect of scattering is to insert the factor (4.18b) into $\bar{C}$ and $\bar{S}$. This corresponds to a change in the velocity of a mean position. If we were to keep only the first terms in the expansions (4.10), the motions of the aggregate of electrons would be fully described by a moving mean position.

The scattering expression (4.18a) corresponds closely to Longmire's obliquity theory, which sets the right side of (4.18a) equal to $(\eta-1)/2$ (see Eq. (22) of Ref. 8, with a slight change in constants). Thus, in some sense, the obliquity theory is a zero*th* approximation to the theory we have developed in this report, with the electron velocity distribution Green's function reduced to the first term of I. We have not attempted to work out the full details of the comparison.

We shall now write out the full expressions for current and ionization density as integrals over $\theta,\varphi,\zeta_1,\zeta_2$. Let $\tau = t'-r'/c$, the retarded time. We define the integral of Eq. (4.14), the equivalent of

the arrival time of the electron, as $T(\zeta',\theta,\varphi)$. Then the condition that Eq. (4.14) vanishes determines a value $\zeta'(\tau,\theta,\varphi)$ such that:

$$\tau = T[\zeta'(\tau,\theta,\varphi),\theta,\varphi] = \frac{m}{eB}\int_o^{\zeta'} \gamma d\zeta \left[1 - \beta\left(\frac{\gamma - 1}{\gamma + 1}\frac{\gamma_o + 1}{\gamma_o - 1}\right)^{2.164} (\cos\theta\cos\zeta - \sin\theta\cos\varphi\sin\zeta)\right] \quad (4.20a)$$

$$= T_1(\zeta',\Theta) + T_2(\zeta',\theta)\cos\varphi \quad (4.20b)$$

The value of ζ' to be used in the current and ionization is to be found by inverting Eq. (4.20). In practice, this must be done by interpolation. The unspecified limits on θ and φ are for all values such that Eq. (4.20) has a solution. We define $U(\zeta)$ as the function of Eq. (4.18b), which we can call the scattering reduction factor, and will write U' for $U(\zeta')$, U_1 for $U(\zeta_1)$, and U_2 for $U(\zeta_2)$. A function D is defined by:

$$D(\zeta',\theta,\varphi) = 1 - \beta' U'(\cos\theta\cos\zeta' - \sin\theta\cos\varphi\sin\zeta') \quad (4.21)$$

D is the ratio between retarded time and local time of the aggregate of electrons. We also define a function $A(\zeta) = a(\beta)\, d\beta/d\zeta$ as:

$$A(\zeta) = \frac{2.164\, K\gamma}{\gamma^2 - 1} \quad (4.22)$$

We introduce the shorthand:

$$w = \cos\theta\sin\zeta' + \sin\theta\cos\varphi\cos\zeta' \quad (4.23a)$$

$$Z_i = \underline{C}_i\cos\theta - \underline{S}_i\sin\theta\cos\varphi \quad (4.23b)$$

We now have the final result for the current and ionization rate:

$$\frac{\mu c}{2} J_c(\tau) = 0.90467 \times 10^{-8}\, g \iint \frac{\sin\theta\, d\theta d\varphi\, K(\cos\theta)\beta' U'}{2\pi D(\zeta',\theta,\varphi)}$$

$$\left[w + \frac{2}{3}\frac{\partial}{\partial\tau}\int_o^{\zeta'} d\zeta_1 A(\zeta_1)\left\{ S_1\left(U_1 + \frac{2}{U_1^2}\right) - 3wZ_1U_1 \right\} \right.$$

$$+ \frac{1}{5}\frac{\partial^2}{\partial\tau^2}\int_o^{\zeta'} d\zeta_1 A(\zeta_1)\left\{ (\underline{C}_1^2 + \underline{S}_1^2)\, w\left(4U_1^3 + \frac{1}{U_1^2}\right) \right.$$

$$\left. + 2S_1Z_1\left(U_1^3 - \frac{1}{U_1^2}\right) - 5wZ_1U_1^3 \right\}$$

$$+ \frac{4}{5}\frac{\partial^2}{\partial\tau^2}\int_o^{\zeta'} d\zeta_1 A(\zeta_1)\int_{\zeta_1}^{\zeta'} d\zeta_2 A(\zeta_2)U_2 \left[-2S_2Z_1\left(U_1^2 - \frac{1}{U_1^3}\right) \right.$$

$$\left. \left. + 10wZ_1Z_2U_1^2 - \left\{ w(\underline{C}_1\underline{C}_2 + \underline{S}_1\underline{S}_2) + S_1Z_2 \right\}\left(2U_1^2 + \frac{3}{U_1^3}\right) \right] \right] \qquad (4.24)$$

$$\dot{N}_S(\tau) = 17435\, g \iint \frac{\sin\theta\, d\theta d\varphi\, K(\cos\theta)}{2\pi D(\zeta',\theta,\varphi)}$$

$$\left[1 + \frac{1}{3}\frac{\partial^2}{\partial\tau^2}\int_o^{\zeta'} d\zeta_1 A(\zeta_1)\left\{ (\underline{C}_1^2 + \underline{S}_1^2)\left(U_1 + \frac{2}{U_1^2}\right) - 3Z_1^2U_1 \right\} \right.$$

$$+ \frac{4}{5}\frac{\partial^3}{\partial\tau^3}\int_o^{\zeta'} d\zeta_1 A(\zeta_1)\int_{\zeta_1}^{\zeta'} d\zeta_2 A(\zeta_2)U_2 \left\{ (\underline{C}_1\underline{C}_2 + \underline{S}_1\underline{S}_2)Z_2 \right.$$

$$\left. \left. \left(2U_1^2 + \frac{3}{U_1^3}\right) + (\underline{C}_2^2 + \underline{S}_2^2)Z_1\left(U_1^2 - \frac{1}{U_1^3}\right) - 5Z_1Z_2^2U_1^2 \right\} \right] \qquad (4.25)$$

The time derivatives in these equations should be placed outside the integral sign, since the limits are functions of τ. We have written them inside to avoid having to repeat the first line of Eqs. (4.24) and (4.25) for each term, but in calculation the time derivatives must be treated last.

This is as far as we have been able to proceed analytically. We have written a computer program to calculate the current and ionization from Eqs. (4.24) and (4.25). This program is purely an exercise in numerical integration, since all the differential equations have been solved. We next discuss the construction of the program.

V. DEVELOPMENT OF THE COMPUTER PROGRAM

Although the programming of Eqs. (4.24) and (4.25) is fairly straightforward, there are some rather tricky numerical points, mostly associated with the zero-velocity cutoff. The program takes as inputs the magnetic field, gamma-ray energy, time-step, and angle-step. Most calculations have been done with a time-step of 0.1 shake and an angle-step of 1 deg in the turn-angle ζ. Double precision arithmetic has been employed throughout.

The program uses different numerical integration routines for the several variables. The integration over θ is a simple summation, since the end-point terms of the trapezoidal rule vanish because of the factor $\sin\theta$. We have used an integration step in θ of 3 deg for θ below 30 deg and 6 deg above. This is a good match to the rate of variation of $K(\cos\theta)\sin\theta$ (see Fig. 6). The innermost integrals over ζ_1 and ζ_2 use the modified trapezoidal rule.[17] This is a four-point rule, which must be specially modified at the end intervals of the integration range or if there are fewer than four points available (as can occur for low-energy electrons). The rule reads

$$\int_0^{\Delta x} f(x)dx = \frac{\Delta x[9f(0) + 19f(1) - 5f(2) + f(3)]}{24} \tag{5.1a}$$

$$\int_{i\Delta x}^{(i+1)\Delta x} f(x)dx = \frac{\Delta x[-f(i-1) + 13f(i) + 13f(i+1) - f(i+2)]}{24} \tag{5.1b}$$

$$\int_{(N-1)\Delta x}^{N\Delta x} f(x)dx = \frac{\Delta x[f(N-3) - 5f(N-2) + 19f(N-1) + 9f(N)]}{24} \tag{5.1c}$$

where the argument $i\Delta x$ of $f(i\Delta x)$ has been abbreviated i and similarly for all other terms. Equation (5.1a) represents the first interval of the integration range, (5.1c) represents the last interval, and it is

assumed that N is at least 3. The error term in Eq. (5.1b) is $11(\Delta x)^5 f^{(4)}(\xi)/720$, where $f^{(4)}(\xi)$ denotes the fourth derivative of f evaluated at some point in the interval. The functions we have to deal with are sufficiently smooth that the error term can be neglected throughout.

The integration over φ is performed by gaussian integration procedures. First, we reconsider the limits of integration. Referring to Eq. (4.20), we define a function $T_m(\theta,\varphi)$ by:

$$T_m(\theta,\varphi) = \frac{m}{eB}\int_o^{\log \gamma_o/K} \gamma_o e^{-K\zeta}\, d\zeta \left[1 - \left(1 - \frac{e^{2K\zeta}}{\gamma_o^2}\right)^{\frac{1}{2}} \left(\frac{\gamma_o\, e^{-K\zeta} - 1}{\gamma_o\, e^{-K\zeta} + 1}\,\frac{\gamma_o + 1}{\gamma_o - 1}\right)^{2.164} (\cos\theta\cos\zeta - \sin\theta\cos\varphi\sin\zeta)\right] \tag{5.2a}$$

$$\gamma_o = \frac{\left[(1+k)^2 + k^2\cos^2\theta\right]}{\left[(1+k)^2 - k^2\cos^2\theta\right]} \tag{5.2b}$$

This is the time for an electron to slow to zero velocity from the initial energy γ_o appropriate to the direction θ,φ. It is clear that T_m can be split into the form:

$$T_m(\theta,\varphi) = T_{m1}(\theta) + T_{m2}(\theta)\cos\varphi \tag{5.3}$$

This lies between the limits $T_{m1} \pm T_{m2}$. Suppose now we choose a value of θ, and let τ increase from zero. As long as τ is less than $T_{m1} - T_{m2}$, there will be a solution of Eq. (4.20), $\tau = T(\zeta',\theta,\varphi)$ for all values of φ. If τ is between $T_{m1} - T_{m2}$ and $T_{m1} + T_{m2}$, there will be a solution for $\cos\varphi \geq (\tau - T_{m1})/T_{m2}$, and not otherwise. If τ exceeds $T_{m1} + T_{m2}$, the integral is zero. Since the integrand depends only on $\cos\varphi$, the integral from 0 to 2π is twice the integral from 0 to π. We thus have the integration forms:

$$\int_o^{\pi} f(\cos\varphi)d\varphi, \qquad \int_o^{\varphi_m} f(\cos\varphi)d\varphi \tag{5.4a}$$

$$\cos \varphi_m = \frac{(\tau - T_{m1})}{T_{m2}} = X_m \tag{5.4b}$$

For the first integral, we have the gaussian approximation,*

$$\frac{1}{\pi}\int_o^{\pi} f(\cos \varphi)d\varphi \sim \frac{1}{n}\sum_1^n f\left(\cos \frac{(2i-1)\pi}{n}\right) \tag{5.5}$$

whereas the second integral leads to a modified version of Eq. (5.5):

$$\frac{1}{\pi}\int_o^{\varphi_m} f(\cos \varphi)d\varphi \sim \frac{1}{n}\sum_1^n \left[1 + \frac{2(1 + X_m)}{(1 - X_m)}\left(1 + \cos \frac{(2i-1)\pi}{n}\right)\right]^{1/2}$$

$$f\left(\frac{1 + X_m}{2} + \frac{1 - X_m}{2}\cos \frac{(2i-1)\pi}{n}\right) \tag{5.6}$$

Here n is the order of the gaussian approximation. We have tested the program with n = 5 and n = 9. The results were indistinguishable to three significant figures, so we used n = 5 to reduce the running time.

Other difficulties at the low-speed range are caused by the function U. As can be seen from its definition (Eq. (4.18b)), near the cut-off turn-angle $\zeta_m = \log \gamma_o/K$, U vanishes like $(\zeta_m - \zeta)^{2.164}$. Inverse cubes of U appear in Eqs. (4.24) and (4.25) multiplied by combinations of C's and S's which make the integrals convergent. However, the calculation requires division of small quantities by small quantities, with attendant loss of accuracy. To avoid this problem, we cut off the ζ integrations one step before the last interval, and then extrapolate to obtain the last value. Testing has shown this procedure to be sufficiently accurate.

With these mathematical problems clarified, we describe the general nature of the program. The integration rule (Eq. (5.1)) has been

*Reference 17, formula 25.4.38, p. 889.

packaged as a subroutine (Part 7) which integrates many functions simultaneously. The atmospheric density $\bar{\rho}$ is stored for altitudes between 18 and 60 km in 2 km steps. Above 60 km, the current per primary gamma is essentially independent of altitude, and the ionization is negligible. The program can take any altitude for which the density has been stored as an input initial altitude.

The program begins at the initial altitude (18 km if not otherwise specified) and the first value of θ (3 deg). It calculates K(cos θ) and h(cos θ) = γ_o. Using Part 7, the functions $\bar{C}$, $\bar{S}$, T_1, and T_2 (see Eqs. (4.19) and (4.20)) are found and stored in Part 2 for all ζ, in 1 deg steps, up to either ζ = log γ_o/K or ζ = 120 deg, whichever occurs first. The value 120 deg was found to cover the full time range of 0 to 10 shakes for all electrons. Next, the functions which appear as single integrals over ζ_1 in Eqs. (4.24) and (4.25) are calculated in Part 3. The expressions $\underline{C}_1$,$\underline{S}_1$, and their powers which involve values of $\bar{C}$,$\bar{S}$ at two values of ζ, are expanded using Eq. (4.12), the separate single-value-of-ζ functions integrated via Part 7, and then the various functions recombined to form the indicated integrals. This procedure, which permits direct step-by-step integration, is much faster than forming the two-values-of-ζ functions, which requires multiple cycling. (If there are N values of ζ, and the relative number of functions after expansion is m, the computing time is reduced by a factor N/2m. Since m is about 2, for N = 120, the usual value, this part of the computation is speeded up by a factor of 30.) In Part 4, the double integrals of Eqs. (4.24) and (4.25) are calculated by the same procedure, which this time uses the already calculated single integrals, some of which appear as the inner integrands in the double integrals. At this point, the factors in brackets in Eqs. (4.24) and (4.25), exclusive of the time differentiations, have been calculated and stored for all ζ.

In Part 5, the time τ is set equal to the first time step, and i (see Eqs. (5.5) and (5.6)) set equal to 1. Equation (4.20) is solved by inverse interpolation to find the value of ζ' to be inserted into Eqs. (4.24) and (4.25). The value is used for direct interpolation to find the values of the terms in the bracketed expressions of Eqs. (4.24) and (4.25), which are then multiplied by the exterior factors $\beta' U'/D$ or

1/D. (The factor 2π disappears--the 2 from halving the φ integration range and π from its use in Eq. (5.5) or (5.6).) The value of i is iterated from 1 to 5 and the expression summed using Eq. (5.5) or (5.6), whichever is appropriate to the selected values of θ and τ. The time is stepped up to either 11 shakes (the extra shake is provided for leeway in later differentiations), or to the last value such that there is contribution to the integral $(T_{m1}(\theta) + T_{m2}(\theta))$, whichever occurs first. The time derivatives are then calculated using five-point Lagrange differentiation.* The results are printed and stored. There are many delicate refinements in these procedures, mostly associated with the transition from Eq. (5.5) to Eq. (5.6) and the angle and time cutoffs.

The angle θ is then stepped to its next value and the calculation returned to the beginning of Part 2. After calculation through Part 5, the results are summed with the previous expressions to form the integration over θ. The successive values are weighted by sin θ K(cos θ) and the variable value of the θ increment. This process is continued to the last value of θ (84 deg), at which point we find the current and ionization rate as functions of time at the indicated altitude. The ionization rate is then inserted into Eq. (2.44) and integrated in Part 6 to obtain the total ionization. A special integration routine is required here because Eq. (2.44) cannot be integrated stepwise but must be cycled in t to take account of the explicit appearance of t in the integrand. Finally, the altitude h is incremented and the entire procedure repeated until the highest altitude is reached. The mean running time is about 70 sec per altitude step on the IBM 370/158.

This completes the description of the computer program, which is included as an appendix to this report. We have executed the program for the representative values B = 0.6 gauss, E_γ = 1.6 MeV, and shall present and discuss the results.

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*Reference 17, formula 25.3.6, p. 883.

VI. RESULTS AND CONCLUSIONS

We shall first discuss the convergence of the series of integrals in Eqs. (4.24) and (4.25). In Figs. 8a, 8b, and 8c, we have plotted the ionization per primary gamma ray versus time at altitudes of 20, 30, and 40 km. The quantity actually plotted is $10^{-3}\ N_S(\tau)/g(r)$, and the three figures have different vertical scales. The terms corresponding to time derivatives of various order in Eq. (4.25) are denoted by subscripts, thus N_2 is the contribution to the total ionization from the second derivative in Eq. (4.25). The curve N is the sum of the three contributions. In Figs. 9a, 9b, and 9c, we have corresponding plots for $10^8\ \mu c J_c(\tau)/2$, designated as J on the figures.

First consider Fig. 8, the ionization. It is apparent that the term N_o is by far the major contributor. Numerically, at 20 km, N_o is at least 84.5 percent of N, the minimum occurring at $\tau = 1$. The second derivative term, N_2, makes a maximum contribution of 9 percent at $\tau = 1$, and N_3 provides a maximum of 7 percent at $\tau = 0.5$. At 30 km, N_o is at least 95.9 percent, N_2 at most 3 percent, and N_3 at most 1.2 percent. The time integration of Eq. (2.44a), combined with the time differentiations of Eq. (4.25), makes the contributions of N_2 and N_3 approach zero at large τ. Figure 8 displays very clearly the rapid convergence of the expansion used in Eqs. (3.19a-d) as far as ionization is concerned.

The convergence of the expansion for the current, presented in Fig. 9, is less good, but still satisfactory. At the lowest altitude, 20 km, the early peak of the current is raised and sharpened by the higher order terms, and the curve decreases more rapidly thereafter. The same effect, only less pronounced, occurs at the higher altitudes. At 20 km, the higher order terms can be as much as 40 percent of the total near the peak, but the qualitative behavior is not changed significantly from J_o and there is no physical reason to suspect major contributions from still higher order terms. In any case, the convergence of the series is somewhat obscured by the coupling between the time derivatives and the delta function gamma source. If a smooth source were used, the series would converge much better.

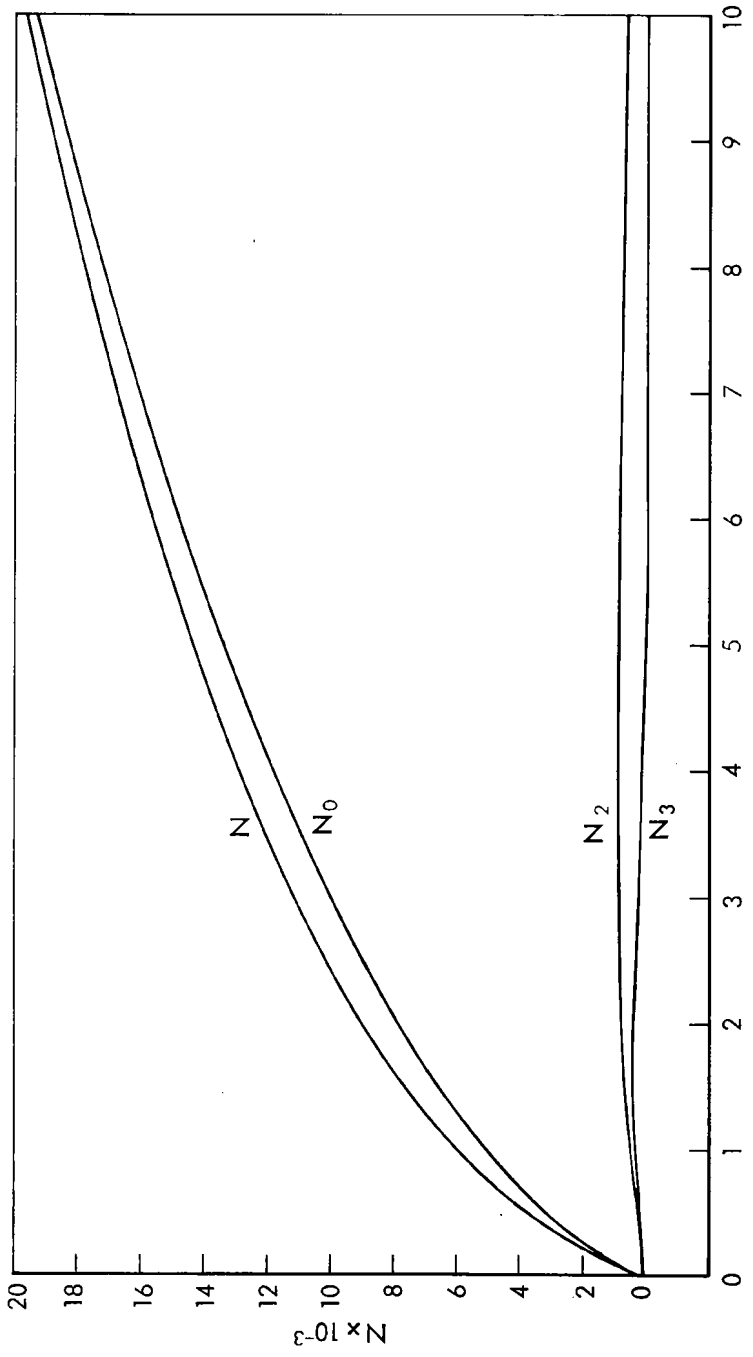


Fig. 8a—Contributions to the ionization; B = 0.6 E_γ = 1.6 MeV H = 20 km

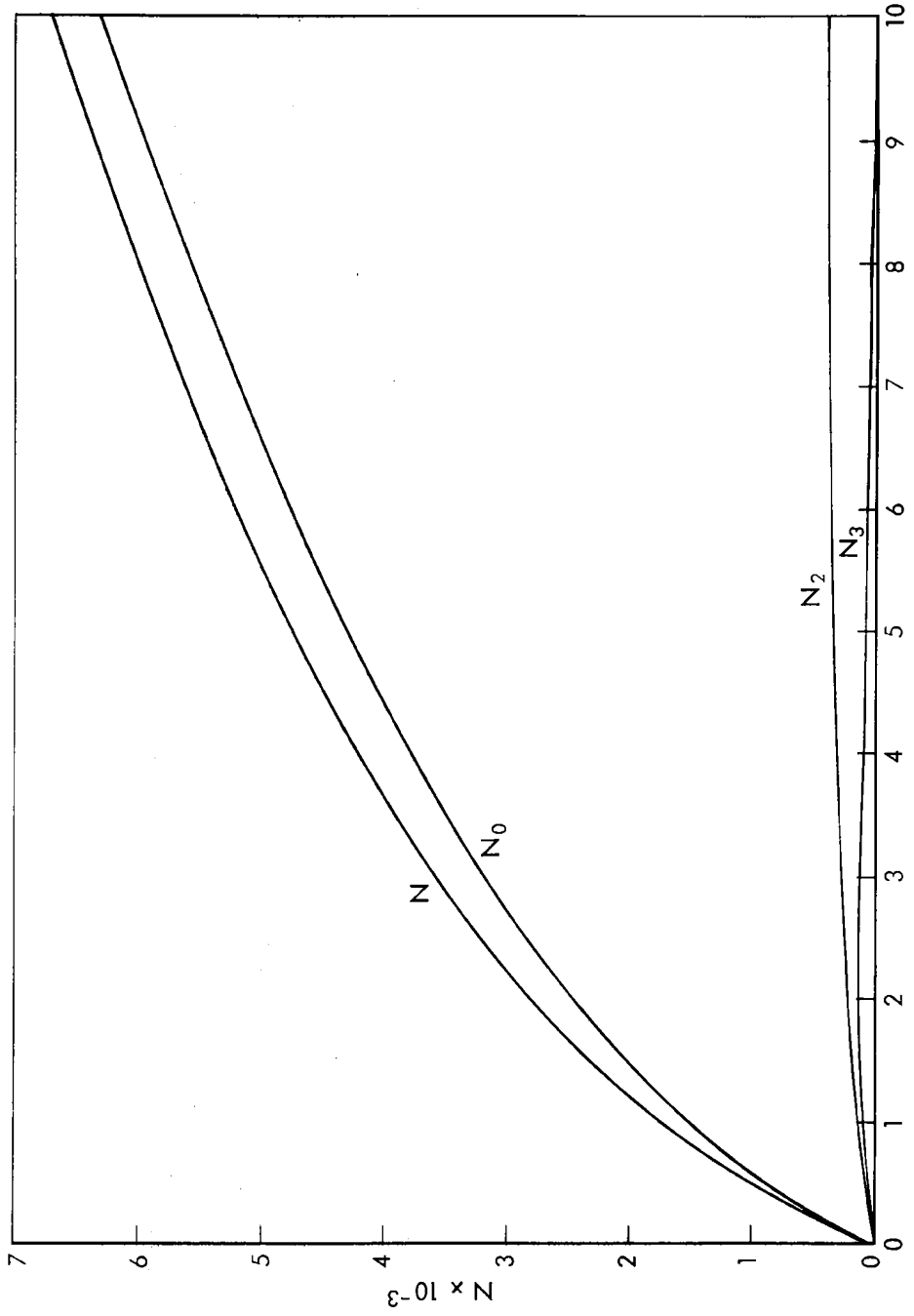


Fig. 8b — Contributions to the ionization; B = 0.6 E_γ = 1.6 MeV H = 30 km

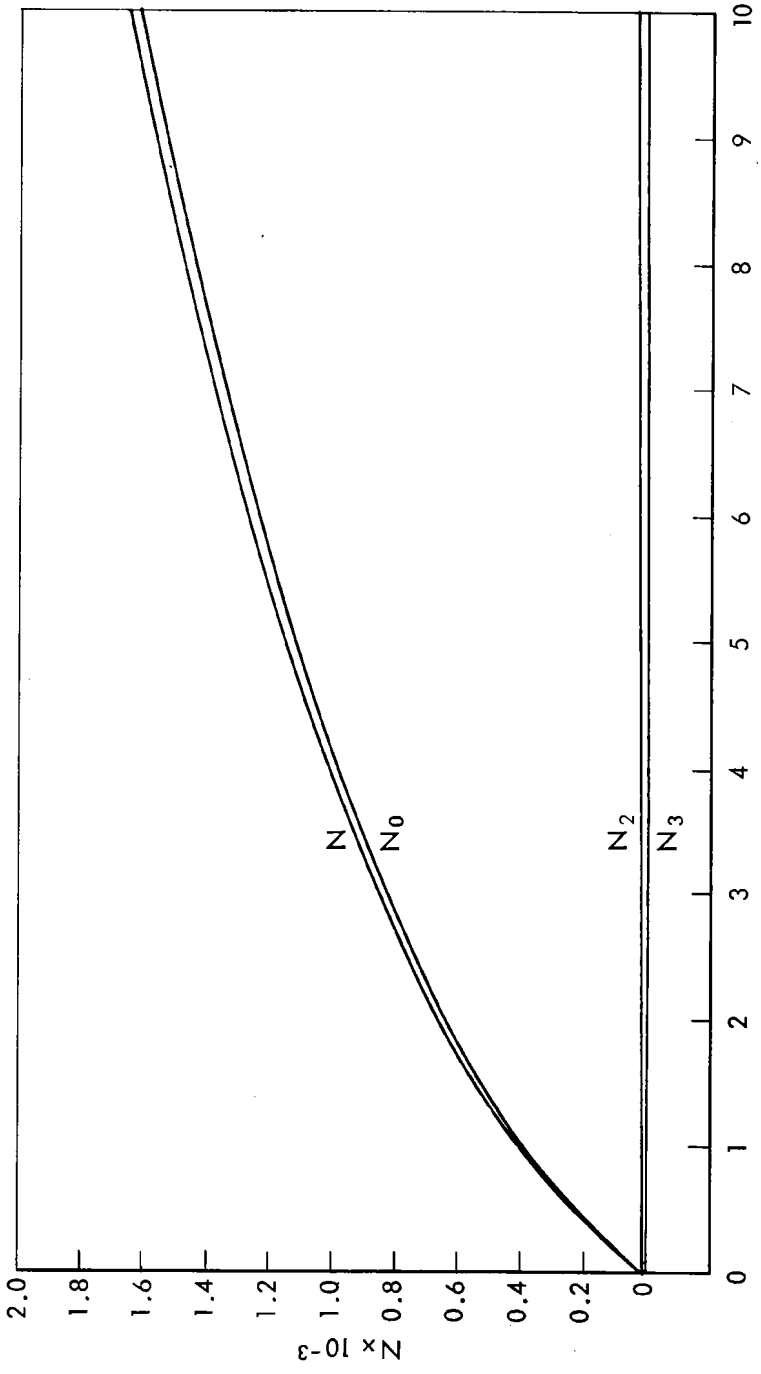


Fig. 8c — Contributions to the ionization; B = 0.6 E_γ = 1.6 MeV H = 40 km

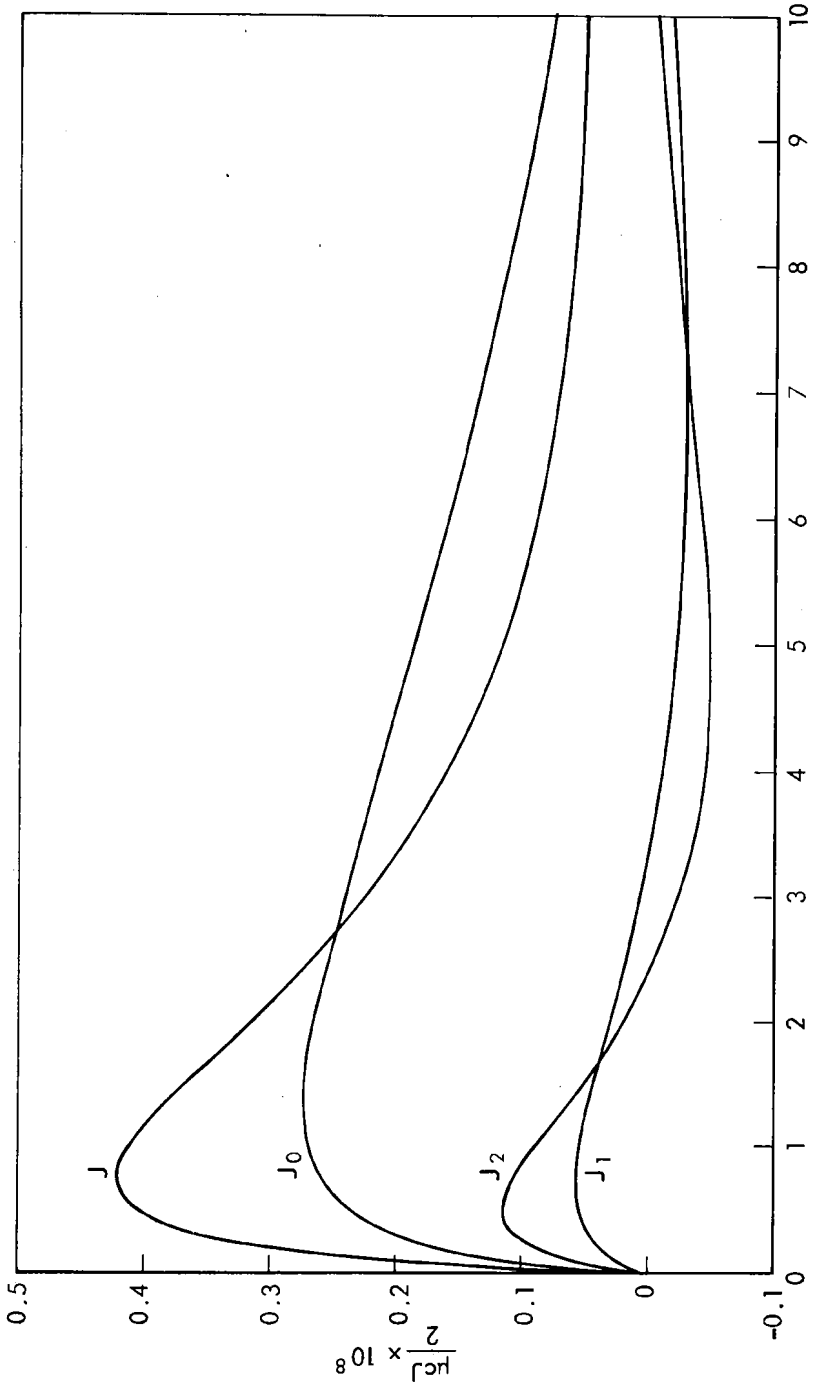


Fig. 9a — Contributions to the current; B = 0.6 E_γ = 1.6 MeV H = 20 km

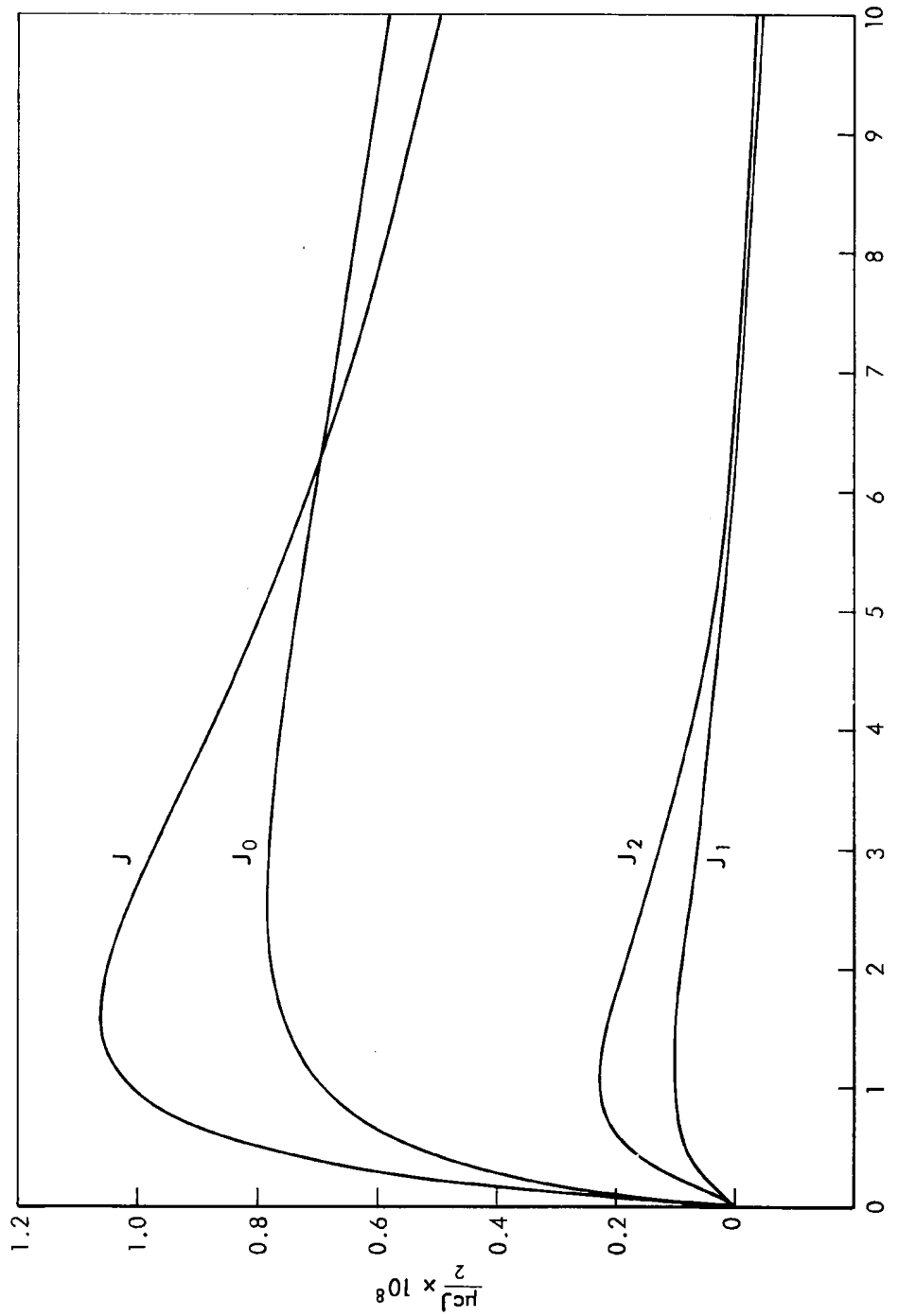


Fig. 9b— Contributions to the current; B = 0.6 E_γ = 1.6 MeV H = 30 km

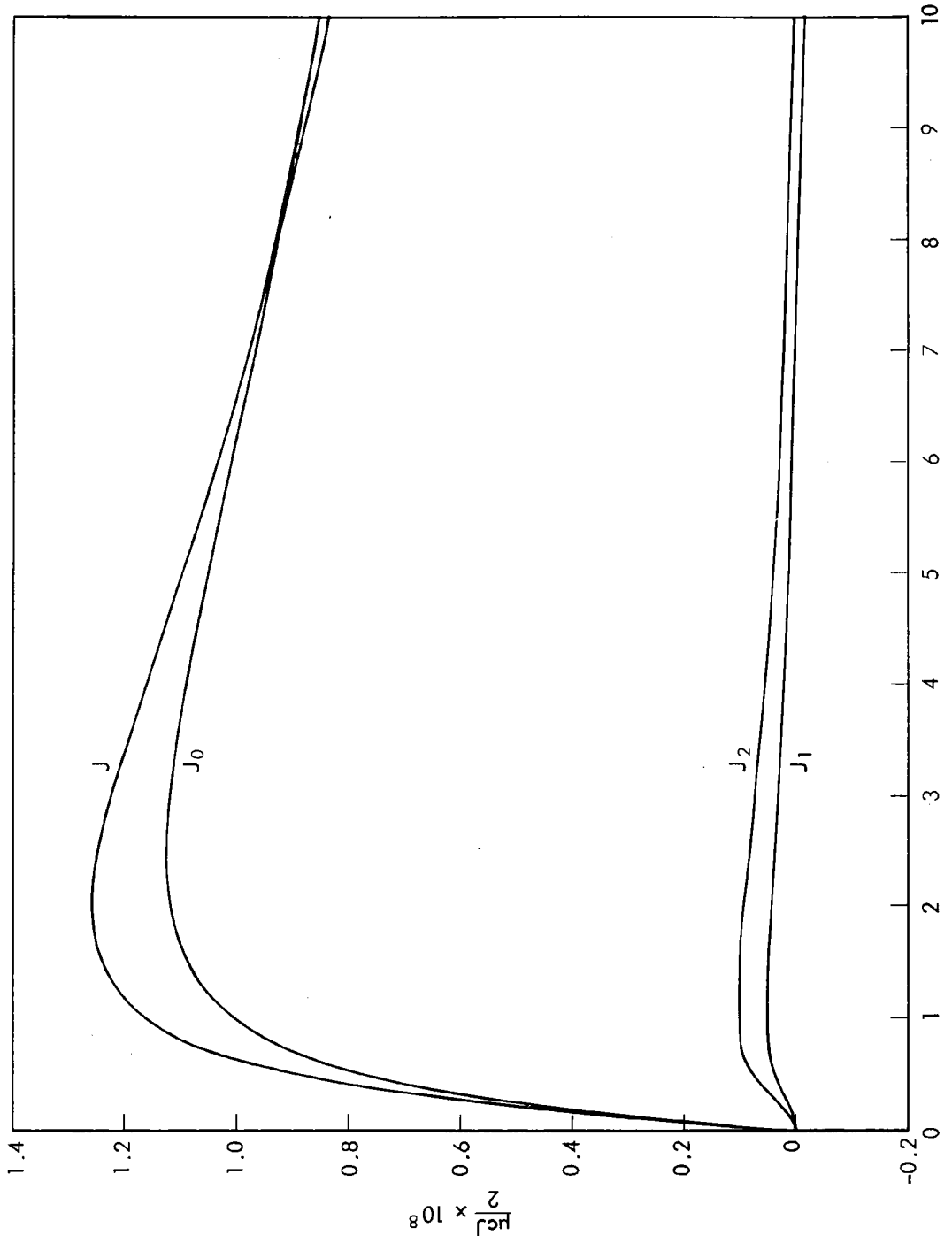


Fig. 9c — Contributions to the current; B = 0.6 E_γ = 1.6 MeV H = 40 km

It is of interest to see what effect the Klein-Nishina distribution of initial velocities has had on the current. In Figs. 10a, 10b, and 10c, we have plotted the current versus initial angle at τ = 1, 2, 5, and 10 shakes for altitudes of 20, 30, and 40 km, with corresponding curves in Figs. 11a, 11b, and 11c for ionization rate (not ionization). The vertical scales are arbitrary but compatible. These curves represent the contribution to current and ionization rate produced by the number of Compton electrons per unit angle at the indicated initial angle. All curves peak near 10 deg, corresponding to the peak of Fig. 6, but the varying behavior at larger angles is interesting. At the highest altitude, the current is most strongly peaked at 10 deg at early times, whereas the peak is considerably broader as time increases and some of the electrons which started at large angles are deflected forward. At lower altitudes, scattering produces a greater broadening of the distribution. Energy loss effects at the lowest altitudes cause the widest angle (lowest energy) electrons to disappear, as can be seen most clearly in Fig. 11a. The changing shape of the angular dependence as time increases makes it unlikely that it would be a reasonable approximation to replace the Klein-Nishina distribution with purely forward-directed electrons, as has been done by other codes.

The principal results of the computer program are presented in Figs. 12 and 13. Figure 12 depicts the current per incident gamma ray (10^8 μcJ/2g is plotted); Fig. 13 shows the ionization in thousands of secondary electrons per incident gamma (10^{-3} N_s/g is plotted). These are shown versus time at altitudes between 18 and 60 km. Below 18 km, the number of gammas is essentially negligible (see Fig. 4). Above 60 km, the ionization is negligible (Fig. 13), and the current is virtually independent of altitude (Fig. 12).

At high altitudes, the effects of scattering and energy loss on the current are slight. Accordingly, the current reduces to that produced by magnetic deflection acting on the Klein-Nishina initial distribution. From Fig. 12, this condition essentially holds at altitudes above 45 km for times up to 10 shakes. As the altitude decreases, the scattering begins to cause the electron distribution to decohere. The peak of the current does not rise as high, and it occurs somewhat earlier

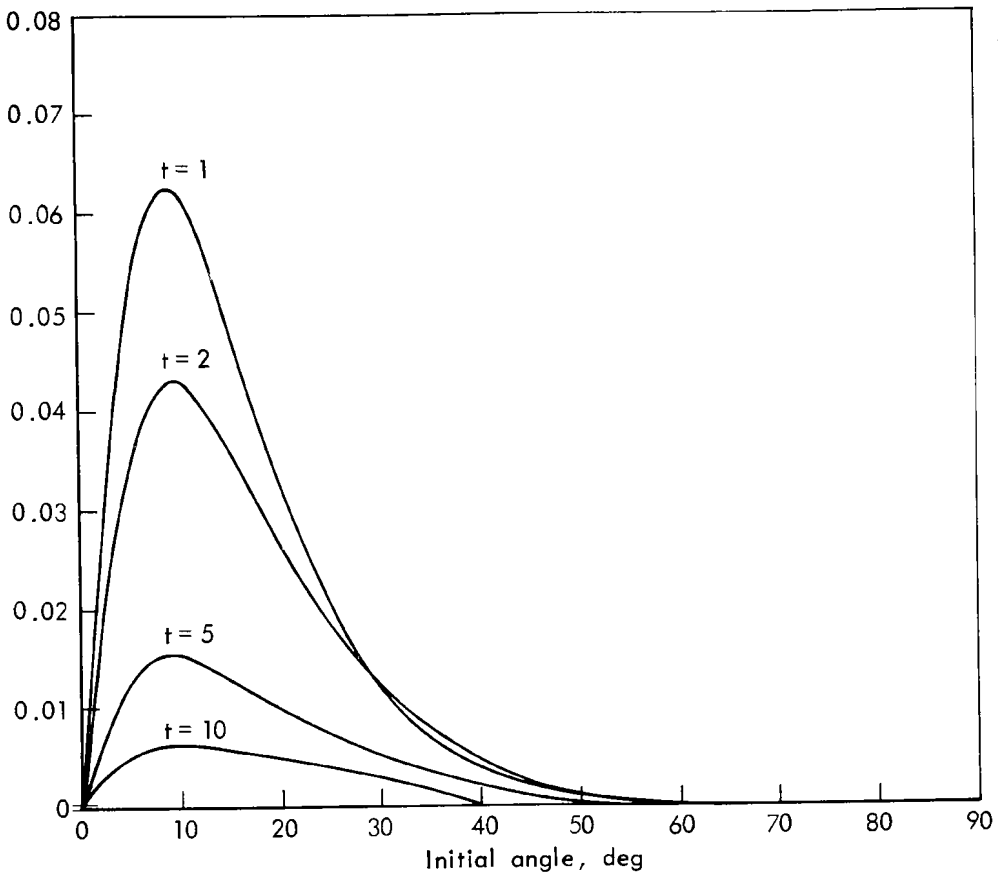


Fig. 10a — Current versus initial angle;
B = 0.6 E_γ = 1.6 MeV H = 20 km

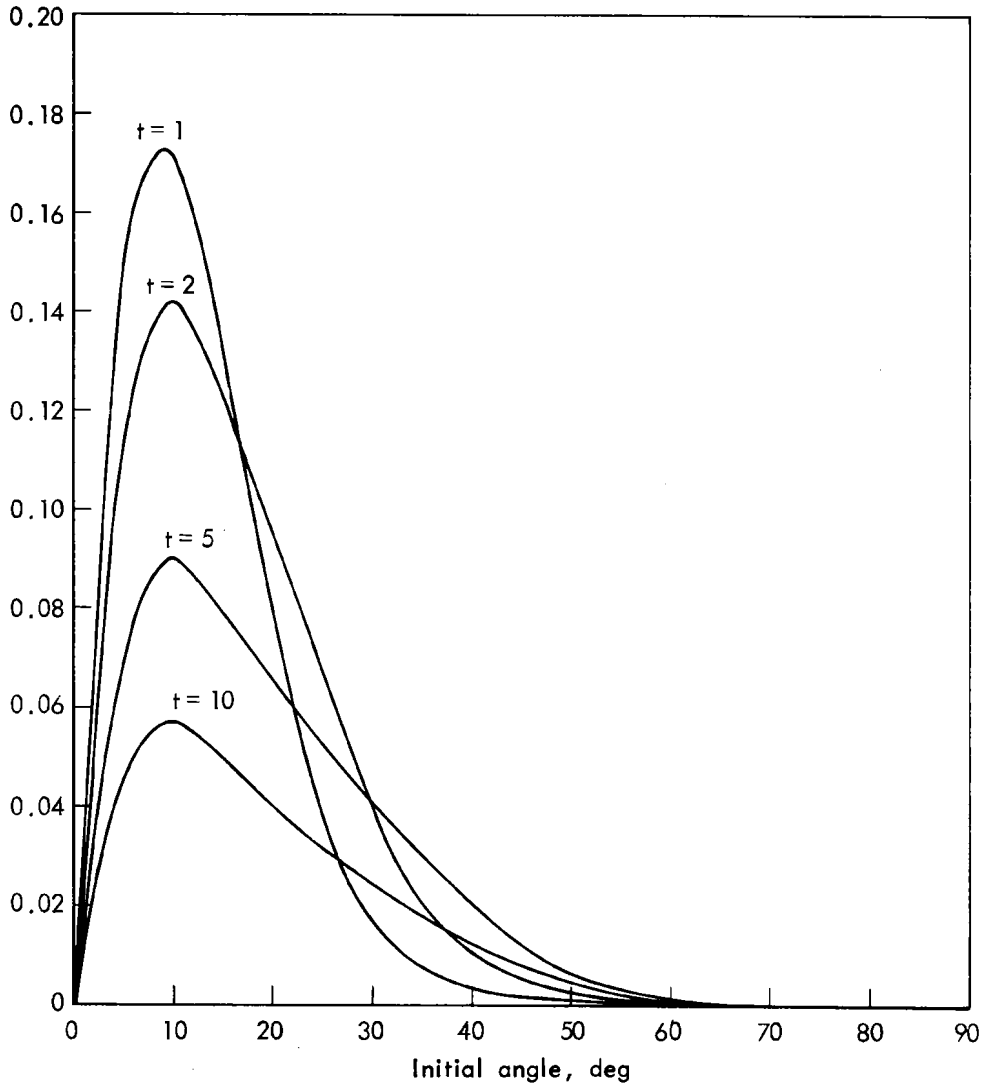


Fig. 10b — Current versus initial angle;
B = 0.6 E_γ = 1.6 MeV H = 30 km

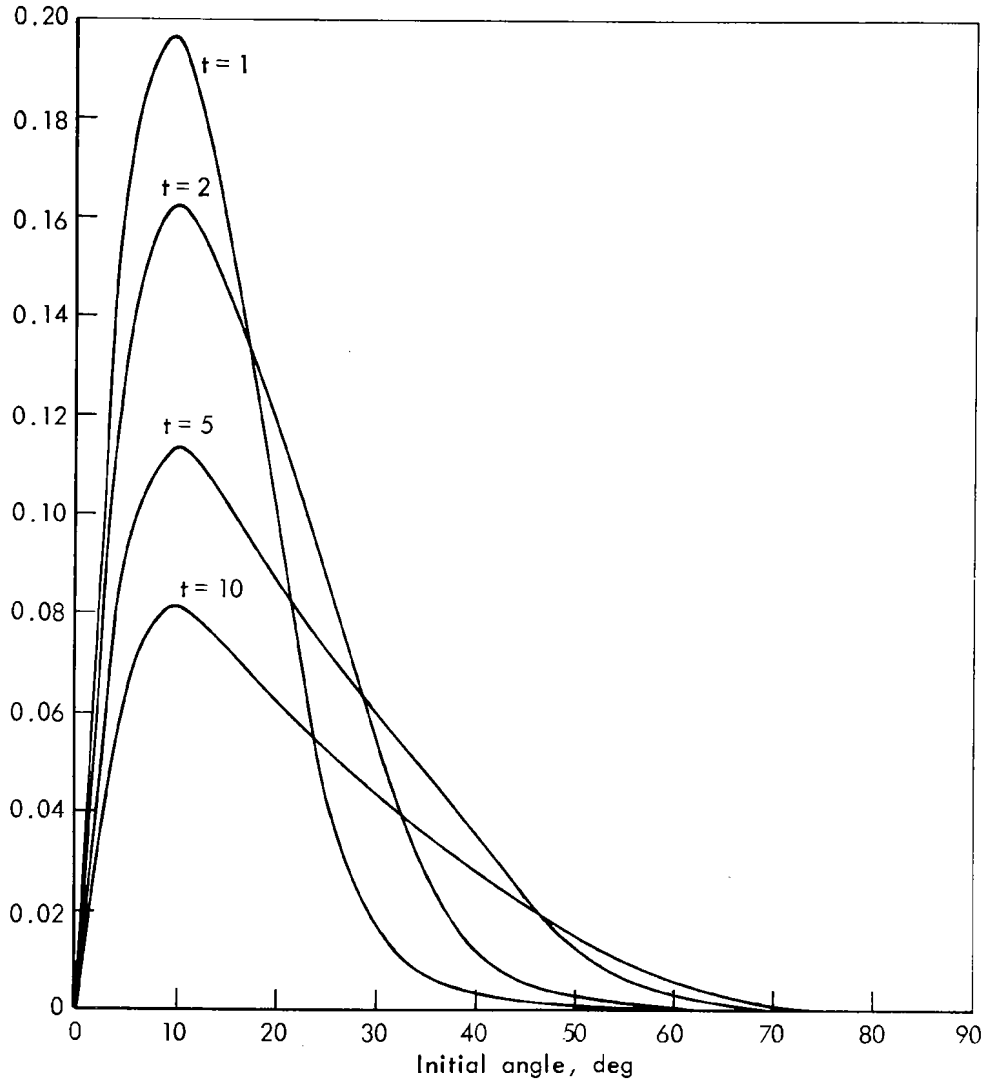


Fig. 10c — Current versus initial angle;
B = 0.6 E_γ = 1.6 MeV H = 40 km

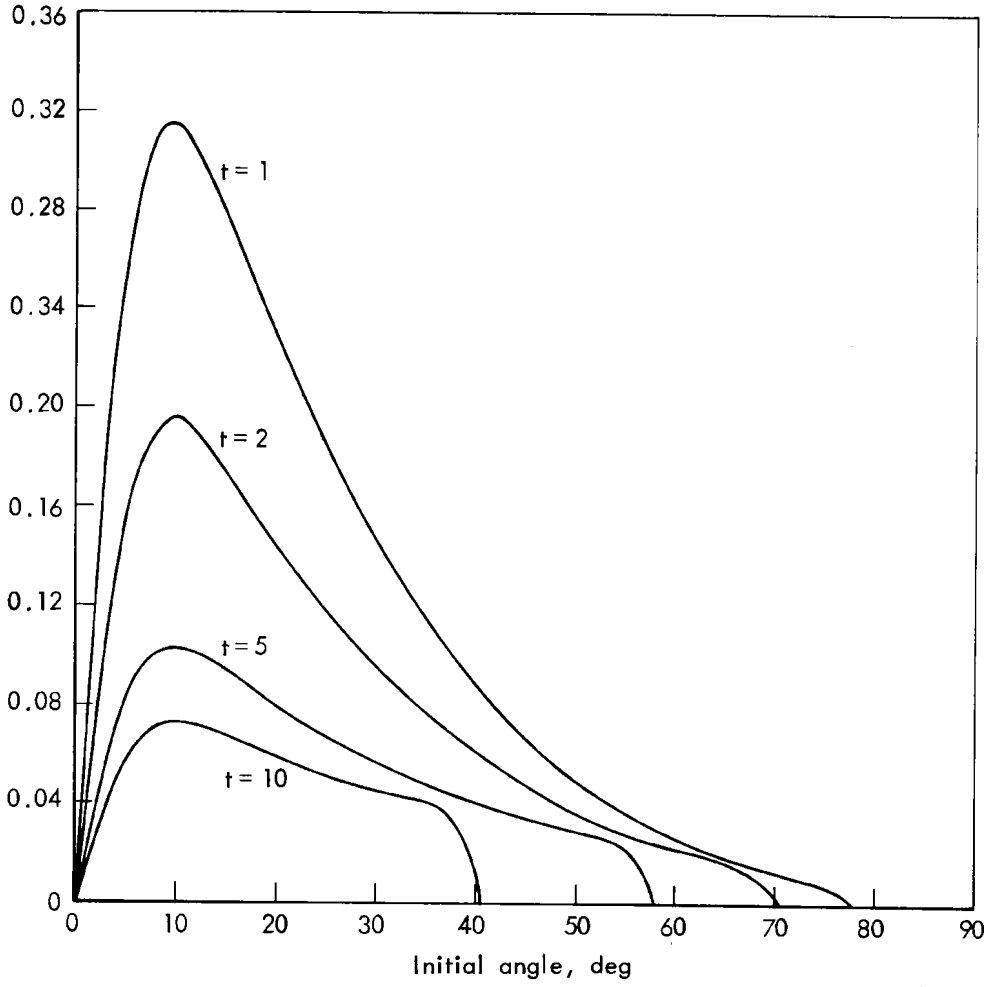


Fig. 11a— Ionization rate versus initial angle;
B = 0.6 E_γ = 1.6 MeV H = 20 km

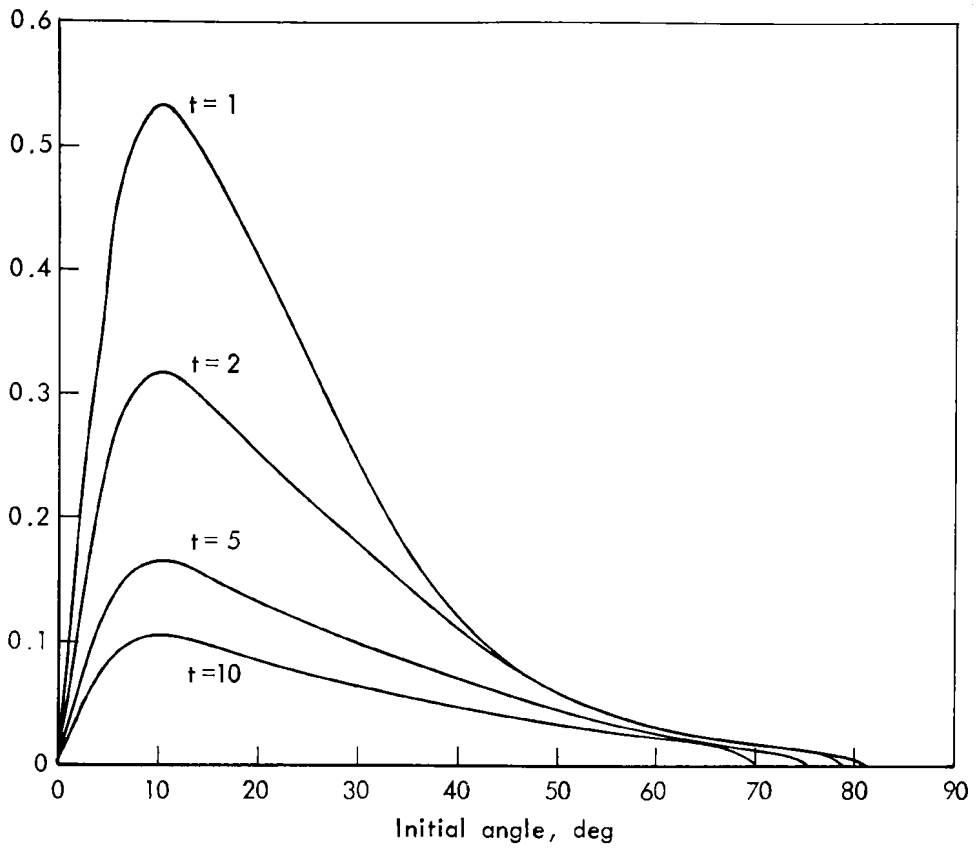


Fig. 11b — Ionization rate vs initial angle;
B = 0.6 E_γ = 1.6 MeV H = 30 km

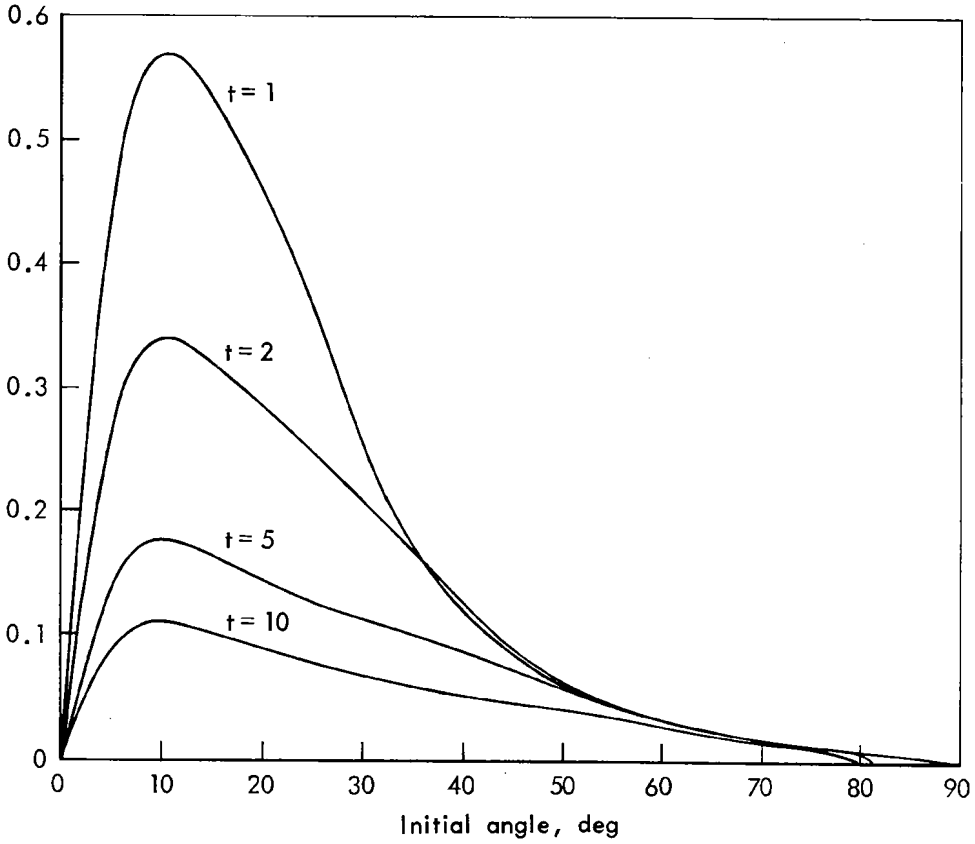


Fig. 11c — Ionization rate vs initial angle;
B = 0.6 E_γ = 1.6 MeV H = 40 km

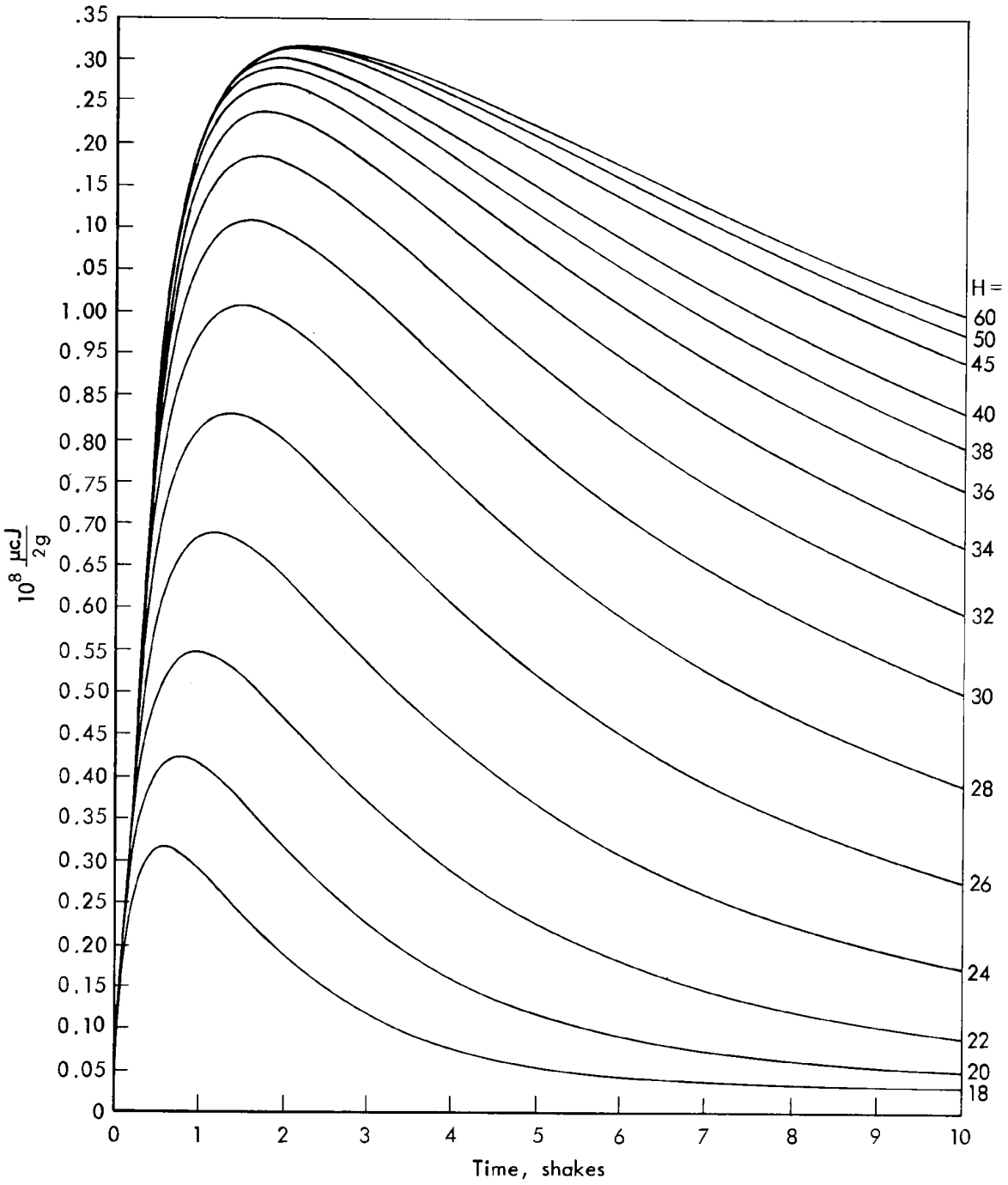


Fig. 12 — Current per primary gamma ray; B = 0.6 E_γ = 1.6 MeV

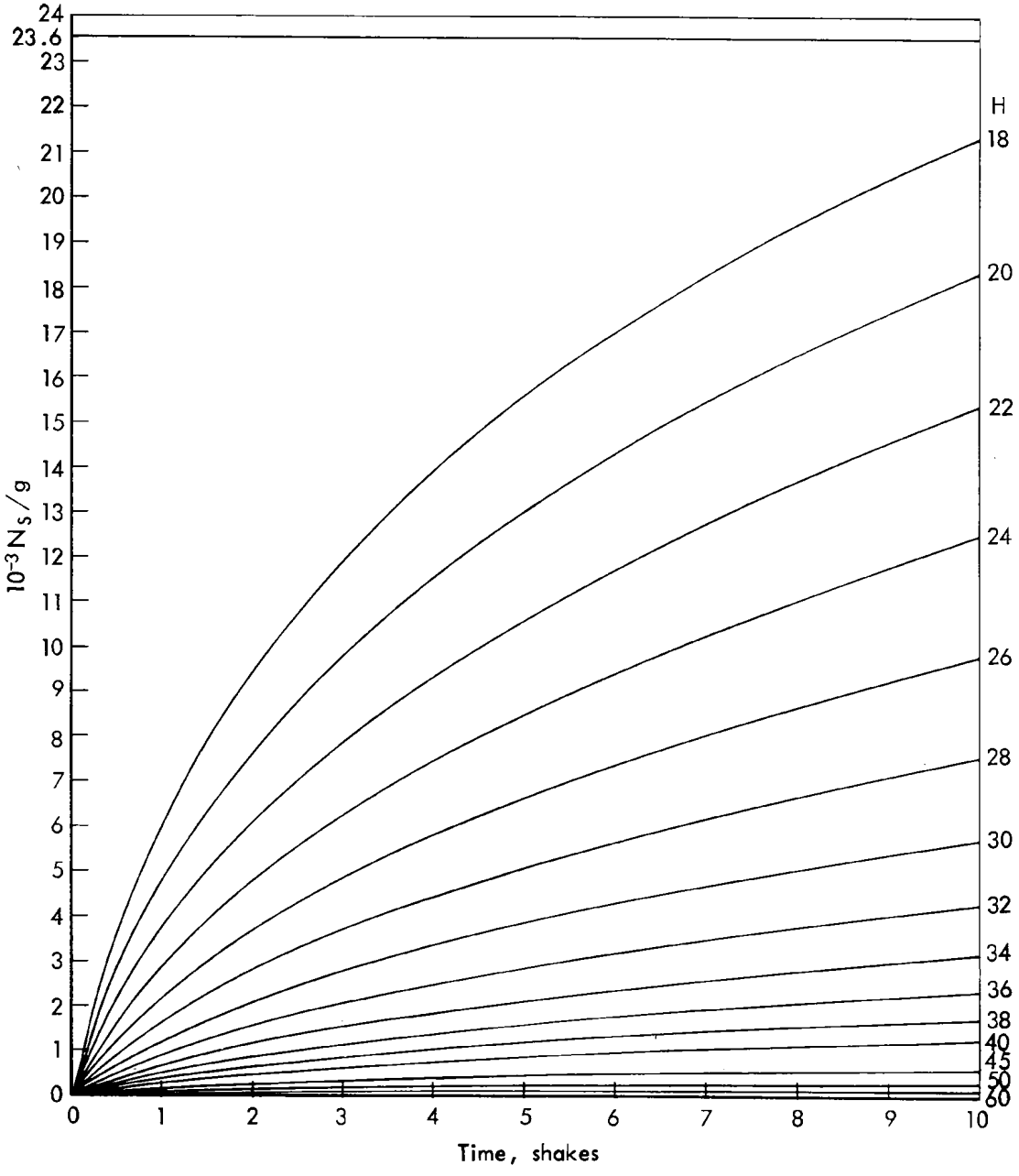


Fig. 13—Ionization per primary gamma ray; $B = 0.6$ $E_\gamma = 1.6$ MeV

in time. From a peak of 1.267 at 2.2 shakes at 60 km, it drops to 1.255 at 2.0 shakes at 40 km, 1.06 at 1.6 shakes at 30 km, and 0.42 at 0.8 shake at 20 km. The peak not only becomes lower and earlier but narrower, with the decay at later times being faster. These effects are exactly what we would qualitatively expect.

A measure of the effectiveness of the scattering in reducing the current is shown in Fig. 14. Here, we have plotted the ratio of the current at a given altitude to the current at high altitude (60 km). The rapid and steady decrease of the relative current with time is evident. Thus, at 20 km, the current is down by more than a factor of four for times later than 2 shakes. This reduction of current at low altitudes is especially important because the fields radiated by these currents are least affected by attenuation. The initial rise of the relative current at high altitudes is real. It corresponds to the fact that those electrons which are scattered forward spend a greater retarded time radiating, with corresponding field enhancement, than those which are scattered away and have their effective radiation reduced. The effect is soon washed out by the general decoherence.

The ionization versus time at various altitudes is displayed in Fig. 13. The horizontal line at 23,600 represents the asymptotic limit of all the curves, corresponding to complete conversion of all the energy of the Compton electrons to secondary ionization. (The mean energy of these electrons produced by the first scattering of the 1.6 MeV gamma rays from the bomb is 800 KeV.) Although it is obvious that the electron energy is lost most rapidly at low altitudes, because the energy loss is proportional to density, the curves do not display any striking behavior, except to note the large range over which the ionization density varies.

If we refer to Eq. (2.38), we see that the ionization density affects the fields by its appearance in conductivity, where it is divided by the density. Therefore, we have plotted $10^{-4}\ N_S/\bar{\rho}g$ in Fig. 15. We call this quantity the equivalent conductivity, since there is no convenient way to characterize the dependence of the collision frequency on field strength, which is required to completely specify the conductivity. Figure 15 is quite remarkable. At low altitudes, the change in

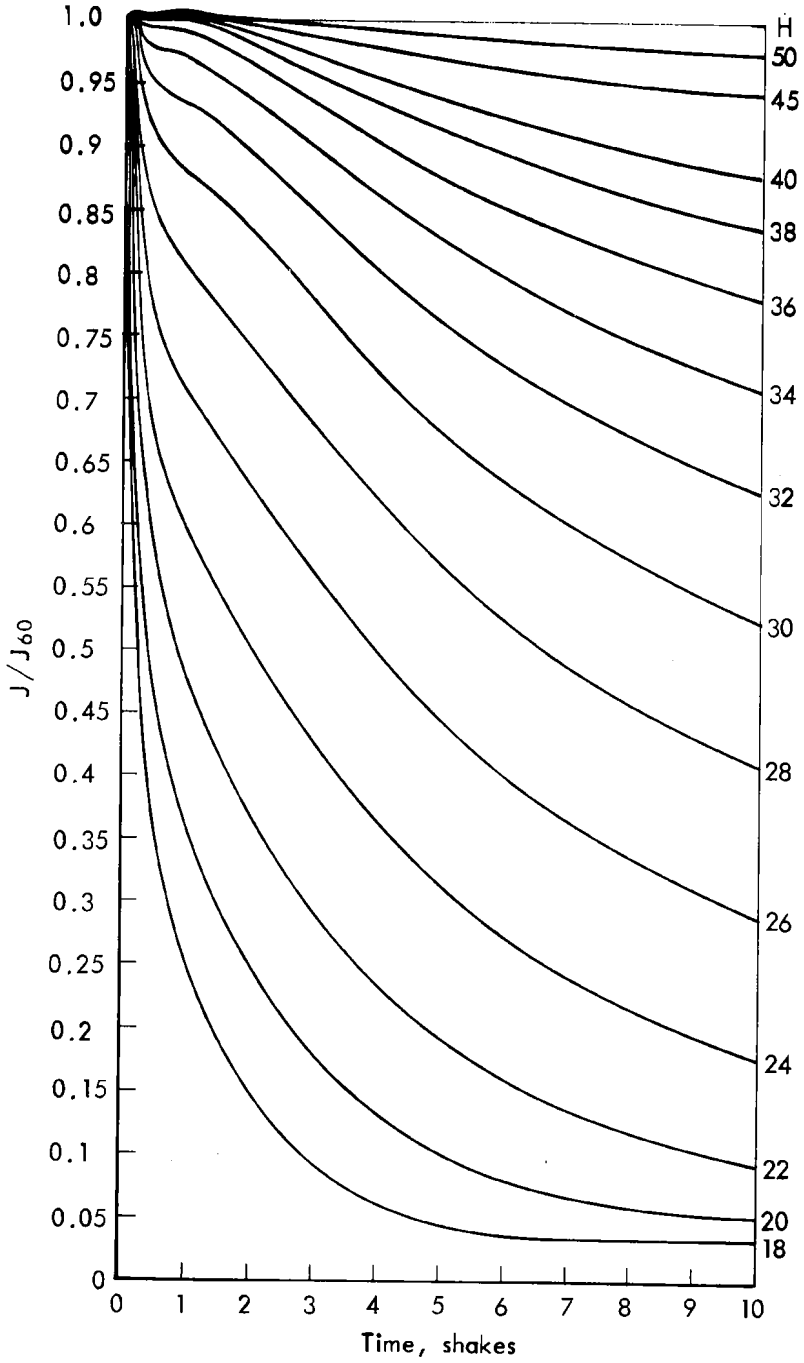


Fig. 14 — Relative current; B = 0.6 E_γ = 1.6 MeV

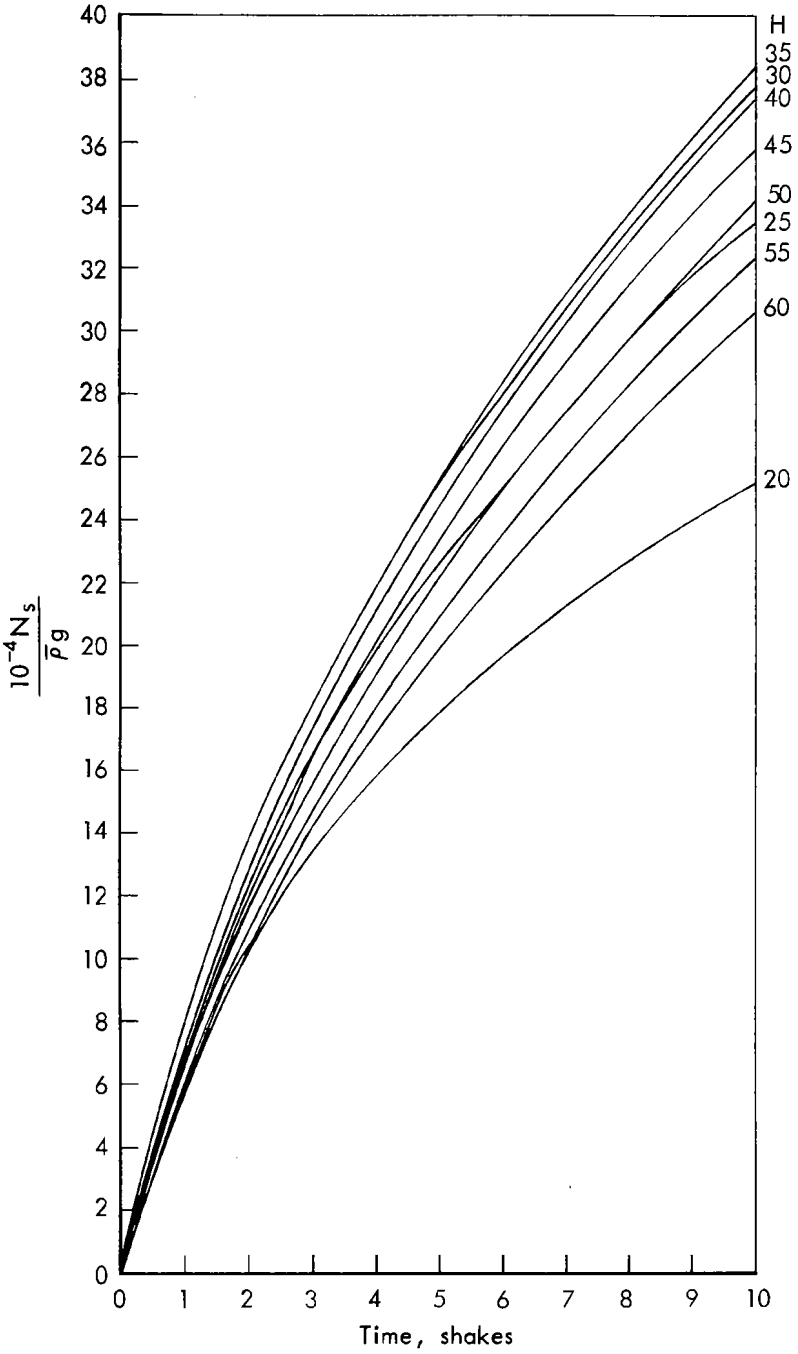


Fig. 15— Equivalent conductivity;
B = 0.6 E_γ = 1.6 MeV

the Compton electron paths due to scattering and energy loss causes the secondary electrons to be produced less rapidly in retarded time at higher densities. Thus, the equivalent conductivity increases with altitude from 20 km up to 35 km. Above 35 km, the ionization lag (Eq. (2.44)) becomes the dominant effect, and the equivalent conductivity decreases as the altitude increases further. The combination of these effects collapses the curves of Fig. 13 into a very limited region. Thus, if the curves of Fig. 15 were replaced by a mean curve, which is essentially the 25 km curve, the departure from this mean of all the curves of Fig. 15 for all altitudes and times is less than 20 percent. It accordingly would be reasonable to replace the equivalent conductivity per electron, regarded as a function of altitude, by an average value which is, of course, a function of time. This approximation clearly is not permissible for the current.

Figures 12 and 13 represent the current and ionization per gamma ray. The total current or ionization at a point is found by multiplying these curves by the function g(r) (Eq. (2.32) in general, Fig. 4 for these values of B and E_γ). We shall not plot these, since the resultant curves display multiple crossings and are very difficult to follow. We note that the curves for total current and total conductivity, regarded as a function of altitude, peak at about 32 to 34 km, corresponding to the peak in g(r).

From the extensive analysis of this report, we draw the following conclusions:

1. The development of the electron velocity distribution with time has been determined in a consistent and satisfactory manner that includes the phenomena of magnetic deflection, energy loss, atmospheric scattering, and the initial Klein-Nishina velocity distribution.
2. The current and ionization have been calculated and both display the proper qualitative and quantitative physical behavior. For a delta function gamma-ray source, the peak of the current is reduced and made both earlier and narrower by the effects of scattering, whereas the ionization is determined primarily

by the proportionality of energy loss to density. The conductivity is insensitive to altitude.

3. A computer program prepared to provide numerical values for the current and ionization is presented in the appendix to this report. The program operates rapidly compared with Monte Carlo simulations, thus demonstrating the virtue of the analytic approach. If the program is augmented by an integration, arbitrary gamma-source time dependence can be included. The current and ionization as calculated here can be used as inputs for calculating fields.

APPENDIX: THE COMPUTER PROGRAM

```
      IMPLICIT REAL*8 (A-H,K,O-Z)
      DIMENSION R(24),    SEC(111),RR(60)
       COMMON    FI(50,121),ENN(111),AJJ(111),PP(5,121)
     &,F(200),CC(121),SS(121),TT(5,121),DD(5,121),QQ(50)
     &,ENNMK(3,111),AJJMK(3,111),EN1(5,121),EN2(5,121),EN3(5,121)
     &,ENNM(3),AJJM(3),AJ1(5,121),AJ2(5,121),AJ3(5,121)
     &,                          ENK(3),AJK(3),ENA(3,9),AJA(3,9)
     &,DDA(9),       QI(9),QC(9),QR(9),QS(9),BU(121),DEQ(5),DER(5)
     &,QK(121),          FF(14,121),GI(5),WI(5),DE(5,4),AK1(5,4)
     &,AK2(5,4),AK3(5,4),EM2(5,4),EM3(5,4),ENM(3),EX2(4),EX3(4)
     &,ENP(111),AJP(111)
      COMMON HH,DELH,YY,YYDEL,VV,H,W,GG,KK,BB,EMY,X,XSN,RHH,B,C,YO
     &,TTP,TTQ,TTL,TTU,TTV
     &,MLIM,MLIMK,IH,LV,IYML,IYU,IYL,LGO,IYV,MVX
      G(ZDUM)=GG*DEXP(-KK*ZDUM)
      UU(ZDUM)=((G(ZDUM)-1D0)*(GG+1D0)/((G(ZDUM)+1D0)*(GG-1D0)))
     1    **2.164D0
      A(ZDUM)=2.164D0*KK*G(ZDUM)/(G(ZDUM)**2-1D0)
      O(ZDUM)=5.68569D0*DSQRT(G(ZDUM)**2-1D0)*UU(ZDUM)/BB
      ELL(MDM)=.395348837209302*(1D0+.16D0*DLOG(1D0+3.31D4*RHH*B*MDM))
      DATA R/.1003D0,.07254D0,.05265D0,.03831D0,.02797D0,.02048D0
     &,.01503D0,.01106D0,.8153D-2,.6024D-2,.4465D-2,.3323D-2,.2486D-2
     &,.1871D-2,.1419D-2,.1084D-2,.8348D-3,.6481D-3,.5066D-3,.3983D-3
     &,.3142D-3,.2483D-3,.1962D-3,.1544D-3/
      DATA ZERO,P,Q/0D0,3.14159265358979D0,1.74532925199433D-2/
     &,QM,QN/0.951056516295150,0.587785252292470/
      FRQ3=4D0/3D0
      DCON=1.74535D4/24D0
C     *****   BF = MAGNETIC FIELD, EE = GAMMA ENERGY,
C     *****   B = TIME-STEP, C = ANGLE-STEP
      READ 1,BB,EE,B,C,YMI
      IYMI=YMI
    1 FORMAT(6D12.4)
      MLIM=11D0/B+1.01D0
      MLIMK=MLIM-2
      K=EE/0.511004D0
      K1=K+1D0
      K2=K+2D0
      KS=K*K
      K1S=K1*K1
      K2S=K2*K2
      S=(2D0+K*(8D0+K*(9D0+K)))/(K*(1D0+2D0*K))**2
     1   -(1D0+K*(1D0-0.5D0*K))*DLOG(1D0+2D0*K)/K**3
      DELH=2D0
      READ 1,HHO,YYO,HH1,YY1
      HH=HHO
      IH=HH1
      IF(IH.NE.0) IH=IH-1
   10 IH=IH+1
      RHH=R(IH)
      KK=6.5954D0*RHH/BB
      PRINT 11,HH,BB,EE,B,C
   11 FORMAT(1H1,'     ALTITUDE =',F4.0,' KM., MAGNETIC FIELD =',F6.2,',
     & GAMMA ENERGY =',F6.2,/20X,'TIME-STEP =',F6.2,' SEC., ANGLE-STEP =
     &',F6.2,' DEG.')
      DO 512 M=1,MLIM
      AJJ(M)=ZERO
      ENN(M)=ZERO
      ENP(M)=ZERO
      AJP(M)=ZERO
```

```
  512 CONTINUE
      TTV=15D0
      MVX=151
      YYDEL=3D0
      YY=YYDEL
   20 CONTINUE

C*******          PART 2
      EMO=ZERO
      ZZ=YY*Q
      XSN=DSIN(ZZ)
      X=DCOS(ZZ)
      XS=X*X
      XSM=1D0-XS
      XS3M=1D0-3D0*XS
      XS4M=1D0-5D0*XS
      XS5M=3D0-5D0*XS
      QI(1)=XSN*QM
      QI(2)=XSN*QN
      QI(3)=ZERO
      QI(4)=-QI(2)
      QI(5)=-QI(1)
      KX=(K*X/K1)**2
      XSK2=K*XS*K2
      GG=(1D0+KX)/(1D0-KX)
      SS(1)=ZERO
      CC(1)=ZERO
      BU(1)=DSQRT(1D0-1D0/(GG*GG))
      QK(1)=ZERO
      DO 21 I=1,5
      TT(I,1)=ZERO
      DD(I,1)=1D0-X*BU(1)
      AJ1(I,1)=ZERO
      AJ2(I,1)=ZERO
      AJ3(I,1)=ZERO
      EN2(I,1)=ZERO
      EN3(I,1)=ZERO
      QII=QI(I)
      GI(I)=QII/XSN
      WI(I)=2D0/(1D0+GI(I))
      QI2=QII*QII
      QR(I)=1D0-5D0*QI2
      QS(I)=1D0-3D0*QI2
   21 QC(I)=QII*(2D0+QR(I))
      EMM2=(K1S-XSK2)/(K1S-KS*XS)
      EMM1=EMM2+1D0/EMM2-(1D0-XS)*(2D0*K1*X/(K1S-XSK2))**2
      EMY=2D0*K1S*X*DSQRT(1D0-XS)*EMM1/S/(K1S-XS*KS)**2
      PRINT 12,YY,EMY
   12 FORMAT(//'     INITIAL ANGLE =',F4.0,' DEG.,  REL INITIAL NUMBER ='
     1,F9.5)
      VV=DLOG(GG)/KK/Q
      VV=VV-C
      IF(VV.GT.119D0) VV=119D0
      IYMX=VV/C+1D0
      IYMXM=IYMX-1
      IF(IYMX.LE.5) GO TO 96
      YO=ZERO
      H=C
      W=VV
      IYML=IYMX
      LV=2
```

```
      LGO=1
C     ASSIGN 602 TO LGO
      CALL PART7
  502 CONTINUE
      Y=H
      IY=2
   25 CONTINUE
      CC(IY)=EI(1,IY)
      SS(IY)=EI(2,IY)
      Z=Y*Q
      GOZ=G(Z)
      QK(IY)=.86207D0*(GG-GOZ )/PHH -X*CC(IY)
      BU(IY)= DSQRT(1D0-1D0/GOZ **2)*UU(Z)
      Y=Y+H
      IY=IY+1
      IF(Y.LE.VV) GO TO 25
      IYU=IY
      IYV=IYU-1
      IY2=IY-2
      IY3=IY-3
      IY4=IY-4
      IY5=IY-5
      QK(IYU)=QK(IY5)-5D0*(QK(IY4)-QK(IYV))+10D0*(QK(IY3)-QK(IY2))
      BU(IYU)=BU(IY5)-5D0*(BU(IY4)-BU(IYV))+10D0*(BU(IY3)-BU(IY2))
      SS(IYU)=SS(IY5)-5D0*(SS(IY4)-SS(IYV))+10D0*(SS(IY3)-SS(IY2))
      CC(IYU)=CC(IY5)-5D0*(CC(IY4)-CC(IYV))+10D0*(CC(IY3)-CC(IY2))
      TTP=QK(IYU)
      TTQ=XSN*SS(IYU)
      TTL=TTP-TTQ
      TTU=TTP+TTQ

      LV=50
      LGO=3
C     ASSIGN 640 TO LGO
      CALL PART7
  540 CONTINUE

C*******          PART 3
      DO 301 J=1,121
      DO 301 I=1,5
  301 PP(I,J)=ZERO
      DO 302 I=1,14
  302 FF(I,1)=ZERO
      Y=H
      IY=2
      IYL=120
      TH=QK(IY)+XSN*SS(IY)
   34 CONTINUE
      TG=TH
      Z=Y*Q
      SINZ=DSIN(Z)
      COSZ=DCOS(Z)
      I=1
   36 CONTINUE
      CCI=CC(IY)
      SSI=SS(IY)
      EIIY=EI(I,IY)
      EI1IY=EI(I+1,IY)
      EI2IY=EI(I+2,IY)
      EI3IY=EI(I+3,IY)
      EI4IY=EI(I+4,IY)
```

```
      F(I)=EIIY*CCI-EI1IY
      FI=F(I)
      F(I+1)=EIIY*SSI-EI2IY
      FI1=F(I+1)
      F(I+2)=SSI*FI-CCI*EI2IY+EI3IY
      F(I+3)=CCI*(FI-EI1IY)+EI4IY
      F(I+4)=SSI*(FI1-EI2IY)+EI(I+5,IY)
      F(I+5)=CCI*(CCI*(FI-2D0*EI1IY)+3D0*EI4IY)-EI(I+6,IY)
      F(I+6)=CCI*F(I+4)-SSI*(SSI*EI1IY-2D0*EI3IY)-EI(I+7,IY)
      F(I+7)=SSI*F(I+3)-CCI*(CCI*EI2IY-2D0*EI3IY)-EI(I+8,IY)
      F(I+8)=SSI*(SSI*(FI1-2D0*EI2IY)+3D0*EI(I+5,IY))-EI(I+9,IY)
      F(I+9)=ZERO
      I=I+10
      IF(I.LT.42) GO TO 36
           PP1=F(1)
      PP(1,IY)=PP1
           PP2=F(2)
      PP(2,IY)=PP2
           PP3=F(3)
      PP(3,IY)=PP3
      PP4=F(4)
      PP(4,IY)=PP4
      PP5=F(5)
      PP(5,IY)=PP5
      DO 31 I=33,49
   31 F(I+20)=F(I)-F(I-20)
      FF(1,IY)=(2D0*(F(14)+F(15))+XS3M*PP4)/3D0
      FF(2,IY)=2D0*X*PP3
      FF(3,IY)=PP5/3D0
      FF(4,IY)=X*(F(31)*(2D0*PP4-PP5)+3D0*F(32)*PP3-F(56)-F(57)
     &     +XS5M*(F(41)*PP4-.5D0*F(66)))
      FF(5,IY)=-3D0*F(31)*PP3-F(32)*(2D0*PP5-PP4)+F(58)+F(59)
     &     +XS4M*(-F(41)*PP3-F(42)*PP4+1.5D0*F(68))
      FF(6,IY)=X*(F(41)*PP5+2D0*F(42)*PP3-1.5D0*F(67))
      FF(7,IY)=-F(42)*PP5+0.5D0*F(69)
      FF(8,IY)=2D0*(-SINZ*(2D0*F(11)+XS3M*PP1)+2D0*COSZ*F(12))/3D0
      FF(9,IY)=2D0*X*(-SINZ*PP2+COSZ*PP1)
      FF(10,IY)=2D0*COSZ*PP2/3D0
      FF(11,IY)=0.2D0*X*(SINZ*(2D0*(F(14)+2D0*F(15))+XS5M*F(24))
     &     +2D0*COSZ*F(13))
     &     -(0.8D0*SINZ*X*(4D0*F(31)*PP1+3D0*F(32)*PP2-2D0*F(54)
     &     -1.5D0*F(55)+XS5M*(2D0*F(41)*PP1-F(64)))
     &     +0.8D0*COSZ*X*(2D0*F(31)*PP2-3D0*F(32)*PP1+0.5D0*F(53)))
      FF(12,IY)=0.4D0*(SINZ*(F(13)-XS4M*F(23))+COSZ*(F(15)+2D0*F(14)
     &     +0.5D0*XS4M*F(24)))
     &     -(0.8D0*SINZ*(-3D0*F(31)*PP2+2D0*F(32)*PP1+0.5D0*F(53)
     &     -XS4M* 2D0*(F(41)*PP2+F(42)*PP1 -F(63)))
     &     +0.8D0*COSZ*(3D0*F(31)*PP1+4D0*F(32)*PP2-2D0*F(55)
     &     -1.5D0*F(54)+XS4M*(2D0*F(41)*PP1-F(64))))
      FF(13,IY)=0.2D0*X*(SINZ*F(25)-2D0*COSZ*F(23))
     &     -(0.8D0*X*(SINZ*(2D0*F(42)*PP2-F(65))
     &     -2D0*COSZ*(F(41)*PP2+F(42)*PP1-F(63))))
      FF(14,IY)=0.2D0*COSZ*F(25)
     &     -(0.8D0*COSZ*(2D0*F(42)*PP2-F(65)))
      Y=Y+H
      IY=IY+1
      TH=QK(IY)+XSN*SS(IY)
      IF(TG.LT.TTL.AND.TH.GT.TTL) IYL=IY
      IF(Y.LE.VV) GO TO 34
      PRINT 853,IYL,IYU,TTL,TTU,TTP,TTQ
  853 FORMAT(/'   IYL =',I4,',  IYU =',I4,',  TTL =',F9.4,',  TTU =',
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```
     &F9.4,',  TTP =',F9.4,',  TTQ =',F9.4)

      LV=42
      LGO=2
C     ASSIGN 632 TO LGO
      CALL PART7
  532 CONTINUE

C*******          PART 4
      DO 401 I=1,200
  401 F(I)=ZERO
      Y=H
      IY=2
   70 CONTINUE
      Z=Y*Q
      SINZ=DSIN(Z)
      COSZ=DCOS(Z)
      I=1
   43 CONTINUE
      SSI=SS(IY)
      CCI=CC(IY)
      EIIY=EI(I,IY)
      EI1IY=EI(I+1,IY)
      EI2IY=EI(I+2,IY)
      EI3IY=CCI*EIIY
      EI4IY=SSI*EIIY
      EI7IY=EI(I+6,IY)
      EI5IY=CCI*EI7IY
      EI6IY=SSI*EI7IY
      EI8IY=EI(I+7,IY)
      EI9IY=EI(I+8,IY)
      F(I)=EI3IY-EI1IY
      F(I+1)=EI4IY-EI2IY
      F(I+2)=CCI*F(I+1)-SSI*EI1IY+EI(I+3,IY)
      F(I+3)=CCI*(F(I)-EI1IY)+EI(I+4,IY)
      F(I+4)=SSI*(F(I+1)-EI2IY)+EI(I+5,IY)
      F(I+5)=EI5IY-EI8IY
      F(I+6)=EI6IY-EI9IY
      F(I+7)=CCI*F(I+6)-SSI*EI8IY+EI(I+ 9,IY)
      F(I+8)=CCI*(F(I+5)-EI8IY)+EI(I+10,IY)
      F(I+9)=SSI*(F(I+6)-EI9IY)+EI(I+11,IY)
      EICIY=EI(I+12,IY)
      F(I+10)=CCI*EICIY-EI(I+13,IY)
      F(I+11)=SSI*EICIY-EI(I+14,IY)
      EIFIY=EI(I+15,IY)
      F(I+12)=CCI*EIFIY-EI(I+16,IY)
      F(I+13)=SSI*EIFIY-EI(I+17,IY)
      EI18IY=EI(I+18,IY)
      F(I+14)=CCI*EI18IY-EI(I+19,IY)
      F(I+15)=SSI*EI18IY-EI(I+20,IY)
      I=I+21
      IF(I.LT.23) GO TO 43
      FF(4,IY)=(-(X*(4D0*F(4)+2D0*F(13)+3D0*(F(5)+F(12))+F(8)-F(15)
     &     +XS5M*(2D0*F(25)+F(34))))+FF(4,IY))*0.8D0
      FF(5,IY)=(3D0*(F(9)+F(11))+F(3)+4D0*F(10)-F(14)+2D0*F(16)
     &     +XS4M*(4D0*F(24)+2D0*(F(30)+F(32))+F(35))+FF(5,IY))*0.8D0
      FF(6,IY)=(-(X*(2D0*(F(26)+F(33))+4D0*F(29)+F(36)))+FF(6,IY))*.8D0
      FF(7,IY)=(2D0*F(31)+F(37)+FF(7,IY))*0.8D0
      FF(11,IY)=-(X*(-SINZ*(4D0*F(1)+3D0*F(7)+2D0*XS5M*F(22))
     &     +COSZ*(3D0*F(2)-2D0*F(6))))*0.8D0+FF(11,IY)
      FF(12,IY)=-(SINZ*(3D0*F(6)-2D0*(F(2)+XS4M*(F(23)+F(27)))))
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```
     &     -COSZ*(4D0*F(7)+3D0*F(1)+2D0*XS4M*F(22)))*0.8D0+FF(12,IY)
      FF(13,IY)=-(2D0*X*(-SINZ*F(28)+COSZ*(F(23)+F(27))))*.8D0+FF(13,IY)
      FF(14,IY)= 2D0*COSZ*F(28)  *0.8D0+FF(14,IY)
      Y=Y+H
      IY=IY+1
      IF(Y.LE.VV) GO TO 70
      DO 411 I=1,14
  411 FF(I,IY) =FF(I,IY5)-5D0*(FF(I,IY4)-FF(I,IYV))
     &     +10D0*(FF(I,IY3)-FF(I,IY2))

C*******          PART 4A
      IY=2
      Y=H
   26 BUIY=BU(IY)
      Z=Y*Q
      SINZ=DSIN(Z)
      COSZ=DCOS(Z)
      QKIY=QK(IY)
      QL=SS(IY)
      DO 49 I=1,5
      QII=QI(I)
      QRI=QR(I)
      QSI=QS(I)
      QCI=QC(I)
      TT(I,IY)=QKIY  +QL*QII
      AJ1(I,IY)=(X*SINZ+QII*COSZ)*BUIY
      DD(I,IY)=1D0-BUIY  *(X*COSZ-QII*SINZ)
      EN2(I,IY)=FF(1,IY)+QII*FF(2,IY)+QSI*FF(3,IY)
      EN3(I,IY)=FF(4,IY)+QII*FF(5,IY)+QRI*FF(6,IY)+QCI*FF(7,IY)
      AJ2(I,IY)=(FF(8,IY)+QII*FF(9,IY)+QSI*FF(10,IY))*BUIY
   49 AJ3(I,IY)=(FF(11,IY)+QII*FF(12,IY)+QRI*FF(13,IY)+QCI*FF(14,IY))
     &     *BUIY
      Y=Y+H
      IY=IY+1
      IF(IY.LE.IYU) GO TO 26

      CALL PART5

      IF(YY.GE.29.9D0) YYDEL=6D0
      YY=YY+YYDEL
      IF(YY.LT.89.9D0) GO TO 20
      PRINT 912,YY
  912 FORMAT(//'    INITIAL ANGLE =',F4.0,' DEG.')
   96 CONTINUE
      M=1
      T=ZERO
  906 CONTINUE
      TVU=.5D0*(TTV-T)/ TTV
      ENN(M)=ENN(M)+TVU*ENP(M)*FRO3
      AJJ(M)=AJJ(M)+TVU*AJP(M)
      IF((2*((M-1)/2).EQ.M-1.AND.T.LE.2.01D0).OR.
     &     (5*((M-1)/5).EQ.M-1.AND.T.GT.2.01D0))
     &PRINT 855,T,AJJ(M),ENN(M)
  855 FORMAT(F8.2,2(37X, F10.5))
      T=M*B
      M=M+1
      IF(M.LE.MVX) GO TO 906

  574 FORMAT(1H )

C*******          PART 6
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```
      D=DCON*PHH*B
      PRINT 661
  661 FORMAT(/8X,'TIME',8X,'PRIDEL',9X,'SECDEL',9X,'JPHIDEL')
      M=1
   93 CONTINUE
      T=M*B
      M=M+1
      MAX=(M-1)/2+1
      DO 95 I=1,MAX
      ELLI=ELL(I-1)
      IF(ELLI.GT.1D0) ELLI=1D0
      ELLM=ELL(M-I)
      IF(ELLM.GT.1D0) ELLM=1D0
      PP(I)=ELLM*PNN(  I)+ELLI*PNN(  M-I+1)
   95 CONTINUE
      IF(M.GT.7) GO TO 106
      IF(M.GT.6) GO TO 105
      IF(M.GT.5) GO TO 104
      IF(M.GT.4) GO TO 103
      IF(M.GT.3) GO TO 102
      IF(M.GT.2) GO TO 101
      PNNJM=12D0*PP(1)
      GO TO 120
  101 PNNJM=8D0*PP(1)+16D0*PP(2)
      GO TO 120
  102 PNNJM=9D0*PP(1)+27D0*PP(2)
      GO TO 120
  103 PNNJM=8D0*(PP(1)+4D0*PP(2)+PP(3))
      GO TO 120
  104 PNNJM=8D0*PP(1)+31D0*PP(2)+21D0*PP(3)
      GO TO 120
  105 PNNJM=8D0*PP(1)+31D0*PP(2)+20D0*PP(3)+13D0*PP(4)
      GO TO 120
  106 PNNJM=8D0*PP(1)+31D0*PP(2)+20D0*PP(3)+25D0*PP(4)
      IF(M.EQ.8) GO TO 120
      DO 110 I=5,MAX
      PNNJM=PNNJM+24D0*PP(I)
  110 CONTINUE
      IF(2*(M/2).NE.M)  PNNJM=PNNJM-12D0*PP(MAX)
  120 CONTINUE
      SEC(M)=D*PNNJM
      PRINT 664,T,PNN(M),SEC(M),   AJJ(M)
  664 FORMAT(F12.2,F15.5,F15.2,2F15.5)
      IF(M.LT.MIIMK) GO TO 93

      HH=HH+DELH
      IF(HH.LE.60D0) GO TO 10
      CALL EXIT
      END



      SUBROUTINE PART5
C*******          PART 5     REVISED
      IMPLICIT REAL*8 (A-H,K,O-Z)
       COMMON    EI(50,121),PNN(111),AJJ(111),PP(5,121)
     &,F(200),CC(121),SS(121),TT(5,121),DD(5,121),QQ(50)
     &,PNNMK(3,111),AJJMK(3,111),EN1(5,121),EN2(5,121),EN3(5,121)
     &,PNNM(3),AJJM(3),AJ1(5,121),AJ2(5,121),AJ3(5,121)
     &,                                PNK(3),AJK(3),PNA(3,9),AJA(3,9)
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```
     &,DDA(9),       QI(9),QC(9),QR(9),QS(9),BU(121),DEQ(5),DEP(5)
     &,QK(121),           FF(14,121),GI(5),WI(5),DE(5,4),AK1(5,4)
     &,AK2(5,4),AK3(5,4),FM2(5,4),EM3(5,4),ENM(3),EX2(4),EX3(4)
     &,ENP(111),AJP(111)
      COMMON HH,DELH,YY,YYDEL,VV,H,W,GG,KK,BB,EMY,X,XSN,RHH,B,C,YO
     &,TTP,TTQ,TTL,TTU,TTV
     &,MLIM,MLIMK,IH,LV,IYML,IYU,IYL,LGO,IYV,MVX
      G(ZDUM)=GG*DEXP(-KK*ZDUM)
      UU(ZDUM)=((G(ZDUM)-1D0)*(GG+1D0)/((G(ZDUM)+1D0)*(GG-1D0)))
     1     **2.164D0
      A(ZDUM)=2.164D0*KK*G(ZDUM)/(G(ZDUM)**2-1D0)
      O(ZDUM)=5.68569D0*DSQRT(G(ZDUM)**2-1D0)*UU(ZDUM)/BB
      DATA ZERO,P,Q/0D0,3.14159265358979D0,1.74532925199433D-2/
     &,QM,QN/0.95105651629515D0,0.58778525229247D0/
      FRO3=4D0/3D0
      PRINT 551
551   FORMAT(/4X,'TIME',4X,'JPHI1',4X,'JPHI2',4X,'JPHI3',5X,'JPHI',4X,
     &'CUMJPHI',           5X,'PRI1',5X,'PRI2',5X,'PRI3',6X,'PRI',5X,'CU
     &MPRI')
      DO 511 I=1,200
511   F(I)=ZERO
      DO 513 I=1,3
      ENNMK(I,1)=ZERO
      AJJMK(I,1)=ZERO
      ENNM(I)=ZERO
      AJJM(I)=ZERO
513   CONTINUE
      FACQM=EMY*YYDEL*Q
      TTUB=TTU-B
      TTUB2=TTUB-B
      TTLB=TTL-B
      TTLB2=TTLB-B
      BDN1=12D0*B
      BDN2=BDN1*B
      BDN3=2D0*B**3
      MLU=TTL/B+1.01D0
      MLX=MLIM
      IF(MLU.LT.MLIM) MLX=MLU
      MUU=TTU/B+1.01D0
      MUX=MLIMK
      IF(MUU.LT.MLIMK) MUX=MUU
      M=1
 80   CONTINUE
      T=M*B
      M=M+1
      DO 81 J=1,3
      ENK(J)=ZERO
 81   AJK(J)=ZERO
      I=1
 82   CONTINUE
      IY=0
 84   IY=IY+1
      IF(T.GT.TT(I,IY)) GO TO 84
      SLU=.2D0
      IF(IY.GT.2) IY=IY-1
      IF(IY.GE.IYV) IY=IY-1
      TTA=TT(I,IY-1)
      TTB=TT(I,IY)
      TTC=TT(I,IY+1)
      TTD=TT(I,IY+2)
      TTE=(-TTA+9D0*(TTB+TTC)-TTD)/16D0
```

```
      TTF=(TTA-27D0*(TTB-TTC)-TTD)/24D0
      TTG=(TTA-TTB-TTC+TTD)/4D0
      TTH=(T-TTE)/TTF
      TTI=(-TTA+3D0*(TTB-TTC)+TTD)/6D0
      TTJ=TTG/TTF
      TTK=2D0*TTJ*TTJ-TTI/TTF
      AP=.5D0+TTH*(1D0-TTH*(TTJ-TTK*TTH))
      APM2=AP-2D0
      APSM=AP*AP-1D0
      AL1=-AP*(AP-1D0)*APM2/6D0
      AL2=0.5D0*APSM*APM2
      AL3=-0.5D0*AP*(AP+1D0)*APM2
      AL4=AP*APSM/6D0
      DDA(I)=AL1*DD(I,IY-1)+AL2*DD(I,IY)+AL3*DD(I,IY+1)+AL4*DD(I,IY+2)
      ENA(1,I)=1D0
      ENA(2,I)=AL1*EN2(I,IY-1)+AL2*EN2(I,IY)+AL3*EN2(I,IY+1)
     &     +AL4*EN2(I,IY+2)
      ENA(3,I)=AL1*EN3(I,IY-1)+AL2*EN3(I,IY)+AL3*EN3(I,IY+1)
     &     +AL4*EN3(I,IY+2)
      AJA(1,I)=AL1*AJ1(I,IY-1)+AL2*AJ1(I,IY)+AL3*AJ1(I,IY+1)
     &     +AL4*AJ1(I,IY+2)
      AJA(2,I)=AL1*AJ2(I,IY-1)+AL2*AJ2(I,IY)+AL3*AJ2(I,IY+1)
     &     +AL4*AJ2(I,IY+2)
      AJA(3,I)=AL1*AJ3(I,IY-1)+AL2*AJ3(I,IY)+AL3*AJ3(I,IY+1)
     &     +AL4*AJ3(I,IY+2)
      J=1
      DDAI=DDA(I)
   86 ENK(J)=ENK(J)+ SLU *ENA(J,I)/DDAI
      AJK(J)=AJK(J)+ SLU *AJA(J,I)/DDAI
      J=J+1
      IF(J.LE.3) GO TO 86
      I=I+1
      IF(I.LE.5) GO TO 82
      DO 87 J=1,3
      AJJMK(J,M)=0.90467D0*FACQM*AJK(J)
   87 ENNMK(J,M)=FACQM*ENK(J)
      IF(M.LT.MLX) GO TO 80

   90 CONTINUE
      T=ZERO
      M=1
      ENNM(1)=FACQM/(1D0-X*BU(1))
      GO TO 85
   91 M=2
      T=B
      ENNM(1)=ENNMK(1,2)
      AJJM(1)=AJJMK(1,2)
      ENNM(2)=(11D0*ENNMK(2,1)-20D0*ENNMK(2,2)+6D0*ENNMK(2,3)
     &     +4D0*ENNMK(2,4)-ENNMK(2,5))/BDN2
      ENNM(3)=-(3D0*ENNMK(3,1)-10D0*ENNMK(3,2)+12D0*ENNMK(3,3)
     &     -6D0*ENNMK(3,4)+ENNMK(3,5))/BDN3
      AJJM(2)=-(3D0*AJJMK(2,1)+10D0*AJJMK(2,2)-18D0*AJJMK(2,3)
     &     +6D0*AJJMK(2,4)-AJJMK(2,5))/BDN1
      AJJM(3)=(11D0*AJJMK(3,1)-20D0*AJJMK(3,2)+6D0*AJJMK(3,3)
     &     +4D0*AJJMK(3,4)-AJJMK(3,5))/BDN2
      GO TO 85
   92 T=M*B
      M=M+1
      ENNM(1)=ENNMK(1,M)
      AJJM(1)=AJJMK(1,M)
      IF(T.GT.TTLB2) GO TO 901
```

```
      ENNM(2)=(-ENNMK(2,M-2)+16D0*(ENNMK(2,M-1)+ENNMK(2,M+1))
     &     -30D0*ENNMK(2,M)-ENNMK(2,M+2))/BDN2
      ENNM(3)=-(ENNMK(3,M-2)-2D0*(ENNMK(3,M-1)-ENNMK(3,M+1))
     &     -ENNMK(3,M+2))/BDN3
      AJJM(2)=-(-AJJMK(2,M-2)+8D0*(AJJMK(2,M-1)-AJJMK(2,M+1))
     &     +AJJMK(2,M+2))/BDN1
      AJJM(3)=(-AJJMK(3,M-2)+16D0*(AJJMK(3,M-1)+AJJMK(3,M+1))
     &     -30D0*AJJMK(3,M)-AJJMK(3,M+2))/BDN2
      GO TO 85
  901 IF(T.GT.TTLB ) GO TO 902
      ENNM(2)=(-ENNMK(2,M-3)+4D0*ENNMK(2,M-2)+6D0*ENNMK(2,M-1)
     &     -20D0*ENNMK(2,M)+11D0*ENNMK(2,M+1))/BDN2
      ENNM(3)=(ENNMK(3,M-3)-6D0*ENNMK(3,M-2)+12D0*ENNMK(3,M-1)
     &     -10D0*ENNMK(3,M)+3D0*ENNMK(3,M+1))/BDN3
      AJJM(2)=(-AJJMK(2,M-3)+6D0*AJJMK(2,M-2)-18D0*AJJMK(2,M-1)
     &     +10D0*AJJMK(2,M)+3D0*AJJMK(2,M+1))/BDN1
      AJJM(3)=(-AJJMK(3,M-3)+4D0*AJJMK(3,M-2)+6D0*AJJMK(3,M-1)
     &     -20D0*AJJMK(3,M)+11D0*AJJMK(3,M+1))/BDN2
      GO TO 85
  902 CONTINUE
      ENNM(2)=(11D0*ENNMK(2,M-4)-56D0*ENNMK(2,M-3)+114D0*ENNMK(2,M-2)
     &     -104D0*ENNMK(2,M-1)+35D0*ENNMK(2,M))/BDN2
      ENNM(3)=(3D0*ENNMK(3,M-4)-14D0*ENNMK(3,M-3)+24D0*ENNMK(3,M-2)
     &     -18D0*ENNMK(3,M-1)+5D0*ENNMK(3,M))/BDN3
      AJJM(2)=(3D0*AJJMK(2,M-4)-16D0*AJJMK(2,M-3)+36D0*AJJMK(2,M-2)
     &     -48D0*AJJMK(2,M-1)+25D0*AJJMK(2,M))/BDN1
      AJJM(3)=(11D0*AJJMK(3,M-4)-56D0*AJJMK(3,M-3)+114D0*AJJMK(3,M-2)
     &     -104D0*AJJMK(3,M-1)+35D0*AJJMK(3,M))/BDN2
      GO TO 85
  801 CONTINUE
      T=M*B
      M=M+1
      IF(T.GT.TTU.OR.M.GT.MLIMK) GO TO 903
      DO 888 J=1,5
  888 DEP(J)=ZERO
      DO 811 J=1,3
      ENK(J)=ZERO
  811 AJK(J)=ZERO
      I=1
  812 CONTINUE
      XM=(T-TTP)/TTQ
      TLU=(T-TTL)/(TTU-T)
      GII=GI(I)
      SLU=.2D0/DSQRT(1D0+WI(I)*TLU)
      QII=XSN*(XM+(1D0-XM)*GII+1D0)/2D0
      QI2=QII*QII
      QRI=1D0-5D0*QI2
      QSI=1D0-3D0*QI2
      QCI=QII*(2D0+QRI)
      IM=3
      IF(T.GT.TTUB2) IM=4
      IF(T.GT.TTUB ) IM=5
      IL=1
  802 IY=IYL-3
      TL=T+(IL-IM)*B
  803 IY=IY+1
      TW=QK(IY)+SS(IY)*QII
      IF(TL.GT.TW) GO TO 803
      IF(IY.GE.IYU) IY=IYV
      TW=QK(IY)+SS(IY)*QII
      IWA=IY-2
```

```
      IWB=IY-1
      IWC=IY+1
      TWA=QK(IWA)+SS(IWA)*QII
      TWB=QK(IWB)+SS(IWB)*QII
      TWC=QK(IWC)+SS(IWC)*QII
      TWD=(-TWA+9D0*(TWB+TW)-TWC)/16D0
      TWE=(TWA-27D0*(TWB-TW)-TWC)/24D0
      TWG=(TWA-TWB-TW+TWC)/4D0
      TWH=(TL-TWE)/TWF
      TWI=(-TWA+3D0*(TWB-TW )+TWC)/6D0
      TWJ=TWG/TWF
      TWK=2D0*TWJ*TWJ-TWI/TWF
      AP=.5D0+TWH*(1D0-TWH*(TWJ-TWK*TWH))
      APM2=AP-2D0
      APSM=AP*AP-1D0
      AL1=-AP*(AP-1D0)*APM2/6D0
      AL2=0.5D0*APSM*APM2
      AL3=-0.5D0*AP*(AP+1D0)*APM2
      AL4=AP*APSM/6D0
      DO 804 J=1,4
      JY=IY-3+J
      Z=(JY-1)*H*Q
      SINZ=DSIN(Z)
      COSZ=DCOS(Z)
      BUJY=BU(JY)
      DE(IL,J)=1D0-BUJY*(X*COSZ-QII*SINZ)
      AK1(IL,J)=(X*SINZ+QII*COSZ)*BUJY
      AK2(IL,J)=(FF(8,JY)+QII*FF(9,JY)+QSI*FF(10,JY))*BUJY
      AK3(IL,J)=(FF(11,JY)+QII*FF(12,JY)+QRI*FF(13,JY)+QCI*FF(14,JY))
     &     *BUJY
      EM2(IL,J)=FF(1,JY)+QII*FF(2,JY)+QSI*FF(3,JY)
804   EM3(IL,J)=FF(4,JY)+QII*FF(5,JY)+QRI*FF(6,JY)+QCI*FF(7,JY)
      DDA(IL)=AL1*DE(IL,1)   +AL2*DE(IL,2)+AL3*DE(IL,3)   +AL4*DE(IL,4)
      DDAI=DDA(IL)
      ENA(1,IL)=1D0/DDAI
      ENA(2,IL)=(AL1*EM2(IL,1)+AL2*EM2(IL,2)+AL3*EM2(IL,3)+AL4*EM2(IL,4)
     &     )/DDAI
      ENA(3,IL)=(AL1*EM3(IL,1)+AL2*EM3(IL,2)+AL3*EM3(IL,3)+AL4*EM3(IL,4)
     &     )/DDAI
      AJA(1,IL)=(AL1*AK1(IL,1)+AL2*AK1(IL,2)+AL3*AK1(IL,3)+AL4*AK1(IL,4)
     &     )/DDAI
      AJA(2,IL)=(AL1*AK2(IL,1)+AL2*AK2(IL,2)+AL3*AK2(IL,3)+AL4*AK2(IL,4)
     &     )/DDAI
      AJA(3,IL)=(AL1*AK3(IL,1)+AL2*AK3(IL,2)+AL3*AK3(IL,3)+AL4*AK3(IL,4)
     &     )/DDAI
      IL=IL+1
      IF(IL.LE.5) GO TO 802
      DO 805 J=1,3
      ENK(J)=ENK(J)+SLU*ENA(J,IM)
805   AJK(J)=AJK(J)+SLU*AJA(J,IM)
      IF(IM.GE.4) GO TO 806
      DEQ(1)=(-ENA(2,1)+16D0*(ENA(2,2)+ENA(2,4))-30D0*ENA(2,3)
     &     -ENA(2,5))/BDN2
      DEQ(2)=(-ENA(3,1)+2D0*(ENA(3,2)-ENA(3,4))+ENA(3,5))/BDN3
      DEQ(3)=( AJA(2,1)-8D0*(AJA(2,2)-AJA(2,4))-AJA(2,5))/BDN1
      DEQ(4)=(-AJA(3,1)+16D0*(AJA(3,2)+AJA(3,4))-30D0*AJA(3,3)
     &     -AJA(3,5))/BDN2
      GO TO 808
806   IF(IM.EQ.5) GO TO 807
      DEQ(1)=(-ENA(2,1)+4D0*ENA(2,2)+6D0*ENA(2,3)-20D0*ENA(2,4)
     &     +11D0*ENA(2,5))/BDN2
```

```
      DEQ(2)=(ENA(3,1)-6D0*ENA(3,2)+12D0*ENA(3,3)-10D0*ENA(3,4)
     &     +3D0*ENA(3,5))/BDN3
      DEQ(3)=(-AJA(2,1)+6D0*AJA(2,2)-18D0*AJA(2,3)+10D0*AJA(2,4)
     &     +3D0*AJA(2,5))/BDN1
      DEQ(4)=(-AJA(3,1)+4D0*AJA(3,2)+6D0*AJA(3,3)-20D0*AJA(3,4)
     &     +11D0*AJA(3,5))/BDN2
      GO TO 808
  807 CONTINUE
      DEQ(1)=(11D0*ENA(2,1)-56D0*ENA(2,2)+114D0*ENA(2,3)-104D0*ENA(2,4)
     &     +35D0*ENA(2,5))/BDN2
      DEQ(2)=( 3D0*ENA(3,1)-14D0*ENA(3,2)+ 24D0*ENA(3,3)- 18D0*ENA(3,4)
     &      +5D0*ENA(3,5))/BDN3
      DEQ(3)=( 3D0*AJA(2,1)-16D0*AJA(2,2)+ 36D0*AJA(2,3)- 48D0*AJA(2,4)
     &     +25D0*AJA(2,5))/BDN1
      DEQ(4)=(11D0*AJA(3,1)-56D0*AJA(3,2)+114D0*AJA(3,3)-104D0*AJA(3,4)
     &     +35D0*AJA(3,5))/BDN2
  808 CONTINUE
      DO 809 J=1,4
  809 DER(J)=DER(J)+SLU*DEQ(J)
      I=I+1
      IF(I.LE.5) GO TO 812
      DO 810 J=1,3
      ENNMK(J,M)=FACQM*ENK(J)
  810 AJJMK(J,M)=0.90467D0*FACQM*AJK(J)
      ENNM(1)=ENNMK(1,M)
      ENNM(2)=DER(1)*FACQM
      ENNM(3)=DER(2)*FACQM
      AJJM(1)=AJJMK(1,M)
      AJJM(2)=0.90467D0*DER(3)*FACQM
      AJJM(3)=0.90467D0*DER(4)*FACQM
   85 CONTINUE
                    SUMN = ENNM(1)+ENNM(2)+ENNM(3)
      ENN(M)=ENN(M)+.5D0*(SUMN+ENP(M))
      ENP(M)=SUMN
      IF(YY.EQ.30D0) ENP(M)=SUMN+SUMN
                    SUMJ = AJJM(1)+AJJM(2)+AJJM(3)
      AJJ(M)=AJJ(M)+.5D0*(SUMJ+AJP(M))
      AJP(M)=SUMJ
      IF(YY.EQ.30D0) AJP(M)=SUMJ+SUMJ
      IF((2*((M-1)/2).EQ.M-1.AND.T.LE.2.01D0).OR.
     &     (5*((M-1)/5).EQ.M-1.AND.T.GT.2.01D0))
     &PRINT 552,T,(AJJM(I),I=1,3),SUMJ,AJJ(M)
     &        ,(ENNM(I),I=1,3),SUMN,ENN(M)
  552 FORMAT(F8.2,2(1X,4F9.5, F10.5))
      IF(M.EQ.1) GO TO 91
      IF(M.GE.MLX) GO TO 801
      IF(M.LT.MLIMK) GO TO 92
  903 CONTINUE

      IF(MUX.EQ.MLIMK) GO TO 905
  904 CONTINUE
      TVU=.5D0*(TTV-T)/(TTV-TTU)
      ENN(M)=ENN(M)+TVU*ENP(M)*FRC3
      AJJ(M)=AJJ(M)+TVU*AJP(M)
      IF((2*((M-1)/2).EQ.M-1.AND.T.LE.2.01D0).OR.
     &     (5*((M-1)/5).EQ.M-1.AND.T.GT.2.01D0))
     &PRINT 855,T,AJJ(M),ENN(M)
  855 FORMAT(F8.2,2(37X, F10.5))
      T=M*B
      M=M+1
      IF(M.LE.MVX) GO TO 904
```

```
  905 TTV=TTU
      MVX=MUX

      RETURN
      END



      SUBROUTINE PART7
C*******          PART 7  REVISED
C     ******      4-POINT VERSION

      IMPLICIT REAL*8 (A-H,K,O-Z)
       COMMON    EF(50,121),ENN(111),AJJ(111),PP(5,121)
     &,F(200),CC(121),SS(121),TT(5,121),DD(5,121),QQ(50)
     &,ENNMK(3,111),AJJMK(3,111),EN1(5,121),EN2(5,121),EN3(5,121)
     &,ENNM(3),AJJM(3),AJ1(5,121),AJ2(5,121),AJ3(5,121)
     &,                                  ENK(3),AJK(3),ENA(3,9),AJA(3,9)
     &,DDA(9),       QI(9),QC(9),QP(9),QS(9),BU(121),DEQ(5),DEF(5)
     &,QK(121),           FF(14,121),GI(5),WI(5),DE(5,4),AK1(5,4)
     &,AK2(5,4),AK3(5,4),EM2(5,4),EM3(5,4),ENM(3),EX2(4),EX3(4)
     &,ENP(111),AJP(111)
       COMMON HH,DELH,YY,YYDEL,VV,U,W,GG,KK,BB,EMY,X,XSN,RHE,B,C,YO
     &,TTP,TTQ,TTL,TTU,TTV
     &,MLIM,MLIMK,IH,LV,IYML,IYU,IYL,LGC,IYV,MVX
       G(ZDUM)=GG*DEXP(-KK*ZDUM)
       UU(ZDUM)=((G(ZDUM)-1D0)*(GG+1D0)/((G(ZDUM)+1D0)*(GG-1D0)))
     1     **2.164D0
       A(ZDUM)=2.164D0*KK*G(ZDUM)/(G(ZDUM)**2-1D0)
       Q(ZDUM)=5.68569D0*DSQRT(G(ZDUM)**2-1D0)*UU(ZDUM)/BB
       DATA ZERO,P,Q/0D0,3.14159265358979D0,1.74532925199433D-2/
     &,QM,QN/0.95105651629515D0,0.58778525229247D0/
  700 CONTINUE
      DO 701 I=1,200
  701 F(I)=ZERO
      DO 702 MY=1,121
      DO 702 I=1,50
  702 FI(I,MY)=ZERO
      LV2=LV+LV
      LV3=LV2+LV
      Y=YO
      IGO=1
      IY=1
      IF(IYML.EQ.1)
     &RETURN
      GO TO (602,632,640),LGC
   30 CONTINUE
      I=1
   32 F(I)=F(I+LV3)
      I=I+1
      IF(I.LE.LV) GO TO 32
   33 Y=H+Y
      IGO=2
      IY=IY+1
      GO TO (602,632,640),LGC
   35 CONTINUE
      I=1
   37 F(I+LV)=F(I+LV3)
      I=I+1
      IF(I.LE.LV) GO TO 37
```

```
      IF(IYML.GT.2) GO TO 703
      I=1
   41 EI(I,2)=H*Q*(F(I)+F(I+LV))*.5D0
      I=I+1
      IF(I.LE.LV) GO TO 41
      RETURN
  703 Y=Y+H
      IGO=3
      IY=IY+1
      GO TO (602,632,640),LGO
   40 CONTINUE
      I=1
   42 F(I+LV2)=F(I+LV3)
      I=I+1
      IF(I.LE.LV) GO TO 42
      IF(IYML.GT.3) GO TO 704
      I=1
   44 EI(I,2)=H*Q*(5D0*F(I)+8D0*F(I+LV)-F(I+LV2))/12D0
      EI(I,3)=H*Q*(F(I)+4D0*F(I+LV)+F(I+LV2))/3D0
      I=I+1
      IF(I.LE.LV) GO TO 44
      RETURN
  704 Y=Y+H
      IGO=4
      IY=IY+1
      GO TO (602,632,640),LGO
   45 CONTINUE
      I=1
   47 EI(I,2)=H*Q*(9D0*F(I)+19D0*F(I+LV)-5D0*F(I+LV2)+F(I+LV3))/24D0
      I=I+1
      IF(I.LE.LV) GO TO 47
   50 CONTINUE
      I=1
   48 EI(I,IY-1)=EI(I,IY-2)+H*Q*(-F(I)+13D0*(F(I+LV)+F(I+LV2))
     &    -F(I+LV3))/24D0
      I=I+1
      IF(I.LE.LV) GO TO 48
      I=1
      IF(Y+H.GT.W) GO TO 60
   52 F(I)=F(I+LV)
      F(I+LV)=F(I+LV2)
      F(I+LV2)=F(I+LV3)
      I=I+1
      IF(I.LE.LV) GO TO 52
      Y=Y+H
      IGO=5
      IY=IY+1
      GO TO (602,632,640),LGO
   55 GO TO 50
   60 CONTINUE
   62 EI(I,IY)=EI(I,IY-1)+H*Q*(F(I)-5D0*F(I+LV)+19D0*F(I+LV2)
     &    +9D0*F(I+LV3))/24D0
      I=I+1
      IF(I.LE.LV) GO TO 62
      RETURN

C*******          PART 8
  602 CONTINUE
      I=1
      Z=Y*Q
      OOZ=O(Z)
```

```
      F(I+6) = OOZ*DCOS(Z)
      F(I+7) = OOZ*DSIN(Z)
      GO TO (30,35,40,45,55),IGO

C*******          PART 38
  632 CONTINUE
      Z=Y*Q
      AOZ=A(Z)
      UUOZ=UU(Z)
      UUZS=UUOZ*UUOZ
      QQ(1) =       AOZ/(UUZS*UUOZ)
      QQ(22)=             AOZ*UUZS
      I=1
   38 CONTINUE
      QQI=QQ(I)
       CCI=CC(IY)
       SSI=SS(IY)
      F(I+126)=QQI*PP(1,IY)
      F(I+127)=F(I+126)*CCI
      F(I+128)=F(I+126)*SSI
      F(I+129)=F(I+127)*SSI
      F(I+130)=F(I+127)*CCI
      F(I+131)=F(I+128)*SSI
      F(I+132)=QQI*PP(2,IY)
      F(I+133)=F(I+132)*CCI
      F(I+134)=F(I+132)*SSI
      F(I+135)=F(I+133)*SSI
      F(I+136)=F(I+133)*CCI
      F(I+137)=F(I+134)*SSI
      F(I+138)=QQI*PP(3,IY)
      F(I+139)=F(I+138)*CCI
      F(I+140)=F(I+138)*SSI
      F(I+141)=QQI*PP(4,IY)
      F(I+142)=F(I+141)*CCI
      F(I+143)=F(I+141)*SSI
      F(I+144)=QQI*PP(5,IY)
      F(I+145)=F(I+144)*CCI
      F(I+146)=F(I+144)*SSI
      I=I+21
      IF(I.LT.23) GO TO 38
      GO TO (30,35,40,45,55),IGO

C*******          PART 46
  640 CONTINUE
      Z=Y*Q
      AOZ=A(Z)
      UUOZ=UU(Z)
      UUZS=UUOZ*UUOZ
      UUZC=UUOZ*UUZS
      QQ(1)=AOZ*UUOZ
      QQ(11)=AOZ/UUZS
      QQ(21)=AOZ*UUZC
      QQ(31)=AOZ/UUZC
      QQ(41)=AOZ*UUZS
      I=1
   46 CONTINUE
      QQI=QQ(I)
      SSI=SS(IY)
      CCI=CC(IY)
      F(I+150)=QQI
      F(I+151)=QQI  *CCI
```

```
F(I+152)=QQI   *SSI
F(I+153)=F(I+151)*SSI
F(I+154)=F(I+151)*CCI
F(I+155)=F(I+152)*SSI
F(I+156)=F(I+154)*CCI
F(1+157)=F(I+155)*CCI
F(I+158)=F(I+154)*SSI
F(I+159)=F(I+155)*SSI
I=I+10
IF(I.LT.42) GO TO 46
GO TO (30,35,40,45,55),IGO
END
```

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