THE DIFFRACTION OF ELASTIC WAVES AND DYNAMIC STRESS CONCENTRATIONS

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PREFACE

This monograph is an outgrowth of continuous research in diffraction of elastic waves and effects of dynamic loadings on underground openings and structures. The authors first became interested in this problem in early 1960 through an Air Force-sponsored study of the survivability of hardened systems. It soon became evident that although the subject of elastic wave diffraction had been under investigation for over a hundred years and problems of stress concentration had been studied nearly as long, there existed relatively little information and few numerical results for the dynamic stress concentration factor. It was also apparent that in order to have a better understanding of how a nuclear-weapon-generated ground shock interacts with underground cavities or openings, essential in survivability analysis, questions of dynamical effects due to wave diffraction must be answered.

Since early 1960 the authors have been supported by the Air Force's Project Rand in the investigation of various aspects of dynamic responses and dynamic stress concentration factors of obstacles of different geometry and substance, with the objective of determining the dynamic response due to both steady-state and transient loadings. Also in this period, because of the support and impetus provided by the Air Force and other governmental agencies, research in this area was greatly accelerated, resulting in a flow of publications. By the mid-1960s a substantial number of published theories and numerical results had become available in journals and in reports from government agencies and industry. It is thought that a systematic presentation of elastic wave diffraction and dynamic stress concentration results is warranted.
because of their wide applicability. These results can be applied not only in the study of hardened systems but in machine design, ultrasonics, structural design, the mechanics of composite materials, and the theory of fractures.

Thus, the objectives of the monograph are twofold: (1) to systematically present methods of solution for both steady-state and transient loading on various obstacles, and (2) to present numerical results of dynamic stress concentration on obstacles of different geometries. An effort was made to collect information from the open literature as well as from government agencies, industry, and individuals. Because of the time constraints of manuscript preparation and the rapid proliferation of research in this field, many of the latest publications could not be cited in this monograph.

In undertaking this monograph, the authors had no idea it would grow to such a mammoth size. They had planned to write a self-contained monograph and to give so complete a discussion that the reader would not be troubled with checking many references. That plan was obviously too ambitious and unwieldy, and it was later modified somewhat. Undoubtedly many equations and derivations could have been omitted if references to standard texts and original papers had been made.

Omitted from the plan were a section in Chapter IV on the application of the integral equation method to the diffraction of P and SV waves, a section in Chapter V on the application of the Wiener-Hopf method to the diffraction of P and SV waves by a semi-infinite strip, a section in Chapter VI on the scattering of SV waves by a sphere, a chapter on spheroidal obstacles (elastic waves in spheroidal coordinates), and a final chapter on experimental methods and observations.
of elastic wave diffraction. However, even if everything originally planned were included, this monograph still would not be a comprehensive treatment on the diffraction of elastic waves. One obvious omission is a detailed discussion of the scattering of sound waves in liquid by an obstacle, with numerical results and graphs. Another omission is the analysis of multiple scattering by many obstacles. The authors wish to be excused if the reader is disappointed by these omissions.

It is said that the publication of a monograph signals the end of active research on the title subject. That will certainly not be true in this case because of the research being done on the diffraction of elastic waves that has not been treated in this monograph. The authors will be most gratified if, in addition to achieving the twofold objectives above, this monograph generates more interest in, and research effort on, the diffraction of elastic waves and dynamic stress concentrations.

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In many cases of modern design, the elementary solutions obtained by the application of the theory of strength of materials are insufficient, and recourse has to be made to the general equation of the theory of elasticity in order to obtain satisfactory results. All problems of stress concentration are of this kind.

Stephen Timoshenko, 1925

(from Transactions of The American Society of Mechanical Engineers, Vol. 47, 1925, p. 237)
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INTRODUCTION

1. A BRIEF HISTORY OF ELASTIC WAVE DIFFRACTION

DIFFRACTION of elastic waves has its origin in the age-old searching for the true nature of light. The name diffraction was given by Fr. Francesco Maria Grimaldi (1618-1663) to the phenomenon that a light beam might be bent slightly while passing the edge of an aperture.

It is now applied to a phenomenon of wave propagation when the rays of waves deviate from rectilinear paths, which cannot be interpreted as reflection or refraction. In the first half of the 19th century, light was interpreted as the propagation of a disturbance in an elastic aether, the dynamics of which were described by what is now called the Theory of Elasticity. Thus, the theory of the propagation of elastic waves was developed long before the application of elasticity theory to stress analysis for structures and machinery components.

One of the major problems in stress analysis is the determination of stress concentration, which is the sharp increase of stress over a nominal value in a localized region of a structural member due to geometric discontinuities such as holes, corners, and notches. During the first half of the 20th century, the subject of stress concentration had evolved from a mathematical curiosity to an important element in engineering design. However, its understanding was limited to the
case of static loading, i.e., when the forces or other sources equivalent to a force are applied gradually and slowly to a structural member in order that the effect of its mass inertia can be neglected, or when the forces have been applied long before the instant of recording such that the observed data show little dependence in time. The investigation of stress concentration under dynamic loading has started only very recently. As in the static case, the analysis of dynamic stress concentration is also based on the theory of elasticity. Hence, it is not surprising to find that dynamic stress concentration is related to the propagation of elastic waves. The effect of a dynamic loading is to generate elastic waves which propagate in a structure or machinery member. When passing through a geometric discontinuity, an elastic wave is diffracted just as the path of a light ray is deviated by the edge of an aperture. Thus, dynamic stress concentration is a result of the diffraction of elastic waves.

After the development of the electromagnetic and quantum theories of light, no one would accept the elastic solid theory. In 1888, Lord Rayleigh in the article "Wave Theory of Light" in the Encyclopaedia Britannica stated that "The elastic solid theory, valuable as a piece of purely dynamical reasoning, and probably not without mathematical analogy to the truth, can in optics be regarded only as an illustration." It is in this spirit that we begin this section with a brief recount of the "Elastic Solid Theory of Light." By traversing through the historical paths, we will discover how elastic wave theory was developed and trace the links which are common to all waves in nature, including the sound (acoustic) waves, electromagnetic waves, and elastic waves.
The first four subsections are concerned with the historical survey of the diffraction and scattering of light, sound, and elastic waves, together with the related mathematical theories and methods. At various stages, we shall call attention to the prominent achievements of many pioneers and their influences on the modern day theory of diffraction of elastic waves. The subject of static stress concentration is introduced in subsection 5 as a separate entity, and then in the final subsection it is correlated with the diffraction of elastic waves and dynamic stress concentrations, the latter including the static stress concentration as a limiting case when frequency approaches zero.

1.1. Elastic Solid Theory of Light

In Grimaldi's book *Physico-Mathesis du Lumine, Coloribus et Iride* which was published posthumously in 1665 at Bologna, the author described an experiment of letting a beam of light pass through two narrow apertures, one behind the other, and then fall on a black surface. He found that the band of light on the surface was a trifle wider than it was when it entered the first aperture. Therefore, he believed that the beam had been bent outward slightly at the edges of the aperture. This was different from the hitherto observed phenomena of reflection and refraction, and it was named *diffraction*.

The same phenomenon was noticed a few years later by Robert Hooke (1635-1703). Although Hooke, and his contemporary Christian Huygens

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* A comprehensive account of the elastic solid theory of light is given by E. T. Whittaker, Ref. 1.1.

† Grimaldi's book is one of the very early scientific books on light. It is perhaps preceded only by G. B. Della Porta's *De Refractioni*, Neapoli, 1593, see Ref. 1.2.
(1629–1695), were the early proponents of the "wave theory" of light, diffraction was one of the phenomena they could not explain. For if light was propagated like sound waves, * in the shadow region bounded by an opaque screen light would spread equally and there would be no darkness. Toward the beginning of the 19th century, diffraction, polarization, and double refraction (in crystals) of light were the major difficulties confronting the wave (longitudinal) theory of light.

In 1801, Thomas Young (1773–1829) discovered the law of interference of light waves, (1.3a, 1.3b) which paved the way for Augustin Jean Fresnel (1788–1827) to discover the real cause of diffraction. An interference can be described simply as two waves that when mixed together destroy (or reinforce) each other, either wholly or partially. In the memoir which won him a prize from the French Academy of Sciences in 1818 on the subject of "Diffraction," (1.4) Fresnel set forth the concept that the diffraction of light is the mutual interference of the secondary waves emitting from an aperture. If the incident waves are conceived to be broken up on arriving at the aperture of a screen, each element of the aperture is then considered as the center of a secondary disturbance according to Huygens' principle. The intensity of the diverging spherical wave does not vary rapidly from one direction to another in the neighborhood of the normal to the incident wave front, and the disturbance at any point of observation is found by taking the aggregate of the disturbances due to all the secondary waves. Since the phase of the motion of each secondary wave is retarded by a quantity corresponding to the distance from its center to the point of

*Sound propagation in the form of a wave was established in Sir Isaac Newton's time (1642–1727).
observation, the arriving secondary waves interfere with each other, resulting in diffraction.

Shortly afterwards, Francois Arago (1786-1853) and Fresnel jointly discovered experimentally that two beams of light polarized in planes at right angles do not interfere with each other. This discovery led Young to believe that light is a transverse wave in an aether,\(^\text{(1.5)}\) and that the motion of the particles in the wave is in a certain constant direction which is at right angles to the direction of propagation of the wave. This phenomenon was called polarization.\(^*\)

Young's explanation of polarization was grasped immediately and expounded further by Fresnel. Based on the concept of transverse waves, Fresnel presented three memoirs\(^\text{(1.7)}\) to the French Academy in 1821 and 1822 discussing double refraction in crystals. He reasoned that light propagating in any direction through a crystal could be resolved into two plane-polarized components, each with a distinct velocity. Lacking a theory for the transverse wave motion in aether at that time, he found from a purely geometric argument that the two velocities must be the roots of a quadratic equation. He derived this equation by considering the relative displacements resulting from a wave motion in an aether. Thus, in a span of little more than one decade, all major difficulties inherent hitherto in the wave theory of light were resolved. The centuries-old question of the nature of light was answered by stating that light was a transverse motion of waves in elastic aether.

*The polarization of sunlight upon reflection was first observed by Stephen Louis Malus (1775-1812) in 1809. Biographical accounts of Arago, Malus, Fresnel, and Young and their scientific contributions are contained in Ref. 1.6 with interesting notes and remarks added by the translators.
Although the equation for sound waves (longitudinal waves) in air, also in aether, was already developed by the end of the 18th century, no general method had been developed for investigating the motion of an elastic aether possessing resistance to both volumetric change and distortion. In 1821, the year Fresnel presented his memoir on crystal optics, Claud Louis-Marie-Henri Navier (1785–1836) presented a molecular theory of an elastic body, giving an equation of motion for the displacement of a particle in elastic solids. His theory immediately drew the attention of other members of the Academy who were searching for an equation governing the transverse motion of elastic aether. In the subsequent years, Augustine-Louis Cauchy (1789–1857) started from an entirely different point of view and developed what is known today as the "Mathematical Theory of Elasticity" (See Section 2). He not only introduced the notion of stress, strain, and stress-strain relations, but also correctly established the number of elastic constants, two for an isotropic solid and 21 for a crystal. The equation of motion in Cauchy's theory agrees with Navier's if the bulk modulus equals 5/3 of the shear modulus of the solid.

Cauchy's theory was contained in a publication in 1828. In the same year, Simeon Denis Poisson (1781–1840) succeeded in solving the differential equation of motion for an elastic solid by decomposing the displacement into an irrotational and a circuital (equivoluminal) component.

Jean le Rond D'Alembert (1717–1783) developed the partial differential equation for a vibrating string in 1750. The same equation was derived as a limiting case of a string of beads by Joseph Louis Lagrange (1736–1813) in 1759.

See Refs. 1.9 and 1.10 for histories of elasticity theory and concise biographical sketches of the scholars' contributions to that theory.
part, each part being a solution of a wave equation. Poisson's analysis has been followed to this day in studying wave motions in solids (see Section 2), but his finding of two waves in solids created a new difficulty in the wave theory of light. For if the illuminous aether behaved like an elastic solid, his analysis showed that two waves* instead of one should be visible.

A multitude of modifications in the elastic solid theory were proposed afterwards. In 1837, George Green (1793-1841) used energy and variational principles to derive the equation of motion and correctly established the boundary conditions at the surface of an elastic solid. His work stimulated James MacCullagh (1809-1847) to postulate in 1839 a solid of which the potential energy depended only on the rotation of a volume element. The equation of motion so derived has the form $\nabla \times \nabla \times \mathbf{u} = \rho \nabla^2 \mathbf{u} / \partial t^2$, $\nabla \cdot \mathbf{u} = 0$, where $\mathbf{u}$ is the vector displacement, $\mathbf{u}$ the shear rigidity and $\rho$ the density. Since in his theory, the longitudinal wave does not exist and the light wave only propagates with one speed, $(\mu/\rho)^{\frac{1}{2}}$, MacCullagh for the first time really solved the problem of devising a medium whose vibrations, calculated in accordance with the established laws of mechanics, should have the same properties as the vibrations of light.†

The elastic solid theory of light was soon replaced by the electromagnetic theory which had been developing independently for over a century. In a series of papers capped by the memoir "A Dynamic Theory

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* The two waves are designated as P- and S-waves in this volume.
† In terms of electromagnetic theory, $\mathbf{u}$ in MacCullagh's equations corresponds to the magnetic field vector, $\nabla \times \mathbf{u}$ to the electric field vector.
of Electromagnetic Field" read to the Royal Society in 1864, \(^{(1.15)}\) James Clerk Maxwell (1831-1879) presented a unified theory of electromagnetism and concluded that "light itself (including radiant heat, and other radiation if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws." The experimental confirmation of Maxwell's theory in 1888 by Heinrich Rudolf Hertz (1857-1894) \(^{(1.16)}\) left no doubt that light was not an elastic wave. However, that did not stop the most illustrious scientists from analyzing the light wave as the propagation of a disturbance in an elastic aether. Toward the end of the 19th century, many important contributions on the diffraction of light—by Gustave Robert Kirchhoff (1824-1887), Lord Rayleigh (John William Strutt, 1842-1919), Horace Lamb (1849-1934), and others—were based upon the theory of elastic waves in solids.

1.2. Diffraction and Scattering of Light

After the elastic solid theory for light was developed, it seemed natural to employ it to investigate the phenomenon of diffraction. The first attempt was made by George Gabriel Stokes (1819-1903) in 1849 when he presented the memoir "On the Dynamical Theory of Diffraction" to the Cambridge Philosophical Society. \(^{(1.18)}\) Following Poisson's approach to initial value problems associated with the wave equation, Stokes derived the general solutions of the dynamic equations for the propagation of a disturbance in an elastic medium. He assumed that the disturbance was produced by a given initial disturbance which was

*Part of the corresponding statical theory was developed by M. W. Weber and C. Neumann in 1858.

†The observational diffraction phenomena together with a history of their discovery are described in Ref. 1.17.
confined to a finite portion of the medium. When light is diffracted by an aperture in a screen, each element of the aperture acts like a source which generates secondary waves. Stokes applied his solution to determine the disturbance corresponding to the secondary waves, and he was able to show the polarization and magnitude of the diffracted light at a point far away (when compared with the wavelength) from the screen.

Stokes' paper and his continuous interest in the light wave, \(^{(1.19)}\) plus an experimental discovery by Tyndall, \(^*\) led Rayleigh to investigate the diffraction of light by small particles and to provide the answer to why the sky is blue.

Starting in 1871, Rayleigh discussed in a sequence of papers the scattering of light by small particles. \(^{\dagger}\) It should be noted that by that time the electromagnetic wave theory of light was beginning to be accepted and the difference (or resemblance) between a sound wave (an elastic wave) and a light wave was understood. Thus, as far as the mathematical analysis was concerned, light could be treated as either an electromagnetic wave or an elastic wave. Using the elastic solid theory, Rayleigh found the important law of scattering in 1871: \(^{(1.22)}\)

When light is scattered by particles which are very small compared with any of the wave lengths, the ratio of the amplitudes of the vibrations of the scattered and

\(^*\) In 1868 Tyndall observed that when a condensed light beam passed through a mixture of air and hydrochloric acid, a cloud was formed which passed in color from the deepest violet through blue. Tyndall remarked in his notebook, "Connect this blue with the colour of the sky." See Ref. 1.20.

\(^{\dagger}\) Rayleigh's contributions to the scattering of sound and light waves have been reviewed by Twersky in Ref. 1.21.
incident light varies inversely as the square of the wave length, and the intensity of the lights themselves as the inverse fourth power.

This law was discovered from a simple dimensional analysis of the wavelength, the amplitude and the size of the particles. It was verified with a mathematical analysis based upon Stokes' work by considering the scattered light as being emitted from a body force in an elastic medium. Since blue has a shorter wavelength in the visible light spectrum, when sunlight is scattered by fine particles (air molecules) in the sky, the blue color with its dominant intensity prevails.

By treating the secondary waves as an emission from a body force in a homogeneous solid, the scattering effect as a consequence of the difference between the refractive power of two media is attributed to a change of density and not to a difference of rigidity. To attack the problem more generally, Rayleigh later assumed a body source in an isotropic but inhomogeneous elastic solid and thus included the difference of rigidity as well as density in the analysis. The result obtained substantiated his law of scattering. (1.23)

Rayleigh followed Stokes' approach to diffraction until 1872 when he treated in detail the scattering of waves (sound) by a spherical obstacle with a finite radius. (1.24) This paper is most important because, aside from its mathematical rigor for a difficult problem, it set the tone for many subsequent analyses of surface scattering. In this paper a velocity-potential of the form \( \varphi = \exp \{ik(x + ct)\} \) is assumed and expanded in a series of spherical harmonics. "The whole motion external to the sphere may be divided into two parts; that belonging to the plane waves supposed to be undisturbed, and secondly a motion due
to the presence of the sphere and radiating outward from it." The incident plane wave has a velocity potential $\varphi$; the velocity potential of the diverging wave $\psi$ may also be expanded in a series of spherical harmonics with unknown coefficients $A_n$. The velocity-potential of the whole motion is found by addition of $\varphi$ and $\psi$, the constants $A_n$ being determined by the boundary conditions, whose form depends upon the character of the obstruction presented by the sphere. He treated the cases of a rigid-fixed sphere and a gaseous sphere with different densities and compressibilities than the surrounding air. He considered incident plane waves as well as spherical waves. By calculating the energy intensity of the scattered waves at large distances from the sphere, he reconfirmed his law of scattering.

This important analysis was reproduced with simplified notations (mainly the use of Legendre functions) in his celebrated treatise *The Theory of Sound* (Vol. 2, Sections 334, 335), the first edition of which appeared in 1878.\(^{(1.25)}\) In the same volume, Rayleigh also discussed the two-dimensional counterpart—the scattering of sound waves by a circular cylindrical obstacle (Section 343).\(^*\) As in the case of a spherical scatterer, the analysis was slanted toward the case when the incident wavelength is larger than the radius of the obstacle—a condition which is now referred to as Rayleigh's Scattering.

The papers cited so far are all based on elastic wave theory. In the paper which appeared in 1881,\(^{(1.27)}\) Rayleigh showed the corresponding results based on the electromagnetic theory of light. In this

\* See also Chapter 8, "The Scattering of Sound," in Morse and Ingard, Ref. 1.26.
paper, in addition to the scattering of electromagnetic waves due to inhomogeneity, two cases of scattering by a circular cylinder are treated in detail—(1) when the electric displacements are parallel to the axis of the cylinder and (2) when the electric displacements are perpendicular to the axis of the cylinder. For the first case, the axial component of the electric displacement vector satisfies the wave equation and it vanishes at a perfect reflector, whereas for the latter, the axial component of the magnetic field vector satisfies the wave equation and its normal derivative vanishes at a perfect reflector. Many years later, Rayleigh returned to the same problem and presented numerical results for the waves diverging from a circular cylinder. (1.28)

Stokes' and Rayleigh's early treatment of scattered phenomenon as secondary waves which are generated by localized sources (body forces) is now classified as Bulk (body) Scattering. (1.29) Rayleigh's later treatment of scattering as caused by the presence of a foreign body with a bounding surface which absorbs, reflects, or diffracts the incident wave is called "surface scattering." In Chapter II, Section 1, Rayleigh's approach to the surface scattering is used to represent the scattering of a horizontally polarized shear wave by a circular cylinder in an elastic medium. We show that the scattering of elastic waves is no different from the scattering of sound or electromagnetic waves. In fact, much of the analysis of elastic waves presented in this book is based upon this method.

We would like to note at this stage that the original definition

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*We would have called it Rayleigh's method were it not for the fact that the second method, the integral equation method, was also originated by him.
of "scattering of waves" is not clear. If we follow Rayleigh's usage that the scattered wave is the difference of the total wave field observed in the presence of an obstacle and the incident wave, then a scattered wave is comprised of the part reflected by the obstacle into the illuminated zone, and the part refracted and diffracted by the obstacle into the shadow zone. In this sense, scattering of waves has a broader implication than the original meaning of "diffraction of waves." However, in the course of studying diffraction of waves, reflection and refraction are integral parts and they are generally implied and disposed very easily. Thus, outside of molecular physics, "scattering" and "diffraction" are often used to describe the same wave phenomenon. In cases when the diffracted part of the scattered waves is an important feature, especially in connection with the passage of waves through an object with sharp edges, the usage of "diffraction of waves" prevails. When the diffracted part has a lesser role, especially in the case of an obstacle without sharp edges, the title of "scattering of waves" is preferred.

1.3. Mathematical Theory of Diffraction

It was mentioned earlier that Fresnel first discovered the real cause of diffraction—the mutual interference of the secondary waves emitted by an aperture and by those parts of the primary wave front which were not obstructed by a screen. He also showed how a disturbance at a point can be calculated by superposing the secondary waves that advance from a surface situated between this point and the light source. In order to arrive at a satisfactory result, he made several rather arbitrary assumptions on the nature of the secondary waves. In
1883, Kirchhoff eliminated all those assumptions and put the whole idea on a sounder mathematical basis. (1.30)

Prior to 1883, the Helmholtz theorem* (1859) was available for expressing the harmonic waves at a point as an integral of a layer of sources plus an integral of a layer of double-sources over a prescribed surface (see Chapter II, Section 2). Kirchhoff generalized this theorem to the case when waves are not necessarily harmonic in time (see Chapter II, Section 2). As an application of his new integral theorem, he investigated the diffraction of light† by a screen with an aperture, still using the elastic solid theory.

The difficulty of applying the Helmholtz-Kirchhoff integral theorems is that the source layer and double-source layer over the surface of the screen are not known in advance, thus one cannot carry out the integration. Kirchhoff assumed the following boundary conditions for the screen: (i) at the luminous side, including the aperture, the total wave and its normal derivatives are equal to those of the incident wave; (ii) at the shadow side of the screen, the total wave and its normal derivative vanish. With these boundary conditions, the wave function (source layer) and its normal derivative (double layer) are known over the surface of the screen and the results after integration are in agreement with Fresnel's.

From experiment it is known that the shadow side of the screen is slightly illuminated, but Kirchhoff assumed it to be perfectly dark.

* Hermann Ludwig Ferdinand Helmholtz (1821-1894).
† For a systematic discussion of Fresnel's and Kirchhoff's theories of diffractions, see Baker and Copson (Ref. 1.32, Chapters 1 and 2) and Chapter 8 in Born and Wolf (Ref. 1.33).
Also, Kirchhoff's assumption makes the boundary values of the wave function and its normal derivative discontinuous across the edge of the aperture, which is certainly not the case physically. This difficulty was not resolved until the method of integral equations was developed by Rayleigh.

The Helmholtz-Kirchhoff theorems had already been discussed in the first edition of *The Theory of Sound*. In 1897, Rayleigh applied the Helmholtz integral theorem to the determination of the diffraction of waves passing through an aperture in a plane screen. (1.34) He treated only the scalar wave and assumed a velocity potential \( \varphi \) satisfying the wave equation. The actual solution consists of two parts, "the first part which would obtain were the screen complete, the second the alteration required to take account of the aperture." The first part \( \chi \) can easily be determined for a given incident wave and boundary conditions at the complete screen (\( d\chi/dn = 0 \)). The second part \( \psi \) is given by the integrals \( \psi = \pm \int \int \varphi e^{-ikr/r} dS \), where \( r \) denotes the distance of the observing point at which \( \psi \) is to be estimated from the element \( dS \) of the aperture, \( \varphi \) is the value of \( \psi \) at the surface \( dS \), and the integration is extended over the entire surface of the aperture. From the continuity conditions of the total wave (\( \psi = \chi + \varphi \)) across the aperture, the values of \( \psi \) on the left-hand side of the above equation are found as the point of observation moves to the surface of the aperture. Rayleigh thus derived an integral equation without actually calling it such for the unknown function \( \psi \). Rayleigh also gave an approximate solution for \( \psi \) and discussed seven related problems including the case of a reflecting blade with boundary conditions of either \( \varphi = 0 \) or \( d\varphi/dn = 0 \), which are mathematically the same as the diffraction of
a horizontally polarized shear wave by a crack or a thin, rigid blade of finite width and infinite length (see Chapter IV, Section 4).

Rayleigh returned to the same problem in 1913, trying to improve the solutions of the integral equation. Many attempts by others followed and to this day no exact solution for that integral equation has been found, even for an aperture with simple geometry like a circular disk (three-dimensional problem) or a narrow slit (two-dimensional problem), without using special wave functions.

A year before the appearance of Rayleigh's paper on integral equations, Arnold Sommerfeld (1868-1951) published another important paper on the diffraction of light. It was mentioned that Kirchhoff's theory of diffraction by a screen with an aperture was criticized for using physically inconsistent boundary conditions, but that it was regarded as a fairly accurate first approximation. A rigorous solution for the correct boundary conditions was first found by Sommerfeld for the diffraction of light by a semi-infinite screen (half-plane).

Sommerfeld did not start from the Helmholtz-Kirchhoff integral theorems—instead he extended the method of multivalued-functions in potential theory first to solve heat equations, then to wave equations.

Sommerfeld's solution, with some simplification due mainly to Carslaw, begins with the concept of constructing mathematical solutions by adding images of a source in the field. Consider a plane harmonic wave with wavelength $2\pi/k$ in polar coordinates $(r,\theta)$ represented by $u(r,\theta,\theta') = \exp \left[ikr \cos (\theta - \theta') \right]$ (time factor omitted). It is obstructed by a half-plane $y = 0, z > 0 (\theta = 0, 2\pi)$. For light with polarization perpendicular to the edge of the semi-infinite reflecting screen (the z-axis), the boundary condition is that the total
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wave \( \nu \), which represents the component of the electric field vector, should vanish at both sides of the half-plane. At first sight, it would appear that the appropriate solution for the total wave ought to be
\[ u = u_0(r, \theta, \theta') - u_0(r, \theta, -\theta') \]
the second term being the image of the incident wave. But this solution vanishes all over the plane \( y = 0 \), and so it solves the problem of reflection by an infinitely extended plane, not the diffraction by a half-plane. Instead, Sommerfeld constructed the two-valued wave function of the form
\[ U(r, \theta, \theta') = \pi^{-\frac{1}{2}} e^{i\pi/4} u_0(r, \theta, \theta') \times \int_\infty^\infty e^{-i\lambda} \, d\lambda \quad \text{where } T = \cos \frac{1}{2}(\theta - \theta')(2kr)^{\frac{3}{2}}. \]
The function is multiple-valued and is of period \( 4\pi \) in the physical plane \( 0 < \theta < 2\pi \), but on a two-sheet Riemann’s surface with the origin (edge of the screen) as branch point, it is uniform and single-valued. The solution
\[ u = U(r, \theta, \theta') - (r, \theta, -\theta'), \]
which satisfied the wave equation \( \nabla^2 u + \kappa^2 u = 0 \), and the boundary conditions \( u = 0 \), at \( 2\theta = 0, 2\pi \), then solves the original problem.

Since Sommerfeld’s solution can be obtained, as discovered later, by many other methods, his technique of constructing multi-valued solutions for the wave equation is not reproduced in this book. Interested readers can find a complete exposition of his method in the book by Baker and Copson (1.32) and in Sommerfeld’s own text (1.39). Nevertheless, the prominence of his achievement should be reemphasized, not only for the skill with which the solution was constructed, but also because his answer is still the only available exact solution in closed form for the diffraction of waves by a screen with an aperture of any kind. By solution in closed form, we mean that it is expressed in terms of elementary functions or a well-studied definite integral—Fresnel’s integral in this case. His solution naturally became a
standard and a test piece for all methods developed afterwards for studying the diffraction of waves, including the use of wave functions in parabolic coordinates (Chapter V, Section 2), the application of homogeneous solutions of wave equations (Chapter V, Section 3), and the method of integral equations coupled with the Wiener-Hopf technique.

1.4. Diffraction of Elastic Waves

Most of the investigations on diffraction and scattering cited so far are for scalar waves, i.e., the wave motion is describable by a single scalar function $\varphi$ which satisfies the wave equation $c_1^2 \varphi = \frac{\partial^2 \varphi}{\partial t^2}$. This is the case of sound waves in air or an inviscid liquid, and electromagnetic waves in two-dimensional space. As it was mentioned before, Poisson showed that the displacement $u$ of waves in elastic solid may be decomposed in two parts, each being a solution of a wave equation. In terms of vector notation and Helmholtz's theorem, Poisson's results are $u = \psi + v \times \omega$ with $c_1^2 \omega = \frac{\partial^2 \omega}{\partial t^2}$ and $c_2^2 \psi = \frac{\partial^2 \psi}{\partial t^2}$, the latter being a vector wave equation. The propagation of electromagnetic waves or light is governed by a single vector wave equation where $\omega$ stands for either the electric field or the magnetic field intensity vector. Mathematically, waves which are governed by a vector wave equation are classified as vector waves.

The earliest mathematical treatment on the diffraction of elastic (vector) waves by a bounded obstacle was probably due to Alfred Clebsch (1833-1872). In 1863, eight years before Rayleigh's first investigation of scattering of light, the Crelle's Journal published a paper by Clebsch "Über die Reflexion an einer Kugelfläche." (1.40) He intended to study the reflection and transmission of light through
lenses by wave theory and used the complete equations of elasticity to calculate the diffraction of elastic waves by a rigid sphere. However, he failed to derive any information from the complex solution which would otherwise serve as a sound basis for a law of reflection of light. Nevertheless, the mathematical analysis of the scattering of waves as a boundary value problem in dynamic elasticity was complete and, in fact, the spherical solid harmonics and what are now known as Spherical Bessel Functions were first studied in that paper. His method of analysis was substantially the same as that used by H. Lamb and other writers afterward, but the paper was for quite a long time overlooked.

The same problem was studied later by Ludwig V. Lorentz (1829–1891) in 1890. (1.42) Following Clebsch's work, he applied the elastic wave theory but excluded the longitudinal waves and solved the vector wave equation in the form \( \sigma^2 \mathbf{v} \times \mathbf{u} = -\beta^2 \mathbf{u}/\beta t^2 \) for the components of displacement vector \( \mathbf{u} \) in spherical coordinates. The boundary conditions were that the \( \mathbf{u} \) and \( \nabla \times \mathbf{u} \) must be continuous across the surface of a sphere. He made use of the complex series solution by first converting the sum of the series into an integral and then evaluating the asymptotic values of the integral which is now a standard technique for extracting information from a series solution at the high frequency limit.

As early as 1882, Lamb was concerned with the vibration of an elastic sphere, which is governed by a vector wave equation. (1.43)

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* Lamb noted this unjust neglect in the third edition of his book *Hydrodynamics* published in 1906. See a historical survey of scattering of plane waves by N. Logan (Ref. 1.41).
In terms of Cartesian components, the vector wave equation was decoupled into three scalar wave equations. The application of the boundary conditions was rather awkward because the Cartesian components of the displacement and stress were grouped together to satisfy the prescribed condition on the spherical surface. This difficulty was avoided by Jaerisch (1.44) when displacement components in spherical coordinates were used.

Being familiar with Rayleigh's earlier papers on scattering, Lamb had the intention of doing a similar analysis based upon elastic solid theory for the purpose of illustrating the selective absorption of light by gas. But he felt that the electromagnetic theory furnished in some respects a much more complete analogy with optical conditions. Therefore, the calculation was suspended and was not finished until 1900 when a companion paper based upon electromagnetic theory was also completed. In the meantime, J. J. Thomson followed Rayleigh and Lamb's early work and analyzed the scattering of electromagnetic waves by a sphere. (1.45)

In the paper on the scattering of waves in an elastic solid embedded with a spherical obstacle (1.46) Lamb considered the following cases: (1) a rigid and fixed sphere, (2) a rigid but moveable sphere, (3) a rigid spherical shell, (4) a spherical cavity, and (5) an elastic sphere. He was interested in the total energy scattered by the sphere and he calculated in detail for all cases the "dissipation ratio" which is the ratio of the energy dissipated by the scattered waves to the energy flux in the incident wave for Rayleigh scattering (long wavelength limit). Since Lamb was interested in illustrating an optical phenomenon, he considered the elastic medium incompressible. Therefore, he
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had only one wave velocity to deal with and the accompanying equation was somewhat simpler than that for a compressible solid.

In the companion paper, the electromagnetic theory was used and a vector wave equation was derived for the electric vector or magnetic vector.\(^{(1.47)}\) The Cartesian components of each vector then satisfy a scalar wave equation whose solution in terms of spherical harmonics and spherical-Bessel functions is the same as the previous paper. The magnitude of the dissipation of energy by secondary waves diverging from the sphere with constant conduction and permeability was found for the limiting case of long wavelength.

A few years later, the same problem was tackled again by Gustav Mie\(^*\) in connection with the energy scattered when light passes through a metallic suspension.\(^{(1.48)}\) He solved the Maxwell equations with the electric and magnetic field vectors being decomposed into components in spherical coordinates so the boundary conditions at the surface of the sphere could be satisfied readily. Since the sphere was assumed to be conductive, Mie's solution is more complicated than Lamb's for an elastic sphere (incompressible). Mie went further than Lamb in that he calculated the electric and magnetic fields on the surface of the metal sphere, the total energy scattered, and the angular distribution of the scattered energy intensity at large distances from the sphere. He also showed the angular variation of the energy intensity on a polar plot, a practice still being followed today.

Mie's work also signified the end of treating the diffraction of light by elastic wave theory. At the same time, the diffraction of

\(^*\)Mie's work is reproduced in Chapter 13 of Born and Wolf (Ref. 1.33).
light by an elliptical cylinder (reflector) was treated in 1908 by Bruno Sieger, based on electromagnetic theory. Sieger devoted a large part of his paper to finding solutions of the wave equation in elliptical coordinates, which are now simply expressed by symbols for Mathieu functions (see Chapter IV). The scattering of electromagnetic waves by a cylinder with a parabolic cross section was treated in 1914 by P. Epstein. Like Sieger, he had to struggle through the solutions of the wave equation in parabolic coordinates, discovering important properties of what are now known as Weber's functions (parabolic cylinder functions; see Chapter V).

With a plane wave impinging on a cylinder whose generator is at a right angle to the wave normal, the problem of diffraction of electromagnetic waves is a two-dimensional one and only one scalar wave equation is involved. Thus any solution for an acoustic wave can be translated to one for an electromagnetic wave. In this connection, Rayleigh's work in scattering of sound by a circular cylinder and Lamb's treatment on the reflection and diffraction of light by a parabolic mirror should also be mentioned. Together with Mie, Sieger, and Epstein, they pioneered the mathematical analysis of diffraction of waves by a boundary which is one of the quadratic surfaces of curvilinear coordinate systems.

Although the investigation of the scattering and diffraction of electromagnetic waves has been an active subject of research for a long time, the interest in the diffraction of waves in an elastic solid (excluding sound waves in air or liquid) subsided for some time. However,

*See Refs. 1.52 and 1.53.
before its revival at about the turn of the second half of this century, research papers on the diffraction of elastic waves appeared occasionally in the literature. In 1927, Katsutada Sezawa published the "Scattering of Elastic Waves and Some Allied Problems." (1.54)
He assumed a general isotropic, homogeneous, elastic solid which is embedded with a circular cylinder, an elliptical cylinder, or a sphere with either rigid or vacuous material. Thus two types of scattered waves (P and S waves) are involved, the same as in Clebsch's work, whereas Lamb considered only one wave for an incompressible solid.
With the addition of one wave, the complexity of the solution is more than doubled. Lacking an electronic computer, Sezawa could only leave the elegantly formulated solutions in terms of special wave functions with brief discussions. He and his associates continued to investigate related problems from time to time, which led Genroku Nishimura (1955), some 30 years afterward, to discover that diffraction of elastic waves is connected with the seemingly unrelated subject of stress concentration. (1.55)

1.5. Stress Concentrations

Compared with investigations into the origin of the diffraction of elastic waves, work on stress concentrations has had a much more modest and recent start. Stress concentration is the localized increase of stress in a structural member due to geometric discontinuities such as holes, cavities, notches, grooves, corners, sudden changes in the cross section, etc. The beginning of research on stress concentration is connected with the study of the causes of failure of railway
axles.* N. J. Macquorne Rankine (1820-1872) noted in 1843 that fracture of railway axles occurred at the reentrant angle where the journal joined the body. He proposed in manufacturing axles to form the journals with a large curve in the shoulder. In the improved form of journals, the destructive action of the vibratory movement was prevented by the continuity of form.

At about the same time railway engineers in France and Germany were considering the same problem. In 1858, A. Wöhler (1819-1914) designed and used a special machine for testing the strength of railway axles under time-varying forces (fatigue test) and found that sharp corners reduced the life of the specimens and that conditions were improved by introducing fillets.

Although the theory of elasticity was fully developed by the middle of the 19th century, it was not applied to the study of stress concentrations until some 50 years later. The first theoretical investigation was due to J. Larmor (1892) who analyzed stress concentration produced in a twisted shaft by an eccentric circular cavity. (1.58) He used the hydrodynamical analogy and concluded that the shearing stress near the circular cavity is twice as great as that at any other place in the shaft.

In 1898, G. Kirsch investigated the stress distribution around a small circular hole in a wide plate subjected to uniform longitudinal tension. By solving the two-dimensional equations of elasticity with the appropriate boundary conditions, he showed that the peak

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*The early development of the research on stress concentrations is reviewed by S. Timoshenko in Ref. 1.56. Recent research works are surveyed by H. Neuber and H. G. Hahn in Ref. 1.57.
circumferential stress at the edge of the hole is three times larger than the applied stress.\(^{(1.59)}\) It is now generally considered that Kirsch produced the first mathematical analysis of stress concentration based on the theory of elasticity. Actually, a year earlier, M. Grüber,\(^{(1.60)}\) while analyzing the stresses due to centrifugal force in a rotating disk, found that the maximum stress would be doubled if a small hole were drilled at the center.\(^*\) Other similar analyses might have appeared earlier, but none would have demonstrated the effect as well as Kirsch's.

Other investigations followed. Distributions of stresses were found in the neighborhood of an elliptic hole in a large plate (Kolosoff, 1909, Inglis, 1913), near a circular hole in a plate with reinforcement (Timoshenko, 1925), near a circular hole in a plate with finite width (Howland, 1930), near a spherical cavity in an infinite solid (Southwell, 1926), etc. In the meantime, experimental methods for determining stresses were rapidly developed, including membrane-analogy for torsion (Prandtl, 1903, Griffith and Taylor, 1918), electric current-analogy for torsion of a shaft with nonuniform cross section (Jacobsen, 1925), various mechanical extensometers and, above all, the method of photo-elasticity (Coker, 1920, Tuzi, 1929). All of these important findings were compiled for the first time in S. Timoshenko’s monograph, Theory of Elasticity in 1934.\(^+\)

\* Also in the supplement of Larmor’s paper, A. E. H. Love calculated the shear strain concentrations at the surface of a spherical cavity in an infinite solid which was subject to pure shear.

to the nominal local stress, evaluated by simple theory, is called the stress concentration factor. It is a measure of the severity of stresses concentrated in a localized region. The nominal stress may be based either on the net cross section of a structural member through the discontinuity or on the gross cross section ignoring the discontinuity, and the simple theory is usually applicable to the member as if the discontinuity were not there. Most theoretical analyses treat discontinuities in an infinitely or semi-infinitely extended member. In that case, it really makes no difference whether the nominal stress is based upon the net or gross section.

The importance of understanding stress concentration in engineering design was stressed by Timoshenko and Dietz in 1925:

It can be concluded that in many practical cases a very high stress concentration is produced by holes, grooves, and shape variations of cross sections. In the case of ductile materials, this stress concentration does not have a weakening effect under static loading. In the case of brittle materials or ductile materials under the action of stress reversal, however, the weakening effect of stress concentration may become of prime importance and it must be taken into account in consideration in actual design. (1.62)

However, this was not accepted nor appreciated then as can be seen from the following criticism which was raised by a well-known consulting engineer and professor in connection with Timoshenko and Dietz's presentation of the solution for stress concentrations in a plate with a circular hole:

In the case of a circular hole... this so-called exact solution is based on two assumptions. The first, of course, is that there is a hole. The second is that the width of the plate is infinite. The first assumption means that there is no hole at all, because a hole of finite diameter d in a plate
of infinite width is the same as a hole of no diameter in a plate of finite width \( w \) since \( \frac{d}{\infty} = \frac{0}{w} \). This one, as above stated, is founded on the assumptions that there is a hole and that there is no hole; in other words, that a thing is and is not at the same time. It gives results for the stress at the edge of the hole which are independent of the diameter. Of course, it is easy for any practical man to see that such results are absolutely worthless as applied to any practical case!

On the other hand the new knowledge of stress concentration found its application in investigating the brittle and fatigue strengths of materials. It was then known that materials always showed a strength much weaker than might be expected from molecular or atomic forces. For brittle materials, A. A. Griffith proposed in 1921 a theory of failure to explain the discrepancy between the theoretical strength and experimental value of a certain glass. (1.63) Assuming that there existed microscopic cracks or flaws in such a material, he showed that whenever the reduction of the strain energy due to an opening equaled the increase of surface energy, the crack might spread spontaneously and the strength of the glass is weakened. In calculating the reduction of strain energy, he assumed the crack was in the form of a narrow ellipse and made use of Inglis' solution for stress concentration around an elliptic hole in a plate under uniaxial tension.

As for a crystalline material, the fatigue of metal was generally recognized. This was slightly different from the early findings of fatigue failure of railroad axles. When a ductile material was subject to alternating load, the maximum stress required to cause failure was much less than the static breaking stress, even for a smooth specimen. The fatigue strength was further reduced if the specimen had a notch or groove on its surface.
From microscopic observations of crystals subject to fluctuating stresses, the fatigue cracks were found to be associated with slips of crystals along certain planes. A number of theories of fatigue were thus put forward.* H. J. Gough summarized in 1933 a theory that during cyclic stressing the repeated plastic deformation gradually decreased because of work-hardening of the material. In 1936, E. Crowan incorporated the concept of stress concentration and restated Gough's strain-hardening theory on a semi-quantitative basis. He considered a metal to consist of a number of structural inhomogeneities imbedded in an elastic matrix. The stress at the inhomogeneities would be higher than in the elastic matrix so that yield would occur first at these points. With continued stress cycling, the inhomogeneities were strain-hardened, hence the stresses on them would gradually increase. If complete strain-hardening could occur without the stress reaching the fracture point at any of these inhomogeneities, then the applied stress would be in the safe range; if not, then failure would occur.

Studies of brittle fractures and fatigue failures widened the scope of investigation of stress concentrations to include the increase of stresses owing to microscopic cracks and inhomogeneities inside a material, which are often known as stress raisers. Undoubtedly, interest in strengthening materials by eliminating stress raisers had stimulated further studies of stress concentrations.

In 1937, H. Neuber published Kerbspannungslehre (Theory of Notch Stress), the first book devoted mainly to the subject of stress

*A brief account of the history of fatigue of metals can be found in Ref. 1.64.
concentration. (1.65) It contains solutions for the stress distribution around holes in a plate under pure tension or under pure bending, stresses near hyperbolic notches in a plate, and stresses at notches in a rod. The last is a three-dimensional problem in elasticity and was solved by making use of the three-function decomposition of the elasticity equations. We note that by cutting a groove around a circular rod, not only the magnitude of the stresses is raised, but also the uniaxial state of stress in the rod due to axial tension is changed to a triaxial state of stress (near-uniform stresses in all three directions) in the notched region. The triaxial tension is one of the main factors in causing brittle fracture in an otherwise ductile material like steel.

On the theoretical side another important advancement is the application of the complex variable theory to the solution of two-dimensional problems of elasticity, first proposed by Kolosoff (1914). (1.66) By using conformal mapping, N. I. Muskhelishvili developed a practical method to find stress distribution in a plane which is weakened by any type of hole. A generalized summary of his work is contained in the monograph Some Basic Problems of the Mathematical Theory of Elasticity (in Russian), the first edition of which appeared in 1933. (1.67)

During the same period, experimental stress analysis took a giant stride—the invention of the SR-4 Bonded Wire Resistance Strain Gauge. The gauge, small as a postage stamp but large in usefulness, was invented independently in 1938 by E. E. Simmons, Jr. and Arthur Ruge (the first letters of their names are used to name the gauge, the numeral 4 representing four persons who participated in its early development). *

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*For a most interesting account of the inception and development of strain gauges, see Ref. 1.68. For the application of strain gauges and other methods of experimental stress analysis, see Ref. 1.69.
With it, measurements of a strain of the order $10^{-6}$ with a gauge length less than 1/3 inch became possible in static tests as well as in the case of dynamic loading. Up to that time most of the stress concentration factors were verified in the laboratory by the photoelasticity method which was then limited mainly to plane problems of elasticity and to static loading. The newly invented strain gauge could be used not only in a laboratory on a specimen with complicated geometry subjected to impact loading, but could also be applied in the field to measure strain in a structural member under severe service conditions.

The general progress in both theoretical analysis and experimental measurements gradually stimulated the interest of the engineering public in stress concentrations. Its understanding was spread wider by the Second World War. During the war, ships and airplanes were produced in large numbers. To speed ship construction, the riveting process was gradually replaced by welding. The first famous Liberty Ships made with extensive use of welding were placed in service near the end of 1941. By the beginning of 1943, 10 fractures in hull structures that were serious enough to endanger the vessels had been reported. In January 1943, a T-2 tanker, also mass-produced by welding, broke spectacularly in two while lying quietly at her outfitting dock. Extensive investigations on both sides of the Atlantic followed.* One source was traced to faulty design, in leaving sharp corners (square hatches in the deck) and abrupt joints (intersections of three plates at right angles) as stress raisers. Another was attributed to defective welds. Any porosity, slag inclusion, or incomplete penetration is equivalent to

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*Many investigations and research reports are summarized in Refs. 1.70 and 1.71.
introducing a sharp notch in the plates and girders, thus changing
entirely the state of stress in the hull structure.

Fatigue failures of aircraft components were recorded frequently
during the war  but nothing was comparable to the catastrophic loss
in 1954 of two Comet jet airplanes—the first commercial jet service
launched. Again, intensive investigations followed. As gathered from
the evidence of full-scale tests of a sister plane in service, the ac-
cident was caused by structural failure of the pressure cabin brought
about by fatigue.† In the test, the complete fuselage was submerged
in a water tank and alternating hydrostatic pressure was applied to
simulate the pressurization of the cabin at high altitude. Cracks
which originated from the corners of the nearly-square-shaped windows
were found after many hours of cyclic compression. Besides, the stress
level around the corners of the cabin windows was so high that even a
mild stress-raiser would initiate fatigue failure.

In the meantime, new theoretical results for stress concentration
factors had been found by many mathematicians and engineers, notably
A. E. Green, J. N. Goodier, C. B. Ling, R. D. Mindlin, M. Sadowsky,
and E. Sternberg.‡ These results combined with the photoelastic
findings contributed by M. M. Frocht, M. Hetenyi, R. D. Mindlin, and

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Fatigue failure and stress concentration in mechanical compo-
nents was briefly reviewed by S. Timoshenko in Ref. 1.72. Fatigue in
aircraft structures was the subject of several international confer-
rences. Ref. 1.73 contains reports of research and practice of various
countries.

† The final report "Comet Accident Investigation" was issued in
1954 by the Royal Aircraft Establishment at Farnborough, directed by
Sir Arnold Hall; see also Ref. 1.74. The fatigue failure of the pres-
surized cabin was reported in Ref. 1.75.

‡ Their works are cited in Ref. 1.61.
many others resulted in a large pool of information. In 1953, R. E. Peterson compiled the important results in one volume which was published with the title *Stress Concentration Design Factors*. The book includes extensive charts and graphs for stress concentrations at grooves and notches, shoulder fillets, holes in plates and shafts, and miscellaneous design elements. Eight years later, W. Kloth gathered more than 250 photographic plates in an *Atlas* showing stress trajectories and empirical values of peak stresses at connections and joints of all kinds of intricate machine and structure elements.

Two more books on stress concentrations appeared in the fifties. G. N. Savin's *Stress Concentration Around Holes* (in Russian) was published in 1951, and was translated into English in 1961 with minor revisions. It contains mainly the two-dimensional problems of elasticity (plane stress, plane strain, and pure bending) solved by the Kolosoff–Muskheilishvili method. The problems of stress concentration in torsion and those of a three-dimensional nature were left for a second volume which has not yet materialized. Instead, the author greatly expanded the original volume in the revised 1966 edition (in Russian) to include some of the problems left out previously, and added the nonlinear and inelastic effects on stress concentrations. H. Neuber revised his book in 1958. Additions were made on the effect of nonlinear and inelastic behavior on the notch stresses, with numerous graphs and charts to facilitate numerical calculations.

*The new edition also contains a chapter on "Dynamic Problems of the Stress Distribution Around Holes."*
1.6 Diffraction of Elastic Waves and Dynamic Stress Concentrations

The last chapter of Neuber's book (Theory of Notch Stress, 2d Ed.) (1.65) with the title "Final Remarks and Future Prospect" contains the following ending (from the English translation):

Thus by means of an exact theory, the first step has been made into the field of changes with respect to time of complex stress distributions under permanent stress. The present author believes that this fact can become significant for a further development of research in strength theory within the next years and raises great hopes for the solution of many unsolved problems of stress concentration variable with respect to time (Probleme der zeitlich veränderlichen Spannungskonzentration) which remained unsolved so far.

In 1955, in a not-so-well-circulated journal, G. Nishimura and Y. Jimbo wrote the article, "A Dynamic Problem of Stress Concentration--Stresses in the Vicinity of a Spherical Matter included in an Elastic Solid under Dynamic Force." (1.55) They followed exactly the earlier work of Sezawa, assuming an incident wave varying harmonically in time in an infinite elastic space embedded with a spherical obstacle, rigid, vacuous, or elastic. The incident wave is scattered by the sphere, and stresses and displacements everywhere are determined from the total wave outside the sphere. Without using an electronic computer, they were able to determine the stresses as a function of the incident wavelength, with the case of infinite wave length (zero frequency) being the one for static loading. The dynamic stress concentration factor is defined as the ratio of stress due to the total wave at a point to the stress due to the incident wave (without the obstacle) at the same point. They showed that at certain wave frequencies, the dynamic stress concentration factors are larger than the static ones.
About the same time, the subject of scattering of elastic waves in solids was revived in the fields of acoustics and geophysics for the purpose of studying energy losses as sound waves pass through obstacles in an elastic matrix. In 1956, Ying and Truett investigated the scattering of plane waves by a spherical obstacle, (1.79) and later, White (1958) studied the scattering at a cylindrical discontinuity with experimental observations. (1.80) Both were dealing with waves varying harmonically in time. The problem of a spherical obstacle has also been treated by Takeuchi (1.81) and by Knopoff (1.82). If the incident wave were a pulse, the analysis becomes more complicated. In 1959, Gilbert and Knopoff provided an asymptotic solution (early time) for the diffraction of a step pulse by a circular cylinder. (1.83)

Once the true cause of stress concentration was identified with the diffraction of elastic waves, this knowledge was immediately applied to new findings. Interestingly, experimental investigation of dynamic stress concentration was initiated independent of theoretical analysis at about the same time. Using the method of dynamic photelasticity, Wells and Post (1958) investigated the dynamic stress distribution surrounding a running crack, (1.84) and Durelli and Dally (1969) measured the dynamic stress concentration factor at a central circular hole in a plate subjected to axial impact. (1.85) Research activity has greatly accelerated in the sixties, resulting inevitably in a flourish of publications, most of which will be reviewed in the subsequent chapters.
2. ELEMENTS OF THE THEORY OF ELASTICITY

AS MENTIONED IN THE HISTORICAL INTRODUCTION, the theory of elasticity had been well developed by the middle of the 19th century. About that time, several treatises on the subject appeared in different languages. In English, the first edition of Mathematical Theory of Elasticity by A. E. H. Love was published in 1892. The book was revised three times and the last edition appeared in 1927. It has since been reprinted many times and is still used widely. Many textbooks on the same subject are also available, of which we mention the one by Timoshenko and Goodier (1969), (2.1) by Sokolnikoff (1956), (2.2) and by Green and Zerna (1968). (2.3) All the books cited above contain excellent accounts of the general theory of elasticity: there is no need to describe it here.

In this section and the next, only the parts of the general theory, and equations which are basic to investigating the scattering of elastic waves, will be presented, and only briefly.

2.1. Deformation, Displacements, and Strains

A continuous body subjected to the action of external forces occupies different regions in space from time to time. Let the regions be referred to a Cartesian coordinates system, fixed in space. Every particle of the body is identified initially by the coordinates $\alpha_i (i = 1, 2, 3)$. If the particle is carried to a new position with coordinate $z_i (i = 1, 2, 3)$, the transformation

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*Thomson (Lord Kelvin) and Tait's Treatise on Natural Philosophy appeared earlier (in 1879) but it was much broader in scope and covered rigid body statics and dynamics, elasticity, and hydrodynamics. W. J. Ibbsenton's Mathematical Theory of Perfectly Elastic Solids (in 1887) was much less in contents.
\[ x_i = x_i(a_1, a_2, a_3, t), \quad i = 1, 2, 3, \quad (2.1a) \]

where \( t \) is considered as a parameter, is called the deformation. We assume that the inverse transformation

\[ a_i = a_i(x_1, x_2, x_3, t), \quad i = 1, 2, 3 \quad (2.1b) \]

uniquely exists. The directed line segment which connects these two positions of the same particle is called the displacement, a vector quantity with components \( u_i \). As can be seen from Fig. 2.1,

\[ u_i = x_i - a_i \quad (2.2) \]

By applying Eq. (2.1), \( u_i \) can be expressed as a function either of the material coordinates \( a_i \), or of the spatial coordinates \( x_i \).

![Fig. 2.1. Geometry of Deformation](image)

The change in length and in relative direction between two adjacent particles accompanying the deformation is called strain. To study strains, it is necessary to consider the local behavior of the
deformation. We define

\[
\frac{dx_i}{da_j} = \frac{\partial x_i}{\partial a_j} \, da_j; \quad \frac{da_i}{dx_j} = \frac{\partial a_i}{\partial x_j} \, dx_j.
\] (2.3)

(The summation convention over repeated indices is adopted throughout.)

By using Eq. (2.2), the coefficients of differentials in (2.3), known as deformation gradients, can be expressed as

\[
\frac{\partial x_i}{\partial a_j} = \delta_{ij} \frac{\partial u_i}{\partial a_j} \quad \frac{\partial a_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j},
\] (2.4)

where \( \delta_{ij} \) is the Kronecker delta which has the value one when \( i = j \), and zero when \( i \neq j \). The quantities \( \partial u_i / \partial a_j \) and \( \partial a_i / \partial x_j \) are the displacement gradients referred to the material coordinates and spatial coordinates respectively. We note that the part \( \delta_{ij} \) represents a translation which shifts \( dx_i \) to \( da_j \) or vice versa.

A body is rigid if the distance between any two particles remains unchanged, i.e.,

\[
dx_i \, dx_i = da_i \, da_i.
\]

The displacement for a rigid body is composed of a translation of a particle of the body plus a rotation of the body about that particle.

The body is said to be strained if the relative positions of any two particles of the body change. A strained body is elastic if, after the external agents that induce the strain have been removed, the body recovers its original shape and size. In other words, the region \( \nu \) occupied by the deformed body (Fig. 2.1) can be brought to coincide with the region \( \nu \) through a rigid-body displacement after
the removal of external forces. Elasticity theory is concerned with the deformation of an elastic body.

In the linear theory of elasticity, it is assumed that the displacement gradients are small, so that the product

\[ \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \]

and

\[ \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \]

can be neglected in comparison with \( \frac{\partial u_i}{\partial x_j} \) and \( \frac{\partial u_i}{\partial x_j} \) respectively. Furthermore, the product

\[ \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_k} \]

is also neglected. Thus

\[ \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_k} \frac{\partial u_k}{\partial x_j} = \frac{\partial u_i}{\partial x_j} \left( \delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \]

which implies that the material and spatial displacement gradients approximate each other. This enables us to establish the kinetic equations for a strained elastic body with reference to either the deformed or undeformed region. Henceforth, we shall use spatial description.

Equations (2.3) and (2.4) show that the local behavior of deformation is characterized by a translation and the displacement gradient \( \frac{\partial u_i}{\partial x_j} \). We decompose the latter into the symmetric and antisymmetric parts:

\[ \frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij}, \]
\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.5) \]

\[ \omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (2.6) \]

The symmetric part \( \varepsilon_{ij} \) is the small strain tensor. It characterizes the change in length and relative position of line elements. The antisymmetric part \( \omega_{ij} \) is the small rotation tensor. It represents the average rotation of two line elements initially at right angles to each other. There are only three nonvanishing components of \( \omega_{ij} \), which can also be derived from the curl of the vector \( \mathbf{u} \) by \( \omega = -\frac{\partial u}{\partial x} \times \mathbf{u} \), or from

\[ \omega_i = \sum_{j} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \quad (2.7) \]

(\( \varepsilon_{ijk} \) is the permutation symbol which has the value +1 when the indices are in cyclic order (123, 231, 312); -1 when in acyclic order (321, 213, 132); and zero otherwise.) The components of the antisymmetric tensor \( \omega_{ij} \) in (2.6) and those of an axial vector \( \omega_i \) in (2.7) are connected by the relation

\[ \omega_{12} = \omega_3, \quad \omega_{23} = \omega_1, \quad \omega_{31} = \omega_2. \]

The three components \( \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33} \) of the small strain tensor are the normal strains, whereas the remaining elements \( \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23} \) are the shear strains. The sum of the three normal strains, which is also the first scalar invariant of the strain tensor, is called cubical dilatation or simply dilatation.
\[ \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{\partial u_k}{\partial x_k}. \] (2.8)

It equals approximately the ratio of the increment of volume to the original volume of a cubical element.

To assure the single-valueness of displacement upon the integration of Eq. (2.5), the six strain components must satisfy the compatibility conditions

\[ \varepsilon_{i,j,k,l} + \varepsilon_{k,l,i,j} - \varepsilon_{i,k,j,l} - \varepsilon_{j,l,i,k} = 0. \] (2.9)

In problems of elastodynamics, solutions are usually found directly in terms of displacements (see subsection 2.4). Thus, if the displacements are continuous and single-valued functions, the strains derived from Eq. (2.5) are always compatible.

2.2. Forces, Tractions, and Stresses

Deformation of an elastic body has among its causes the action of a force, a vector quantity. Forces that act throughout the body and are generally proportional to the masses of the particles are called body forces. Those that act over a surface of the body are surface forces.

If a deformed body is divided into two parts by a fictitious plane, the action of one part of the body on the other can be represented by a force and a couple. Since the action varies from point to point on the plane, we introduce the surface traction and the surface couple to represent the action at a point on the plane. The integral of the surface traction over the entire plane equals the total force. Surface traction, a vector, will be denoted by \( T^{(n)} \), where \( n \)
is a unit vector normal to the plane. The total couple on the plane then equals an integration of the surface couple over the entire area. In the classical theory of elasticity, the surface couple is assumed to be of negligible importance. When two bodies are mutually in contact, the action of one body on the other through the contact surface can also be represented by surface traction \( \mathbf{t}^{(n)} \).

The traction not only changes from point to point, it also depends on the orientation of the plane through a given point. It can be shown that at a point, if the tractions acting on three mutually perpendicular planes \( (\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \text{ and } \mathbf{t}^{(3)}) \) are known, the traction over any other plane with direction normal \( \mathbf{n} \) can be calculated by

\[
\mathbf{t}^{(n)} = n_1 \mathbf{t}^{(1)} + n_2 \mathbf{t}^{(2)} + n_3 \mathbf{t}^{(3)}.
\]

The \( n_i \) are the components of \( \mathbf{n} \) along the coordinate axes \( (x_1, x_2, x_3) \) formed by the three planes. Since each vector may be resolved into components along the three axes, the components of the traction on a plane with normal \( \mathbf{n} \) are given by

\[
T_{ij}^{(n)} = n_1 T_{ij}^{(1)} + n_2 T_{ij}^{(2)} + n_3 T_{ij}^{(3)} = n_i T_{ij}^{(i)}, \quad i, j = 1, 2, 3.
\]

When expressed in terms of a new symbol \( \sigma_{ij} \)

\[
T_{ij}^{(i)} = \sigma_{ij}, \quad i, j = 1, 2, 3,
\]

we have

\[
T_{ij}^{(n)} = n_i \sigma_{ij}.
\]

The nine components \( \sigma_{ij} \) of tractions at three surfaces form a stress...
tensor which specifies completely the force action at a point in a deformed body.

The components $\sigma_{11}$, $\sigma_{22}$, $\sigma_{33}$ are normal stresses. Each is the normal component of the traction acting on the surface perpendicular to $x_i$-axis. The remaining six tangential components are shear stresses.

2.3. Equations of Motion and Constitutive Equations

The equations of motion for an elastic body are derived from the principles of balance of momentum, balance of angular momentum, and balance of energy. Consider an infinitesimal element of an elastic body with mass density $\rho$ being subjected to a body force per unit mass $f$. From the balance of momentum is deduced the equation of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j = \rho \frac{\partial^2 u_i}{\partial t^2}. \quad (2.12)$$

The balance of angular momentum leads to the result

$$\epsilon_{ijk} \sigma_{jk} = 0,$$

which implies that

$$\sigma_{ij} = \sigma_{ji} \quad (2.13)$$

or that the stress tensor is symmetric.

The principle of balance of energy states that the rate of change of energy equals the rate of work done by the external forces acting in the elastic body. Let $\mathcal{W}$ be the internal energy per unit of volume of a strained body. By applying this principle it can be shown that
\[ \frac{dW}{dt} = \sigma_{ij} \frac{d\epsilon_{ij}}{dt}. \]  

(2.14)

If in addition, the internal energy density \( W \) is a function of nine strain components,

\[ W = \tilde{W}(\epsilon_{ij}), \]  

(2.15)

it then follows from the equation of balance of energy (2.14) that

\[ \sigma_{ij} = \frac{\partial \tilde{W}(\epsilon_{ij})}{\partial \epsilon_{ij}}. \]

Since both strain and stress tensors are symmetric, we write

\[ \sigma_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{W}}{\partial \epsilon_{ij}} + \frac{\partial \tilde{W}}{\partial \epsilon_{ji}} \right). \]  

(2.16)

Equation (2.15) defines the property of the material and it is known as the constitutive equation for an elastic medium. The linear theory further assumes that \( W \) is a quadratic function of \( \epsilon_{ij} \)

\[ \tilde{W} = \frac{1}{2} c_{ijkl} \epsilon_{ij} \epsilon_{kl}, \]  

(2.17)

where \( c_{ijkl} \) are constant coefficients if the material is homogeneous. Substitution of (2.17) in (2.16) results in a stress-strain relation

\[ \sigma_{ij} = c_{ijkl} \epsilon_{ij} \epsilon_{kl}, \]  

(2.18)

which is known as the generalized Hooke's law for an anisotropic, homogeneous elastic material. Because of the assumed quadratic form (2.17) and the symmetry of \( \sigma_{ij} \) and \( \epsilon_{ij} \), the elastic constants \( c_{ijkl} \) possess the following symmetry.
\[ c_{ijkl} = c_{klij} = c_{ijkl} = c_{ijkl} \]  \hspace{1cm} (2.19)

Thus out of 81 components of \( c_{ijkl} \), only 21 are independent.

If the material is also isotropic

\[ c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) , \]

where \( \lambda \) and \( \mu \) are known as Lamé's constants and the stress-strain relation assumes the simple form

\[ \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2 \mu \epsilon_{ij} \] \hspace{1cm} (2.20)

Instead of \( \lambda \) and \( \mu \), other constants are also used to represent the elastic property of an isotropic, homogeneous material. The common ones are Poisson's ratio \( \nu \), Young's modulus \( E \), and bulk modulus \( k \).

Any one of these constants can be expressed in terms of the other two as shown in Table 2.1. Since \( \nu \) relates the shearing stresses \( \sigma_{ij}(i \neq j) \) to shearing strains, it is also called shear modulus.

2.4. Boundary Value Problems of Elasticity

In the previous subsections, we have shown that the components of stress tensor \( \sigma_{ij} \), strain tensor \( \epsilon_{ij} \), and displacement vector \( u_j \), a total of 15 unknown quantities, are related by the following 15 equations:

\[ \begin{align*}
\frac{\partial \sigma_{ij}}{\partial x_k} + \rho f_{ij} &= \rho \frac{\partial^2 u_j}{\partial t^2}, \\
\sigma_{ij} &= c_{ijkl} \epsilon_{kl} ,
\end{align*} \hspace{1cm} (2.18) \hspace{1cm} (2.12) \]
Table 2.1

EQUIVALENCE OF ELASTIC CONSTANTS

<table>
<thead>
<tr>
<th>Constants</th>
<th>(\lambda, \mu)</th>
<th>(\mu, \nu)</th>
<th>(\nu, \lambda)</th>
<th>(\nu, E)</th>
<th>(E, \nu)</th>
<th>(E, \kappa)</th>
<th>(\nu, \kappa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>(\frac{\lambda(1 - 2\nu)}{2\nu})</td>
<td>(\frac{\mu(E - 2\nu)}{3\mu - E})</td>
<td>(\frac{3k - 2\nu}{3\kappa + \mu})</td>
<td>(\frac{2\nu(1 + \nu)}{3(1 - 2\nu)})</td>
<td>(\frac{E}{2(1 + \nu)})</td>
<td>(\frac{3kE}{9k - E})</td>
<td>(\frac{3k(1 - 2\nu)}{2(1 + \nu)})</td>
</tr>
<tr>
<td>(\mu)</td>
<td>(\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu})</td>
<td>(\frac{\lambda(1 + \nu)(1 - 2\nu)}{\nu})</td>
<td>(\frac{9k(k - \lambda)}{3k - \lambda})</td>
<td>(\frac{2\nu(1 + \nu)}{3(1 - 2\nu)})</td>
<td>(\frac{E}{2(1 + \nu)})</td>
<td>(\frac{3kE}{9k - E})</td>
<td>(\frac{3k(1 - 2\nu)}{2(1 + \nu)})</td>
</tr>
<tr>
<td>(E)</td>
<td>(\frac{\lambda}{2(\lambda + \mu)})</td>
<td>(\frac{\lambda(1 + \nu)}{3\nu})</td>
<td>(\frac{\lambda}{3k - \lambda})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\kappa)</td>
<td>(\frac{\lambda + 2\nu}{3})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ \varepsilon_{i,j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(2.5)

The body force \( f_{i,j} \), density \( \rho \), and elastic constants \( c_{ijkl} \) are assumed to be given. Elimination of \( \varepsilon_{i,j} \) and then \( \sigma_{i,j} \) from the above equations leads to

\[ \sigma_{i,j} = c_{ijkl} \frac{\partial u_k}{\partial x_l}, \]  

(2.21)

and finally,

\[ c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + \rho f_{i,j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \]  

(2.22)

This is the equation of motion in terms of the displacement vector for a homogeneous elastic body.

For an isotropic material, \( c_{ijkl} \) is replaced by two constants, \( \lambda, \mu \), as in (2.20). From the displacement components, stresses can be computed directly by

\[ \sigma_{i,j} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{i,j} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \]  

(2.23)

and the equation of motion is

\[ (\lambda + \mu) \frac{\partial^2 u_i}{\partial x_i \partial t^2} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_i} + \rho f_{i,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \]  

(2.24)

The above equations are expressed in terms of Cartesian components of vectors \( u_i \) and \( f_i \), and the stress tensor \( \sigma_{ijkl} \). They can also be expressed in terms of the vectors and the tensor themselves as
\[ g = \lambda (v \cdot u) I + \nu (v u + u v), \quad (2.25) \]

\[ (\lambda + \nu) vv \cdot u + \nu v^2 u + \rho f = \rho \ddot{u}. \quad (2.26) \]

Here dots over a quantity mean the partial derivative with respect to
time, and the dyadic notation is used. (See Chapter 1 of Ref. 2.4)
Thus \( v \) is the vector differential operator and \( I \) is the unitary dyadic.

Over the surface, \( s \), of a body (Fig. 2.1) with volume, \( v \), the
action of external sources is described in terms of forces or geometri-
cal constraints, or more precisely, by means of prescribing tractions
\( t^{(n)} \) or displacements \( u \). Thus the determination of the deformation of
an elastic body can be formulated as a boundary value problem.

For a homogeneous, isotropic elastic body with volume \( v \) and sur-
faced \( s = s_1 + s_2 \), given body force \( \rho f \) in \( v \), surface traction \( \mathbf{T}^{(n)} \) over
\( s_1 \) with normal \( n \) and displacement \( \mathbf{w} \) over \( s_2 \), determine the displacement
\( u(x, t) \) in \( v \) that satisfies the equations

\[ (\lambda + \nu) vv \cdot u + \nu v^2 u + \rho f = \rho \ddot{u} \quad \text{in } v \quad (2.26) \]

with

\[ \mathbf{T}^{(n)} = [\lambda (v \cdot u) I + \nu (v u + u v)] \cdot n = \mathbf{T}^{(n)} \quad \text{on } s_1 \quad (2.27a) \]

\[ u(x, t) = \mathbf{w} \quad \text{on } s_2. \quad (2.27b) \]

In addition, since the displacements are reckoned from some conven-
tiently chosen time which is taken as \( t = 0 \), the displacement \( u_0 \) and
velocity \( \dot{u}_0 \) at time \( t = 0 \), known as the initial conditions, must also
be prescribed. Thus to the above equations, we add
\( u(x, 0) = u_0 \)
\( \dot{u}(x, 0) = \dot{u}_0 \)

This completes the mathematical formulation of the dynamic problems of elasticity.

Even if a solution is found which satisfies Eqs. (2.26), (2.27), and (2.28), whether it is the one and only solution can be answered by the uniqueness theorem. Dispensing the complete theorem and proof, we merely note the sufficient conditions for a uniqueness theorem (Chapter 7 of Ref. 2.5):

(a) Specification of the initial displacement and velocity throughout the body;

(b) Specification, at each and every point of the surface of any one of the eight combinations formed by choosing one member of each of the three products:

\[ \sigma_{nn} u_n, \quad \sigma_{ns} u_s, \quad \sigma_{nt} u_t, \]

where \( n, s, \) and \( t \) indicate three mutually perpendicular directions and \( n \) is normal to the surface.

With the above conditions satisfied, a solution found for the boundary value problem which is formulated by equations (2.26) through (2.28) is ensured to be unique. The above theorem also suggests that boundary conditions other than (2.27) can be used. For instance, along a surface normal to the coordinate axis \( x_1 \), we can specify \( u_1 = \sigma_{12} = \sigma_{13} = 0 \), a condition sometimes called the rigid-smooth boundary. Such a boundary condition seems to be further removed from reality than the
condition for a rigid surface (2.27b) or a traction-free (stress-free) surface (2.27a). However, they are encountered occasionally in practice when one is not sure of the real physical bounding conditions at the surface.

2.5. Reduction to Wave Equations

In the absence of body force, the displacement equation of motion is

\[(\lambda + \mu)\nabla \cdot \mathbf{u} + \nu \nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}}.\]  \hspace{1cm} (2.29)

According to the Helmholtz theorem, any vector field can be expressed as the sum of the gradient of a scalar field \(\varphi\) and the curl of a vector field \(\psi\):

\[\mathbf{u} = \nabla \varphi + \nabla \times \psi, \quad \nabla \cdot \psi = 0.\]  \hspace{1cm} (2.30)

We call the \(\varphi\) and \(\psi\) the scalar and vector displacement potentials respectively. Substitution of the above to (2.29) leads to

\[\nabla [(\lambda + 2\mu)\nabla^2 \varphi + \rho \ddot{\varphi}] + \nabla \times [\nu \nabla^2 \psi - \rho \dot{\psi}] = 0,\]

which is satisfied if

\[\sigma_p^2 \nabla^2 \varphi = \ddot{\varphi}, \quad \sigma_p^2 = (\lambda + 2\mu)/\rho,\]

\[\sigma_s^2 \nabla^2 \psi = \ddot{\psi}, \quad \sigma_s^2 = \nu/\rho.\]  \hspace{1cm} (2.31)

It is seen that \(\varphi\) and \(\psi\) satisfy a scalar and a vector wave equation respectively.

Since the wave equations are much simpler than the original
equations of motion, solutions for \( u \) will be constructed from (2.30) in which the potentials satisfy the wave equations (2.31) and the boundary and initial conditions. A question arises as to whether every solution of (2.31) is included in the above procedure. This is answered by the completeness theorem (Ref. 2.5) stating that every solution of (2.29) admits a decomposition of (2.30) with \( \varphi \) and \( \psi \) satisfying the equations (2.31).

The completeness theorem also assures that there are only two types of waves in an elastic solid, the one given by \( \varphi \) propagating with wave speed \( c_p \), and the other given by \( \psi \), which propagates with a speed \( c_s \). Since \( \lambda > 0 \) and \( \mu > 0 \), \( c_p \) is always larger than \( c_s \).

The ratio of these two wave speeds is a function of Poisson's ratio, \( \nu \), of the material only. Because of its frequent occurrence in elastodynamics, we denote it by \( \kappa \) with

\[
\kappa = \frac{c_p}{c_s} = \left( \frac{\lambda + 2\mu}{\mu} \right)^{\frac{1}{2}} = \left( \frac{2 - 2\nu}{1 - 2\nu} \right)^{\frac{1}{2}}. \tag{2.32}
\]

The constant \( \kappa \) may be treated as another material constant in lieu of \( \nu \).

Because of its faster speed, the wave arising from \( \varphi \) is called the primary wave (\( P \)-wave). The one from \( \varphi \times \psi \) is called the secondary wave (\( S \)-wave). Since each type of wave can be identified by other physical characteristics, primary waves are known also as dilatational, irrotational, compressional, longitudinal waves, etc. The corresponding names for secondary waves are rotational, equivoluminal, shear, transverse waves, etc. In the ensuing discussion, the two types of wave will be identified simply as \( P \)-wave (Pressure or Primary wave) and \( S \)-wave (Shear or Secondary wave).
2.6. Plane Waves

In a solid, plane waves are represented by

\[ u_i = A_i f(\sigma_k v^k - ct), \]

or

\[ \psi = A_f (r \cdot v - ct), \]

where \( A \) is the amplitude; \( f \) an arbitrary function; \( v \), the wave normal, which indicates the direction of wave propagation; and \( c \) is the wave speed. Not every \( c \) and every amplitude vector \( A \) are feasible solutions of (2.29). Substitution shows that only when

\[ (\lambda + \mu)(A \cdot v)v + (u - \rho c^2)A = 0 \]

does the form given by (2.33) represent plane waves in elastic solids. The equation directly above is satisfied if

1. \[ A = |A|v \quad \text{and} \quad c^2 = (\lambda + 2\mu)/\rho = c_p^2, \]

2. \[ A \cdot v = 0 \quad \text{and} \quad c^2 = u/\rho = c_s^2. \]

The first is a plane P-wave with the displacement vector parallel to the wave normal; the second is a plane S-wave where the displacement vector is always at right angles to the wave normal.

Instead of (2.33) and the condition (2.34), we can also represent plane elastic waves in terms of displacement potentials. A P-wave is given by

\[ \psi = \psi_o f(\sigma^i v_i - c_p t), \]

\[ \psi = 0, \]
with corresponding displacements

$$u_p = \nabla \varphi = \varphi_0 \nu \varphi' (r \cdot \nu - c_p t),$$

(2.36)

where prime indicates differentiation with respect to the argument.

Shear waves can be represented by

$$\varphi = 0,$$

$$\psi = \psi_0 \nu \varphi' (r \cdot \nu - c_s t),$$

where \( \psi_0 \) is any constant vector perpendicular to \( \nu \). Let \( p \), a unit vector, indicate the polarization of a shear wave. We can write

$$\psi = \psi_0 (p \times \nu) g (r \cdot \nu - c_s t),$$

and represent the shear wave by

$$\varphi = 0,$$

(2.37)

$$\psi = \psi_0 (p \times \nu) g (r \cdot \nu - c_s t),$$

where \( \psi_0 \) is a scalar coefficient. The potentials in (2.37) yield the displacement vector

$$u_s = \nabla \times \psi = \psi_0 p \varphi' (r \cdot \nu - c_s t), \quad p \cdot \nu = 0.$$

(2.38)

There is no ambiguity about the polarization of a plane P-wave, as the displacement vector \( u_p \) is always parallel to the wave normal \( \nu \). However, unless \( p \) in (2.38) is specified, the polarization of a shear wave is uncertain because \( u_s \) may be along any one of the infinite unit vectors normal to \( \nu \) (Fig. 2.2). As a matter of convenience, we choose, in a medium, a straight line as the vertical axis (\( x_2 \)-axis in Fig. 2.2) and refer to any plane perpendicular to the straight line as the horizontal plane. Shear waves with polarization parallel to
the horizontal $x_3$-axis are called SH waves; the ones with polarization parallel to a vertical plane ($x_1$-$x_2$ plane) are SV waves. This reference system is very convenient in dealing with seismic waves as the ground surface provides a natural horizontal base. For other problems, we can judiciously choose a particular direction as the vertical axis, and the rest of the axes follow.

The functions $f$ and $g$ in the above are arbitrary functions, including the generalized functions. Familiar examples are the unit step function

$$h(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi > 0, \end{cases}$$

and the delta function

$$\delta(\xi) = h'(\xi).$$
If in (2.36), \( f'(t) = h(t) \) and

\[
\mathbf{u}_p = \varphi_o \psi(h(c_p t - r \cdot v)),
\]

(2.39) represents a step plane P-wave. At a given position \( r = r_o \), there is no disturbance for \( t < (r_o \cdot v/c_p) \). At time \( t = r_o \cdot v/c_p \), the displacement suddenly jumps to the magnitude \( \varphi_o \) and remains constant thereafter. If \( f'(t) \) in (2.36) is a delta function, the displacement at the station \( r_o \) increases suddenly to a large value at \( t = r_o \cdot v/c_p \) and dies out as the pulse passes.

Another function of special interest is the harmonic function \( \exp(it\omega t) \) where \( \omega \) is the circular frequency (radians per unit time). It is understood that whenever such a function is used, only the real or the imaginary part represents the motion. Thus, if in (2.35) we let

\[
\varphi = \varphi_o e^{i k (r \cdot v - c_p t)} = \varphi_o e^{i k \Theta},
\]

(2.40)

\[
\psi = 0,
\]

\[
k_p = \omega, \quad k_v = k, \quad \Theta = r \cdot v - c_p t,
\]

the displacement has the form

\[
\mathbf{u}_p = i k \varphi_o v e^{i k \Theta},
\]

(2.41)

where the coefficient \( \varphi_o = a + ib \) is taken as a complex number. The actual motion is represented by either the real part

\[
\mathbf{u}_p = -v \psi (a \sin k \Theta + b \cos k \Theta) = -v k \sqrt{a^2 + b^2} \cos (k \Theta - \theta_1)
\]

(2.42a)
or the imaginary part

\[ u_p = \nu k (a \cos k \theta - b \sin k \theta) = \nu \sqrt{a^2 + b^2} \cos (k \theta + \epsilon_2) \]  \hspace{1cm} (2.42b)

with \( \epsilon_1 = \tan^{-1}(a/b) \) and \( \epsilon_2 = \tan^{-1}(b/a) \). Since

\[ k \theta = k r \cdot \frac{v}{\lambda} - \omega t = k \cdot \frac{r}{c_p} - \omega t, \]

the motion is simple harmonic in time and sinusoidal in space. We call \( \theta \) the phase of the wave, \( k \) the vector wave number, and \( \epsilon \) the phase constant. The wave number is related to the wave length \( \lambda \) by \( k = 2\pi/\lambda \) and to the circular frequency \( \omega \) by \( k = \omega/c_p \) for the P-wave.

As can be seen from (2.41) and (2.42) we could use the sine and cosine functions directly, in lieu of the exponential form, to represent the waves.

Wave propagations that are represented by harmonic function

\[ \exp [i(k \cdot r - \omega t)](-\infty < t < \infty) \] as in (2.41) are designated as the steady state. The motion continues, and no time can be reckoned for when the motion started, nor for when it is to end. If the body is at rest before a certain time \( t = t_0 \) and the motion is initiated at \( t = t_0 \), the waves are in the transient state. Wave motion represented by (2.39) is clearly a transient one; so is the motion

\[ u_p = \begin{cases} v_p e^{i(k \cdot r - \omega t)}, & t > |r|/c_p, \\ 0, & t < |r|/c_p. \end{cases} \]

The latter, although a sinusoidal motion, does not start at a given position until time \( t = t_0 = |r|/c_p \).

In Section 4, we shall discuss these two types of motion in detail.
It is sufficient to note here that they are related through the principle of superposition. Knowing the steady state wave propagation, the transient phenomena can be determined by superposing steady waves of all frequencies.

2.7. Equations in Orthogonal Curvilinear Coordinates

From Eqs. (2.30) and (2.31) it is seen that the displacement field of an elastic solid in motion can be represented by a scalar potential $\Phi$ which satisfies a scalar wave equation, and by a vector potential $\mathbf{W}$ satisfying a vector wave equation. The vector wave equation is actually a composition of three equations for the three components of $\mathbf{W}$. Except in Cartesian coordinates, all three components may occur in all three equations when they are expressed in terms of orthogonal curvilinear coordinates. The coupling of all three unknown components of $\mathbf{W}$ in three equations causes immense difficulties in tracking the solution of the vector wave equation.

Even if the solutions of the simultaneous equations are obtainable, there is still the difficulty of meeting the boundary conditions as the components of the potential $\mathbf{W}$ must be combined with $\Phi$ to yield displacements and stresses which are the usual quantities specified on the boundary of an elastic solid. If, however, the boundary is one of the orthogonal curvilinear coordinate surfaces, some simplification is possible. A general approach is to decompose a vector wave field into three parts, with each part determinable by a scalar wave function. (See Chapter 13 of Ref. 2.3.)

Let $\xi_1$, $\xi_2$, and $\xi_3$ be the three orthogonal curvilinear coordinates,
with scale factors \( h_1, h_2, \) and \( h_3 \) defined by

\[
d s^2 = \sum_{i=1}^{3} (d x_i) \, ^2 = \sum_{i=1}^{3} (h_i \, d \xi_i) \, ^2,
\]

where \( x_i \) are the Cartesian coordinates and \( ds \) is the length of a line element. When the transformation of the coordinates is specified by

\[
x_i = x_i(\xi_1, \xi_2, \xi_3), \quad i = 1, 2, 3,
\]

or its inverse

\[
\xi_i = \xi_i(x_1, x_2, x_3),
\]

the scale factors can be computed by

\[
h_{ij} = \left( \frac{\partial x_i}{\partial \xi_j} \right)^2 + \left( \frac{\partial x_j}{\partial \xi_i} \right)^2 = \left[ \left( \frac{\partial x_i}{\partial \xi_1} \right)^2 + \left( \frac{\partial x_j}{\partial \xi_2} \right)^2 + \left( \frac{\partial x_j}{\partial \xi_3} \right)^2 \right]^{-1}.
\]

The direction of each coordinate curve is indicated by a unit vector \( \mathbf{e}_i (i = 1, 2, 3) \). For scattering problems, one or several of the coordinate surfaces may form the boundary of a scatterer (Fig. 2.3).

The dilatational part of the displacement \( \mathbf{u} \) is given by the gradient of a scalar potential \( \varphi \) with

\[
\mathbf{u} \text{ (dilatational)} = \mathbf{L} = \nabla \varphi
\]

and the rotational part by \( \nabla \times \varphi \) with \( \nabla \cdot \varphi = 0 \). On account of the condition of zero divergence, only two of the three components of \( \varphi \) are independent. Since \( \varphi \) gives rise to a shear wave, the resolution of a plane shear wave into SV-wave and SH-wave suggests that \( \varphi \) may be decomposed into two parts: One is a vector along a preferred direction,
Fig. 2.3. Curvilinear Coordinate System \((\xi_1, \xi_2, \xi_3)\)

say \(e_3\), and the other is at right angles to the first vector. Thus we set

\[
\gamma = e_3(\omega \psi) + \xi \nabla \times (e_3 \omega \chi),
\]

where \(\psi(\xi_1, \xi_2, \xi_3, t)\) and \(\chi(\xi_1, \xi_2, \xi_3, t)\) are two unspecified scalar functions, \(\omega(\xi_3)\) is a function of coordinate \(\xi_3\), and \(\xi\) is a scalar factor having the dimension length. The factor \(\xi\) is introduced to give \(\chi\) the same dimension as \(\psi\).

The two displacements corresponding to \(\psi\) and \(\chi\) are represented by \(M\) and \(N\) respectively, with

\[
\psi_{(\text{rotational})} = M + N
\]

\[
= \nabla \times \gamma
\]

\[
= \nabla \times (e_3 \omega \psi) + \xi \nabla \times \nabla \times (e_3 \omega \chi).
\]

It is clear that the first component \(M = \nabla \times (e_3 \omega \psi) = \nabla(\omega \psi) \times e_3\) is
perpendicular to \(e_3\) and is tangent to the surface \(\xi_3 = \text{constant}\). For the case of plane wave propagation Cartesian coordinates are used. If \(e_3\) is taken in the direction of wave normal, the coordinate surfaces \(\xi_3 = \text{constant}\) are then the wave fronts, and \(\psi\) and \(x\) give rise to the plane SV- and SH-wave respectively. (See Chapter II, Section 2.) However, when the curvilinear coordinates are used for the general wave propagation, the coordinate surfaces are not necessarily the wave fronts. Thus the shear wave displacement may or may not be tangent to the wave surfaces.

As a displacement potential each part of \(\psi\) must be a solution of the vector wave equation \((2.31)\). In other words, \(\psi\), \(x\), and \(\omega\) should be chosen so as to satisfy the following equations:

\[
\sigma_s^2 \nabla^2 (e_3 \psi) = e_3 \omega \psi, \\
\sigma_s^2 \nabla^2 (\nabla \times (e_3 \omega x)) = \nabla \times (e_3 \omega x).
\]  
(2.45)

The restriction on the choice of \(\psi\) could be somewhat relaxed if we recall that in the derivation of \((2.31)\), the condition that \(\psi\) must meet is actually

\[
\nabla \times [\mu \nabla \psi - p \psi] = 0.
\]

Hence, instead of \((2.45)\), the \(\psi\) could be chosen to satisfy

\[
\nabla \times [\sigma_s^2 \nabla (e_3 \psi) - e_3 \nabla \psi] = 0. 
\]  
(2.46)

We proceed first with the determination of \(\psi\) and note that for any vector field \(\mathbf{F}\),

\[
\nabla \times \nabla \psi = \nabla (\nabla \cdot \mathbf{F}) - \nabla \times \nabla \times \mathbf{F}.
\]
Applying the well-known formulas of vector calculus in curvilinear coordinates, we find

\[ \nabla \cdot (e_3 \varphi) = \nabla \left[ \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi_3} \left( h_1 h_2 \varphi \right) \right] \]

\[ \nabla \times \nabla \times (e_3 \varphi) = \frac{e_1}{h_2 h_3} \frac{\partial}{\partial \xi_3} \left[ \frac{h_2}{h_1} \frac{\partial (h_3 \varphi)}{\partial \xi_1} \right] + \frac{e_2}{h_2 h_3} \frac{\partial}{\partial \xi_3} \left[ \frac{h_1}{h_2} \frac{\partial (h_3 \varphi)}{\partial \xi_2} \right] \]

\[ - \frac{e_3}{h_1 h_2} \left\{ \frac{\partial}{\partial \xi_1} \left[ \frac{h_2}{h_1} \frac{\partial (h_3 \varphi)}{\partial \xi_1} \right] + \frac{\partial}{\partial \xi_2} \left[ \frac{h_1}{h_2} \frac{\partial (h_3 \varphi)}{\partial \xi_2} \right] \right\} . \]

If \( h_3 \) is constant and \( h_2/h_1 \) is independent of the coordinate \( \xi_3 \), the above expressions can be simplified considerably with the result

\[ \nabla^2 (e_3 \varphi) = \frac{1}{h_3} \nabla \left[ \frac{\varphi}{h_1 h_2} \frac{\partial (h_1 h_2)}{\partial \xi_3} \right] \]

\[ + e_3 \left[ \nabla^2 (\varphi) - \frac{1}{h_1 h_2 h_3} \frac{\partial^2 \varphi}{\partial \xi_3^2} \right]. \]  \hspace{1cm} (2.47)

It is comfortable to note that in addition to the Cartesian coordinate system, all the cylindrical coordinate systems (circular, elliptical, and parabolic), the spherical coordinate and the conical coordinate systems also belong to this category.

For the three cylindrical coordinates, \( h_1 h_2 \) is also independent of the \( \xi_3 \) coordinate. We thus set \( \varphi = \text{constant} \) and reduce Eq. (2.47) to

\[ \nabla^2 (e_3 \varphi) = \omega e_3 \nabla^2 \varphi. \]

It is seen that the vector wave equation (2.45) is satisfied if
which is a scalar wave equation for the wave potential function
\[ \psi(\xi_1, \xi_2, \xi_3, t). \]

For spherical and conical coordinates, \( h_1 h_2 = \xi_3^2 f(\xi_1, \xi_2) \), the choice of \( \omega = \text{constant} \) is to no advantage. Substituting the following result
\[ \nabla^2 (\omega \psi) = \omega \psi^2 + 2(\nabla \omega) \cdot (\nabla \psi) + c\psi^2 \]
into (2.47), we obtain
\[ \nabla^2 (e_3 \omega \psi) = \frac{1}{h_3} \left[ \frac{\omega}{h_1 h_2} \frac{\partial (h_1 h_2)}{\partial \xi_3} \right] \]
\[ + e_3 \left\{ \omega \psi^2 + \frac{1}{h_3^2} \left[ 2 \frac{\partial \omega}{\partial \xi_3} \frac{\partial \psi}{\partial \xi_3} + \psi \frac{\partial^2 \omega}{\partial \xi_3^2} - \frac{\omega}{h_1 h_2} \frac{\partial \psi}{\partial \xi_3} \frac{\partial (h_1 h_2)}{\partial \xi_3} \right] \right\} \]
(2.49)

Of the coefficients of \( e_3 \) the terms inside the brackets will cancel each other if \( \omega(\xi_3) \) is linear in \( \xi_3 \). Setting \( \omega_3 = \xi_3 \), we obtain
\[ \nabla^2 (e_3 \omega \psi) = (2/h_3) \nabla \psi + e_3 (\omega \psi^2). \]

The first term with \( \text{grad} \psi \) need not concern us here because
\( \text{curl}(\text{grad} \psi) = 0 \). Hence if \( \psi \) is a solution of the scalar wave equation as in (2.48), the condition (2.46) is satisfied.

In either of the above two cases, the task of finding the first part of the potential \( \Psi \) is reduced to solving the scalar wave equation (2.48). Admittedly, this procedure works for only six systems of curvilinear coordinates.
The second scalar potential \( \chi \) can be determined in exactly the same manner. It is to be a solution of the scalar wave equation

\[
\alpha^2 \nabla^2 \chi = \ddot{\chi}.
\] (2.50)

In summary, the displacement vector \( \mathbf{u} \) may be resolved into a dilatation part (P-wave) \( \mathbf{L} = \nabla \varphi \) and a rotational part \( \nabla \times \mathbf{\psi} \) (S-wave) where \( \varphi \) satisfies a scalar wave equation and \( \mathbf{\psi} \) a vector wave equation. Out of the eleven coordinate systems (confocal quadric surfaces) for which the solution of a scalar wave equation is separable into product of three factors, each dependent on only one coordinate, there are only six coordinates for which the vector wave equations are separable. They are: Cartesian, circular cylindrical, elliptic cylindrical, parabolic cylindrical, spherical, and conical coordinates. For those systems with coordinates \( \xi_i (i = 1, 2, 3) \) and unit vectors \( \mathbf{e}_i \), the rotational part may be decomposed further into two components \( \mathbf{M} \) and \( \mathbf{N} \) with

\[
\mathbf{u} = \mathbf{L} + \mathbf{M} + \mathbf{N}
\] (2.51)

and

\[
\mathbf{L} = \nabla \varphi,
\]

\[
\mathbf{M} = \nabla \times (\mathbf{e}_3 \omega \psi) = \nabla (\omega \psi) \times \mathbf{e}_3,
\] (2.52)

\[
\mathbf{N} = \mathbf{L} \times \nabla \times (\mathbf{e}_3 \omega \chi) = \nabla \left[ \frac{\partial (\omega \chi)}{\partial \xi_3} \right] - \mathbf{e}_3 \nabla^2 (\omega \chi),
\]

where \( \mathbf{L} \) is a scalar factor with length dimension and \( \varphi, \psi, \chi \) satisfy the following scalar wave equations [Eqs. (2.31), (2.48), and (2.50)]:
\[ \begin{align*}
\sigma_p^{2} v^2 \phi &= \ddot{\phi}, \\
\sigma_\theta^{2} v^2 \psi &= \ddot{\psi}, \\
\sigma_\phi^{2} v^2 \chi &= \ddot{\chi}.
\end{align*} 
\tag{2.53} \]

The vector \( \mathbf{M} \) is always perpendicular to the unit vector \( \mathbf{e}_3 \), and \( \mathbf{N} \) is at right angles to \( \mathbf{M} \) when \( \nabla(\omega x) \) is proportional to \( \nabla(\omega \psi) \).

For the Cartesian system, \( \mathbf{e}_3 \) may be directed along any one of the coordinates and \( \omega(\xi_3) \) is taken as 1.

For the three cylindrical coordinate systems, \( \mathbf{e}_3 \) is taken along the axis of the cylinder and \( \omega = 1 \).

For the spherical or conical coordinate systems \( \mathbf{e}_3 \) is directed toward the radial direction and \( \omega = \xi_3 \).

For the other five coordinate systems, this resolution fails to apply. However, for problems with axial symmetry, one of the displacement components vanishes and the rotational part of a displacement vector is determined by a single wave potential. Hence the number of separable coordinate systems for a vector wave field is increased to nine with the addition of parabolic, prolate spheroidal, and oblate spheroidal coordinates. Solutions for vector wave equations in prolate and oblate spheroidal coordinates and with axial symmetry are discussed in Refs. (2.6) and (2.7).

3. TWO-DIMENSIONAL APPROXIMATIONS OF ELASTICITY

In the most general case, the displacement components in Eq. (2.26) are functions of time \( t \) and three spatial coordinates \( x_1, x_2, \) and \( x_3. \)
The equations, even when they can be reduced to wave equations, have four independent variables. Their solutions are indeed very difficult.

In many applications, the elastic body of concern has a characteristic length in one direction either much longer or much shorter than the others. Furthermore the cross sections along that direction may be uniform. Typical examples are the footing of a wall resting on a large foundation, a long tunnel under a flat surface, and a thin plate with constant thickness. When bodies with such geometry are subjected to a special distribution of external forces, various approximations can be made of the displacement components to simplify the equations of motion.

Two types of geometry are of special interest. One is that of a bulky solid with uniform cross section along one direction. A long cylindrical underground tunnel is a familiar example (Fig. 3.1). Problems of this type can be classified as presenting anti-plane strain or plane strain, according to how the external forces are applied. When a distributed force is applied parallel to the lengthwise direction ($x_3$ in Fig. 3.1) such that the dominant displacement is in that direction and has constant magnitude along the lengthwise axis, the problem falls into the category of anti-plane strain. If the force is applied perpendicularly to the characteristic length, and distributed uniformly in the lengthwise direction, there is then functionally no displacement along that direction. It is a problem of plane-strain.

The second type of geometry is that of a thin plate of uniform thickness. The plate could be perforated, or it may have cracks. Plates with external load applied parallel to the mid-surface are
grouped as presenting problems of generalized plane stress. When the forces are applied transversely to the plate, the problem is one of bending.

These four types of problems will be defined more precisely in the following subsections. A two-dimensional approximation of each problem is given therein. Whether an actual problem can be treated by one of these approximations will become clear after the assumptions are made evident in each formulation.

3.1. Anti-Plane Strain

Refer a bulky elastic body to a Cartesian coordinate system \( x_1, x_2, x_3 \), and let \( x_3 \) be the special direction along which the cross-sectional areas of the body are constant (Fig. 3.1). Anti-plane strain is defined as

\[ u_1 = 0. \]
\[ u_2 = 0, \] (3.1)

\[ u_3 = w(x_1, x_2, t). \]

From (2.23), the stresses for an isotropic elastic solid are

\[ \sigma_{13} = \mu \frac{\partial w}{\partial x_1}, \]

\[ \sigma_{23} = \mu \frac{\partial w}{\partial x_2}, \] (3.2)

\[ \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0. \]

The assumption (3.1) reduces the third of the equation of motion (2.26) to a single scalar wave equation

\[ \mu \nabla^2 w(x_1, x_2, t) = \rho \frac{\partial^2 w}{\partial t^2}, \] (3.3)

where

\[ c^2 = \frac{a^2}{\frac{\partial^2}{\partial x_1^2}} + \frac{a^2}{\frac{\partial^2}{\partial x_2^2}}. \]

The other two equations are satisfied identically. If the elastic body is bounded by the surface \( S \) defined by

\[ f(x_1, x_2) = 0 \] (3.4)

the boundary condition for a rigid surface will be

\[ w = 0 \text{ on } S, \] (3.5a)
whereas for a traction-free surface, with outernormal \( \mathbf{n} \), the boundary condition is

\[
\eta_1 \sigma_{13} + \eta_2 \sigma_{23} = \mu \left[ \cos (x_1, \mathbf{n}) \frac{\partial \omega}{\partial x_1} + \cos (x_2, \mathbf{n}) \frac{\partial \omega}{\partial x_2} \right] = 0 \quad \text{on } S.
\]  

(3.5b)

Although the equations above are formulated in terms of Cartesian coordinates, their generalization to cylindrical coordinates is immediate. For this generalization, \( x_1 \) and \( x_2 \) are considered as plane curvilinear coordinates, perpendicular to the \( x_3 \)-axis, and equation (3.1) still holds. There are still only two nonvanishing stress components, \( \sigma_{13} \) and \( \sigma_{23} \). Their relation to displacement \( \omega \) should then be calculated from (2.25) where \( \nabla \) is a two-dimensional gradient operator for the particular plane curvilinear coordinates. The Laplacian operator in the scalar wave equation should also be converted accordingly.

Under the assumption of anti-plane strain, the dilatation \( \nabla \cdot \mathbf{u} \) is zero, and the waves are rotational (S-waves). Because the displacement vector of the wave is always parallel to the \( x_3 \)-axis, which for convenience can be taken as lying on a horizontal plane, we shall call the waves of anti-plane strain SH waves. Strictly speaking, the name manifests itself only when there is a direction which can be clearly labelled as horizontal.

The SH wave so defined is mathematically analogous to sound waves in air. If \( \varphi \) denotes a velocity potential such that the velocity vector of sound wave \( \mathbf{v} = \nabla \varphi \), then \( \varphi \) satisfies also a scalar wave equation

\[
\sigma^2 \varphi^2 = \varphi, \quad \sigma^2 = (\partial^2 / \partial \omega^2), \tag{3.6}
\]
where the pressure $p$ and condensation $s$ (the relative change of density) are related to the potential by

$$
p = -\rho \frac{\partial \phi}{\partial t}, \quad s = \frac{p}{\rho c^2}.
$$ (3.7)

Thus $\omega$ in (3.3) corresponds to the velocity potential $\phi$; a rigid surface in anti-plane strain ($\omega = 0$) corresponds to an opening of an acoustic conduit where the pressure is taken to be zero. A stress-free surface would be an unmovable wall to a sound wave, at which the velocity would be zero.

It is to be emphasized that the analogy of sound waves to SH waves is purely a mathematical one. Physically, the sound wave is of the dilatation type, and for plane waves, the motion of air particles is in the direction of propagation. It is a longitudinal wave. The plane SH wave is a transverse wave.

3.2. Plane Strain

Referring to the same geometry as illustrated in Fig. 3.1, plane strain is defined as

$$
\begin{align*}
\sigma_1 &= \sigma(x_1, x_2, t), \\
\sigma_2 &= \sigma(x_1, x_2, t), \\
\sigma_3 &= 0.
\end{align*}
$$ (3.8)

Because of this assumption, the shearing stresses along the $x_3$-axis $\sigma_{13}, \sigma_{23}$ are always zero, and the normal stress $\sigma_{33}$ is related to the others by
\[ \sigma_{33} = -\nu(\sigma_{11} + \sigma_{22}). \]

In Cartesian coordinates, the other nonvanishing stresses are related to displacements by
\[ \sigma_{\alpha\beta} = \lambda \frac{\partial u_\gamma}{\partial x_\gamma} \delta_{\alpha\beta} + \mu \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right), \quad \alpha, \beta, \gamma = 1, 2. \quad (3.9) \]

They resemble (2.23) in general form except that the indices only take the value of 1 and 2. Similarly, the equations of motion reduce to two scalar equations:
\[ (\lambda + \mu) \frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\beta} + \mu \frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\alpha} + \rho \ddot{r}_\beta = \rho \frac{\partial^2 u_\beta}{\partial t^2}, \quad \alpha, \beta = 1, 2; \quad (3.10) \]

the third one vanishes.

The decomposition of the displacement vector into two potentials still applies. The condition (3.8) is satisfied if in (2.30)
\[ \varphi = \psi \varepsilon_3. \]

Thus, instead of (2.30) and (2.31), we have for a plane strain:
\[ \psi = \nu \varphi(x_1, x_2, t) + \nabla \times [\varepsilon_3 \Psi(x_1, x_2, t)] \quad (3.11) \]

and
\[ \frac{\sigma^2}{\rho} \nabla^2 \varphi = \ddot{\varphi}, \quad \frac{\sigma^2}{\rho} = (\lambda + 2\mu)/\rho, \quad (3.12) \]
\[ \frac{\sigma^2}{\rho} \nabla^2 \psi = \ddot{\psi}, \quad \frac{\sigma^2}{\rho} = \nu/\rho. \]

The above equations, together with the stress-displacement relation
\[ \sigma = \lambda (\nabla \cdot \mathbf{u}) I + \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T), \]  

(3.13)

complete the field equations for plane strain. All quantities in (3.11) to (3.13) are independent of \( x_3 \) coordinates and \( \mathbf{u} \) and \( \sigma \) are two-dimensional vector and dyadics, respectively.

From (3.11), we find that the shear wave derived from the potential \( \psi \) in plane strain has displacements

\[ u_1 = \frac{\partial \psi}{\partial x_2}, \quad u_2 = -\frac{\partial \psi}{\partial x_1}. \]

The displacement vector is always transverse to the \( x_3 \)-axis. We shall call the shear wave defined by (3.11) and (3.12) an \( SV \)-wave in the following studies.

3.3. Generalized Plane Stress

For an elastic plate under the action of external forces applied parallel to the mid-plane (Fig. 3.2), calculations of the displacement

\[ \text{Fig. 3.2. Geometry of Generalized Plane Stress} \]
or stress variation across the plate thickness are very difficult. In a case where the plate thickness is small in comparison with the wavelength and other dimensions, a set of two-dimensional equations can be derived for the average values of displacement and stresses across the thickness. The appropriate equations were obtained by Poisson (1829)\(^{(1.12)}\) and Cauchy (1828)\(^{(3.1)}\) by expanding the displacement components into power series of the thickness coordinates and truncating the series at various orders of approximation. Given here is a brief account of the equations with a derivation similar to that by Filon (1903)\(^{(3.2)}\) for generalized plane stress in elastostatics.

Let the \(x_1-x_2\) coordinate plane be the mid-surface of the plate, and denote the average stress and displacements by \(\bar{u}_j, \bar{\sigma}_{ij}\)

\[
\bar{u}_j(x_1, x_2, t) = \frac{1}{2b} \int_{-b}^{b} u_j(x_1, x_2, x_3, t) dx_3, \quad (3.14)
\]

\[
\bar{\sigma}_{ij}(x_1, x_2, t) = \frac{1}{2b} \int_{-b}^{b} \sigma_{ij}(x_1, x_2, x_3, t) dx_3, \quad (3.15)
\]

where \(2b\) is the thickness of the plate. Since the plate is thin and free from stresses at surface \(x_3 = \pm b\), we assume that

\[
\bar{\sigma}_{31} = \bar{\sigma}_{32} = \bar{\sigma}_{33} = 0. \quad (3.16)
\]

In the stress equation of motion (2.12), if the same average procedure is performed, we have, with \(f_3 = 0\)

\[
\frac{\partial \bar{\sigma}_{ij}}{\partial x_i} + \rho \bar{f}_j = \rho \frac{\partial^2 \bar{u}_j}{\partial t^2}, \quad \alpha, \beta = 1, 2. \quad (3.17)
\]

Similar average processes are performed for the strain displacement
relation and (2.5) and the stress-strain relation (2.20)

\[ \tilde{\sigma}_{ij} = \lambda \tilde{\varepsilon}_{ij} + 2\mu \varepsilon_{ij} \]

\[ i, j = 1, 2, 3. \quad (3.18) \]

\[ \tilde{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right). \]

Since \( \tilde{\sigma}_{33} = 0 \) by assumption, the average normal strain \( \bar{\varepsilon}_{33} \) is related to the other two by

\[ \bar{\varepsilon}_{33} = -\frac{1}{\lambda + 2\mu} \left( \varepsilon_{11} + \varepsilon_{22} \right). \quad (3.19) \]

Substitution of the above to (3.18) results in

\[ \tilde{\sigma}_{ij} = \lambda' \varepsilon_{ij} + 2\mu \varepsilon_{ij} \]

or

\[ \tilde{\sigma}_{\alpha\beta} = \lambda' \left( \frac{\partial \tilde{u}_\gamma}{\partial x_\gamma} \right)_{\alpha\beta} + \mu \left( \frac{\partial \tilde{u}_\alpha}{\partial x_\beta} + \frac{\partial \tilde{u}_\beta}{\partial x_\alpha} \right), \quad \alpha, \beta, \gamma = 1, 2. \quad (3.20) \]

where

\[ \lambda' = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{2\mu}{1 - \nu}. \quad (3.21) \]

Combining (3.20) with (3.17) gives

\[ (\lambda' + \mu) \partial^2 \tilde{u}_\alpha \partial x_\alpha \partial x_\beta + \mu \partial^2 \tilde{u}_\beta \partial x_\alpha \partial x_\alpha + \rho f_\beta = \rho \partial^2 \tilde{u}_\beta \partial x_\alpha \partial x_\alpha \quad \alpha, \beta = 1, 2. \quad (3.22) \]

With \( \tilde{u}_\alpha \), \( \tilde{\sigma}_{\alpha\beta} \) defined by Eqs. (3.14) through (3.16), Eqs. (3.20) and (3.22) complete the generalized plane stress equation. All quantities in the above equations are independent of the \( x_3 \)-coordinate.
Comparison of Eqs. (3.20) and (3.22) with Eqs. (3.9) and (3.10) for plane strain shows that these two formulations are mathematically identical, the only difference being that the material constant $\lambda$ in plane strain is replaced by $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$ in plane stress. Thus the solution for a long bulk solid can be converted to the one for a thin disk and vice versa, providing all other conditions are the same. From plane strain solution to plane stress solution, we simply replace $\lambda$ by $\lambda'$ and leave the constant $\mu$ unchanged. If a solution for a plane stress is known, changing $\lambda$ to $2\lambda'\mu/(2\mu - \lambda')$ will convert it to the one for plane strain. Conversion of other elastic constants are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Plane Strain to Plane Stress</th>
<th>Plane Stress to Plane Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\frac{2\lambda\mu}{\lambda + 2\mu}$</td>
<td>$\frac{2\lambda\mu}{2\mu - \lambda}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\frac{\nu}{1 + \nu}$</td>
<td>$\frac{\nu}{1 - \nu}$</td>
</tr>
<tr>
<td>$E$</td>
<td>$E \frac{1 + 2\nu}{(1 + \nu)^2}$</td>
<td>$\frac{E}{1 - \nu^2}$</td>
</tr>
</tbody>
</table>

Unlike the plane strain solution, the plane stress solution is formulated in terms of the average stresses and average displacements across the thickness of the plate, not in terms of the pertinent values at each field point. It may thus lead to erroneous results when the
displacements vary sharply across the thickness. Such is the case when the plate vibrates longitudinally at high frequencies. A comparison with the results based on three-dimensional elasticity theory shows that the generalized plane stress approximation for a plate is valid when the frequency is less than \( \pi c_s/2b \) and when the wavelength is longer than \( 2\pi b \). (3.3)

Under the plane stress assumption, the displacement, a two-dimensional vector, can be determined from two potentials \( \varphi \) and \( \psi \) as in (3.11). The field associated with \( \psi \) is still the SV wave, but that part from \( \varphi \) is no longer the P-wave. The waves of \( \varphi \), still dilatational, travel with the speed

\[
\sigma_p' = [(\lambda + 2\mu)/\rho]^{\frac{1}{2}} = [2\mu/\rho(1 - \nu)]^{\frac{1}{2}}.
\]

They are known as the extensional waves in plates.

3.4. Bending of a Plate

If the plate discussed in the previous subsection is subjected to external forces applied perpendicular to the plate, waves of an entirely different nature are generated. To analyze the dominant feature of the plate motion under such forces, we assume that (Fig. 3.3)

\[
\begin{align*}
&u_1(x_1, x_2, x_3, t) = x_3 \psi_1(x_1, x_2, t), \\
&u_2(x_1, x_2, x_3, t) = x_3 \psi_2(x_1, x_2, t), \\
&u_3(x_1, x_2, x_3, t) = \omega(x_1, x_2, t)
\end{align*}
\]
and define

\[ M_{\alpha \beta}(x_1, x_2, t) = \int_{-b}^{b} x_3 \sigma_{\alpha \beta}(x_1, x_2, x_3, t) \, dx_3, \]
\[ \alpha, \beta = 1, 2. \quad (3.25) \]

\[ Q_{\beta}(x_1, x_2, t) = \int_{-b}^{b} \sigma_{3 \beta}(x_1, x_2, x_3, t) \, dx_3, \]

**Fig. 3.3. Bending of a Plate**

Assuming further that

\[ M_{33}(x_1, x_2, t) = \int_{-b}^{b} x_3 \sigma_{33}(x_1, x_2, x_3, t) \, dx_3 = 0, \quad (3.26) \]

we obtain, upon multiplying the stress displacement equations (2.29) by \( x_3 \) and integrating over the thickness,

\[ \sigma_{11} : M_{11} = D \left( \frac{\partial \psi_1}{\partial x_1} + \nu \frac{\partial \psi_2}{\partial x_2} \right), \]

\[ \sigma_{22} : M_{22} = D \left( \frac{\partial \psi_2}{\partial x_2} + \nu \frac{\partial \psi_1}{\partial x_1} \right), \quad (3.27a) \]

\[ \sigma_{12} : M_{12} = \mu \left( \frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} \right), \]
where

\[ I = 2b^3/3, \quad D = 2\mu I/(1 - \nu) = 2EI^3/3(1 - \nu^2), \]

and use has been made of (3.24) and (3.26) to express \( \int x_3 \varepsilon_{13} \, dx_3 \) in terms of \( \int x_3 \varepsilon_{11} \, dx_3 \) and \( \int x_3 \varepsilon_{22} \, dx_3 \). Integrating the remaining two equations from \(-b\) to \(b\), we obtain

\[ \sigma_{13} : Q_1 = 2ub \left( \psi_1 + \frac{\partial \omega}{\partial x_1} \right), \]

\[ \sigma_{23} : Q_2 = 2ub \left( \psi_2 + \frac{\partial \omega}{\partial x_2} \right). \]

Equations (3.27), which relate the bending moment \( M_{12} \) and shear force \( Q_\alpha \) to the rotation of a cross section \( \psi_\alpha \) and the deflection of the plate \( \omega \), are a set of two-dimensional equations. They replace the general stress displacement relations.

The stress equation of motion can be modified accordingly. Multiplying the first two (\( j = 1,2 \)) or (2.12) by \( x_3 \) and then integrating over the plate thickness for all three equations, we obtain

\[ \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} - Q_1 = \rho I \frac{\partial^2 \psi_1}{\partial x_1^2}, \]

\[ \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - Q_2 = \rho I \frac{\partial^2 \psi_2}{\partial x_2^2}, \]

\[ \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + q = 2ub \frac{\partial^2 \omega}{\partial x_1^2}. \]

The quantity \( q = (\tau_{33})_{-b}^b \) is the result of normal stresses applied to the upper and lower surfaces of the plate.
TWO-DIMENSIONAL APPROXIMATIONS

In the classical theory of the bending of plates, the effect of shear force on deflections is neglected, or equivalently the plate rigidity in resisting shear force is assumed infinite. This reduces (3.27b), with $Q_\alpha/2b\mu \to 0$, to

$$\psi_\alpha = -\frac{3w}{3x_\alpha}, \quad \alpha = 1,2.$$ 

Physically, this means the angle of rotation of a cross section equals the slope of the mid-plane. Through this assumption, the moments are related directly to the curvature of the deflected plate by

$$M_{11} = -D \left( \frac{\partial^2 w}{\partial x_1^2} + \nu \frac{\partial^2 w}{\partial x_2^2} \right),$$

$$M_{22} = -D \left( \frac{\partial^2 w}{\partial x_2^2} + \nu \frac{\partial^2 w}{\partial x_1^2} \right), \quad (3.29)$$

$$M_{12} = -2\mu I \frac{\partial^2 w}{\partial x_1 \partial x_2} = -(1-\nu) D \frac{\partial^2 w}{\partial x_1 \partial x_2}.$$ 

Furthermore, in the classical theory, the effect of rotatory inertia, the term $\rho I\dot{\omega}_\alpha$ in (3.28), is neglected. Thus the first two equations of (3.28) give rise to the magnitudes of shear force due to bending,

$$Q_1 = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} = -D \frac{\partial}{\partial x_1} \nu^2 w,$$

$$Q_2 = \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = -D \frac{\partial}{\partial x_2} \nu^2 w, \quad (3.30)$$
where
\[ v^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \]

Substitution of the above into the third of (3.28) results in
\[ Dv^2\omega^2 + 2bp + \frac{\partial^2 w}{\partial t^2} = q. \tag{3.31} \]

This is the classical equation of plates in bending.

If the motion is harmonic in time with frequency \( \omega \), the deflection has the form
\[ w(x_1, x_2, t) = \tilde{w}(x_1, x_2) e^{-i\omega t} \]

and \( \tilde{w} \) satisfies the equation \((q = 0)\)
\[ (v^2 + \gamma^2)\tilde{w} = 0, \quad \gamma^4 = \frac{2bp}{D} \omega^2. \tag{3.32} \]

Let \( \tilde{w} = \tilde{w}_1 + \tilde{w}_2 \). The above equation is satisfied if
\[ (v^2 + \gamma^2)\tilde{w}_1 = 0, \tag{3.33} \]
\[ (v^2 - \gamma^2)\tilde{w}_2 = 0. \]

The first is a Helmholtz equation, and \( \tilde{w}_1 e^{-i\omega t} \) represents the wave motion in the plate. The \( \tilde{w}_2 \) represents a wave attenuating as it progresses. Combination of these two parts constitutes the flexural wave in plate.

The phase velocity of plane harmonic flexural waves is
\[ \sigma_f = \frac{\omega}{\gamma} = \gamma \frac{\nu}{2bd}\frac{b}{c_s} = \gamma \frac{b c_s}{2 \sqrt{3(1-\nu)}}. \]  

The wave is dispersive, as the wave velocity depends on the wavelength \(2\pi/\gamma\). As in the case of generalized plane stress approximation, the classical theory for the bending of plate is also limited to low frequencies \(<0.3c_s/b\) and long wavelength \(>\pi b\). (3.4)

Derived above are the basic equations of motion for the different approximations. These equations will be used to solve specific problems for various geometries and boundary conditions, and for steady-state, as well as for transient responses.

In the following section, a method for determining steady-state response and transient response will be presented.

4. STEADY-STATE AND TRANSIENT RESPONSES OF AN ELASTIC SOLID

In the preceding sections we derived the displacement equations of motion for a homogeneous, isotropic linear elastic solid and the various two-dimensional approximations. Symbolically, the equations (2.29), (3.3), (3.10), (3.22), and (3.31) can be represented as

\[ L[u(x, t)] + \sigma_f^b(x, t) = \rho \ddot{u}(x, t), \]  

where \(u\) is a scalar or a vector, \(\sigma_f^b\) is the body force, and \(L\) is a linear differential operator. When the body force is neglected, Eq. (4.1a) becomes

\[ L[u(x, t)] = \rho \ddot{u}(x, t). \]  

It might be of interest to note here that we have not mentioned
any external force other than the body force. Of course the motion of the elastic body will depend on the nature of the external force, \( f(x, t) \). The applied force may arise as an internal or surface source, or in the form of boundary conditions and initial conditions. Thus we may consider the problem from two different points of view. If the external forces are treated as sources, then the governing equation becomes

\[
L\{u(x, t)\} = \rho \ddot{u} - f(x, t),
\]

where \( f(x, t) \) describes the source density, giving not only the distribution of sources in space but also the time dependence at each point in space. Since we have considered all applied forces as sources, Eq. (4.1c) must be accompanied by a homogeneous boundary condition.

Contrary to the above, let us now consider the applied force \( f(x, t) \) as boundary and initial conditions. Then, the governing equation is the same as Eq. (4.1b)—in other words, a homogeneous equation. However, the boundary conditions which are associated with the problem now become inhomogeneous.

A fundamental property for the linear differential Eq. (4.1) is that, if \( u' \) and \( u'' \) are solutions, i.e., \( L(u') = \rho \ddot{u} \) and \( L(u'') = \rho \ddot{u}'' \), then \( u = c_1u' + c_2u'' \) is also a solution, where \( c_1 \) and \( c_2 \) are arbitrary constants. In general, if

\[
L\{u^{(n)}\} = \rho \ddot{u}^{(n)} - f, \quad n = 1, 2, 3, \ldots
\]

then

\[
u = \sum_{n=1}^{\infty} c_n u^{(n)}
\]

(4.2)
also satisfies Eq. (4.1). We shall on occasion replace the sum by an integral.

We have seen through the formulation above that the motion \( u(x, t) \)
is caused by an externally applied force \( f(x, t) \), viewed either as
sources or as boundary conditions. In both cases, we shall call all
the applied forces source or input, and the consequent motion of the
elastic body response or output. Determination of the response for
a given input is our primary concern.

The input-output relationship for an elastic body is complicated,
because \( u \) and \( f \) are functions of spatial coordinates \( x \), as well as
functions of time \( t \). However, in our present discussion on the steady-
state responses or transient responses, we need not include such com-
plications. We shall focus our attention on the response of a fixed
spatial point, and examine its time history as a result of the applied
sources.

With regard to time dependence of the response, an elastic body
is assumed to follow two basic conditions: the condition of causality
and the condition of time invariance. A function \( f(t) \) is causal if

\[
f(t) = \begin{cases} 
0, & t < 0, \\
f(t), & t > 0.
\end{cases}
\]  

(4.3)

The condition of causality states that if the input \( f(t) \) is causal,
the output \( u(t) \) is also causal, i.e.,

\[
u(t) = \begin{cases} 
0, & t < 0, \\
u(t), & t > 0.
\end{cases}
\]
As is clear from this condition, the problem of an elastic body with
time dependent initial stresses is ruled out.

The condition of time invariance is that if $u(t)$ is the response
for the source $f(t)$, then the response to $f(t - t_1)$ is $u(t - t_1)$ where
$t_1$ is any constant. This condition is satisfied if the coefficients
of the linear operator $L$ are time independent, as is true for an elas-
tic body with constant material constants.

As a function of time, the response will be classified as either
steady-state or transient, as mentioned briefly in Section 2.6. We
shall elaborate the properties and relationships of these responses in
the subsequent discussion. We have left out much of the mathematical
rigor in the analysis of the transient response, such as the inte-
grability of a function and the approach to singular functions from
the theory of distributions. We shall present only the techniques
which will be used repeatedly in the rest of the book.

4.1. Steady-State Response

Let us consider first the steady-state response, since it is
mathematically less complex than the transient response. By steady
state we mean a response that is simple harmonic in an unbounded time
domain. Usually, it is represented by

$$u(t) = Ae^{i\omega t}, \quad -\infty < t < \infty, \quad (4.4)$$

where $A$ is a complex number. As mentioned in subsection 2.6, only the
real or the imaginary part of (4.4) is taken to represent the motion.

Simple harmonic motion of an elastic solid can arise either during
free or during forced vibrations. For a bounded medium, it is possible to start a system by giving it an initial displacement or an initial velocity (impulse) which is compatible with one of the principal modes of the body. The subsequent free vibration is simple harmonic with a frequency equal to the corresponding principal frequency. Since the principal modes are usually very complicated for an elastic solid, excitation of this type seldom occurs. On the other hand, a simple harmonic source or simple harmonic forces applied at boundaries will generate forced motion of the system. After sufficient time has elapsed, the irregular initial disturbance of the forced motion dies out owing to a small damping inherent in all systems. What remains is a simple harmonic motion of the same frequency as the source.

In either case, the dependence on time for the response may be separated off as

$$u(x, t) = \phi(x, \omega)e^{i\omega t}, \quad (4.5)$$

where $\phi(x, \omega)$ is a function of the spatial coordinates for a given frequency. The problem of determining the steady-state response for a source

$$f(x, t) = F(x, \omega)e^{i\omega t}, \quad (4.6)$$

or the corresponding excitation appearing in the boundary condition, is then reduced to solving the equation

$$L\{\phi(x, \omega)\} = -\omega^2 \phi(x, \omega) \quad (4.7)$$

with the appropriate boundary conditions.
When the displacement potentials $\varphi$ and $\psi$ are used as in (2.3.1), the wave equations reduce to the familiar Helmholtz equation for steady-state response. We note that no initial conditions arise here due to the steady-state motion. We have chosen the zero time of observation long after the initiation of the motion. As far as the observer is concerned, the harmonic motion was started at time $t = -\infty$.

When the source has a magnitude of unity, i.e., $f(t) = e^{\pm i\omega t}$, the coefficient of $e^{\pm i\omega t}$ in the steady-state response is called the admittance of the elastic system. In terms of Eqs. (4.5) through (4.7), we may define the admittance as

$$\chi(x_i, \omega) = \frac{U(x_i, \omega)}{F(x_i, \omega)}.$$  \hspace{1cm} (4.8)

Knowing the admittance $\chi$ of an elastic body and the magnitude of the external source $F$, the steady-state response is simply

$$u(x_i, t) = F(x_i, \omega)\chi(x_i, \omega)e^{\pm i\omega t}.$$  \hspace{1cm} (4.9)

If we are interested only in the steady-state response to simple harmonic forces there would be nothing more to say, and we would go directly to the specific problems. However, in many problems of practical interest, we are more interested in the response of an elastic system to an aperiodic disturbance, or in cases when the source is applied to the body suddenly. Therefore, we shall now address ourselves to the transient response problem.

The Fourier transform, and the related Laplace transform, are techniques used frequently for solving the transient problem. In the following subsections we shall discuss both techniques.
4.2. Fourier Transform

The Fourier transform \( F(\omega) \) is defined by

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \tag{4.9a}
\]

(See Refs. (2.3), (4.1), and (4.2) for a class of \( f(t) \) such that \( F(\omega) \) exist.) The function \( f(t) \) can be determined from \( F(\omega) \) according to the Fourier integral theorem:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega \int_{-\infty}^{\infty} e^{i\omega(\tau-t)} f(\tau) d\tau. \tag{4.10}
\]

Introducing \( F(\omega) \), the above may be written as

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega. \tag{4.9b}
\]

Equation (4.9b) is often referred to as Fourier's inversion formula, and \( f(t) \) is called the inverse Fourier transform of \( F(\omega) \).

The pair of equations (4.9) then represents the fundamental set of relations between the time-dependent function \( f(t) \) and the frequency-dependent function \( F(\omega) \). In general, the function of \( F(\omega) \) is complex:

\[F(\omega) = R(\omega) + iX(\omega) = A(\omega)e^{i\phi(\omega)}.\]

\( A(\omega) \) is called the Fourier spectrum of \( f(t) \) and \( \phi(\omega) \) is its phase angle.

In what follows, we shall list some of the useful simple theorems and formulas for the Fourier transform.
(a) Linearity. If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transform of $f_1(t)$ and $f_2(t)$, respectively, and $a_1$ and $a_2$ are arbitrary constants, then
\[ a_1f_1(t) + a_2f_2(t) \leftrightarrow a_1F_1(\omega) + a_2F_2(\omega). \] (4.11)

The notation $\leftrightarrow$ is used here to indicate that function $f(t)$ and $F(\omega)$ are related by Eq. (4.9).

(b) Time Scaling. If $a$ is a real constant, then
\[ f(at) \leftrightarrow \frac{1}{|a|} \frac{F(\omega)}{a}. \] (4.12)

(c) Time and Frequency Shifting. If the function $f(t)$ is shifted by a constant $t_0$, then its Fourier spectrum remains the same, but a term $t_0 \omega$ is added to the phase angle
\[ f(t - t_0) \leftrightarrow F(\omega)e^{it_0 \omega} = A(\omega)e^{-it_0 \omega}, \] (4.13)

and if the $F(\omega)$ is shifted by a real constant $\omega_0$, then
\[ e^{-i\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0). \] (4.14)

(d) Time and Frequency Differentiation. The Fourier transform of the $n^{th}$ derivative of $f(t)$ is
\[ \frac{d^n f(t)}{dt^n} \leftrightarrow (-i\omega)^n F(\omega). \] (4.15)

Here $f(t)$ and all its derivatives up to $(n-1)^{th}$ order are assumed to vanish as $|t| \to \infty$. This class of functions is particularly interesting since in almost all physical problems the disturbance
dies out in time. Likewise, differentiating $F(\omega)$ with respect to \(\omega\) gives

$$
(i\omega)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}.
$$

(4.16)

(e) **Moment Theorem.** This theorem relates the derivatives of $F(\omega)$ at $\omega = 0$ to the moments of its inverse transform. The $n^{th}$ moment $m_n$ of $f(t)$ is defined by

$$
m_n = \int_{-\infty}^{\infty} t^n f(t) dt, \quad n = 0, 1, 2, 3, \ldots.
$$

(4.17)

and the theorem states that

$$
(i\omega)^n m_n = \frac{d^n F(\omega)}{d\omega^n} \bigg|_{\omega=0}, \quad n = 0, 1, 2, 3, \ldots.
$$

(4.18)

It follows that the $0^{th}$ moment is the area under the curve of $f(t)$ and represents the value of $F(\omega)$ at $\omega = 0$. Thus, the slope $df/d\omega$ at $\omega = 0$ is represented by the first moment of $f(t)$, etc.

Similarly we may define the $n^{th}$ moment $M_n$ of $F(\omega)$ as

$$
M_n = \int_{-\infty}^{\infty} \omega^n F(\omega) d\omega.
$$

(4.19)

With Eq. (4.19), the derivatives of $f(t)$ at the origin, $t = 0$, are related to the moments of the Fourier transform $F(\omega)$ by

$$
(-i)^n \frac{d^n f}{dt^n} \bigg|_{t=0} = \frac{d^n F(\omega)}{d\omega^n} \bigg|_{\omega=0}, \quad n = 0, 1, 2, 3, \ldots.
$$

(4.20)
Analogous to the previous discussion of \( m_n \), the value of \( f(t) \) at \( t = 0 \) is then determined by the area under the curve \( P(\omega) \), and the slope \( df/dt \), at \( t = 0 \), is determined by the first moment. That is

\[
\left. \frac{df}{dt} \right|_{t=0} = M_1 = \int_{-\infty}^{\infty} \omega P(\omega) d\omega. \tag{4.21}
\]

Equations (4.18) and (4.20) furnish powerful tools for evaluating the behavior either of \( P(\omega) \) or \( f(t) \) at \( \omega = 0 \) or \( t = 0 \), if one of the functions is completely defined. Furthermore, they are used sometimes as criteria in the approximate techniques. (See Chapter IV.)

(f). The Time and Frequency Convolution Theorem (Faltung). Last is the convolution theorem. Let \( F_1(\omega) \) and \( F_2(\omega) \) be the transforms of \( f_1(t) \) and \( f_2(t) \), respectively. Then the inverse transform of the product \( F_1(\omega) \cdot F_2(\omega) \) is

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} F_1(\omega) F_2(\omega) d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau
\]

\[
= f_1(t) \ast f_2(t). \tag{4.22}
\]

Alternatively, the results can be written as

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} F_1(\omega) F_2(\omega) d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(\tau) f_1(t-\tau) d\tau
\]

\[
= f_2(t) \ast f_1(t). \tag{4.23}
\]

An integral of this form is called a convolution integral of \( f_1 \) and \( f_2 \) (or a Faltung of \( f_1 \) and \( f_2 \)).
In a similar manner, the Fourier transform $F(\omega)$ of the product of two functions $f_1(t)f_2(t)$ is equal to the convolution $F_1(\omega) * F_2(\omega)$ of their respective transforms $F_1(\omega)$ and $F_2(\omega)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t)f_2(t)e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega - \lambda) d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(\lambda)F_1(\omega - \lambda) d\lambda. \quad (4.24)$$

For convenience in future reference, the equations discussed above are presented in Table 4.1.

**Table 4.1**

**PROPERTIES OF FOURIER TRANSFORM**

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t} d\omega \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt$$

$$a_1f_1(t) + a_2f_2(t) \quad a_1F_1(\omega) + a_2F_2(\omega)$$

$$f(at) \quad \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$f(t - t_0) \quad F(\omega)e^{i\omega t_0}$$

$$e^{-i\omega t}f(t) \quad F(\omega - \omega_0)$$

$$\frac{d^n f(t)}{dt^n} \quad (-i\omega)^n F(\omega)$$

$$(it)^n f(t) \quad d^n F(\omega)/d\omega^n$$
\[ \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \]

\[ h(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \]

Convolution Theorem:

\[ f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t) f_2(t - \tau) d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t - \tau) f_2(t) d\tau \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} F_1(\omega) F_2(\omega) d\omega \]

\[ F_1(\omega) * F_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(\lambda) F_2(\omega - \lambda) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(\omega - \lambda) F_2(\lambda) d\lambda \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t) f_2(t) e^{i\omega t} dt \]

Moment Theorem:

Define

\[ m_n = \int_{-\infty}^{\infty} t^n f(t) dt \quad \text{and} \quad M_n = \int_{-\infty}^{\infty} \omega^n F(\omega) d\omega \]

Then

\[ t^n m_n = \left. \frac{d^n F(\omega)}{d\omega^n} \right|_{\omega=0}, \quad (-i)^n M_n = \left. \frac{d^n f}{dt^n} \right|_{t=0}, \quad n = 0, 1, 2, \ldots \]
4.3. Special Functions and Their Fourier Transforms

So far we have listed only basic formulas of the Fourier transform; no mention has been made of the type of function \( f(t) \) that will be of most interest. In dealing with transient responses of elastic bodies which follow the condition of causality, the causal function is of primary interest.

First, we note that a function can always be decomposed into a sum of an even and part:

\[
f(t) = \frac{1}{2} [f(t) + f(-t)] + \frac{i}{2} [f(t) - f(-t)] = f_e(t) + f_o(t).
\]

If a Fourier transform is decomposed into a real and an imaginary part,

\[
F(\omega) = R(\omega) + iX(\omega), \quad (4.25)
\]

\[
R(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos \omega t dt,
\]

\[
X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin \omega t dt.
\]

An even function \( f_e(t) \) satisfying the condition

\[
f_e(t) = f_e(-t),
\]

has the transform \( X(\omega) = 0 \) and

\[
R(\omega) = \sqrt{2} \int_{0}^{\infty} f_e(t) \cos \omega t dt. \quad (4.26a)
\]

The inverse transform is

\[
f_e(t) = \sqrt{2} \int_{0}^{\infty} R(\omega) \cos \omega t d\omega. \quad (4.26b)
\]
Similarly, an odd function \( f_o(t) = -f_o(-t) \) has the transform pair

\[
X(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_o(t) \sin \omega t \, dt,
\]

\[
f_o(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty X(\omega) \sin \omega t \, d\omega,
\]

and \( R(\omega) = 0 \).

A causal function \( f(t) \) (Eq. 4.3) may be treated either as an even function \( f_e(t) \) or as an odd function \( f_o(t) \) over the range \(-\infty < t < \infty\) (Fig. 4.1). For the time interval of interest, \( t > 0 \),

\[
f(t) = 2f_e(t) = 2f_o(t), \quad t > 0,
\]

and the transform of the causal function can be calculated from either (4.26a) or (4.27a), with \( f_e(t) \) or \( f_o(t) \) replaced by \( \frac{1}{2} f(t) \).

---

**Fig. 4.1.** Even and Odd Parts of \( f(t) \)
Furthermore, as can be seen from (4.28), (4.26b), and (4.27b),

$$f(t) = 2\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} R(\omega) \cos \omega t d\omega = 2\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X(\omega) \sin \omega t d\omega, \quad t > 0.$$  

(4.29)

Thus the real and imaginary parts of the Fourier transform of a causal function are not independent of each other, but one of them can be computed from the other.

Next, we shall present some special properties of two singular functions. These functions are of significance because once the response of a system to these functions is known, the responses of the system to arbitrary inputs are determinable, according to the convolution theorem, by a simple integration.

(a) Dirac Delta Function \(\delta(t)\):

One may describe \(\delta(t)\) loosely as a function which is zero everywhere except at \(t = 0\), where it is infinite, and which has a unit area under its graph.

$$\delta(t) = 0, \quad |t| > 0,$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$  

(4.30)

We shall define the delta function \(\delta(t)\) by the following integral for an arbitrary function \(\varphi(t)\):

$$\int_{-\infty}^{\infty} \delta(t - \tau)\varphi(t) dt = \varphi(\tau);$$  

(4.31)
\( \varphi(t) \) is assumed to be continuous at \( t = 0 \). Similarly, the \( n \)-th derivative of \( \delta(t) \) is defined by

\[
\int_{-\infty}^{\infty} \frac{d^n \delta(t - \tau)}{d\tau^n} \varphi(t) d\tau = (-1)^n \frac{d^n \varphi(t)}{d\tau^n} \bigg|_{\tau = 0} 
\]  

(4.32)

The Fourier transform of \( \delta(t) \) is:

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}}
\]

(4.33)

\[ \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega. \]

(b) Heaviside Unit Step Function \( h(t) \):

A function closely related to the delta function is the Heaviside unit function \( h(t) \), defined by

\[
h(t) = \begin{cases} 
0, & t < 0, \\
1, & t > 0,
\end{cases}
\]

(4.34)

and

\[
h(t) = \int_{-\infty}^{t} \delta(t) d\tau, \quad h'(t) = \delta(t).
\]

(4.35)

The Fourier transform of \( h(t) \) is

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt = -\frac{1}{i\omega}
\]

(4.36)

\[
h(t) \leftrightarrow -\frac{1}{i\omega}.
\]

(4.37)
4.4. Transient Response

At the end of subsection 4.1 we showed that for a linear elastic system, the steady-state response to a simple harmonic force \( F(\omega) e^{-i\omega t} \) is \( F(\omega) \chi(x,\omega) e^{-i\omega t} \), where \( \chi(x,\omega) \) is the admittance of the system.

Now, to find the transient response of an aperiodic disturbance, \( f(t) \), we first break up \( f(t) \) into its simple harmonic components by means of the Fourier integral

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega, \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.
\]

Then, having solved the steady-state problem to obtain the admittance, \( \chi(x,\omega) \), we superimpose the components to obtain the response of the system resulting from the original aperiodic force \( f(t) \):

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x,\omega) F(\omega) e^{-i\omega t} d\omega.
\] (4.38)

If the input is a unit impulse, recall that

\[
\delta(t) \leftrightarrow \frac{1}{\sqrt{2\pi}}.
\]

Then the systems response to \( \delta(t) \) is simply

\[
u_\delta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(x,\omega) e^{-i\omega t} d\omega.
\] (4.39)

Due to a delta function input, we refer to \( u_\delta(x, t) \) as the impulse response. It should be apparent now that the admittance function \( \chi(x,\omega) \) and the impulse response \( u_\delta(x, t) \) form a Fourier transform pair.
For an arbitrary input, we may determine the system response $u(x, t)$ by Eq. (4.38), or if we wish, use the convolution theorem as given by Eq. (4.23). Thus the response to arbitrary input $f(t)$ is:

$$u(x, t) = \int_{-\infty}^{\infty} f(\tau) u_\delta(x, t - \tau) d\tau,$$

(4.40)

or

$$u(x, t) = \int_{-\infty}^{\infty} u_\delta(x, \tau) f(t - \tau) d\tau.$$

(4.41)

Since the delta function is causal, from the condition of causality the impulse response is also causal. Thus

$$u_\delta(x, t - \tau) = 0 \quad \text{when } t < \tau.$$

It then follows that Eqs. (4.40) and (4.41) can be rewritten as

$$u(x, t) = \int_{-\infty}^{t} f(\tau) u_\delta(x, t - \tau) d\tau,$$

(4.42)

and

$$u(x, t) = \int_{0}^{\infty} f(t - \tau) u_\delta(x, \tau) d\tau.$$

(4.43)

If, in addition, $f(\tau)$ is causal, with $f(t - \tau) = 0$ when $t < \tau,$ then the above equations are reduced to the familiar Duhamel integral:

$$u(x, t) = \int_{0}^{t} f(\tau) u_\delta(x, t - \tau) d\tau,$$

(4.44)
or

\[ u(x_i, t) = \int_0^t u_\delta(x_i, \tau)f(t - \tau)d\tau. \]  \hfill (4.45)

Equations (4.44) and (4.45) are a statement of the principle of superposition in time as illustrated in Fig. 4.2.

![Diagram showing Duhamel's integral and unit impulse response](image)

**Fig. 4.2. Illustration of the Duhamel Integral and Unit Impulse Response**

Using the formulas given above we may now derive the other familiar form of Duhamel's integral, that of the response due to arbitrary input in terms of the response to a Heaviside unit function, \( h(t) \), Eq. (4.34). Letting \( u_h(x_i, t) \) denote the indicial response, that is, the response to a unit step function \( h(t) \), then according to (4.45) we have

\[ u_h(x_i, t) = \int_0^t u_\delta(x_i, \tau)f(t - \tau)d\tau, \]  \hfill (4.46)

or

\[ u_h'(x_i, t) = \frac{du_h(x_i, t)}{dt} = u_\delta(x_i, t). \]  \hfill (4.47)
It follows from (4.44) that the response to an arbitrary \( f(t) \) can be written as

\[
u(x, t) = \int_0^t f(\tau)u_n'(t - \tau)d\tau.
\]

(4.48)

or after integrating by parts once, we have

\[
u(x, t) = f(0)u_n(t) + \int_0^t f'(\tau)u_n(t - \tau)d\tau.
\]

(4.49)

Equation (4.49) is another form of Duhamel integral. Again it is merely a statement of the method of superposition in time as illustrated by Fig. 4.3.

![Fig. 4.3. Illustration of Duhamel's Integral and Step Response](image)

To summarize, the transient response \( u(t) \) of an elastic body due to an arbitrary source \( f(t) \) can be determined by one of the following three ways:

1. Finding the steady-state response or the admittance \( \chi(\omega) \) (4.38) on the same body and the Fourier spectrum \( F(\omega) \) of \( f(t) \) (4.9), then using (4.38):
\[ u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\chi(\omega)e^{-i\omega t}d\omega; \quad (4.50) \]

2. Finding the impulse response \( u_\delta(t) \) due to a delta function source, then using (4.43):

\[ u(t) = \int_{0}^{t} f(t - \tau)u_\delta(\tau)d\tau; \quad (4.51) \]

The functions \( u_\delta(t) \) and \( \chi(\omega)/\sqrt{2\pi} \) form a Fourier transform pair (4.39)

\[ u_\delta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \chi(\omega)e^{-i\omega t}d\omega, \quad (4.52a) \]

\[ \frac{1}{\sqrt{2\pi}} \chi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_\delta(t)e^{i\omega t}dt; \quad (4.52b) \]

3. Finding the indicial response \( u_\eta(t) \) due to a unit step function source, then using (4.49) for the causal function \( f(t) \):

\[ u(t) = f(0)u_\eta(t) + \int_{0}^{t} f'(\tau)u_\eta(t - \tau)d\tau. \quad (4.53) \]

Indicial response and impulse response are related by (4.46) and (4.47).

4.5. Laplace Transform

In dealing with transient problems, the source functions in most cases are causal. By the condition of time invariance, the response is also a causal function. For problems of this type, the Laplace transform is frequently used. The Laplace transform \( F_\xi(s) \) of a function \( f(t) \) is defined as
\[ F_k(s) = \int_0^\infty f(t)e^{-st}dt. \tag{4.54} \]

The inversion formula is
\[ f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F_k(s)e^{st}ds, \tag{4.55} \]

where \( \gamma \) is a real constant which can have any value such that the path of integration on the complex \( s \)-plane lies to the right of all singularities of \( F_k(s) \).

The above pair of transforms can be derived formally from the Fourier integral. Consider the function
\[ f_1(t) = \begin{cases} e^{-\gamma t}f(t), & \text{if } \gamma > 0, \quad t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{4.56} \]

Replacing \( f_1(t) \) in the Fourier integral formula (4.10)
\[ f_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(\tau - t)f_1(\tau)d\tau}, \]

by (4.56), we have
\[ f(t) = \frac{1}{2\pi} e^{\gamma t} \int_{-\infty}^{\infty} e^{-i\omega t}d\omega \int_{0}^{\infty} f(\tau)e^{-(\gamma - i\omega)\tau}d\tau. \]

If in the above we set \( s = \gamma - i\omega \) and
\[ F_k(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau, \]
then

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\gamma(s) e^{(\gamma-i\omega)t} \, d\omega \]

\[ = \frac{1}{2\pi} \int_{\gamma-i\omega}^{\gamma+i\omega} F_\gamma(s) e^{st} \, ds. \]

The last equation is the inversion formula (4.55). It is thus seen that the Laplace transform is but another form of Fourier transform. In fact, for causal function \( f(t), F_\gamma(s) \) in (4.54) is just \( \sqrt{2\pi} \) times the Fourier transform \( F(\omega) \) when \( s \) is identified with \( -i\omega \).

Equations analogous to (4.11) through (4.24) can also be derived for the Laplace transform. They are listed in Table 4.2.
DIFFRAC TION AND STRESS CONCENTRATIONS

Table 4.2

PROPERTIES OF LAPLACE TRANSFORM

\[ f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} \, ds \]

\[ F(s) = \int_0^\infty f(t) e^{-st} \, dt \]

\[ a_1 f_1(t) + a_2 f_2(t) \]

\[ a_1 F_1(s) + a_2 F_2(s) \]

\[ f(at) \]

\[ F\left(\frac{s}{a}\right) \]

\[ f(t - t_0) \]

\[ e^{-ts} F(s) \]

\[ e^{st} f(t) \]

\[ F(s - s_c) \]

\[ \frac{d^n f(t)}{dt^n} \]

\[ s^n F(s) - s^{n-1} F(+0) \ldots \]

\[ \ldots F^{(n-1)}(+0) \]

\[ t^n f(t) \]

\[ (-1)^n \frac{d^n F}{ds^n} \]

\[ \delta(t) \]

\[ 1 \]

\[ h(t) \]

\[ \frac{1}{s} \]

Convolution Theorem:

\[ f_1(t) * f_2(t) = \int_0^t f_1(t - \tau) f_2(\tau) \, d\tau = \frac{1}{s} F_1(s) F_2(s) \]
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Chapter II

METHODS OF ANALYSIS

HAVING OUTLINED THE BASIC EQUATIONS OF MOTION for an elastic body, we shall present the methods to be used in analyzing the diffractions of elastic waves and dynamic stress concentrations. In an unbounded homogeneous medium, waves propagate without interruption at a constant speed and along a fixed path. But with the insertion of an obstacle in the medium, the path of the wave propagation is changed, and the obstacle, when excited by the otherwise undisturbed incident wave, acts as a secondary source which emits waves radially outward from itself. The deviation of the wave from its original path is known as the diffraction, and the radiation of secondary waves from the obstacle is called scattering. In an elastic medium, the obstacle may be in the form of a cavity or a crack, or a rigid body—generally, a substance with elastic moduli and density different from the medium's. As a result of diffraction and scattering, the stresses near the obstacle are different, sometimes higher, than the stresses at the same point produced by uninterrupted waves, a phenomenon known as dynamic stress concentration.

Wave diffractions are encountered also in acoustics, in optics, and in other branches of physics. Familiar examples would be the scattering of light by dust particles, diffractions of light by grating, and the scattering of sound by fog. The method of studying the diffraction of elastic waves is not much different from methods used with
other waves, owing to physical similarity and mathematical analogy. There is additional difficulty in the analysis, however, because of the presence in an isotropic elastic solid of two waves with different speeds, as opposed to one acoustical or electromagnetic wave in air.

Another distinction, as a matter of interest, is that with elastic waves we are often dealing with waves at low frequencies (long wavelengths), and in the region near the obstacle (the "near region"). Thus the asymptotic expansion methods developed in the study of the scattering of electromagnetic waves at high frequencies (short wavelengths) and far regions are not discussed here. Also, in connection with dynamic stress concentrations, the displacements and stresses associated with a wave at a point, instead of the overall wave scattering pattern, are what is sought. This may require in many cases a different computing technique.

Three methods of analysis will be discussed here: The method of wave functions expansion; The method of integral equation; and The integral transform. All of these have been tried for elastic waves.

To show the essence of the methods without undue mathematical complexity, the diffraction of the SH wave (see subsection 3.1 of Chapter I) by a cylindrical obstacle is chosen as the illustrative example. Because the polarization of the SH waves is assumed to be parallel to the length of a cylindrical insert, the scattered waves have the same polarization and wave velocity. As a result, only one type of wave, the SH wave, needs to be considered in the complete analysis.

In this regard, the methods of analysis presented here are similar to those used with diffraction and scattering of sound waves and electromagnetic waves by a cylindrical obstacle, which have been discussed
in many texts. Lamb's *Hydrodynamics* (1932)\(^{0.1}\) contains an excellent account of earlier works on the scattering and diffraction of waves of expansion (acoustic waves). The same subject is treated in the recent text *Theoretical Acoustics* (Chapter 8) by Morse and Ingard (1968), \(^{0.2}\) and in the treatise *Methods of Theoretical Physics* (Chapter 11) by Morse and Feshbach (1953). \(^{0.3}\) In *The Theory of Electromagnetism* by Jones (1964), \(^{0.4}\) more than two chapters are devoted to the scattering of electromagnetic waves; the same is true of the *Principles of Optics* by Born and Wolf (1965). \(^{0.5}\) Based on Huygens' principle and Kirchhoff's integral formulas, diffraction theories are treated in Baker and Copson's *The Mathematical Theory of Huygens' Principle* (1950)\(^{0.6}\) and in *Die Bewegungsweise in der Kirchhoffschen Theorie der Beugung* by von A. Rubinowicz (1966). \(^{0.7}\) The *Handbuch der Physik* \(^{0.8}\) (Vol. 25/1) contains the article *Theorie der Beugung* by Höhl, Maue, and Westphahl (1961), a very comprehensive treatment of the diffraction theory.

1. **SCATTERING OF SH WAVES BY A CIRCULAR CYLINDER**

CONSIDER A CIRCULAR CYLINDRICAL INCLUSION in an infinitely extended solid which is referred to a coordinate system as shown in Fig. 1.1. The cylinder is bounded by \(r = \alpha\) and it can either be made of a rigid and infinitely dense material, or else be vacuous. A shear wave defined by

\[
\begin{align*}
\varepsilon_x &= 0, & \varepsilon_y &= 0, & \varepsilon_z &= \varepsilon(x,y,t),
\end{align*}
\]

\(1.1\)
and

\[ \psi^i(t) = \omega_o e^{i(kx-\omega t)}, \quad \omega = k\sigma, \quad (1.2) \]

propagates in the positive \( z \)-direction with constant velocity \( \sigma \) and frequency \( \omega \). It has a constant amplitude \( \omega_o \) and wavelength \( 2\pi/k \). We use the superscript \( (i) \) to indicate an incident wave. Waves given by Eq. (1.2) may be generated by tangential forces distributed over a large plane (plane source) located far from the cylinder.

The problem so defined can be approximated by anti-plane strain (subsection 3.1, Chapter 1). If the \( z \)-axis is taken as the horizontal axis, the incident wave is a plane harmonic SH wave. Upon impinging on the surface of the cylinder, part of the incident wave is reflected. The scattered wave is also of the SH type and is represented by

\[ u_x = 0, \quad u_y = 0, \quad u_z = \omega(s)(x, y, t), \quad (1.3) \]

with a superscript \((s)\). Furthermore, the motion is at steady state as a consequence of the assumed simple harmonic source function in (1.2). The function \( \omega(s) \) is to be found from the solution of the wave equation.
(1-3.3) and boundary conditions. (When reference to equations in other chapters is made, the Roman numeral designates the chapter, followed by the equation number in Arabic numerals.)

Since the geometry of the scatterer is definable by one coordinate surface \((r = a)\) of circular cylindrical coordinate system, all field quantities will be expressed accordingly. First we collect the field equations in circular cylindrical coordinates \((r, \theta, z)\) for anti-plane strain. The nonvanishing displacement and stress components (see I-3.1 through I-3.5) are

\[
\begin{align*}
\varepsilon_z &= \omega(r, \theta, t), \\
\sigma_{rz} &= \mu \frac{\partial \omega}{\partial r}, \\
\sigma_{\theta z} &= \mu \frac{1}{r} \frac{\partial \omega}{\partial \theta},
\end{align*}
\]

and the wave equation is

\[
V^2 \omega = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \omega = \frac{1}{c^2} \frac{\partial^2 \omega}{\partial t^2}.
\]

For a rigid, immovable cylinder, the boundary condition is

\[
\omega(r, \theta, t) = 0, \quad \text{at } r = a.
\]

For a cavity, there is no stress acting on the surface and the boundary condition is

\[
\sigma_{rz} = \mu \frac{\partial \omega(r, \theta, t)}{\partial r} = 0, \quad \text{at } r = a,
\]
or simply

\[ \frac{\partial \omega}{\partial r} = 0, \quad \text{at } r = a. \]  \hspace{1cm} (1.7)

Besides, there should be no disturbance at \( r = \infty \) other than the incident waves.

1.1 The Method of Wave Functions Expansion

Following the Huygens' principle, each particle on the boundary of the cylinder acts as a secondary source generating wavelets after it has been struck by the incident wave. The waves so generated constitute the scattered waves and are represented by an unknown function \( \omega^{(s)}(r, \theta, t) \) as in (1.3). To determine the \( \omega^{(s)} \) for a circular cylinder, we note first that the wave functions (Ref. 1.1)

\[ \bar{h}^{(1),(2)}(kr) \cos n\theta, \quad k = \omega/c_s, \quad n = 0, 1, 2, \ldots, \]  \hspace{1cm} (1.8)

satisfy the Helmholtz equation in plane polar coordinates

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k^2 \right) \omega = 0. \]

The Hankel functions of the first kind \( H_n^{(1)}(z) \), and of the second kind \( H_n^{(2)}(z) \), are related to the Bessel functions of the first kind \( J_n(z) \) and the second kind \( Y_n(z) \) by

\[ H_n^{(1),(2)}(z) = J_n(z) \pm iY_n(z). \]  \hspace{1cm} (1.9)

Wave functions like (1.8) are usually obtained by the method of separation of variables. If \( \omega(r, \theta) = R(r) \Theta(\theta) \), where \( R \) and \( \Theta \) are
functions of \( r \) and \( \theta \) respectively, the Helmholtz equation can be separated into two ordinary differential equations

\[
r^2 P'' + r P' + (k^2 r^2 - \mu^2) P = 0, \\
\theta'' + \mu^2 \theta = 0,
\]

where \( \mu \), an integer in this case, is the separation constant. From these ordinary equations, solutions \( H^{(1),(2)}_\mu(kr) \) and \( \cos \mu \theta, \sin \mu \theta \) are derived.

When the wave functions in (1.8) are combined with the time factor \( e^{-i\omega t} \), they represent cylindrical waves generated by a line source along the \( z \)-axis. From the asymptotic behavior of \( H^{(1),(2)}_\mu(z) \)

\[
H^{(1),(2)}_\mu(z) \to \frac{2}{\pi z} e^{\mp i(z - \frac{1+2\mu}{4} \pi)},
\]

we find the function \( H^{(1)}_\mu(kr) \cos \mu \theta e^{-i\omega t} \) representing a diverging or outgoing cylindrical wave; that is, the waves as generated by the line source propagating away from the center. Similarly, the function \( H^{(2)}_\mu(kr) \cos \mu \theta e^{-i\omega t} \) represents a converging or incoming cylindrical wave. The boundary condition at \( r = \infty \) in this problem eliminates the latter from the scattered waves, hence only the function with \( H^{(1)}_\mu(kr) \) will be used.

A linear combination of the wave functions in (1.8) is also a wave function. A special case is

\[
1/2 \left[ H^{(1)}_\mu(kr) + H^{(2)}_\mu(kr) \right] \cos \mu \theta = J_\mu(kr) \cos \mu \theta,
\]
which when combined with $e^{-i\omega t}$ represents a standing shear wave inside a circular cylinder:

$$\omega(r, \theta, t) = \omega_0 \sum_{n=0}^{\infty} C_n J_n(kr) \cos n\theta e^{-i\omega t}. \quad (1.10)$$

The scattered wave function $\omega(s)$ is expressed as

$$\omega(s)(r, \theta, t) = \omega_0 \sum_{n=0}^{\infty} A_n H_n^{(1)}(kr) \cos n\theta e^{-i\omega t}, \quad (1.11)$$

where $A_n$ and $C_n$ are unknown coefficients. The total wave field is then given by the addition of $\omega(z \theta)$ in (1.2) and $\omega(e)$ in (1.11),

$$\omega = \omega(r, \theta, t) = \omega(z \theta) + \omega(e), \quad (1.12)$$

what remains is the determination of the coefficients $A_n$ from the boundary conditions.

To achieve that purpose, we first write

$$\omega(z \theta) = \omega_0 e^{-ikr \cos \theta} e^{-i\omega t}.$$

The exponential function $e^{-ikr \cos \theta}$ is periodic in $\theta$ with period $2\pi$.

It can be expanded into Fourier series in complex form

$$e^{-ikr \cos \theta} = \sum_{n=-\infty}^{\infty} c_n(r) e^{in\theta},$$

with

$$c_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ikr \cos \theta} e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ikr \cos \theta} \cos n\theta d\theta.$$
From the integral definition of the Bessel function

\[ 2\pi n J_n(z) = \int_0^{2\pi} e^{iz\cos\theta} \cos n\theta d\theta, \quad (1.13) \]

we obtain

\[ c_n(r) = i^n J_n(kr), \]

and

\[ e^{ikr \cos \theta} = \sum_{n=-\infty}^{\infty} e^{in\theta} \frac{e^{in\theta}}{n!} n! J_n(kr) \cos n\theta, \quad (1.14) \]

where

\[ \alpha_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \ldots \end{cases} \]

The incident plane SH wave in polar coordinates is then represented by

\[ \psi(t) = \psi_0 \sum_{n=0}^{\infty} \alpha_n i^n J_n(kr) \cos n\theta e^{-i\omega t}. \quad (1.15) \]

This equation may be regarded as a superposition of the infinite number of cylindrical standing waves given by (1.10), resulting in a pattern of wave propagation along the \( x \)-axis.

Omitting the time factor \( e^{-i\omega t} \) from here on, we represent the total wave in (1.12) by an infinite series
\[ \omega = \omega_0 \sum_{n=0}^{\infty} \left( e_n \gamma_n^2 (kr) + A_n H_n^{(1)} (kr) \right) \cos n\theta. \]  

(1.16)

The corresponding stresses are

\[ \sigma_{rr} = \mu \frac{\omega}{r} \sum_{n=0}^{\infty} \left( e_n \gamma_n^2 (kr) - k r J_{n+1} (kr) \right) \cos n\theta, \]

\[ + A_n [2 J_n (kr) - k r J_{n+1} (kr)] \cos n\theta, \]  

(1.17)

\[ \sigma_{\theta \theta} = -\mu \frac{\omega}{r} \sum_{n=0}^{\infty} \left( e_n \gamma_n^2 (kr) + A_n H_n^{(1)} (kr) \right) \sin n\theta, \]

where use has been made of the formulas for the derivatives of Bessel functions (Eq. III-2.36). The unknown coefficients \( A_n \) will now be determined for four types of boundary conditions.

(a) Fixed-Rigid Cylinder.

At the surface of a rigid cylinder \( r = a \), which is held immobile by some external constraints, the boundary condition is given by (1.6). Thus

\[ \omega(a, \theta) = \omega_0 \sum_{n=0}^{\infty} \left( e_n \gamma_n^2 (ka) + A_n H_n^{(1)} (ka) \right) \cos n\theta = 0. \]

Since the \( \cos n\theta \) are orthogonal in the interval \((0, 2\pi)\), for all \( n \)'s, coefficients in each term of the above series must vanish. It then follows

\[ A_n = -e_n \gamma_n \frac{J_n (ka)}{H_n^{(1)} (ka)}, \quad n = 0, 1, 2, \ldots \]  

(1.18)

Substitution of (1.18) into (1.16) and (1.17) completes the result.
(b) Cavity.

For a stress-free cavity $\sigma_{rr} = 0$ at $r = a$. The first of (1.17) then leads to

$$A_n = -\epsilon_n i^n \frac{j_n'(ka)}{H_n^{(1)'}(ka)} \quad n = 0, 1, 2, \ldots$$

$$= -\epsilon_n i^n \frac{n j_n(ka) - k a j_{n+1}(ka)}{n j_n^{(1)}(ka) - k a H_{n+1}^{(1)}(ka)}.$$  \(1.19\)

The rest is the same as the case of a rigid cylinder.

(c) Elastic Cylinder.

If the scatterer is made of elastic material with shear modulus $\mu_f$ and density $\rho_f$, which are different from the surrounding medium, part of the incident wave is refracted by the cylindrical surface and generates a standing wave inside the cylinder. Following (1.10), we represent the refracted wave by

$$\omega^{(f)}(r, \theta, t) = -\omega_0 \sum_{n=0}^{\infty} \xi_n j_n(k_f r) \cos n \theta e^{-i\omega t},$$  \(1.20\)

with $k_f = \omega/\sigma_{eff}$ and $\sigma_{eff} = (\mu_f/\rho_f)^{1/2}$. The total wave outside the cylinder is still given by $\omega(t) = \omega(\theta) + \omega(\phi)$ and the total wave inside it is $\omega^{(f)}$.

Boundary conditions for elastic inclusion are the continuity of stresses and displacements across the interface. For an SH wave scattered by a circular elastic cylinder, they are, at $r = a$,

$$\omega(\theta) + \omega(\phi) = \omega^{(f)},$$

$$\mu \frac{\partial}{\partial r} [\omega(\theta) + \omega(\phi)] = \nu_f \frac{\partial \omega^{(f)}}{\partial r}.$$  \(1.21\)
Substitution of (1.11), (1.15), and (1.20) into the above conditions leads to the following result:

\[
A_n = (-i^n \epsilon_n / \lambda) [\nu \omega_j f_n(k \alpha) J_n'(k \alpha) - \nu k J_n'(k \alpha) J_n(k \alpha)],
\]

\[
C_n = (-i^n \epsilon_n / \lambda) \nu k [J_n'(k \alpha) H_n(k \alpha) - H_n'(k \alpha) J_n(k \alpha)],
\]

\[
\Delta = \nu \omega_j f_n J_n'(1)(k \alpha) - \nu k J_n'(1)(k \alpha).
\]

(1.22)

This result reduces to (1.18) when \( \nu_j = \infty \) and to (1.19) when \( \nu_j = 0 \).

(d) Movable-Rigid Cylinder.

The result in case (a) shows that with the boundary condition (1.6), there is a net force

\[
S(t) = \int_0^{2\pi} [g_{pa}(\alpha, \theta, t)] d\theta,
\]

(1.23)

acting on the surface of the rigid cylinder. This force is balanced by some external constraint force which fixes the cylinder. In the absence of such a constraint force, the cylinder will translate as a rigid body with its motion \( \dot{W}(t) \) being governed by the usual kinetic equation for a rigid body. Hence the boundary conditions for a movable-rigid cylinder are

\[
\omega(t) + \omega(t) = \dot{W}(t),
\]

\[
m \ddot{W}(t) = S(t),
\]

(1.24)

where \( m = \pi a^2 \rho \) is the mass of the rigid cylinder per unit length and \( S \) is given by (1.23). In this formulation, both \( S \) (and therefore the \( \dot{W}(t) \) which it includes) and \( \omega(t) \) contain the unknown coefficient \( A_n \).
For steady-state wave motion, $W(t) = W_0 e^{-i\omega t}$, and a substitution of (1.17) in the second part of (1.24) gives rise to

$$-\rho_f \omega^2 W_0 = \mu \omega \int_0^{2\pi} \left[ e^{i\alpha \xi_j \beta n_n} (k\alpha) + A_n \bar{B}_n (k\alpha) \right] \cos \theta \, d\theta \begin{cases} 0, & n \neq 0, \\ \{2\pi i \omega \xi_j \beta n_n \}, & n = 0. \end{cases}$$

Hence, the $A_n$ in the series solution in the movable-rigid case is the same as that for the fixed-rigid case when $n \neq 0$. For $n = 0$, the first part of (1.24) and the result above combined, yield

$$A_0 = \frac{-[k\alpha \nu_0 (k\alpha) + 2\gamma \nu_0 (k\alpha)]}{[k\alpha B_0 (k\alpha) + 2\gamma B_0 (1) (k\alpha)]},$$

$$A_n = -2i \beta j_\beta n_n (k\alpha) \bar{H}_n (1) (k\alpha), \quad n \neq 0, \quad (1.25)$$

where $\gamma = \rho / \rho_f$ is the density ratio of the matrix material to the cylinder. For a very dense material inclusion, $\gamma \rightarrow 0$, and $A_0$ becomes the same as that in (1.18).

We have thus illustrated the method of wave-function expansion as applied to the diffraction of an SH wave by a circular cylinder. The essence of this method is in expressing the scattered waves in terms of a series of wave functions. The coefficients in the series expansion are determined from the boundary conditions and the magnitude of the incident waves. In this method, success hinges on the finding of the wave functions for particular curvilinear coordinates which are suitable for the geometry of the scatterer.

Only a few regular-shaped scatterers such as circular, elliptical,
and parabolic cylinders, spheres, etc., can be described by the familiar orthogonal curvilinear coordinates. The corresponding wave functions are usually derived by separating the variables in the wave equations. This results in several ordinary differential equations, one for each coordinate and the time. We shall speak of a solution such as $H_n^{(1)}(kr)$ for the equation with variable $r$ or an equivalent radial coordinate, as the radial function, and the others, such as $\cos n\theta$, as angular functions. The product of the radial and angular functions constitutes the wave functions.

For a scalar wave equation $\nabla^2 \varphi + k_1^2 \varphi = 0$, solutions by separation of variables are obtainable for eleven coordinates systems (Chapter 5 of Ref. 0.3). They are: (1) rectangular, (2) circular cylindrical, (3) elliptic cylindrical, (4) parabolic cylindrical, (5) spherical, (6) conical, (7) parabolic, (8) prolate spheroidal, (9) oblate spheroidal, (10) ellipsoidal, and (11) paraboloidal. For a vector wave equation $\nabla^2 \mathbf{w} + k_2^2 \mathbf{w} = 0$, the number of separable coordinates is further limited to the first six, as discussed in subsection I-2.7. With the exception of a few axial symmetric problems, this just about accounts for the shapes of scatterers which can possibly be treated by this method. For problems of anti-plane strain and plane strain, this method is confined to scatterers that have the cross section of a circle, an ellipse, or a parabola, because the obstacle must be in the shape of a long cylinder.

 Discussions on the wave functions for the elliptical and parabolic cylinder coordinates and the spherical coordinates are postponed until later chapters. Here we just mention that even when the coordinates
are separable, there is still a major difficulty in the scattering of P or SV waves when the method of wave-function expansion is applied. This will be elaborated upon in Chapter IV. For a scatterer with its shape not describable by one of the coordinates mentioned above, appeal to other methods becomes necessary.

Before taking up the second method of analysis, we shall discuss a few features of scattering phenomena. Although we use the method of wave-function expansion to carry out the analysis whenever it becomes necessary, the discussion is not confined to any particular method of analysis.

1.2. Rigid Body Scattering and Radiation Scattering

The examples in the previous section show that the scattered waves produced by an elastic inclusion differ from those produced by an otherwise identical but rigid inclusion in that the elastic cylinder undergoes additional forced vibrations when excited by the waves outside the cylinder. On top of the usual scattering of an incident wave \( \omega(z) \) by a rigid obstacle, these vibrations give rise to radiation of waves from the elastic cylinder. Thus we may approach the elastic cylinder problem by considering the total scattered wave as a superposition of a rigid body scattering, \( \omega_1^{(e)} \), which is determined by assuming the surface of the elastic cylinder to be rigid, and a radiation scattering, \( \omega_2^{(e)} \), which is generated by the vibrating surface, the magnitude of radiation being proportional to the unspecified amplitude of vibration. The total wave outside the cylinder is \( \omega(z) = \omega(z) + \omega_1^{(e)} + \omega_2^{(e)} \) which contains a yet-unknown amplitude of vibration at the
surface of the cylinder. It is this total wave which exerts a surface force \( \int [i \omega (t) / \partial r] dA \) and excites the elastic cylinder to vibrate. By analyzing the forced vibration of an elastic cylinder whose amplitude of vibration at the surface must be compatible with that assumed in calculating \( \omega_2(s) \), the unknown amplitude of surface vibration and, hence, the magnitude of the radiation scattering, may be calculated.

As an illustration, we shall treat by this procedure the same problems in the previous section that were originally proposed by Junger (1953) \(^{(1.2)}\) for investigating the scattering of sound waves by a cylindrical shell. This was extended by Thau (1967) \(^{(1.3)}\) for the scattering by a movable-rigid inclusion in an elastic medium where the total wave is made of the scattered wave (by a rigid but fixed inclusion) and the radiation from the same rigid inclusion in translation and rotation.

(a) Elastic Cylinder.

Instead of solving the scattering by a circular cylinder as in Eqs. (1.20) through (1.22), we decompose the scattered wave \( \omega_2(s) \) into two parts: \( \omega_1(s) \) and \( \omega_2(s) \). The incident wave \( \omega_1(t) \) is still represented by (1.15) and \( \omega_2(s) \) by (1.11), with \( A_n = -e^{-i \eta_n^2 \eta} (k \alpha) / H_\eta(k \alpha) \) as in (1.18), for a fixed rigid cylinder. (The superscript \( (l) \) for Hankel functions will be omitted henceforth.) Assume the elastic cylinder is undergoing a forced vibration simultaneously, and let its motion at the surface be represented by

\[
\delta = \omega_0 \sum_n \cos \eta \alpha \cos \omega t, \quad r = \alpha, \tag{1.26}
\]

where \( \alpha_n \) are to be specified later. Waves radiated by a vibrating circular cylinder are also represented by series of wave functions
like (1.11) where the unknown coefficients are determined from the boundary conditions at the vibrating surface, which is assumed in (1.26). Thus the radiation scattering part is given by

$$\omega_2^{(s)}(r, \theta, t) = \omega_o \sum_{n \geq 0} \left[ H_n(kr) / H_n(ka) \right] \cos n \theta e^{-i \omega t}. \quad (1.27)$$

The total scattered wave is $\omega^{(s)} = \omega_1^{(s)} + \omega_2^{(s)}$, and the total wave outside the cylinder is $\omega^{(t)} = \omega^{(i)} + \omega_1^{(s)} + \omega_2^{(s)}$.

To determine the unspecified coefficients $a_n$, we investigate the forced vibration of an elastic circular cylinder. Mathematically the problem is formulated as to find the solution of the wave equation

$$\frac{\partial^2}{\partial t^2} \omega = c_{sf}^2 \nabla^2 \omega, \quad c_{sf}^2 = \left( \frac{v_f}{c_f} \right)^2$$

and boundary condition

$$\mu_f \frac{\partial \omega}{\partial r} = \mu \frac{\partial \omega^{(t)}}{\partial r}, \text{ at } r = a. \quad (1.28)$$

We first write a solution for $\omega$ as

$$\omega(r, \theta, t) = \omega_o \sum_{n \geq 0} \lambda_n J_n(k_r r) \cos n \theta e^{-i \omega t}. \quad (1.29)$$

where $\lambda_n$ are unknowns and $k_r = \omega / c_{sf}$. Since this motion must be compatible with that assumed in (1.26), we have

$$\lambda_n J_n(k_r a) = a_n.$$

From the boundary condition (1.28) with $\omega$ given by (1.29) and $\omega^{(t)}$ by the sum of (1.15), (1.18), and (1.27), the unspecified amplitude coefficients $a_n$ are found as
\begin{equation}
\alpha_n = \nu e^{i \eta k (H_n'(ka)J_n'(ka) - J_n'(ka)H_n'(ka))} / \Delta, \tag{1.30}
\end{equation}

where \( \Delta \) is the same as in (1.22). The rest of the solution also agrees with (1.22).

We note that this procedure, which is based on a physical intuition, is also mathematically rigorous. What we have done is to replace \( \omega^{(s)} \) by \( \omega^{(s)}_1 + \omega^{(s)}_2 \) and to satisfy the boundary conditions (1.21) in three steps:

(i) \( \omega^{(s)}_1 + \omega^{(s)}_2 = 0 \),

(ii) \( \omega^{(s)}_2 = 0 \), 

(iii) \( \frac{3}{2r} \left[ \omega^{(s)}(r) + \omega^{(s)}_1 + \omega^{(s)}_2 \right] = \mu f \frac{\partial \omega}{\partial r} \),

The \( \omega \) in (ii) takes over the role of \( \omega^{(f)} \) in (1.21) and (iii) is analogous to the second equation in (1.21). For the simple example chosen, the previous method is much more direct and elegant and there is no reason to follow this seemingly laborious procedure. However for a more complicated problem where the continuity conditions at the interface, like (1.21), are difficult to satisfy in the mathematical analysis, this procedure is advantageous, as one can treat the forced vibration problem in the third step by applying some results that are already well known, or by using an approximation method. A case of interest is determining the scattering of an elastic cylinder reinforced by stiffening rings placed at regular intervals along the length. A direct approach would be very difficult, if not impossible. But by this decomposition procedure, one is able to calculate the rigid-body scattering and radiation scattering as usual, except that the latter
contains unspecified amplitudes. These amplitudes may be calculated from an approximate analysis of the forced vibration of a cylinder with stiffness. Junger in his original paper suggested setting up the equations of vibrations in Lagrangian form, with the unspecified amplitudes as the generalized coordinates and the surface tractions generated by the total wave as the generalized forces, and then proceeding to solve a system of Lagrangian equations.

(b) Movable-Rigid Cylinder.

The solution (1.25) for a rigid but movable circular cylinder can also be derived by this procedure. The rigid-body scattering part \( \omega_1^{(s)} \) is determined from the boundary condition

\[
\omega(t) + \omega_1^{(s)} = 0, \quad \text{at } r = a, \tag{1.32a}
\]

and the radiation scattering part from the condition

\[
\omega_2^{(s)} = \omega(t), \quad \text{at } r = a, \tag{1.32b}
\]

where \( \omega(t) \) is the motion at the surface of the cylinder with an unspecified amplitude. Finally, the motion of the cylinder is governed by the kinetic equation for the translation of a rigid cylinder

\[
m \ddot{\omega} = S_1 + S_2, \tag{1.32c}
\]

where

\[
S_1 = \int_0^{2\pi} u \left[ \frac{\partial}{\partial r} (\omega(t) + \omega_1^{(s)}) \right]_{r=a} a \, d\theta,
\]

\[
S_2 = \int_0^{2\pi} u \left[ \frac{\partial \omega_2^{(s)}}{\partial r} \right]_{r=a} a \, d\theta.
\]
are respectively the surface shearing force due to the rigid body scattering and the radiation scattering waves. We note that the combination of these three boundary conditions is equivalent to (1.24).

The \( \omega^{(i)} \) and \( \omega^{(s)}_1 \) are the same in the case of a fixed rigid cylinder. When the cylinder translates as a rigid body along its axis, its motion is represented by

\[
\omega(t) = \alpha \omega_o e^{-i \omega t},
\]

where \( \alpha \) is an unknown factor. Thus, the radiated wave from the cylinder is

\[
\omega^{(s)}_2 = \alpha \omega_o \left[ H_0^0(kx) / H_0^0(kx) \right] e^{-i \omega t}.
\]  

(1.33)

The corresponding shearing forces are:

\[
S_1 = 2 \pi \omega_o k \left[ J_0'(ka) - J_0(ka) H_0^0(ka) / H_0(ka) \right] e^{-i \omega t},
\]

\[
S_2 = 2 \pi \omega_o k \alpha \left[ H_0'(ka) / H_0(ka) \right] e^{-i \omega t}.
\]

After substituting the pertinent quantities in (1.32c) we can solve for the unknown factor \( \alpha \) with the result

\[
\alpha = 2 \gamma \left[ J_0'(ka) H_0^0(ka) - J_0(ka) H_0'(ka) \right] \left[ k \alpha H_0^0(ka) + 2 \gamma H_0'(ka) \right]^{-1},
\]

(1.34)

where \( \gamma = \sigma / \rho \) was introduced in (1.25). The total wave \( \omega(t) = \omega^{(i)} + \omega^{(s)}_1 + \omega^{(s)}_2 \) with \( \alpha \) being given above is the same as (1.25).

We emphasize again that this decomposition procedure has the advantage of being more amenable to approximate analysis, since each part of the scattered waves may be treated separately.
1.3. Dynamic Stress Concentrations and Intensity of Scattered Energy

All results of scattering of waves obtained by the wave function expansion method are expressed in complicated series of special functions. The results are of little use until the special functions and the series are evaluated. Before the invention of automatic electronic computers the computation itself became a major task, and despite many researchers' effort throughout the world the information compiled had been rather meager. One of the major interests in studying the scattering of waves has been the determination of the intensity of energy carried by the scattered wave, which as mentioned in the Historical Introduction was begun by Lord Rayleigh. Another area of interest has been understanding the diffraction of waves by an obstacle, the original object of Clebsch's investigation. Only recently has the dynamic stress concentration—the rise of stresses in a localized region near or at the scatterer when it obstructs the passage of an otherwise uninterrupted elastic wave—become a subject of interest.

We shall illustrate the calculation of dynamic stress concentrations near the scatterer, and calculation of the scattered energy intensity at a distance far from the obstacle for the case where the wavelength of the incident wave is long in comparison with the dimension of the scatterer. The limiting case as the ratio of the dimension to the wavelength approaches zero is usually referred to as Rayleigh scattering. When the incident wavelength is much shorter, the series solution often converges too slowly to be useful. In that case, the series of wave functions can be evaluated first by replacing the sum of the series like $\sum f(n)$ by a contour integral of the type $e^{-\sqrt{2}(z) \cot (\pi z) dz}$ and then calculating the asymptotic value of the integral.
(Section 4.5 of Ref. 0.3). For studying diffraction phenomena with very short wavelengths and high frequencies, other asymptotic methods (e.g., Ref. 1.4) are more useful.

(a) Dynamic Stress Concentration.

Consider an elastic medium with a cylindrical cavity under the impact of an SH wave. Without the cavity, the only nonzero stress everywhere in the medium as given by the incident wave \( \omega(i) \) in (1.2) is

\[
\frac{\sigma}{x^2} = \mu \omega(i) / \partial x = \nu \kappa \omega \dot{e}^{i(kx-\omega t+\pi/2)}.
\]

(1.35)

The maximum value is \( \sigma = |\sigma| = \kappa \omega \). In polar coordinates,

\[
\begin{align*}
\sigma_{r^2} &= \sigma \cos \theta \dot{e}^{i(kx-\omega t+\pi/2)}, \\
\sigma_{\theta z} &= -\sigma \sin \theta \dot{e}^{i(kx-\omega t+\pi/2)}.
\end{align*}
\]

In the presence of a cavity, the stresses are given by (1.17), with \( A_n \) being determined in (1.19). At the surface of the cavity \( r = a \),

\[
\sigma_{rz} = 0
\]

(1.36)

\[
\sigma_{\theta z} = -\sigma \dot{e}^{-i(\omega t-\pi/2)} \sum_{n=0}^{\infty} \frac{\epsilon_n^{n-1}}{\kappa a} \left[ J_n(ka) - \frac{n\dot{J}_n(ka)}{\dot{H}_n(ka)} \frac{H_1(ka)}{H_n(ka)} \right] \sin \theta.
\]

The series when summed is a complex number of the form \( A + iB \) for a given angle \( \theta \). As in Eq. 1-2.41, we take the real part of the product \( (A + iB)e^{-i(\omega t-\pi/2)} \) or \( A \cos(\omega t-\pi/2) + B \sin(\omega t-\pi/2) \) as the solution.

Thus during one complete cycle with a period \( T = 2\pi/\omega \), the real part
\[ A \text{ is the stress value at } t = T/4, \text{ and the imaginary part } B \text{ the stress at } t = T/2. \] The peak stress is given by \((A^2 + B^2)^{1/2}\) which is the absolute value of the complex number \(A + iB\) and will be denoted as 

\[ |\sigma_{\theta z}|. \]

As \(ka \to 0\), \(J_n(ka) \to (ka)^n/2^nn!\), and \(B_n(ka) \to -i2^n(n-1)!/\pi(ka)^n\); thus, in (1.17),

\[ \sigma_{\theta z}(r, \theta) = \sigma_o(1 + a^2/r^2) \sin \theta, \quad (1.37) \]

as the terms with \(n > 1\) in the series are of the order \((ka)^{n-1}\). Since \(\omega = (ka)(\sigma_o/\alpha)\), the zero limit for \(ka\) is also the static limit of the dynamic problem. Our analysis shows that for a cylindrical cavity in an elastic solid which is subjected to statically applied shearing stress \(\sigma_{zz} = \sigma_o\), the \(\sigma_{\theta z}\) at the locations \(\theta = \pm \pi/2\), \(r = a\), are \(2\sigma_o\), a concentration of stress with a factor 2.

The corresponding problem in elastostatics is usually solved by finding a solution satisfying the static equation in anti-plane strain

\[ \nabla^2 \omega(r, \theta) = 0, \]

and the boundary conditions

\[ \sigma_{\theta z} = \mu \omega/r \theta = -\sigma_o \sin \theta, \quad \text{as } r \to a, \]

\[ \sigma_{rr} = \mu \omega/r \theta = 0, \quad \text{at } r = a. \]

The solution can be found easily as

\[ \omega = (\sigma_o/\mu)(1 + a^2/r^2) r \cos \theta. \quad (1.38) \]

For other values of \(ka\), the series (1.36) is evaluated with the aid of a computer. We normalize all stresses by \(\sigma_o\) and define \(\left| \sigma_{\theta z}/\sigma_o \right|\) as
as the dynamic stress concentration factor. At the boundary \( r = a \), the values of \( |\sigma_{\theta_2}/\sigma_0| \) as a function of angle \( \theta \) are shown in Fig. 1.2 for three different normalized wave numbers \( k\alpha \). We note that when \( k\alpha = 0.1 \), the angular distribution for \( |\sigma_{\theta_2}/\sigma_0| \) is nearly the same as

\[ \text{Fig. 1.2} \quad |\sigma_{\theta_2}| \text{ vs } \theta \text{ at Various } k\alpha \]

that for the static value and is symmetric about the axis \( \theta = \pi/2 \). For \( k\alpha = 2.0 \) the distribution is asymmetric and stresses are larger on the illuminated side \( (\pi < \theta < \pi/2) \) than that on the shadow side \( (\pi/2 < \theta < 0) \). At \( \theta = \pi/2 \), the real, imaginary part as well as the absolute values of \( \sigma_{\theta_2}(\alpha, \pi/2)/\sigma_0 \), are shown in Fig. 1.3 as the wave number \( k\alpha \) increases. At \( k\alpha = 0.4 \), the dynamic stress concentration factor is about 2.1, which is 5 percent larger than the static value. As the frequency increases, the stress concentration factor is reduced. This is the reason that low frequency and long wavelength limit analysis is an important consideration in the study of dynamic stress concentrations.
(b) Scattered Energy.

Across a surface $A$ with a unit normal vector $\mathbf{n}$, the rate of energy flow is given on page 19, Ref. 1.5.

$$\dot{E} = -\iiint_A \mathbf{n} \cdot \mathbf{\sigma}_{ij} \mathbf{u}_j \, dA,$$

(1.39)
in which $E$ is the total energy (strain energy and kinetic energy), $\mathbf{n} \cdot \mathbf{\sigma}$ is the stress vector at the surface element $dA$, and $\dot{\mathbf{u}}$ is the particle velocity. The integrand is the actual rate of work done by the surface traction per unit area, and by the conservation law of energy it equals the flow of energy across a unit area per unit time.

In the case of steady-state waves, both $\sigma_{ij}$ and $\mathbf{u}_j$ are harmonic functions of time which can be written as

$$\sigma_{ij}(x_k, t) = \overline{\sigma}_{ij}(x_k)e^{-i\omega t},$$
\[ u_j(x_k, t) = \bar{u}_j(x_k)e^{-i\omega t}, \quad (1.40) \]

where \( \sigma_{ij} \) and \( \bar{u}_j \) are complex functions. For calculating the energy, we take either the real or the imaginary parts of \( \sigma_{ij} \) and \( u_j \). Since \( \text{Re} u_j = (u + u^*)/2 \) where an asterisk indicates the complex conjugate of the corresponding quantity, we have, in place of (1.39),

\[ \dot{E} = -\frac{1}{4} \iint_A \eta (\sigma_{ij} \dot{u}_j + \sigma_{ij} \dot{u}_j^\star + \sigma_{ij}^\star \dot{u}_j + \sigma_{ij}^\star \dot{u}_j^\star) dA. \]

Furthermore, for periodic waves it is the average flow of energy over one period \( T = 2\pi/\omega \) which has any physical significance. Because

\[ \frac{1}{T} \int_0^T e^{-i\omega t} dt = 0, \]

we obtain the time average of energy flow as

\[ \text{Ave}(\dot{E}) = \frac{1}{T} \int_0^T \dot{E} dt = -\frac{1}{4} i\omega \iint_A \eta (\sigma_{ij} \dot{u}_j^\star - \sigma_{ij}^\star \dot{u}_j) dA. \quad (1.41) \]

The above equation is the basic one in calculating time average of energy flow for elastic waves.

For the plane incident SH wave (1.2), the time average of energy flux per unit area is

\[ \text{Ave}(\dot{e}) = \frac{\text{Ave}(\dot{E})}{A} = -\frac{1}{4} i\omega (\sigma_{xz}^\star w - \sigma_{xz} w^\star), \]

where \( \sigma_{xz} = \bar{\sigma}_{xz} e^{-i\omega t} \) is given in (1.35). After substitution, we find
that the time average of the energy carried by the incident wave per unit area is

$$\text{Ave}(\hat{\mathcal{E}}) = \frac{1}{2} \mu k \omega_o \omega = \frac{1}{2} \sigma_o \omega_o,$$

(1.42)

where $\sigma_o = k \omega_o$ is the maximum stress of the incident wave.

For a circular cylindrical obstacle, the scattered waves are represented by (1.11). Hence the energy flux across a cylindrical surface $r = R$ (per unit length of the cylinder) is

$$\text{Ave}(\hat{\mathcal{E}}) = -\frac{1}{2} i \omega \int_0^{2\pi} \left[ \sigma_{\tau \zeta}^{s} \sigma_{\tau \zeta}^{*} \right]_{r=R} Rd\theta.$$

(1.43)

Substituting the scattered wave displacement $\omega^{(s)}$ and stress $\sigma^{(s)}_{\tau \zeta}$ into the above equation leads to

$$\text{Ave}(\hat{\mathcal{E}}) = -\frac{1}{2} \pi \omega_o \omega^2 \sum_{n=0}^{\infty} \left( 2/\varepsilon_n \right) A_n \mathcal{A}_n^{*} k R \left[ H_n^{'}(k R) H_n^{*}(k R) - H_n^{*}(k R) H_n^{'}(k R) \right],$$

where use has been made of the orthogonality condition

$$\int_0^{2\pi} \cos m \theta \cos \eta \theta d\theta = \pi \delta_{mn}, \quad m \neq n \neq 0,$$

As $k R \to \infty$,

$$H_n^{'}(k R) \to (2/\pi R)^{1/2} e^{i [k R - (n+1/2) \pi]/2},$$

$$H_n^{*}(k R) \to i H_n(k R),$$

we thus find at large distance $R$,
Ave(\dot{E}) = \mu k c_S \nu^2 \left[ 2A_0^* A_0 + \sum_{n=1}^{\infty} A_n^* A_n \right]. \quad (1.44)

A natural reference to compare this total scattered energy against is the time average of the energy flux per unit area by the incident wave (1.42). Denoting the ratio of these two energies by \( \gamma \), we have

\[ \gamma = \frac{\text{Ave}(\dot{E})}{\text{Ave}(\dot{\epsilon})} = \frac{2}{k} \left[ 2A_0^* A_0 + \sum_{n=1}^{\infty} A_n^* A_n \right]. \quad (1.45) \]

Since \( \gamma \) has the dimension of an area—per unit length is implied in Ave(\dot{E})—it is now referred to as the scattering cross section. The values for the coefficients \( A_n \) depend on the boundary conditions of the obstacle.

For the case of a cylindrical cavity, the \( A_n \)'s are given by (1.19). As \( k \alpha \to 0 \)

\[ A_0 = -i \pi (k \alpha/2)^2, \]

\[ A_n = 2\pi i^{n+1} [n! (n - 1)!]^{-1} (k \alpha/2)^{2n}, \quad n \geq 1. \]

The most important terms in (1.44) for small \( k \alpha \) correspond to \( n = 0, \ n = 1 \). Omitting the rest, we have the scattered energy for a circular cylindrical cavity,

\[ \text{Ave}(\dot{E}) = \sigma_0 \sigma_S \nu^2 \left[ \frac{3}{8} \pi^2 (k \alpha)^4 \right]. \quad (1.46) \]

Since \( 2\pi/\lambda \) is the wavelength of the incident wave, the scattered energy is indeed inversely proportional to the fourth power of the wavelength, a law first discovered by Rayleigh.
If one is interested in the change in directionality of the scattered energy as the wavelength is changed, the energy flux per unit area which is given by the integrand in Eq. (1.43) is of interest. For SH waves, it is given by

\[
\text{Ave}(\phi) = -\frac{i\omega}{4 \rho_0 c_0} \sum_{m} \sum_{n} \left( A_n^m k \bar{h}_n^m(kR) \bar{h}_m^*(kR) \right) \cos \theta \cos \phi. \quad (1.47)
\]

Calculations of the time average of the total scattered energy (1.46) as a function of \(ka\) as well as calculations of the angular distribution of the scattered energy density (1.47) have been carried out in analogous problems of sound scattering (Ref. 0.2). We reproduce the results in Fig. 1.4 for the scattering of sound waves by a rigid cylinder.

![Diagram showing scattering](image)

Fig. 1.4 The scattering of SH waves from a cylindrical cavity of radius \(a\). Polar diagrams show the distribution in angle of the intensity of the scattered wave, and the lower graph shows the dependence of the total scattered intensity on \(ka = 2\pi a/\lambda\). (From Vibration and Sound by P. M. Morse. 1948, McGraw-Hill Book Company. Used with permission of McGraw-Hill Book Company.)
which is mathematically analogous to the scattering of SH waves by a
 cylindrical cavity. In an elastic solid with microscopic defects and
 inhomogeneities, the energy carried away by scattered waves is one of
 the major causes for the attenuation of an incident stress wave (Ref.
 1.5).

2. METHOD OF INTEGRAL EQUATION

In this section another method for solving the wave diffraction prob-
lem is discussed—the method of integral equation. From a theoretical
point of view, this method is more direct and basic, as it is based
on the Helmholtz and Kirchhoff's mathematical interpretation of Huygens'
principle (Baker and Copson). (0.6) Huygens' principle may be stated
by saying that each surface-element of an existing wave front at time
$t_0$ may be regarded as the source of a secondary wave which will be
propagated outward from the source in the form of a sphere with a
velocity, say, $c$; and that the wave front which represents the whole
disturbance at a later time $t$ is simply the envelope of the spherical
secondary waves which have the radius $c(t - t_0)$. Helmholtz-Kirchhoff's
integral formula yields mathematically what the disturbance at time $t$
should be if the wave velocity $c$ and the waves at time $t_0$ were known.

For the diffraction of waves, the boundary of the scatterer upon
the impinging of an incident wave acts like a layer of secondary
sources which emits the scattered waves. Thus the Helmholtz-Kirchhoff
formula is immediately applicable. From this formula, an integral
equation is derived which determines precisely the scattered waves.
Only diffraction of scalar waves is discussed in this section for the purpose of illustrating the method. Thus for a three-dimensional problem, the medium is assumed to have zero shear rigidity and the waves in the solid are determined by a single potential \( \psi \) in Eq. (I-2.30). For a two-dimensional problem, our discussion, as in the previous section, is directed at the diffraction of an SH wave by a prismatic cylinder. Applications of this method to elastic waves are discussed in Chapter IV of this text.

2.1. Helmholtz's and Kirchhoff's Formulas

We start with Green's identity for two functions \( \psi \) and \( G \) (Section 7.2 of Ref. 0.3).

\[
\iiint_V (\psi \nabla^2 G - G \nabla^2 \psi) dV = \iint_A \left( \psi \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} \right) dA,
\]

where \( A \) is the surface bounding a volume \( V \) and \( \partial / \partial n \) denotes differentiation along the outward normal to \( A \). The bounding surface \( A \) may be finite or infinite in extent; it may also be composed of two or more surfaces, one bounding the volume \( V \) from the outside and the others bounding it from the inside.

Consider two special functions \( \psi(r) \) and \( G(r, r_o) \) which satisfy the following Helmholtz equations, respectively.

\[
\nabla^2 \psi(r) + k^2 \psi(r) = 0,
\]

\[
(\nabla^2 + k^2)G(r, r_o) = (\nabla^2 + k^2)G(r_o, r) = -\delta(r - r_o)
\]

where \( r(x, y, z) \) and \( r_o(x_o, y_o, z_o) \) are the position vectors of the
"observation points" and "source points" respectively. \( \nabla^2 \) is the Laplacian operator with respect to the "observing coordinates" \( x, y, z \); \( \nabla_o^2 \) the operator in "source coordinates" \( x_o, y_o, z_o \). Substituting (2.2) and (2.3) in Eq. (2.1) and carrying out the integration with respect to source coordinates, we obtain

\[
\iiint_V \psi(r_o) \delta(r - r_o) dV_o = \iint_A \left( \psi \frac{\partial G}{\partial n_o} - G \frac{\partial \psi}{\partial n_o} \right) dA_o,
\]

where \( dV_o = dx_o dy_o dz_o \) (Fig. 2.1). Since the delta function \( \delta(r - r_o) = \delta(x - x_o) \delta(y - y_o) \delta(z - z_o) \) has the following integral property

\[
\iiint_V F(r_o) \delta(r - r_o) dV_o = \begin{cases} 0, & r \text{ outside } V, \\ F(r), & r \text{ inside } V, \end{cases}
\]

the following formula follows,

\[
\iiint_A \left[ G(r, r_o) \frac{\partial \psi(r_o)}{\partial n_o} - \psi(r_o) \frac{\partial G(r, r_o)}{\partial n_o} \right] dA_o = \begin{cases} \psi(r), & r \text{ inside } A, \\ 0, & r \text{ outside } A. \end{cases} \tag{2.4}
\]

\[\text{Fig. 2.1 Geometry of Observation Points } P(r) \text{ and Source Points } Q(r_o) \text{ for the Interior Problem}\]

In (2.4), the function \( \psi \) and its first and second derivatives are assumed to be continuous in the volume \( V \), and the function \( G(r, r_o) \) is
known as Green's function for the steady state wave equation (2.3).

In an unbounded three-dimensional space, the solution for equation (2.3) is

\[ G(r, r_0) = \frac{e^{ik|\mathbf{r} - \mathbf{r}_0|}}{4\pi|\mathbf{r} - \mathbf{r}_0|} = \frac{e^{ik\rho}}{4\pi\rho} = G(r_0, r) \]  

\[ \rho \equiv |r - r_0| = [(z - z_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{1}{2}}. \]

Equation (2.4) is the Helmholtz first (interior) formula (Ref. 0.6).

Helmholtz's first formula is applicable in the case when all the singularities of the function \( \psi(r) \) lie outside the surface \( A \). (By a singularity of \( \psi \) we mean a point at which \( \psi \) or one of its first and second partial derivatives is discontinuous.) If on the other hand, all the singularities of \( \psi(r) \) lie within a closed surface \( A \), we can apply Green's identity to the region \( V \) bounded internally by \( A \) and externally by another closed surface \( B \), a sphere with the center at the origin and large radius \( R \) (Fig. 2.2). The surface in Eq. (2.1)

\[ \text{Fig. 2.2 Geometry for the Observation Point } P(r) \text{ and Source Point } Q(r_0) \text{ for the Exterior Problem} \]
is now composed of \( A \) and \( B \). Since \( \psi(r) \) is assumed to be continuous outside \( A \), application of Green's identity leads, as in (2.4), to

\[
\iint_{A+B} \left[ \frac{\partial \psi(r)}{\partial n^O} - \frac{\partial G(r, r_o)}{\partial n^O} \right] dA^O = \begin{cases} \psi(r), & r \text{ inside } V, \\ 0, & r \text{ outside } V. \end{cases}
\]

On the large surface \( B \), \( r_o = R \) and \( \partial / \partial n^O = \partial / \partial R \), also \( dA = R^2 \sin \theta d\theta d\phi \). Thus the above integral over the surface \( B \) becomes, as \( R \to \infty \),

\[
\lim_{R \to \infty} \iint_B \left( G \frac{\partial \psi}{\partial R} - \psi \frac{\partial G}{\partial R} \right) dA^O = \lim_{R \to \infty} \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} e^{ikR} \left[ \left. \left( \frac{3 \psi}{3r_o} - i k \psi \right) \right|_{r_o = R} + \psi \right] \sin \theta d\theta d\phi.
\]

The integral vanishes if, for any finite value \( M \),

\[
|r\psi| < M, \quad \text{as } r \to \infty
\]

(2.6)

\[
r \left( \frac{3 \psi}{3r} - i k \psi \right) \to 0, \quad \text{as } r \to \infty
\]

for all values of angular coordinates \( \theta \) and \( \varphi \). Equations (2.6) are known as the Sommerfeld radiation conditions. Thus for a function \( \psi(r) \) being regular in \( V \), and satisfying Sommerfeld radiation conditions, its value at an observing point \( P(r) \) is given by the surface integral over the source points \( Q(r_o) \) as

\[
\iint_A \left[ \frac{\partial \psi(r)}{\partial n^O} - \frac{\partial G(r, r_o)}{\partial n^O} \right] dA^O = \begin{cases} \psi(r), & r \text{ inside } V, \\ 0, & r \text{ outside } V. \end{cases}
\]

As shown in Fig. 2.2, the unit normal \( n^O \) is away from the region \( V \), and
is an inward normal to the closed surface $A$. If an outer normal $n^o$ to $A$ is used, we have

$$\iiint_A \left[ \psi(r_o) \frac{\partial G(r, r_o)}{\partial n_o} - G(r, r_o) \frac{\partial \psi(r_o)}{\partial n_o} \right] dA_o = \begin{cases} \psi(r), & r \text{ outside } A, \\ 0, & r \text{ inside } A. \end{cases}$$

This is the Helmholtz second (exterior) formula.

Green's function as given by (2.5a) represents a spherical wave generated by a point source at the location $Q(r_o)$. Helmholtz formulas state that the waves $\psi(r)$ in the region $V$ are determined from a layer of source $\psi(r_o)$ and a layer of doublet $\partial \psi(r_o)/\partial n_o$ over the surface $A$, each point on the layers emitting a spherical wave.

If the surface $A$ is a cylindrical surface with its generics parallel to the $z$-axis (Fig. 2.3), and if the source functions $\psi(r_o)$ and $\partial \psi(r_o)/\partial n_o$ are independent of the coordinate $z$, the waves $\psi(r)$ in the region $V$ are then also independent of $z$ and the problem reduces to a two-dimensional one. For two-dimensional waves, the Green function in (2.5a) can be integrated over the $z$-coordinate

$$G(r, r_o) = \int_{-\infty}^{\infty} \frac{e^{ikd}}{4\pi d} dz = \int_{-\infty}^{\infty} \frac{e^{ik\sqrt{d^2 + z^2}}}{4\pi \sqrt{d^2 + z^2}} dz,$$

where $d$ is the projection of $o$ onto the $x$-$y$ plane. Except for a constant multiplier, the above integral is the same as the integral representation of the Hankel function of the first kind and 0th order $H^{(1)}_0(kd)$ (see Chapter III, Section 1). Thus, in two-dimensional problems,
Fig. 2.3 Line Source (QQ') and Observation Point (P or P')

\[ G(r, r_o) = (i/4)H_0^{(1)}(kr) = (i/4)H_0^{(1)}(k|r - r_o|), \]  

(2.5b)

where \( r \) and \( r_o \) are position vectors to the observing point \( P \) and source point \( Q \) respectively, all being on the \( xy \) plane as shown in Fig. 2.3. Green's function given by (2.5b) represents a continuous distribution of point sources with equal strength located along the line QQ', parallel to the z-axis. It is sometimes referred to as a line source.

For two-dimensional interior problems, in view of (2.5b), Eqs. (2.4) can be written explicitly as

\[
\frac{i}{4} \int_{\Gamma} \left[ H_0^{(1)}(kr - r_o) \frac{\partial \psi(r_o)}{\partial n_o} - \psi(r_o) \frac{\partial H_0^{(1)}(k|r - r_o|)}{\partial n_o} \right] ds 
\]

\[
= \begin{cases} 
\psi(r), & \text{r inside } \Gamma, \\
0, & \text{r outside } \Gamma. 
\end{cases} 
\]  

(2.8)
Similarly, when \( \psi(r) \) has singularities inside \( \Gamma \), we substitute (2.5b) into (2.7) and obtain the exterior formula

\[
\frac{i}{4} \int_{\Gamma} \left[ \psi(r_o) \frac{\partial H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_o|)}{\partial n_o} - H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_o|) \frac{\partial \psi(r_o)}{\partial n_o} \right] ds
\]

\[
= \begin{cases} 
\psi(r), & r \text{ outside } \Gamma, \\
0, & r \text{ inside } \Gamma.
\end{cases}
\tag{2.9}
\]

Equations (2.8) and (2.9) are known as Weber's first (interior) and second (exterior) formulas for waves in two-dimensional space. The integral is over a closed curve \( \Gamma \), the circumference of a cross section of the cylinder, with element length \( ds \), and \( \partial / \partial n_o \) is the derivative along the outer normal to curve \( \Gamma \). The wave function \( \psi(r) \) satisfies the two-dimensional Helmholtz equation:

\[
(\nabla^2 + k^2) \psi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) \psi(x,y) = 0.
\]

For the exterior problem, the corresponding Sommerfeld radiation conditions are

\[
\sqrt{r} \psi \to 0, \quad \text{as } r \to \infty, \tag{2.10}
\]

\[
\sqrt{r} \left( \frac{\partial \psi}{\partial r} - ik \psi \right) \to 0, \quad \text{as } r \to \infty.
\]

The Helmholtz and Weber formulas are derived for steady-state waves, and the time dependence is \( e^{-i\omega t} \). The generalization of these formulas to waves of arbitrary time dependence are due to Kirchhoff. Let us start the derivation of Kirchhoff's formula by applying the Fourier transform to the wave equation in three dimensions:
\[ \sigma^2 \nabla^2 \psi(r,t) = \tilde{\psi}(r,t), \quad (2.11) \]

and obtain the familiar Helmholtz equation

\[ \psi^2 \psi(r, \omega) + \kappa^2 \psi(r, \omega) = 0, \quad \kappa = \omega / \sigma, \]

where \( \psi(r, \omega) = 1 / \sqrt{2\pi} \int_{-\infty}^{\infty} \psi(t)e^{i\omega t} dt \). Using the result shown in (2.7), we find for the observing point \( P \), inside the region \( V \),

\[ -4\pi \psi(r, \omega) = \iint_A \left[ \frac{e^{ikp}}{p} \frac{\partial \psi(r)}{\partial n_o} - \psi(r) \frac{\partial}{\partial n_o} \frac{e^{ikp}}{p} \right] dA_o, \]

\[ p = |r - r_o|. \]

By the Fourier inversion transform formula, we have

\[ -4\pi \psi(r, t) = \iint_A \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{e^{ikp}}{p} \left( \frac{\partial \psi}{\partial n_o} \right) - \psi \frac{\partial}{\partial n_o} \left( \frac{e^{ikp}}{p} \right) \right] e^{-i\omega t} \, dw \, da_o. \]

\[ = \iint_A \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{p} \frac{\partial \psi}{\partial n_o} - \psi \frac{\partial}{\partial n_o} \left( \frac{1}{p} \right) + i\kappa \frac{1}{p} \frac{\partial}{\partial n_o} \right] e^{-i\omega(t-\rho/\sigma)} \, dw \, da_o. \]

Applying the time shift theorem in Eq. (1-4.13) and writing

\[ [\psi] = \psi(r, t - \rho / \sigma) \quad (2.12) \]

for any function \( \psi(r, t) \), we complete the inversion of the Fourier transform and obtain

\[ 4\pi \psi(r, t) = \iint_A \left\{ -\frac{1}{p} \left[ \frac{\partial \psi}{\partial n_o} \right] + [\psi_o] \frac{\partial}{\partial n_o} - \frac{1}{p} \left[ \frac{\partial \psi}{\partial t} \right] \frac{\partial}{\partial n_o} \right\} dA_o. \quad (2.13) \]

The above is Kirchhoff's formula for the exterior problem (Fig. 2.2).
METHOD OF INTEGRAL EQUATION

The $[\psi_0]$, $[\partial \psi / \partial n_0]$, and $[\partial \psi / \partial t]$ inside the integrals above are the corresponding quantities over the surface $A$ at the time $t - \rho / c$. They are the "retarded" values as compared to $\psi$, $\partial \psi / \partial n_0$, and $\partial \psi / \partial t$ at time $t$. Kirchhoff's formula states that the waves $\psi(r)$ at the time $t$ are determined by summing the contributions from a layer of simple sources $\psi(r)$ and $\partial \psi / \partial n$ and a layer of doublets $\partial \psi(r)/\partial n$ over a surface $A$ at the time $t - \rho / c$. The time-lag $\rho / c$ is precisely the time required for the wavelets to travel from the source points $Q$ to the observing point $P$. When the surface $A$ is taken as the wave front, and the layers of simple sources and doublets on $A$ are regarded as secondary sources emitting secondary waves, the Kirchhoff formula is then a mathematical proof of Huygens' principle for scalar waves.

In the subsequent subsections, we shall show how the Helmholtz formulas are applied to investigate the diffraction of steady-state waves. Analogously, the Kirchhoff formula forms the basis for studying the diffraction of pulses (transient waves). We shall not, however, pursue the latter subject, which was discussed in a recent series of papers by Freidman and Shaw. (2.1, 2.2, 2.3)

2.2. Integral Equations

In the remaining part of this section we shall treat the Weber formulas as a special case of Helmholtz's equations (2.4) and (2.7). For three-dimensional wave diffractions, Green's function $e^{ikr}/4\pi r$ given by (2.5a) is to be used. For two-dimensional problems, the area integral over $A$ is to be replaced by a line integral over $\Gamma$ as in (2.8) and (2.9), and the Green function should be $(i/4)H_0^{(1)}(ko)$ as defined in (2.5b).
The total wave $\psi^{(t)}$ in a medium is composed of two parts; the incident wave $\psi^{(i)}$ and scattered wave $\psi^{(s)}$:

$$\psi^{(t)} = \psi^{(i)} + \psi^{(s)}. \quad (2.14)$$

Each wave function satisfies the Helmholtz formula, (2.4) or (2.7).

Let $A$ be the surface of a scatterer with volume $V_A$ (Fig. 2.4). We seek the solution for the total wave $\psi^{(t)}$ in the region $V$ outside the surface $A$.

![Diagram showing the approach of the observation point $P(r)$ to the source point $Q(r)$ on the surface of a scatterer with volume $V_A$ and bounding surface $A$.]

**Fig. 2.4** Approach of the Observation Point $P(r)$ to the Source Point $Q(r_0)$ on the Surface of a Scatterer with Volume $V_A$ and Bounding Surface $A$

The scattered wave function $\psi^{(s)}$, which represents physically the waves radiated by secondary sources on or inside the surface $A$, usually is singular inside $V_A$. Thus Helmholtz's second formula is applicable with

$$\iint_A \left[ \psi^{(s)}(r_0) \frac{\partial G(r, r_0)}{\partial n_0} - G(r, r_0) \frac{\partial \psi^{(s)}(r_0)}{\partial n_0} \right] dA_0 = \psi^{(s)}(r), \quad r \in V. \quad (2.15)$$
Since \( G(r, r') \) is given by either (2.5a) or (2.5b), the equation above states that the scattered waves in the region \( V \), outside the boundary surface \( A \) of the scatterer, can be found from a surface integration over \( A \) if the values of \( \psi_o^{(s)} \) and \( \partial \psi_o^{(s)} / \partial n_o \) on the surface are known. The derivative of \( \psi_o^{(s)}(r) \) in \( V \) is obtained simply by differentiating the above equation:

\[
\frac{\partial}{\partial n} \iint_A \left[ \psi_o^{(s)}(r) \frac{\partial G(r, r')}{\partial n_o} - G(r, r') \frac{\partial \psi_o^{(s)}(r)}{\partial n_o} \right] dA_o = \frac{\partial}{\partial n} \psi_o^{(s)}(r),
\]

\( r \in V. \) \( (2.16) \)

Unfortunately, \( \psi_o^{(s)}(r) \) and \( \partial \psi_o^{(s)}(r) / \partial n_o \) are usually unknown for a given problem.

To find \( \psi_o^{(s)} \) and its normal derivative on the surface \( A \), we let the observation point \( P(r) \) approach the source point \( Q(r_o) \) on the surface. With \( r \to r_o \), Eq. (2.15) reduces to an integral equation for \( \psi_o^{(s)}(r) \) or \( \partial \psi_o^{(s)}(r) / \partial n_o \). However, because \( \partial G(r, r') / \partial n_o \) is discontinuous across the surface \( A \), the limits must be carried out with care. A general theorem for the continuity of \( \psi^{(s)} \) and \( \partial \psi^{(s)} / \partial n \) along a line normal to \( A \) can be constructed in a manner analogous to the integral theorems of potential functions. (See Chapter VI, Sections 5 and 6 of Ref. 2.4 or Section 38 of Ref. 2.5.) The following is a formal evaluation of the limits. (0.8)

Consider at first the limit of the leading term on the left-hand side of (2.15),

\[
\lim_{r \to r_o} \iint_A \psi(r) \frac{G(r, r_o)}{\partial n_o} dA.
\]
The suffix \( \sigma \) is dropped for the moment and \( r^+_{\sigma} \) indicates that the limit is approached from the positive side of the normal \( n_{\sigma} \) (Fig. 2.4). Since \( \partial G(r, r_{\sigma}) / \partial n_{\sigma} \) is singular at \( r = r_{\sigma} \), we exclude the source point from the surface integral by encircling it with a small area \( \Sigma \). In the neighborhood of \( \Omega(r_{\sigma}) \), the Green's function for the wave equation can be approximated by its static value

\[
G(r, r_{\sigma}) = e^{\pm k|\frac{r-r_{\sigma}}{4\pi}|r - r_{\sigma}}
\]

Hence

\[
\lim_{r \to r^+_{\sigma}} \int_{A} \psi(r_{\sigma}) \frac{\partial G(r, r_{\sigma})}{\partial n_{\sigma}} \, dA_{\sigma} = \frac{1}{4\pi} \lim_{r \to r^+_{\sigma}} \int_{\Sigma} \psi(r_{\sigma}) \frac{\partial}{\partial n_{\sigma}} \frac{1}{|r - r_{\sigma}|} \, dA_{\sigma}
\]

\[
+ \lim_{r \to r^+_{\sigma}} \int_{A-\Sigma} \psi(r_{\sigma}) \frac{\partial G(r, r_{\sigma})}{\partial n_{\sigma}} \, dA_{\sigma}.
\]

The limit of the second term on the right can be evaluated directly because \( \partial G / \partial n_{\sigma} \) is continuous at \( A - \Sigma \). For the first term, one notes that

\[
\frac{\partial}{\partial n_{\sigma}} \frac{1}{|r - r_{\sigma}|} \, dA_{\sigma} = \frac{n_{\sigma} \cdot (r - r_{\sigma})}{|r - r_{\sigma}|^2} \, dA_{\sigma} = d\omega(r, r_{\sigma}),
\]

where \( d\omega \) is the solid angle subtended by the surface \( dA_{\sigma} \). With a smooth surface at \( r_{\sigma} \), we then obtain

\[
\lim_{r \to r^+_{\sigma}} \int_{\Sigma} \psi(r_{\sigma}) \frac{\partial}{\partial n_{\sigma}} \frac{1}{|r - r_{\sigma}|} \, dA_{\sigma} = \psi(r_{\sigma}) \lim_{r \to r^+_{\sigma}} \int_{\Sigma} d\omega = 2\pi \psi(r_{\sigma}).
\]
The final answer is

\[
\lim_{r \to r^+} \iint_A \psi(r_o) \frac{\partial G(r, r_o)}{\partial n_o} \, dA_o = \frac{1}{2} \psi(r_o) + \text{p.v.} \iint_A \psi(r_o) \frac{\partial G(r, r_o)}{\partial n_o} \, dA_o,
\]

\[r = r_o, \quad (2.17)\]

where P.V. designates the principal value of the integral as defined by

\[
\text{p.v.} \iint_A F(x, y) \, dxdy = \lim_{\Sigma \to 0} \iint_{A-\Sigma} F(x, y) \, dxdy. \quad (2.18)
\]

The limit of the second integral in (2.15) as \(r \to r^+\) can be evaluated directly if the unknown function \(\partial \psi(r_o)/\partial n_o\) satisfies the Hölder condition. Thus as \(r \to r^+_o\), Eq. (2.15) reduces to

\[
\frac{\partial \psi(r)}{\partial n} = \iint_A \left[ \psi(r_o) \frac{\partial G(r, r_o)}{\partial n_o} - G(r, r_o) \frac{\partial \psi(r_o)}{\partial n_o} \right] \, dA_o,
\]

\[r \text{ on } A. \quad (2.19)\]

The statement "\(r \text{ on } A\)" means that \(r\) is set equal to \(r_o\) after integration where \(r_o\) are the coordinates of the surface points, and the principal value of the integral is to be taken whenever it becomes necessary. Applying the same limiting process to Eq. (2.16), we obtain

---

*A function \(f(r)\) is said to satisfy the Hölder condition at \(r_o\) if there are three positive constants \(a, b,\) and \(\alpha\) such that

\[|f(r) - f(r_o)| \leq a |r - r_o|^\alpha\]

for all points \(r\) for which \(|r - r_o| < b\). When \(0 \leq \alpha \leq 1\), this is known as the Lipschitz condition.
The integral equation (2.19) which is of the second kind of Fredholm-type integral equation, or the integral-differential equation (2.20). On the other hand, if the \( \psi^{(s)} \) is prescribed at the surface \( A \), \( \partial \psi^{(s)}/\partial n \) is then determined by Eq. (2.19), which becomes a Fredholm integral equation of the first kind.

In many problems, the boundary values are prescribed in terms of the total wave function \( \psi^{(t)} \) or \( \partial \psi^{(t)}/\partial n \). It is then more convenient to derive a set of integral equations for the total wave. This can be done easily by noting that the incident wave \( \psi^{(i)} \), which has no singularity inside the boundary \( A \), satisfies the Helmholtz first formula:

\[
\iint_{A} \left[ G(r, r_{o}) \frac{\partial \psi^{(i)}(r_{o})}{\partial n_{o}} - \psi^{(i)}(r_{o}) \frac{\partial G(r, r_{o})}{\partial n_{o}} \right] dA_{o} = 0,
\]

\( r \) in \( V \).

Adding it to Eq. (2.15) and using (2.14), we find

\[
\psi^{(i)}(r) + \iint_{A} \left[ \psi^{(t)}(r_{o}) \frac{\partial G(r, r_{o})}{\partial n_{o}} - G(r, r_{o}) \frac{\partial \psi^{(t)}(r_{o})}{\partial n_{o}} \right] dA_{o} = \psi^{(t)}(r),
\]

\( r \) in \( V \). (2.21)
By letting \( r \) approach \( r_o \) as in (2.19), or by differentiating it and then taking the limit as in (2.20), we obtain

\[
\psi^{(t)}(r) + \iint_A \left[ \psi^{(t)}(r_o) \frac{\partial G(r, r_o)}{\partial n_o} - G(r, r_o) \frac{\partial \psi^{(t)}(r_o)}{\partial n_o} \right] dA_o = \frac{1}{2} \psi^{(t)}(r),
\]

\( r \) on \( A; \) \hspace{1cm} (2.22)

\[
\frac{\partial \psi^{(t)}(r)}{\partial n} + \frac{\partial}{\partial n} \iint_A \left[ \psi^{(t)}(r_o) \frac{\partial G(r, r_o)}{\partial n_o} - G(r, r_o) \frac{\partial \psi^{(t)}(r_o)}{\partial n_o} \right] dA_o = \frac{1}{2} \frac{\partial \psi^{(t)}(r)}{\partial n}, \hspace{1cm} r \) on \( A. \) \hspace{1cm} (2.23)

Again the integrals are evaluated in the sense of principal values.

Solutions of Eqs. (2.19), (2.20), (2.22), or (2.23) yield the values of \( \psi^{(s)} \) or \( \psi^{(t)} \), or their normal derivatives, at the boundary \( A \), from which the values of the corresponding quantities in the region \( V \) outside \( A \) can be found from the integrals in (2.15) or (2.21).

Two special boundary conditions are to be noted. One is that the total field \( \psi^{(t)} \) at \( A \) vanishes, or what is equivalent, \( \psi^{(s)} = \psi^{(t)} \) at \( A \). This is usually referred to as Dirichlet's condition. In three-dimensional elasticity, if the medium is assumed to be irrotational \((\nu = 0 \text{ in } I-2.29)\), then the displacement \( \mathbf{u} = \nabla \varphi \) and stress \( \sigma = \lambda \nabla^2 \varphi + \mu \nabla (\nabla \cdot \varphi) = \rho \omega^2 \varphi \) for steady-state waves. Thus when the wave function \( \varphi \) is taken to be the displacement potential \( \varphi \), the Dirichlet condition may be considered as the boundary condition for a stress-free cavity in an "irrotational" medium. On the other hand, in the anti-plane strain approximation \((I-3.1)\), the displacement component \( u(x,y,t) \) along the
The z-axis satisfies the wave equation (I-3.3). For such cases, we interpret $\psi(t)$ as the total displacement $\omega$ in the field and the Dirichlet condition represents the one for a clamped (fixed) surface. We note that waves in "irrotational medium" are analogous to sound waves in air (I-3.7).

The second special boundary condition is that the normal derivative of $\psi(t)$ vanishes at the surface $A$, or equivalently, $\partial \psi(t)/\partial n = -\partial \psi(z)/\partial n$ at $A$. This is known as Neumann's condition, and represents a clamped surface $A$ in an "irrotational" medium or a stress free surface in the two-dimensional anti-plane strain approximation (I-3.5b). For either of these two types of boundary conditions, the integral equations are greatly simplified and are listed below (Ref. 2.6):

1. **Dirichlet Condition** $\psi(t) = 0, \psi(z) = -\psi(z)$ at $A$. From (2.22):

   $$\psi(z)(r) = \iint_A G(r, r_o) \frac{\partial \psi(t)(r_o)}{\partial n} \, dA_o, \quad r \text{ on } A; \quad (2.24)$$

   From (2.23):

   $$\frac{\partial \psi(z)(r)}{\partial n} = \frac{\partial \psi(t)(r)}{\partial n} + \iint_A \frac{\partial G(r, r_o)}{\partial n} \frac{\partial \psi(t)(r_o)}{\partial n_o} \, dA_o, \quad r \text{ on } A. \quad (2.25)$$

2. **Neumann Condition** $\partial \psi(t)/\partial n = 0$ at $A$. From (2.23):

   $$-\frac{\partial \psi(t)(r)}{\partial n} = \frac{\partial}{\partial n} \iint_A \psi(t)(r_o) \frac{\partial G(r, r_o)}{\partial n_o} \, dA_o, \quad r \text{ on } A; \quad (2.26)$$

   From (2.22):

   $$\psi(t)(r) = \frac{1}{2} \psi(t)(r) - \iint_A \psi(t)(r_o) \frac{\partial G(r, r_o)}{\partial n_o} \, dA_o, \quad r \text{ on } A. \quad (2.27)$$
Two sets of equations for $\psi^{(s)}$ and $\partial \psi^{(s)}/\partial n$ can be derived analogously from (2.19) and (2.20). Equations (2.24) and (2.26) are Fredholm's integral equations of the first kind, which have the unknown function $\partial \psi_{O}^{(t)}/\partial n_{O}$ or $\psi^{(t)}(r_{O})$ only inside the integrals; Eqs. (2.25) and (2.27) are Fredholm's integral equations of the second kind. Because of the complexity of the kernel $G(r, r_{O})$ or $\partial G(r, r_{O})/\partial n_{O}$, no general method is available for solving these integral equations analytically for an arbitrary surface $A$. At this moment, the common approach for finding the unknown functions $\psi^{(t)}$ is to approximate the integral by a summation process and calculate $\psi^{(t)}$ at discrete points on the boundary. This will be illustrated by an example in subsection 2.4. However, if the boundary $A$ is a surface of a curvilinear coordinate system for which the original wave diffraction problem can be solved by the method of series expansion, the integral equations can also be solved by the so-called Hilbert-Schmidt method. Two examples illustrating the method are given in the following subsection.

2.3. Method of Hilbert-Schmidt

The Fredholm integral equation of the first kind for $\psi(r)$

$$ f(r) = \iint_{A} G(r, r_{O}) \psi(r_{O}) dA_{O}, \quad r \text{ on } A, \tag{2.28} $$

can be solved if the kernel $G(r, r_{O})$ can be expanded into series of orthogonal functions suitable for the surface $A$. Let $\varphi_{n}(r)$ ($n = 1, 2, \ldots$) be the orthogonal functions which satisfies the wave equation and the orthogonality condition.
\[ \iint_A \varphi_n(r) \varphi_m(r) \omega(r) dA = \begin{cases} 1(m = n), \\
0(m \neq n). \end{cases} \tag{2.29} \]

The functions \( \varphi_n \) have been properly normalized and \( \omega(r) \) is a weighting function. Suppose also \( G(r, r') \) admits the following series expansion

\[ G(r, r') = \sum_{n=1}^{\infty} b_n \varphi_n(r) \varphi_n(r'). \]

We then expand the given function \( f(r) \) and the unknown function \( \psi(r) \) into two series

\[ f(r) = \sum_n a_n \varphi_n(r), \]

\[ \psi(r) = \left[ \sum_m a_m \varphi_m(r) \right] \omega(r). \]

By substituting the three series in the integral equation, we find

\[ \sum_n a_n \varphi_n(r) = \sum_n \sum_m a_m b_n \varphi_n(r) \int_A \omega(r') \varphi_n(r') \varphi_m(r') dA', \quad \text{on } A, \]

which, in view of the orthogonality condition, fixes the unknown coefficient \( a_n \) as

\[ a_n = \frac{a_n}{b_n}, \quad n = 1, 2, \ldots \tag{2.30} \]

This, in essence, is Hilbert-Schmidt's method. It will now be applied to investigate the diffraction of sound waves (irrotational medium) by a rigid sphere (Neumann condition) and SH waves as scattered by a rigid circular cylinder (Dirichlet condition).
(1) Scattering of SH Wave by a Rigid Cylinder

This problem has been treated in detail in Section 1. We refer to Fig. 2.3 with the origin 0 being placed at the center of the circular cylinder. The total wave \( \omega(t) \) at the surface \( \Gamma \) satisfies the integral equation (2.24)

\[
\omega(t)(r) = \int_{\Gamma} G(r, r_o) \frac{\partial \omega(t)(r_o)}{\partial n_o} \, ds, \quad r \text{ on } \Gamma,
\]

because \( \omega(t) = 0 \) at the surface \( \Gamma \). In the above, \( r \) and \( r_o \) are two dimensional position vectors on \( x-y \) plane, and as in (2.5b)

\[
G(r, r_o) = \frac{1}{i/4} j^{(1)}_0(k|r - r_o|).
\]

In cylindrical coordinate \((r, \theta)\) — see page 1372, Ref. 0.3 —

\[
G(r, r_o) = \frac{i}{4} \sum_{m=0}^{\infty} \cos m(\theta - \theta_o) \begin{cases} J_m(kr_o) h^{(1)}_m(kr), & r > r_o, \\ J_m(kr) h^{(1)}_m(kr_o), & r < r_o, \end{cases}
\]

and an incident SH wave along the \( z \)-axis is represented by

\[
\omega(t)(r) = \sum_{m=0}^{\infty} \varepsilon_m J_m(kr) \cos m\theta.
\]

We now assume the normal derivative of the total wave on the boundary \( \Gamma(r_o = a) \) to be

\[
\frac{\partial \omega(t)(r_o)}{\partial n_o} = \sum_{n=0}^{\infty} B_n \cos n\theta_o.
\]

With \( ds = ad\theta_o \), the integral equation takes the form
\[ \omega^{(t)}(r) = \left( \frac{i}{4} \right) \sum_{n} B_n \cos \theta_o \left[ \sum_{m} J_m(k\alpha) H^{(1)}_m(kr) \cos m(\theta - \theta_o) \right] d\theta_o \]

\[ = \left( \frac{i\pi}{4} \right) \sum_{n} B_n J_n(k\alpha) H^{(1)}_n(kr) \cos \theta, \quad r \text{ on } \Gamma, \]

where the following result has been utilized

\[ \int_0^{2\pi} \cos m(\theta - \theta_o) \cos \theta_o d\theta_o = \begin{cases} 0 & m \neq n, \\ \pi \cos \theta & m = n. \end{cases} \]

Replacing \( \omega^{(t)}(r_o) \) by its series representation, we have

\[ \sum_{m=0}^{\infty} \epsilon_m J_m(k\alpha) \cos m\theta = \left( \frac{i\pi}{4} \right) \sum_{n=0}^{\infty} B_n J_n(k\alpha) H^{(1)}_n(k\alpha) \cos \theta. \]

By applying the orthogonality conditions for \( \cos m\theta \), we find the value for the undetermined coefficients \( B_n \) as

\[ B_n = -4i \epsilon_n \frac{\pi a H^{(1)}_n(k\alpha)}{\gamma_n}. \]

Once the value for \( \omega^{(t)}/\partial n \) at the surface \( \Gamma(r = a) \) is known, the scattered waves in the field \( r > a \) can be calculated from (2.21) which is written as

\[ \omega^{(s)}(r) = \int_{\Gamma} G(r, r_o) \frac{\omega^{(t)}(r_o)}{\partial n_o} \, ds, \quad r \text{ in } \Omega. \]

The integration can be carried out in exactly the same manner as above, and the result is
\[ \omega(\theta)(r) = -(i\pi/4) \sum_{n=0}^{\infty} \frac{\bar{J}_n(ka)H_n^{(1)}(kr)}{n\pi} \cos n\theta \]

\[ = -\sum_{n=0}^{\infty} \frac{\epsilon_n [J_n(ka)/H_n^{(1)}(ka)]H_n^{(1)}(kr) \cos n\theta} {n\pi} . \]

It checks with (1.18) obtained by the series expansion method.

(2) Scattering of Plane Sound Waves by a Rigid Sphere

We take \( \psi^{(2)}(r) \) in (2.22) or (2.23) to be the displacement potential which gives rise to the total wave scattered by a rigid sphere in an irrotational medium (sound wave). The incident plane wave is propagating along the z-axis as shown in Fig. (2.5) with frequency \( \omega \), wave number \( k = \omega/c_0 \):

\[ \psi^{(2)}(r) = e^{i(kz - \omega t)} . \]

The time factor will be omitted henceforth. The boundary condition

![Fig. 2.5 Scattering of a Plane Wave by a Sphere](image-url)
is that on the surface \( r = a, \ u = \psi(t) = 0, \) thus

\[
\frac{\partial \psi(t)}{\partial r} = 0, \quad \text{at } r = a.
\]

For this condition, (2.26) is the appropriate integral equation:

\[
- \frac{\partial \psi(t)}{\partial r} = \frac{\partial}{\partial r} \iint_A \psi(t)(r) \frac{\partial G(r,r_o)}{\partial r_o} \, dA_o, \quad r = r_o,
\]

In spherical coordinates, Green's function \( G(r,r_o) = e^{ikp}/4\pi \rho \)
can be expressed as a double series of orthogonal functions (p. 1466, Ref. 0.3).

\[
e^{ikp}/4\pi \rho = (ik/4\pi) h^{(1)}(k\rho)
= \frac{ik}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_{mn}(0) X_n^m(0,\phi) \bar{X}_n^m(0,\phi) \begin{cases} j_n(kr) h_n^{(1)}(kr), & r > r_o, \\ j_n(kr) h_n^{(1)}(kr), & r < r_o, \end{cases}
\]

(2.32)

where

\[
\epsilon_{mn} = \epsilon_m(2n+1) \frac{(n-m)!}{(n+m)!}, \quad \epsilon_m = \begin{cases} 1 & m = 0, \\ 2 & m \geq 1, \end{cases}
\]

\[
X_n^m(0,\phi) = e^{im\phi} P_n^m(\cos \theta),
\]

\[
\bar{X}_n^m(0,\phi) = e^{-im\phi} P_n^m(\cos \theta),
\]

\[
X_n^m(0,\phi) = P_n(\cos \theta),
\]

\[
\bar{X}_n^m(0,\phi) = P_n(\cos \theta),
\]
and \( j_n(z) \) and \( h_n^{(1)}(z) \) are spherical Bessel functions, and \( P_n^m(\theta) \) are the associate Legendre polynomials. The \( \chi_n^m(\theta, \phi) \) is known as the complex spherical harmonic. In the above series representation, only the real part of the product

\[
\chi_n^m(\theta, \phi) \bar{\chi}_n^m(\theta_o, \phi_o) = P_n^m(\theta_o) \bar{P}_n^m(\theta) \cos m(\phi - \phi_o)
\]

should be used.

The incident wave can be represented by a similar series

\[
\psi^{(i)}(r) = \sum_{n=0}^{\infty} (2n+1) j_n^2 (kr) P_n(\cos \theta), \quad (2.33)
\]

which in fact can be derived from (2.32) by setting \( m_0 = 0, \quad \theta_o = 0, \)

and \( r_o \to \infty \). Note that \( \psi^{(i)}(r) \) is independent of the angular coordinates \( \phi \). Since the geometries of both the scatterer and the incident wave have azimuth symmetry, the scattered wave, and hence the total wave, is also independent of \( \phi \). Thus we assume

\[
\psi^{(i)}(r) = \sum_{n=0}^{\infty} B_n^P (\cos \theta_o), \quad \text{on } A, \quad (2.34)
\]

with unknown coefficients \( B_n \). Substituting the above into the integral equation, we find for \( r \) on \( A \)

\[
-\frac{\partial \psi^{(i)}(r)}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \left[ \sum_n B_n^P (\cos \theta_o) \right] d^2 \sin \theta_o d\theta_o d\phi_o.
\]
The result for integration with respect to $\phi_o$ is

$$
\int_0^{2\pi} \frac{1}{\lambda m(\theta_o, \phi_o)} d\phi_o = \begin{cases} 0 & \xi > 0, \\ 2\pi m \cos \theta_o & \xi = 0, \end{cases}
$$

and for $\theta_o$ is

$$
\int_0^{\pi} P_n(\cos \theta_o) P_m(\cos \theta_o) \sin \theta_o d\theta_o = \begin{cases} 0, & n \neq m, \\ \frac{2}{2n + 1}, & n = m. \end{cases}
$$

Thus

$$
\frac{\partial \psi(t)(r)}{\partial r} = \frac{3}{2} \int \frac{ik^2 \alpha^2 B_n k(1)}{k^2 \alpha^2 h_n(k)} (kr) j_n(k) P_n(\cos \theta), \quad r \text{ on } A,
$$

and $\psi(t)(r)$ given by (2.33) and $r = r_o = \alpha$ on $A$, we find from the above equation

$$
B_n = S \frac{n+1(2n+1)}{k^2 \alpha^2 h_n(k)},
$$

Substitution of $B_n$ in Eq. (2.34) completes the solution of the integral equation.

From the known values of $\psi(t)(r_o)$, scattered waves in the field outside $r_o = \alpha$ are calculated from the integral (2.21)

$$
\psi(s)(r) = \int_0^{\pi} \int_0^{2\pi} \psi(t)(r_o) \frac{\partial G(r, r_o)}{\partial r_o} \alpha^2 \sin \theta_o d\theta_o d\phi_o
$$

$$
= \sum_n B_n ik^2 \alpha^2 j_n(k) h_n^{(1)}(kr) P_n(\cos \theta)
$$
\begin{equation}
\omega^{(i)}(r) = \sum_{n=0}^{\infty} \frac{\hat{r}_n^{(1)}(ka)}{\hat{h}_n^{(1)}(ka)} h_n^{(1)}(kr) P_n(\cos \theta), \quad r \text{ in } V.
\end{equation}

The total wave \( \psi^{(t)}(r) \) is given by \( \psi^{(s)}(r) + \psi^{(i)}(r) \). Again this is precisely the same solution one would obtain by using the method in Section 1 (Ref. 0.2.).

2.4. Approximate Solutions of Integral Equations

The examples in the previous subsection demonstrate that the solutions which are derived by applying Hilbert-Schmidt's method to an integral equation could have been obtained directly from the wave-function expansion method; in fact, this method would have been much easier. However, for a problem with a particular geometry such that no existing wave functions are suitable, an integral equation for the scattered wave can still be formulated, but it cannot be solved exactly. Recourse must be made to some approximation procedures. One effective procedure is to approximate the integral by a finite sum and then calculate the unknown quantities at many discrete points by solving a system of algebraic equations. From a few published results, as discussed below, this method seems to be very promising.

Consider again the diffraction of an SH wave by a rigid cylinder, not necessarily circular. On the boundary, the normal derivative of the total wave satisfies the following integral equation

\begin{equation}
\omega^{(t)}(r) = \frac{i}{4} \int_{\Gamma} \hat{h}_0^{(1)}(k|r - r_o|) \frac{\partial \omega^{(t)}(r_o)}{\partial n_o} ds, \quad r \text{ on } \Gamma. \quad (2.36)
\end{equation}
We rewrite the above equation as

\[ \omega(r) = \frac{i}{4} \int_{\Gamma} H_0^{(1)}(k\sigma)U(r)ds, \quad r \text{ on } \Gamma, \]

with \( U(r) = \frac{\partial \omega(t)}{\partial n_0}, \) the unknown function; \( \omega(r) = \omega(t)(r_0), \) a given quantity, and

\[ \sigma = [(x - x_0)^2 + (y - y_0)^2]^\frac{1}{2}. \]

A smooth curve on an \( x-y \) plane is defined by one parameter with

\[ x = x(t), \quad y = y(t) \]

Thus, on the curve \( \Gamma, \) \( r = r[x(t), y(t)]; \quad r_0 = r[x(t_0), y(t_0)]. \)

\( ds = [(dx/dt)^2 + (dy/dt)^2]^{1/2}dt = s'(t)dt, \) and

\[ \sigma = [(x(t) - x(t_0))^2 + (y(t) - y(t_0))^2]^{1/2} = \sigma(t, t_0). \]

In terms of the parameter \( t, \) the integral equation takes the new form

\[ \omega(t) = \frac{i}{4} \int_{\Gamma} H_0^{(1)}[k\sigma(t, t_0)]U(t_0)s'(t_0)dt_0. \]  \hspace{1cm} (2.37)

The boundary \( \Gamma \) will now be subdivided by placing \( N \) points on \( \Gamma \)

with the index \( j \) indicating the end of the \( j^{th} \) segment. The above integral is then approximated by means of one of the numerical quadrature rules:

\[ \omega(t) = \frac{i}{4} \sum_{j=1}^{N} A_j H_0^{(1)}[k\sigma(t, t_j)]U(t_j)s'(t_j). \]
where \( A_j \) depends on how the arc length is divided and on which quadrature rule is used. When the trapezoidal rule is adopted, \( A_j = 2\pi/N \) because the \( \Gamma \) is equally divided and \( t \) changes from 0 to \( 2\pi \) for a closed curve. Obviously, this relationship holds for any point \( r(t) \) on the curve. Setting \( t = t_j, \quad l = 1, 2, \ldots, N \), we have (Fig. 2.6)

\[
\omega(t_j) = \frac{i}{4} \sum_{j=1}^{N} A_j H_0^{(1)} \left[ k_j \sigma(t_j, t_j') U(t_j) s_j'(t_j) \right], \quad l = 1, 2, \ldots, N.
\]

This equation will be abbreviated as

\[
\omega_j = \frac{i}{4} \sum_{j=1}^{N} A_j H_0^{(1)} \left( k_j \sigma_j \right) U_j s_j', \quad l = 1, 2, \ldots, N, \quad (2.38)
\]

where \( \omega_j \) is the value of \( \omega \) at the \( l \)th point, \( \sigma_{l,j} \) the chord distance between the \( l \)th and the \( j \)th point, \( U_j \) the unknown value of function \( U \) at the \( j \)th point, and \( s_j' \) is the value of \( ds/dt \) at \( j \). We now have reduced the integral equation (2.37) to a system of \( N \) linear algebraic equations for \( U_j \), and presumably they can be solved numerically by using digital computers.

Here, however, as in the above equations, there is one major difficulty. The Hankel function has a singular point at \( \sigma = 0 \).

From its definition

\[
H_0^{(1)}(z) = J_0(z) + iY_0(z),
\]

\[
J_0(z) = 1 - \frac{2}{z^2} + \frac{4}{z^4} - \ldots,
\]
\[ Y_0(z) = \frac{2}{\pi} \left[ \gamma + \log \frac{z}{2} \right] J_0(z) - \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{(-1)^m(z/2)^{2m}}{(m!)^2} (1 + \frac{1}{2} + \ldots + \frac{1}{m}), \]

we see that \( H_0^{(1)}(k\sigma \jmath \xi) \) has a logarithmic singularity in its imaginary part when \( \xi = j(\sigma \jmath \xi = 0) \). This difficulty can be overcome by excluding the term for \( j = \xi \) from the summation, and approximating the integration along the curve \( \Gamma \) in the neighborhood of \( r(t_\ell) \) by other means.

Since \( J_0(k\sigma) \to 1 \) as \( \sigma \to 0 \), the real part of \( H_0^{(1)}(k\sigma) \) in the sum causes no difficulty. For the \( Y_0(k\sigma) \) part, the predominant contribution is from the first term in its series expansion. In the neighborhood of \( t_\ell \), the integral (2.37) is then given approximately by

\[
I_\ell = \frac{i}{k} \int_{t_\ell - \epsilon}^{t_\ell + \epsilon} \left\{ 1 + i \frac{\sigma}{\pi} \left[ \gamma + \log \frac{k\sigma(t_\ell - t_0)}{2} \right] \right\} U(t_0) s'(t_0) dt_0.
\]

Reverting to integration with respect to the arc length \( ds_\ell \) where \( s_\ell \) is measured from the point \( \ell \), and then making no distinction between the arc length and the chord length \( R_\ell \) between the point \( \ell \) and \( \ell + \frac{1}{2} \) and the same for the arc from \( \ell - \frac{1}{2} \) to \( \ell \), we find (Fig. 2.6)

\[
I_\ell = \frac{i}{4} \int_0^{R_\ell} \left[ 1 + i \frac{\sigma}{\pi} \left( \gamma + \log \frac{k\sigma}{2} \right) \right] U(s_\ell) ds_\ell
\]

\[
= \frac{i}{2} U(t_\ell) \int_0^{R_\ell} \left[ 1 + i \frac{\sigma}{\pi} \left( \log \frac{k\sigma}{2} + \gamma \right) \right] ds_\ell
\]

\[
= U(t_\ell) \left[ \frac{1}{2} \left( \frac{1}{2} R_\ell - \frac{1}{\pi} \left( \gamma - 1 + \log \frac{kR_\ell}{2} \right) R_\ell \right) \right].
\]
We assume here that the points are equally divided, otherwise the integration to the left and to the right of the point \( r(t_{\ell}) \) should be carried out separately. With the singularity at \( j = \ell \) so separated, Eq. (2.38) is modified to

\[
\omega_\ell = \frac{1}{4} \sum_{j \neq \ell}^N \int_{\sigma_{\ell j}}^{(1)} (k \sigma_{\ell j}) U_j s_j' + \left[ \frac{i}{2} - \frac{1}{\pi} \left( \gamma - 1 + \log \frac{KR_\ell}{2} \right) \right] \epsilon_{\ell j}' \]

\[\ell = 1, 2, \ldots N. \quad (2.40)\]

To recapitulate, in the equations above the subscript \( j \) or \( \ell \) stands for the corresponding quantities at the \( j^{th} \) or \( \ell^{th} \) point on the curve \( \Gamma \); \( \omega = \omega^{(1)} \) is the given incident wave and \( U = \partial \omega^{(t)}/\partial n \) is the unknown normal derivative of the total wave on \( \Gamma \); \( A_j \) is a constant in a numerical quadrature formula; \( \sigma_{\ell j} \) is the chord distance between the \( j^{th} \) and \( \ell^{th} \) points; \( R_\ell \) is the chord length between the \( \ell \) and \((\ell + \frac{1}{2})\)
points; and \( \gamma = 0.5772... \) is the Euler constant.

The numerical procedure above was used by Banyaugh and Goldsmith (1963)\(^{(2.7)}\). They calculated the scattering of sound by a circular cylinder by dividing the circle into \( N = 36 \) segments. The real and imaginary parts of \( U_j \) \((j = 1, \ldots, 36)\) are then calculated by solving a system of 36 equations with complex coefficients. Numerical results so obtained agree very well with those calculated from the series solution, with a maximum difference of 4 percent. For the far field the total wave functions can be calculated by using the finite sum approximation of the integral \((2.21)\). They also treated the cases where the curve \( \Gamma \) is an ellipse and even cases where \( \Gamma \) is a curve like a three-leafed rose.

This example clearly indicates the advantage of this method—that it is not restricted to geometrical configurations to which the method of separation of variables may be applied. In fact, it can be extended to a boundary containing a cusp or corner. There is, however, an inherent difficulty in solving a large order linear system of equations, even with the aid of digital computers. Also, other types of singular kernels must be handled with special care when the integral equation is to be reduced to a system of algebraic equation, because the singular part quite often has a dominant role in the final solution.

For a three-dimensional problem, the representation of a boundary surface \( A \) in \((2.22)\) and \((2.23)\) requires two parameters. The approximation procedure is to replace the double integration by a double summation. For problems with axial symmetry, Brundrit (1965)\(^{(2.8)}\)
developed a procedure for numerical solution using a quite different approach to handle the logarithmic singularity. Comparison shows that the error is within 1 percent in a testing case for the diffraction of sound by a sphere. However, when the method is applied to a narrow prolate spheroid, large errors occur at the points along the axis of symmetry.

3. METHOD OF INTEGRAL TRANSFORMS

THE FIRST TWO SECTIONS of this chapter were devoted mostly to steady state problems. We shall now turn our attention to methods of solving transient problems. We recall that a discussion has already been given in the first chapter on how the transient response of a linear elastic system can be determined from the steady state response through Fourier synthesis. The method of integral transforms is presented at this stage because many recent publications on the subject of wave diffraction have employed this method. As readers shall note in the ensuing discussion, the two methods are essentially the same, except that the initial approaches to the problem are slightly different.

This method is best illustrated by an example. Let us again, for the sake of continuity, consider the diffraction of SH waves by a cylindrical cavity.

Consider a unit step incident disturbance propagating in the positive \( x \)-direction. This can be represented as

\[
\omega(t) = \omega_0 H \left( t - \frac{x + a}{c_s} \right), \quad (3.1)
\]
where \( w_0 \) is the amplitude of the pulse, \( H\left(t - \frac{x + \alpha}{c_s}\right) \) is the unit step function in which \( t = 0 \) is the time when the incident wave first arrived at \( x = -\alpha \), and \( c_s \) is the wave velocity. As the incident pulse impinges upon the cavity, an additional shear wave will be generated due to reflection and diffraction. It is governed by:

\[
\alpha^2 w(S) = \frac{1}{c_s^2} \frac{\partial^2 w(S)}{\partial t^2}.
\] (3.2)

Before the incident pulse reaches the edge of the cavity, no scattering takes place. Thus if we reckon the time \( t = 0 \) to be the instant that the front of the pulse reaches \( \theta = \pi \), then for the scattered waves, the velocity \( \omega_w(S)(r, \theta, t)/\partial t \) and the displacement \( w(C)(r, \theta, t) \) are both zero at \( t = 0 \).

We now apply the Laplace transform to Eq. (3.2) in order to eliminate the independent time variable. Denote \( W(r, \theta, \rho) \) by \( \mathcal{W}(r, \theta, \rho) \). From the zero initial condition we obtain

\[
\alpha^2 \mathcal{W}(S) - \mathcal{h}^2 \mathcal{W}(S) = 0, \quad \mathcal{h}^2 = \frac{\rho^2}{c_s^2}.
\] (3.3)

Equation (3.3) is the modified Helmholtz equation.

As a function of \( r \) and \( \theta \), \( \mathcal{W}(S) \) is periodic in \( \theta \). One could therefore expand \( \mathcal{W}(S) \) as a Fourier series in \( \theta \). Although in this approach much of the detailed information near the wave front would be lost, an accurate solution would be yielded nevertheless for the late-time phenomenon. We shall discuss this a little later in this section.
In the case of cylindrical coordinates, the appropriate solution for an outward propagating disturbance is

$$W'(\theta) = \sum_{n=0}^{\infty} A_n K_n(\lambda r) \cos n\theta,$$

(3.4)

where $K_n$ is the modified Bessel function of second kind. The coefficients $A_n$ of the expansion in Eq. (3.4) are to be determined by the traction-free boundary condition, $\sigma_{rz} = 0$, at $r = a$.

Transforming the incident pulse $w_0 H(t - \frac{x + a}{c_s})$, we obtain

$$W(t) = \int_{0}^{\infty} w_0 H(t - \frac{x + a}{c_s}) e^{-\frac{p}{c_s}t} dt = \frac{\omega_0 e}{p} \frac{-(x+a/c_s)}{\lambda r}.$$

(3.5)

To facilitate the computations for the coefficients $A_n$, we now expand the right-hand side of Eq. (3.5) in cylindrical coordinates:

$$W(t) = \frac{\omega_0 e}{p} e^{-\frac{xp}{c_s}} \frac{-(x+a/c_s)}{\lambda r} \int_{0}^{\infty} (-1)^n e^{i n (\lambda r) \cos n\theta}.$$

(3.6)

where $I_n$ is the modified Bessel function of the first kind, $\lambda = p/c_s$,

$e_n = 1$ when $n = 0$, and $e_n = 2$ when $n \geq 1$.

The boundary condition requires, at $r = a$,

$$s_{rz}(t) + S'(S) = 0,$$

(3.7)

where $S_{rz}$ denotes the Laplace transform of the stress $\sigma_{rz}$. It follows
then, that

$$\nu \left( \frac{3w(i)}{3r} + \frac{3w(S)}{3\theta} \right) = 0, \quad \text{at } r = a, \quad (3.8)$$

or

$$\nu \sum_{n=0}^{\infty} \left[ \left( \frac{w_o}{p} \right) e^{-\frac{ha}{r}} (-1)^n e_h I_{n+1}(ha) + A_n hK'(ha) \right] \cos n\theta = 0, \quad (3.9)$$

which gives

$$A_n = -\frac{w_o e^{-\frac{ha}{r}} (-1)^n e_h I_{n+1}(ha)}{pk'(ha)} = -\frac{w_o e^{-\frac{ha}{r}}}{p} e_h (-1)^n \frac{I_{n+1}(ha) + hai_{n+1}(ha)}{k_n(ha) - hai_{n+1}(ha)} \quad (3.10)$$

Having determined $A_n$, we can calculate the transformed displacement $\hat{w}$, or the transformed stresses $S_{02}$ and $S_{22}$, anywhere in the field. They are:

$$\hat{w}(r, \theta, p) = w(i) + w(S)$$

$$= \frac{w_o e^{-\frac{ha}{r}}}{p} \sum_{n=0}^{\infty} (-1)^n e_h \left[ I_n(kr) - \frac{I_{n+1}(ha) + hai_{n+1}(ha)}{k_n(ha) - hai_{n+1}(ha)} K_n(hr) \right] \cos n\theta; \quad (3.11)$$

$$S_{22}(r, \theta, p) = \frac{w_o e^{-\frac{ha}{r}}}{pr} \sum_{n=0}^{\infty} (-1)^n e_h \left\{ \left[ I_n(hr) + hai_{n+1}(hr) \right] - \left[ i_n(ha) + hai_{n+1}(ha) \right] \right\} \cos n\theta; \quad (3.12a)$$
\[ S_{\theta z}(r, \theta, p) = \frac{-\omega \epsilon_e^{-ha}}{pr} \sum_{n=0}^{\infty} n(-1)^n \epsilon_n \left\{ I_n(hr) \right\} \]

\[ -\left[ \frac{nI_n(ha) + haI_{n+1}(ha)}{nK_n(ha) - haK_{n+1}(ha)} \right] K_n(hr) \right\} \sin n\theta. \] (3.12b)

The transient responses are determined by inverting the Laplace transform. The points of most interest for dynamic stress concentrations are at the boundary, \( r = a \). Since \( S_{r z} \) vanishes identically, the only remaining stress is \( S_{\theta z} \). Letting \( r = a \) in Eq. (3.12b), and using the relationship

\[ I_n(a)K_{n+1}(a) + I_{n+1}(a)K_n(a) = -\frac{1}{a} \]

we obtain the expression for the transformed stress at the boundary as

\[ S_{\theta z} \bigg|_{r=a} = \frac{-\omega \epsilon_e^{-ha}}{pa} \sum_{n=0}^{\infty} n(-1)^n \epsilon_n \left\{ \frac{1}{nK_n(ha) - haK_{n+1}(ha)} \right\} \sin n\theta. \] (3.13)

The transformed stress concentration factor is determined by normalizing \( S_{\theta z} \) by \( \frac{\omega \epsilon_e}{\sigma_p} \), which is the magnitude of the transformed incident stress. Defining \( S_{\theta z}/(\omega \epsilon_e/\sigma_p) = S_{\theta z}^* \), then the normalized transient stress becomes

\[ \overline{\sigma}_{\theta z}(a, \theta, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} S_{\theta z}(a, \theta, p)e^{pt}dp \quad t > 0, \] (3.14)

Before we proceed to discuss the inversion procedure, it might
be well at this stage to show that the approach used here is basically identical to the Fourier synthesis method which was discussed in Chapter I.

In Chapter I, Section 4, it was shown that when the function is causal, the Laplace transform is the same as the Fourier transform if we identify \( p \) with \( -i\omega \). If we now substitute \( p = -i\omega \) into the arguments of \( I_n \) and \( K_n \) and note that

\[
I_n(-zi) = e^{-\frac{\pi}{2}ni} J_n(z), \tag{3.15a}
\]

\[
K_n(-zi) = \frac{\pi}{2} e^{\frac{\pi}{2}(n+1)i} H_n^{(1)}(z). \tag{3.15b}
\]

Equation (3.6) can be reduced to

\[
\omega(i) = \omega_0 \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos n\theta, \tag{3.16}
\]

where \( k \) is the familiar wave number, \( \omega/\omega_0 \). Equation (3.16) is identical to Eq. (1.15). With the time factor \( e^{-i\omega t} \) omitted, substitution of (3.15b) into (3.4) gives

\[
\omega(S) = \sum_{n=0}^{\infty} A_n H_n^{(1)}(kr) \cos n\theta. \tag{3.17}
\]

Again, Eq. (3.17) is identical to (1.11) except for a factor of \( (\pi/2)e^{(\pi/2)(n+1)i} \) which can be absorbed by \( A_n \).

These results shown in Eqs. (3.16) and (3.17) can also be readily obtained if one uses the Fourier transform at the outset; that is, by using the Fourier transform on Eqs. (3.1) and (3.2), the solution
to the transformed wave equation is precisely that given by (3.17).

Returning now to Eqs. (3.13) and (3.14), we note that the transient stress \( \sigma_{0z} \) is evaluated by assuming that the order of summation and integration can be interchanged, i.e.,

\[
\sigma_{0z} = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \varepsilon_n \left[ \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{p(t-ha)}}{(ha)^2 \kappa_n'(ha)} \, dp \right] \sin \pi \theta. \tag{3.18}
\]

It is apparent that the transient stress, as expressed by (3.18), can be viewed as a sum of the stress due to the responses of each mode, i.e., \( n = 0, 1, 2, \ldots \) etc. If we desire to determine the "exact" solution, this will entail evaluating the inverse transform as shown by the integral in (3.18) for all \( n \)'s, which is difficult. Various approximations can be made for the inversion, however, depending on what phenomenon we are seeking in the scattering problem.

Thus if we are interested in "early time" stress behavior, for example—that is, in the time shortly after the wave arrivals—then according to the Initial-Value Theorem we need to investigate the behavior of the integrand when \( p \) is large. On the other hand, if we wanted to know the maximum stress one might be guided by the "long time" solution (static solution) of the problem.

Let us examine the integral in Eq. (3.18). The path of the

---

*Initial-value theorem states that the values of the Laplace transform \( F_k(p) \) of a function \( f(t) \), for large \( p \), depends on the behavior of \( f(t) \) near the origin \( t = 0 \):

\[
F_k(p) \sim \frac{1}{p^{n+1}} \frac{d^n f(0)}{dt^n}, \quad p \to \infty.
\]
integral is $\gamma - i\omega + \gamma + i\omega$, which is known as the Bromwich path.

Straightforward integration along the Bromwich path is, in general, difficult; results can often be obtained, however, by suitably modifying the path of integration. The choice of a path of integration will depend, naturally, on the integrand. More specifically, the path of integration is governed by the types of singularities of the integrand.

For the present problem, the integrand $F_{2n}(p)$ in the inversion integral [Eq. (3.18)] is

$$F_{2n}(p) = \frac{e^{-ha}}{(ha)^2 K_n'(ha)}$$

The modified Bessel function of the second kind can be defined as:

$$K_n(z) = \frac{1}{2} \sum_{r=0}^{\infty} \frac{(-1)^r (n-r-1)!}{r!} \left( \frac{z}{2} \right)^{n-2r}$$

$$+ (-1)^{n+1} \sum_{r=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{n+2r}}{r! (n+r)!} \left\{ \ln \frac{z}{2} + \frac{1}{2} [\psi(r+1) + \psi(n+r+1)] \right\}$$

(3.19)

and

$$\psi(r+1) = \left( 1 + \frac{1}{2} + \ldots + \frac{1}{r} \right) - \gamma,$$

$$\psi(n+r+1) = \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n+r} \right) - \gamma,$$

(3.20)

$$\psi(1) = \gamma,$$

$$\gamma = 0.5772$$, Euler's constant.
Formulas for derivatives of $K_\eta(z)$ are:
\[
\begin{align*}
zK'_n(z) &= \eta K_n(z) - zK_{n+1}(z), \quad (3/21) \\
zK'_n(z) &= -\eta K_n(z) - zK_{n-1}(z).
\end{align*}
\]

Asymptotic expansions of $K_\eta(z)$ for $|z| \gg 1$, $|z| \gg |\eta|^2$, and $-\pi < \arg z \leq \pi$ are:

\[
K_\eta(z) \approx \sqrt{\frac{x}{2\pi}} e^{-\eta} e^{-2\pi x - \frac{1}{12\eta}(1 - 1)}
\]

\[
K'_\eta(z) \approx \sqrt{\frac{x}{2\pi}} e^{-\eta} \left[ 1 + \frac{(4\eta^2 - 1)(4\eta^2 - 3)}{1152} \right]^{1/2} \left[ \frac{(r - 1)!}{(8x)^{r-1}} \right], \quad \text{for } \eta \geq 1. \quad (3.22)
\]

First of all, it is noted from Eq. (3.19) that $K_\eta$ and $K'_\eta$ both possess a logarithmic singularity. Thus, they have a branch point at the origin. Secondly, $K_\eta$ and $K'_\eta$ have complex zeros. The number of zeros which they have are dependent upon the order $\eta$. It was shown at page 511 of Ref. 3.1 that for any order $\nu \geq 0$ (not necessarily an integer) $K_\nu(z)$ has no real zeros for which $|\arg z| \leq \frac{\pi}{2}$. The number of zeros in $\pi/2 < |\arg z| < \pi$ is the even integer nearest to $(\nu - \frac{1}{2})$, unless $(\nu - \frac{1}{2})$ is also an integer, in which case the number is $(\nu - \frac{1}{2})$. The number of zeros for $K'_\nu(z)$ is shown to be equal to the larger of the two consecutive even numbers that bracket the number $(\nu - \frac{1}{2})$.

From the discussion of the modified Bessel function, the integrand $F_{k\eta}(p)$ possesses both simple poles and a branch point. In what
follows, the inversion of $F_{\xi\lambda}(p)$ will be discussed to illustrate the inversion process. The appropriate contour $C$ for the inversion of $F_{\xi\lambda}(p)$ is the contour shown in Fig. 3.1. There are two poles which are the roots of $K'_1(\eta a) = 0$, one branch point at the origin, and a branch cut extending from 0 to $\infty$ along real $p$.

![Diagram of contour C for $F_{\xi\lambda}(\zeta)$](image)

*Fig. 3.1 Contour C for $F_{\xi\lambda}(\zeta)$*

Since $F_{\xi\lambda}$ is analytic in the interior of $C$ the integration along the contour $C$ is zero. Denoting $\zeta = \eta a$ and $\tau = (a_0/a)\xi$, then we have

$$\int_{C} \frac{e^{\xi(\tau-1)}}{\tau^2 K'_1(\xi)}\, d\xi = 0. \quad (3.23)$$

It follows
METHOD OF INTEGRAL TRANSFORMS

\[ \int \frac{e^{\zeta(t-1)}}{L \zeta K'_1(\zeta)} \, d\zeta = -\left[ \int \frac{e^{\zeta(t-1)}}{\zeta K'_1(\zeta)} \, d\zeta \right]_{AB+CD+EF+GH} + 2\pi i \sum_1^2 \text{Residue.} \quad (3.24) \]

In the limit as \( \beta \to \infty \), the path \( L \) is the Bromwich path and the left-hand side is proportional to the inversion transform of \( F_{11} \),

\[ f_1(\tau) = \left( \frac{c_B}{\alpha} \right) \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} F_{11}(\zeta) e^{\tau \zeta} \, d\zeta, \]

\[ = -\left( \frac{c_B}{\alpha} \right) \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{A-B} A-F e^{\zeta(t-1)} \zeta K'_1(\zeta) \zeta e^{-\zeta} + \frac{2}{L} \sum_1^2 \text{Residue.} \quad (3.25) \]

Let us now examine the contributions of the various segments of the contour. Along the segments \( A-B \) and \( C-H \), \( \zeta = Re^{i\theta} \). When \( R \) is large

\[ K_1(\zeta) \sim \sqrt{\frac{\pi}{2\zeta}} e^{-\zeta} \quad (3.26a) \]

\[ \zeta K'_1(\zeta) \sim -\sqrt{\frac{\pi}{2}} e^{-\zeta}. \quad (3.26b) \]

Substituting these asymptotic expansions into the integrand in (3.25) gives

\[ \frac{e^{\zeta(t-1)}}{\zeta^{2} K'_1(\zeta)} + \frac{\sqrt{\frac{2}{\pi}} e^{\zeta \tau}}{\zeta^{(3/2)}}. \quad (3.27) \]

Furthermore, on the arc \( AB \) the real part of the exponent \( \zeta \tau \) is not greater than \( \alpha t \) where \( \alpha \) is an arbitrary constant for which \( \zeta \) is in
the half-plane of convergence. Hence

\[ \left| \frac{e^{\frac{\zeta t}{\xi (3/2)}}}{\zeta^{(3/2)}} d\zeta \right| \leq e^{\alpha \tau} d\theta, \]

and if \( \theta_A \) is angle \( \theta \) at \( A \), and \( I_{AB} \) denotes the integral along \( AB \), then

\[ |I_{AB}| \leq e^{\alpha \tau} \int_{\theta_A}^{\pi/2} d\theta = e^{\alpha \tau} \left( \frac{\pi}{2} - \theta_A \right). \tag{3.28} \]

Since \( \theta_A \to \pi/2 \) as \( R \to \infty \), it follows that \( I_{AB} \) vanishes as \( R \to \infty \). Likewise, \( I_{GH} \) tends to zero as \( R \) tends to infinity.

On the arcs \( \Gamma_1 \) and \( \Gamma_2 \) the Re \( (\zeta) \) is always negative. Hence, for \( \tau > 0 \) the integrand \( \to 0 \) as \( R \to \infty \). Then according to Jordan's lemma, the integrals along \( \Gamma_1 \) and \( \Gamma_2 \) tend to zero also. (Jordan's lemma: If \( f(z) \to 0 \) as \( z \to \infty \) then \( \int_{\Gamma} e^{tx} f(z) dz \to 0 \) as \( R \to \infty \) for \( t > 0 \), where \( \Gamma \) is \( \Gamma_1 \) or \( \Gamma_2 \) in Fig. 3.1.)

On the small circle \( \Gamma_3 \), \( \zeta = \rho e^{i\theta} \). When \( \rho_0 \ll 1 \), we use the small argument approximation for \( K_\nu(\zeta) \), (Eq. 3.19), and we find for \( |\zeta| \ll 1 \)

\[ K_1(\zeta) \sim \frac{1}{\zeta}, \]
\[ \zeta^2 K_1(\zeta) \sim -1. \]

It follows
\[ \int \frac{e^{\xi(\tau-1)}}{\frac{2}{\zeta^2}K_1(\zeta)} \, d\xi \leq \int_{\pi}^{\pi} e^{\rho(\tau-1)} \rho \, d\theta \]  

(3.29)

and it tends to zero as \( \rho \to 0 \).

The two integrals along \( CD \) and \( EF \) in (3.25) remain to be examined. Along the path \( CD, \zeta = re^{i\pi} = -\zeta \), and on the path \( EF, \zeta = re^{-i\pi} = -\zeta \).

When the argument \( \zeta \) in \( K_n(\zeta) \) is real and less than zero, \( K_n(\zeta) \) is complex. It can be expressed in terms of \( K_n(\zeta) \) and \( I_n(\zeta) \) with real positive arguments through analytic continuation, since for real positive \( r \),

\[ K_{\nu}(re^{m\pi i}) = e^{-m\pi i} K_{\nu}(r) - im \frac{\sin m\pi \nu}{\sin \nu} I_{\nu}(r), \quad m \text{ an integer}. \]  

(3.30)

Thus along \( CD, \zeta = -\zeta \) (\( m = 1 \) in Eq. 3.30) and we have

\[ K_n(-\zeta) = -K_n(\zeta) - i\pi I_n(\zeta). \]  

(3.31)

and on \( EF, \zeta = -\zeta \) (\( m = -1 \) in Eq. 3.30)

\[ K_n(-\zeta) = -K_n(\zeta) + i\pi I_n(\zeta). \]  

(3.32)

Using (3.21) and these expressions for \( K_n(-\zeta) \) along the branch lines, the branch integrals for \( n = 1 \) as \( R \to -\infty \) become

\[ I_{CD+EF} = \int_{CD+EF} \frac{e^{\xi(\tau-1)}}{\xi[K_1(\xi) - iK_2(\xi)]} \, d\xi \]

\[ = -\int_{-\infty}^{0} \frac{e^{-\xi(\tau-1)} \, d\xi}{(-\xi)[(-K_1(\xi) - i\pi I_1(\xi)) + \xi(-K_2(\xi) - i\pi I_2(\xi))]} \]
\[
\int_{0}^{\infty} \frac{e^{-\xi(\tau-1)} d\xi}{(-\xi)\{[-K_1(\xi) + i\pi I_1(\xi)] + \xi[-K_2(\xi) + i\pi I_2(\xi)]\}}
\]

\[
= -2\pi i \int_{0}^{\infty} \frac{[I_1(\xi) + \xi I_2(\xi)]e^{-\xi(\tau-1)} d\xi}{\xi([K_1(\xi) + i\pi K_2(\xi)]^2 + \pi^2[I_1(\xi) + \xi I_2(\xi)]^2)}.
\]  \tag{3.33}

Thus the inverse Laplace transform of \( F_{i1} \) in (3.25) is

\[
\varphi_1(t) = \zeta = \left(\frac{a}{a}\right) \left[ \frac{2}{1} R_{1,k} - \frac{1}{2\pi i} I_{CD+EF} \right].
\]  \tag{3.34}

where \( R_{1,k} \) is the residue. Subscript 1 indicates that it is from the \( n = 1 \) term and subscript \( k \) refers to a root of \( K_1'(\zeta) \) associated with a particular \( n \). When \( n = 1 \), there are only two roots of \( K_1'(\zeta) = 0 \). The roots of \( K_1'(\zeta) = 0 \) as a function of \( \nu \) are shown in Fig. 3.2 and Table 3.1.

---

Fig. 3.2 Zeros of \( K_1'(\zeta) \)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \circ )</td>
<td>1</td>
</tr>
<tr>
<td>( \square )</td>
<td>2</td>
</tr>
<tr>
<td>( \triangle )</td>
<td>3</td>
</tr>
<tr>
<td>( \bullet )</td>
<td>4</td>
</tr>
<tr>
<td>( \mathbb{H} )</td>
<td>5</td>
</tr>
<tr>
<td>( \nabla )</td>
<td>6</td>
</tr>
<tr>
<td>( \times )</td>
<td>7</td>
</tr>
<tr>
<td>( \diamond )</td>
<td>8</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>9</td>
</tr>
<tr>
<td>$k/n$</td>
<td>1</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>-0.64354</td>
</tr>
<tr>
<td></td>
<td>$\pm i\cdot 50118$</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\pm i\cdot 44079$</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\pm i\cdot 43637$</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\pm i\cdot 43515$</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\pm i\cdot 43464$</td>
</tr>
</tbody>
</table>
The general expression for the residues is given by

\[ R_{n,k} = \frac{\zeta_{n,k}(\tau-1)}{e^{(\zeta_{n,k})^2K''_n(\zeta_{n,k})}}. \tag{3.35} \]

Finally, we have the stress contributed by the \( \kappa = 1 \) term in the series as

\[
f_1(t) = \frac{\sigma_0}{a}
\left[ \sum_{k=1}^{2} e^{\zeta_{1,k}(\tau-1)} \frac{e}{(\zeta_{1,k})^2K''_1(\zeta_{1,k})}
\right] \cdot \frac{\left[ I_1(\xi) + \xi I_2(\xi) \right] e^{-\xi(\tau-1)} d\xi}{\xi \left[ K_1(\xi) + \xi K_2(\xi) \right]^2 + \pi^2 \left[ I_1(\xi) + \xi I_2(\xi) \right]^2}, \tag{3.36} \]

\( \tau > 0. \)

Each of the remaining terms in the series can be inverted like the \( \kappa = 1 \) term, and the dimensionless stress \( \bar{\sigma}_{0z} = \sigma_{0z}/(\omega_0/\alpha) \) due to all \( \kappa \)-modes can be expressed as follows:

\[
\bar{\sigma}_{0z} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} n \left[ \sum_{k=1}^{M} e^{\zeta_{n,k}(\tau-1)} \frac{e}{(\zeta_{n,k})^2K''_n(\zeta_{n,k})} \right]
\]

\[
+ \int_0^\infty \frac{\left[ I_n(\xi) + \xi I_{n+1}(\xi) \right] e^{-\xi(\tau-1)} d\xi}{\xi \left[ nK_n(\xi) + \xi K_{n+1}(\xi) \right]^2 + \pi^2 \left[ nI_n(\xi) + \xi I_{n+1}(\xi) \right]^2} \sin n\theta \tag{3.37} \]

\( \tau > 0 \)

where \( M \) is the total number of roots of \( K_n'(\xi) = 0 \) for a particular \( n \).

The stress \( \bar{\sigma}_{0z} \) as given by Eq. (3.37) is due to an incident wave
characterized by a unit step in displacement. The stress time history of the incident wave, therefore, is a unit impulse. Hence, we may consider \( \sigma_{0z} \) as the response due to an incident wave characterized by a unit impulse in stress.

Equation (3.37) is valid for \( \tau > 0 \). It should be pointed out, however, that this approach is not accurate during the interval \( 0 < \tau < 1 \). Notwithstanding the early-time difficulties, numerical results for \( \sigma_{0z} \) are obtained by evaluating the first seven terms of Eq. (3.37). The composite response of the cavity with seven terms in the series is shown in Fig. 3.3. Tables 3.2 and 3.3 give the contribution to the response due to the residue part and the branch integral part of the solution, respectively.

![Graph](image)

**Fig. 3.3** \( \sigma_{0z} \) as a Function of Half Transit Time for Delta Input and \( \theta = \pi/4, \pi/2, 3\pi/4 \)
Table 3.2
CONTRIBUTION TO $\sigma_{\theta_2}^n(t)$ DUE TO RESIDUES FOR $n^{th}$ MODE

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>.4</th>
<th>.8</th>
<th>1.6</th>
<th>3.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1.227</td>
<td>-1.006</td>
<td>-.791</td>
<td>-.433</td>
<td>-.084</td>
<td>-.028</td>
</tr>
<tr>
<td>2</td>
<td>.274</td>
<td>-.319</td>
<td>-.524</td>
<td>-.293</td>
<td>.084</td>
<td>-.012</td>
</tr>
<tr>
<td>3</td>
<td>.686</td>
<td>.317</td>
<td>-.235</td>
<td>-.263</td>
<td>.052</td>
<td>.006</td>
</tr>
<tr>
<td>4</td>
<td>.183</td>
<td>.757</td>
<td>.257</td>
<td>-.211</td>
<td>-.020</td>
<td>-.004</td>
</tr>
<tr>
<td>5</td>
<td>-1.081</td>
<td>.084</td>
<td>.460</td>
<td>-.182</td>
<td>-.037</td>
<td>.003</td>
</tr>
<tr>
<td>6</td>
<td>.178</td>
<td>-.539</td>
<td>.403</td>
<td>-.158</td>
<td>-.003</td>
<td>-.002</td>
</tr>
<tr>
<td>7</td>
<td>.744</td>
<td>-.556</td>
<td>.102</td>
<td>-.141</td>
<td>.023</td>
<td>.002</td>
</tr>
</tbody>
</table>

Table 3.3
CONTRIBUTION TO $\sigma_{\phi_2}^n(t)$ DUE TO BRANCH INTEGRAL FOR $n^{th}$ MODE

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>.4</th>
<th>.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.162</td>
<td>.098</td>
<td>.062</td>
<td>.040</td>
<td>.027</td>
<td>.019</td>
</tr>
<tr>
<td>2</td>
<td>-.145</td>
<td>-.066</td>
<td>-.032</td>
<td>-.016</td>
<td>-.008</td>
<td>-.004</td>
</tr>
<tr>
<td>3</td>
<td>.129</td>
<td>.045</td>
<td>.017</td>
<td>.006</td>
<td>.002</td>
<td>.001</td>
</tr>
<tr>
<td>4</td>
<td>-.116</td>
<td>-.031</td>
<td>-.009</td>
<td>-.003</td>
<td>-.001</td>
<td>$0^+$</td>
</tr>
<tr>
<td>5</td>
<td>.107</td>
<td>.022</td>
<td>.005</td>
<td>.001</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>-.099</td>
<td>-.015</td>
<td>-.002</td>
<td>$0^+$</td>
<td>$0^+$</td>
<td>$0^+$</td>
</tr>
<tr>
<td>7</td>
<td>.093</td>
<td>.011</td>
<td>.001</td>
<td>$0^+$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
It is apparent from the tables and Fig. 3.3 that the residue part of the solution gives a damped oscillation behavior while the value of the branch integral decreases, monotonically, as a function of time. The frequencies of the oscillation are determined by the imaginary part of the roots of \( K_n'(\xi) = 0 \) (see Fig. 3.2 or Table 3.1), and the damping factors are determined by the real part of the root.

In other words, one can determine the natural frequencies and the associated damping factor, (in this case the radiation damping), of a cavity for the anti-plane strain case by observing the behavior of the roots of \( K_n'(\xi) = 0 \). Examination of Fig. 3.2 shows that the natural frequencies of the cavity increase as the order \( n \) increases; on the other hand, the damping factor tends to level off as \( n \) increases. In addition to this observation, it is noted that there exist multiple roots of each \( n \), which is analogous to several nodes for each mode \( n \) of a vibrating circular membrane.

Also of physical interest is another type of input which occurs when the incident stress is characterized by a unit step function, \( H(t) \). The response \( \sigma_{\delta z} \) due to an incident stress \( H(t) \) is readily determined from the relationship of \( H(t) \) and \( \delta(t) \) as indicated in Chapter 1. The stress \( \sigma_{\delta z} \) due to \( H(t) \) is given as:

\[
\sigma_{\delta z} = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\infty}^{\infty} \frac{1}{\beta} \overline{S}_{\delta z} e^{\beta t} d\beta
\]

where the term \( 1/\beta \) is the Laplace transform of \( H(t) \) and \( \overline{S}_{\delta z} \) was used previously for the case of \( \delta(t) \) input.

Values of the integral along different segments of the path \( C \)
(Fig. 3.1) for $\sigma_{\theta z}$ are determined as before, the only difference being in the value of the integral around the small circle $\Gamma_3$ as $\rho_0 \to 0$. Here this integral becomes

$$\lim_{\rho_0 \to 0} \int_{\Gamma_3} \frac{e^{\tau(t-1)}}{\zeta^3 \tilde{K}_1(e)} \, d\zeta = -2\pi i, \quad \tau > 0.$$ 

Hence the contribution made by integral around the origin yields the solution $-2 \sin \theta$, which is the long-term solution or the static solution.

Shown in Figures 3.4 and 3.5 are the modal and composite behaviors of $\sigma_{\theta z}$, respectively. It is evident, from Fig. 3.4, that all modes

![Graph of $\sigma_{\theta z}$ vs $\theta$ for modes $n=0, 1, 2, 4, 6, 7$ at $\theta = \pi/2$](image)

Fig. 3.4 Modal behavior of $\sigma_{\theta z}$ at $\theta = \pi/2$ ($n = 0, ..., 7$)
other than \( n = 1 \) contribute primarily during the first transit time \((\tau \leq 2)\). For all practical purposes, these modal vibrations have disappeared after \( \tau = 4 \). Figure 3.5 shows the stress time history at three different locations on the boundary of the cavity. Here we noticed the same difficulty as for the impulse input is encountered in the illuminated side of the cavity \((\pi/2 < \theta < 3\pi/2)\) when \( 0 < \tau < 1 \). We shall shortly discuss a method which resolves this difficulty as first shown by Friedlander.

For time \( \tau > 1 \), we have used Eq. (3.38) to evaluate \( \sigma_{\theta z} \) at \( \theta = 3\pi/4, \pi/2, \pi/4 \). We noted that there is an overshoot of the static value by approximately 10 percent.

In the foregoing discussion, we have focused our attention on the long-time solution because studies on dynamic stress concentratons
have shown that the maximum stress usually develops at a relatively late time. As we indicated earlier, a modal approach is not appropriate for early-time behavior. We shall present a method for overcoming this difficulty first presented by Friedlander in 1954 (Ref. 3.2), in his investigation of "diffraction of pulses by a cylinder."

This method has proven extremely useful in many similar investigations (Refs. 3.3 through 3.6).

Returning now to Eq. (3.13) expressed in dimensionless form,

we have

$$\bar{S}_{0z} \bigg|_{r=a} = e^{-\text{ha}} \sum_{n=0}^{\infty} \frac{n(-1)^{n+1}}{(\text{ha})^{2}K_{n}'(\text{ha})} \cdot \frac{1}{\pi} \cdot \sin n\theta,$$

which can be also expressed as

$$\bar{S}_{0z} \bigg|_{r=a} = -e^{-\text{ha}} \sum_{n=-\infty}^{\infty} \frac{\text{in} \cdot \text{in} \pi}{(\text{ha})^{2}K_{n}'(\text{ha})} e^{-\text{in} \pi} = - \sum_{n=-\infty}^{\infty} f(n) e^{-\text{in} \theta}. \quad (3.39)$$

The Fourier series in Eq. (3.39) which sums over $n$, can be changed into another series which will allow us to examine the early-time behavior. This is accomplished by using the Poisson summation formula. The procedures are described briefly:

$$\bar{S}_{0z} = - \sum_{n=-\infty}^{\infty} f(n) e^{-\text{in} \theta} = - \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{\text{in} \xi} d\xi \cdot e^{-\text{in} \theta}. \quad (3.40)$$

where $F(\xi)$ is the Fourier transform of $f(n)$.

Interchanging integration and summation we obtain
\[ S_{\theta z} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) d\xi \sum_{\eta=-\infty}^{\infty} e^{i\eta(\xi-\theta)}, \] (3.41)

where the summation under the integral yields for \((-\pi \leq \xi \leq \pi)\),

\[ \sum_{\eta=-\infty}^{\infty} e^{i\eta(\xi-\theta)} = 2\pi \delta(\xi-\theta). \] (3.42)

By noting that the left-hand side of Eq. (3.42) is periodic, and by making the right-hand side periodic, we obtain a relation which is valid for all values of \(\xi\),

\[ \sum_{\eta=-\infty}^{\infty} e^{i\eta(\xi-\theta)} = 2\pi \sum_{\eta=-\infty}^{\infty} \delta(\xi-\theta-2\eta\pi). \] (3.43)

Substituting Eq. (3.42) into Eq. (3.40) we obtain the following relationship

\[ S_{\theta z} = -\sum_{\eta=-\infty}^{\infty} f(\eta) e^{-i\eta\theta} = -\frac{1}{\sqrt{2\pi}} \sum_{\eta=-\infty}^{\infty} F(\theta + 2\eta\pi). \] (3.44)

This is a form of the well-known Poisson summation formula. According to our previous definition of \(F(\xi)\), it follows

\[ F(\theta) = \frac{e^{-i\theta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i\nu}{(ha)^{2} K_{\nu}(ha)} e^{-i\nu(\theta-\pi)} d\nu. \] (3.45)

Equation (3.45) is often referred to as the wave sum. Physically, it implies that the stress at any given point \(\theta\) on the circumference of the cavity is the sum of all those parts of \(F(\theta)\) which have over-
lapped the given point as the waves wind around the cavity. If we are merely interested in the early-time solution, then we need only consider the \( m = 0 \) term, or to investigate the integral shown in Eq. (3.45):

\[
\overline{S}_{xz} = -e^{-\frac{\pi}{ha}} \int_{-\infty}^{\infty} \frac{i v}{(ha)^{2} K'(ha)} e^{-iv(\theta-\tau)} dv.
\]

The evaluation of similar integrals for large values of \( ha \) was discussed in great detail in Refs. (3.2) and (3.3); it is beyond the scope of this book. It is of interest to note, however, that available numerical results, obtained for the early time have shown consistently that the stress is much lower for early time than for late time. Thus for our purposes we shall not further labor over the early time solution. For an excellent account of the early-time phenomenon the reader is referred to the book *Sound Pulses* by Freidlander (Ref. 3.7).

This section has shown one method for solving transient problems. Other methods for solving transient problems will be demonstrated later. We might add that the two types of input used in this section are very useful because once their solutions are known, the solutions to any arbitrary input can be determined readily through a simple quadrature, as shown in Section 4 of Chapter I.

4. A PERTURBATION METHOD FOR ELASTIC WAVES

SO FAR IN THIS CHAPTER we have discussed only the scattering of SH waves. The methods illustrated will be applied directly to P and
SV waves in the following chapters. One will soon find out that the method of wave functions expansion as well as the integral transform technique are limited to even fewer coordinate systems than those enumerated at the end of Section 1. The main obstacle is the coexistence of two types of scattered waves each with a distinct wave speed. Since the wave speeds appear explicitly in the wave equations, the angular wave functions will contain two different wave speeds in their arguments. This is true for all curvilinear coordinates except the circular cylindrical with angular function $e^{in\theta}e^{imz}$, and the spherical with $P_n^m(\cos \theta) \exp (im\phi)$. If one of the radial coordinate surfaces is taken as the boundary of a scatterer, and if the displacements and stresses are expressed as series of the products of radial and angular wave functions with different arguments, the series does not form orthogonal sets at the boundary of the scatterer. Thus the unknown coefficients in the series cannot be determined exactly from the boundary conditions.

This difficulty in analysis will be discussed again in Chapter IV via a concrete example. In this section a recently developed perturbation method is outlined (Refs. 4.1, 4.2), which partially overcomes this difficulty. The essence of this method is replacing the two unequal wave numbers (frequency/wave speed) which appear in the two steady state wave equations (I-2.31), with their root-mean-squared (rms) value.

The difference between the actual wave numbers and the root-mean-square average happens to be a number less than $\frac{1}{2}$, which can be used as a perturbation parameter. The wave equations and boundary condi-
tions are then all perturbed with the result that, for each order of the perturbation expansion, only the rms wave number is involved. Thus the boundary conditions for each order of expansion can be satisfied by series-wave function solutions.

In this section, we set forth the perturbation procedure and the general solutions of each order of the perturbed equations. Applications of this method are illustrated in Chapter IV, Section 3, for the elliptic coordinate system, and in Chapter V, Sections 3 and 4, for the parabolic coordinate system.

4.1. Perturbation of Displacement Potentials

The basic field equations are given in Chapter I:

\[ \mathbf{u} = \nabla \varphi + \nabla \times \psi, \] (I-2.30)

\[ \sigma_p \varphi = \varphi, \quad \frac{c^2}{\sigma_p} = (\lambda + 2\mu)/\rho, \] (I-2.31)

\[ \sigma_s \psi^2 = \psi, \quad \frac{c^2}{\sigma_s} = \mu/\rho, \]

\[ \mathbf{g} = \lambda(\nabla \mathbf{u}) \mathbf{e} + \mu(\nabla \mathbf{u} + \mathbf{u} \nabla). \] (I-2.25)

For the remaining discussion we confine the method to steady-state waves. The time factor \( \exp(-i\omega t) \) will be separated from all field variables. Hence \( \mathbf{u}, \varphi, \psi \) and \( \mathbf{g} \) are functions of spatial coordinates only and (I-2.31) reduces to a scalar and a vector Helmholtz equation.

\[ (\nabla^2 + \alpha^2)\varphi = 0, \quad \alpha = \omega/c_p, \] (4.1)

\[ (\nabla^2 + \beta^2)\psi = 0, \quad \beta = \omega/c_s. \]
PERTURBATION METHOD FOR ELASTIC WAVES

The wave number $\alpha$ differs from $\beta$ for all real solids and their ratio is a function of Poisson's ratio (cf. I-2.32)

$$\kappa = \frac{c_p}{c_s} = \frac{\beta}{\alpha} = \left(\frac{2-2\nu}{1-2\nu}\right)^{\frac{2}{3}}.$$

The method of perturbation begins by noting that the root-mean-squared wave number

$$k^2 = \frac{1}{2} \sqrt{\alpha^2 + \beta^2}$$

is related to the two distinct wave numbers $\alpha$ and $\beta$ by

$$\alpha^2 = k^2(1 - 2\epsilon), \quad \beta^2 = k^2(1 + 2\epsilon),$$

where

$$\epsilon = \frac{\beta^2 - \alpha^2}{2(\alpha^2 + \beta^2)} = \frac{1}{2(3 - 4\nu)}.$$

Since Poisson's ratio, $\nu$, has a numerical value between zero and one-half, $\epsilon$ is limited between $1/6$ and $1/2$. It thus can be used as the parameter in regular perturbation expansions for $\varphi$, $\psi$, $u$, and $g$:

$$\varphi = \varphi^{(0)} + \epsilon\varphi^{(1)} + \ldots + \epsilon^n\varphi^{(n)} + \ldots;$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \ldots + \epsilon^n\psi^{(n)} + \ldots;$$

$$u = u^{(0)} + \epsilon u^{(1)} + \ldots + \epsilon^n u^{(n)} + \ldots;$$

$$g = g^{(0)} + \epsilon g^{(1)} + \ldots + \epsilon^n g^{(n)} + \ldots.$$
where \(n\)'s are integers.

In terms of \(k\) and \(\epsilon\), equations (4.1) are written as

\[
[v^2 + k^2(1-2\epsilon)]\varphi = 0,
\]

\[
[v^2 + k^2(1+2\epsilon)]\psi = 0.
\]

Substituting the perturbation series in the above equations and collecting coefficients of like powers of \(\epsilon\), we obtain the field equations for the \(n\)th order potentials

\[
(v^2 + k^2)\varphi^{(n)} = 2k^2 \varphi^{(n-1)},
\]

\[
(v^2 + k^2)\psi^{(n)} = -2k^2 \psi^{(n-1)},
\]

where \(\varphi^{(-1)}\) and \(\psi^{(-1)}\) are taken as zero. Similarly, the \(n\)th order displacements and stresses are related to \(\varphi^{(n)}\) and \(\psi^{(n)}\) by

\[
u^{(n)} = v\varphi^{(n)} + v\times\psi^{(n)}; \tag{4.6}
\]

\[
\sigma^{(n)} = \lambda(v\cdot\nu^{(n)}) + \mu(\nabla\nu^{(n)} + \nu^{(n)}\nu), \tag{4.7}
\]

\[
= \mu[(\kappa^2-2)(v\cdot\nu^{(n)}) + \nabla\nu^{(n)} + \nu^{(n)}\nu].
\]

Note that the constant \(\kappa\) in (4.7) is unperturbed. The boundary conditions are treated in like manner. Thus the field equation (4.5) and boundary conditions for each order of perturbation depend on the common wave number \(k\).

Solutions for the 0th order equations
\[(\nabla^2 + k^2)\varphi^{(0)} = 0, \quad (4.8)\]
\[(\nabla^2 + k^2)\psi^{(0)} = 0,\]
and the corresponding \(n\)th order boundary conditions can be found in a manner similar to solving problems of SH waves. Proceeding successively to higher-order equations, we decompose the solutions of (4.5) into two parts: the complementary solutions \(\varphi^{(n)}_c\) and \(\psi^{(n)}_c\), and the particular solutions \(\varphi^{(n)}_p\) and \(\psi^{(n)}_p\). The former satisfy the homogeneous equations
\[\begin{align*}
(\nabla^2 + k^2)\varphi^{(n)}_c &= 0, \\
(\nabla^2 + k^2)\psi^{(n)}_c &= 0,
\end{align*}\]
while the latter can be constructed from the solutions of lower orders. General solutions are then given by
\[\begin{align*}
\varphi^{(n)} &= \varphi^{(n)}_c + \varphi^{(n)}_p, \\
\psi^{(n)} &= \psi^{(n)}_c + \psi^{(n)}_p. \quad (4.9)
\end{align*}\]

Consider the first order perturbation which has the following field equations:
\[\begin{align*}
(\nabla^2 + k^2)\varphi^{(1)} &= 2k^2\varphi^{(0)} = -2\nabla^2 \varphi^{(0)}, \\
(\nabla^2 + k^2)\psi^{(1)} &= -2k^2\psi^{(0)} = 2\nabla^2 \psi^{(0)}. \quad (4.10)
\end{align*}\]
A particular solution for each equation is
\[ \varphi_P^{(1)} = -r \cdot \varphi^{(0)}, \]
\[ \psi_P^{(1)} = r \cdot \psi^{(0)}, \]

which can be verified by direct substitution.

The second order equations follow from (4.5), (4.9), and (4.11):

\[ (\nu^2 + k^2) \varphi^{(2)} = -2k^2 r \cdot \varphi^{(0)} - 2\varphi^{(1)}, \]
\[ (\nu^2 + k^2) \psi^{(2)} = -2k^2 r \cdot \psi^{(0)} + 2\psi^{(1)}. \]

Particular solutions are

\[ \varphi_P^{(2)} = -r \cdot \varphi(2) + \varphi^{(1)} - \frac{1}{2} k^2 r^2 \varphi^{(0)}, \]
\[ \psi_P^{(2)} = -r \cdot \psi(2) - \psi^{(1)} - \frac{1}{2} k^2 r^2 \psi^{(0)}, \]

where \( m = 2 \) or \( 3 \) according to \( \varphi \) and \( \psi \) being two-dimensional or three-dimensional variables.

Higher order solutions can be generated in analogous manner, but the particular solutions become rather lengthy. Furthermore, the lengthy particular solutions will modify the boundary conditions such that the corresponding complementary solutions become very difficult. In practice, we shall be content with results of the first few orders.

This completes formally the method of perturbation for boundary value problems in elastodynamics. For plane strain problems with \( \mathbf{u}(x_1, x_2, t) = (u_1, u_2, 0) \) as defined in Section 3 of Chapter I, we may take \( \psi^{(n)} = (0, 0, \varphi^{(n)}) \) and consider both potentials \( \varphi^{(n)}, \psi^{(n)} \).
as functions being independent of \( x_3 \). The rest of the formulation is unchanged.

For problems of generalized plane stresses, \( u(x_1, x_2, t) \) represents the average value of displacement across the thickness coordinate \( x_3 \) of a thin plate (\( u \) in Eq. I-3.14). The rest is the same as the plane-strain formulation except \( \kappa \) in (4.7) is replaced by \( \kappa' \) (see Eq. I-3.23) with

\[
\kappa' = \frac{\omega}{\rho} = \frac{\beta}{a'} = \left( \frac{2}{1-\nu} \right)^{1/2},
\]

and \( a' = \omega/\rho \). The two wave numbers \( \alpha' \) and \( \beta \) are related to their root-mean-squared value \( k' \) through a new parameter \( \epsilon' \) with

\[
\epsilon' = \frac{1+\nu}{2(3-\nu)}.
\]  

(4.14)

We take comfort in noting that \( \epsilon' \), having values from 1/6 to 3/10 for all actual values of Poisson's ratio, is still suitable to be a perturbation parameter.

4.2. Perturbation of Displacements

The use of displacement potentials in the development of the perturbation method clearly demonstrates the way of uncoupling the two types of elastic waves in the analysis of a boundary value problem. However, for problems with prescribed displacements on the boundary, it is more convenient to apply the same perturbation scheme directly to the displacement-equation of motion, thus avoiding the use of potentials altogether. Note that Eq. (I-2.26) can be rewritten
as

\[(\kappa^2 - 1)\nabla^2 u + \varphi^2 u = c_g^{-2} \ddot{u}. \tag{4.15}\]

Letting \(u(x_j, t) = u(x_j) \exp(-i\omega t)\) with \(j = 1, 2, 3\) and applying Eq. (4.3), we have the following form of the displacement equation of motion:

\[4\epsilon \nabla^2 u + (1-2\epsilon)\varphi^2 u + k^2(1-4\epsilon^2)u = 0.\]

Expanding \(u\) into the perturbation series and collecting terms of like powers of \(\epsilon\), we obtain the various order-displacement equations:

\[(\varphi^2 + k^2)u^{(0)} = 0, \]

\[(\varphi^2 + k^2)u^{(1)} = 2\varphi^2 u^{(0)} - 4\nabla^2 u^{(0)}, \tag{4.16}\]

\[(\varphi^2 + k^2)u^{(n)} = 2\varphi^2 u^{(n-1)} - 4\nabla^2 u^{(n-1)} + 4k^2 u^{(n-2)}, \quad n > 1.\]

The stress components and stress-displacement relations for various orders are still given by (4.4) and (4.7) respectively.

The particular solution for the first order equation is

\[u_{p}^{(1)} = (r \cdot \nabla)u^{(0)} - 2\varphi (r \cdot u^{(0)}),\]

or

\[u_{p}^{(1)} = (r \cdot \nabla)u^{(0)} - 2r \cdot (u^{(0)} \varphi), \tag{4.17}\]

and the particular solution for the second order equation is

\[u_{p}^{(2)} = r \cdot \nabla(u^{(1)}_{c} + (2-\frac{2m}{k^2})u^{(0)}) - 2\varphi \cdot (u^{(1)}_{c} + u^{(0)}).\]
where \( m \) has the same meaning as in (4.13). The rest of the analysis is no different from the previous one, using displacement potentials. The radiation conditions and uniqueness of solutions for each order of perturbation are discussed in Ref. 4.1.

Application of the perturbation method to the diffraction of P or SV waves by a rigid circular cylinder and a rigid-smooth semi-infinite plane has been carried out in Ref. 4.2. Since the exact solutions are available (cf. Chapters III and V), they serve as a check to the accuracy of a two-term (0th and first order) perturbation solution for each problem. Additional examples utilizing this method are given in Chapters IV and V of this text.

When applied to diffraction of elastic waves, the accuracy of this method can be seen by examining the perturbation of an incident plane P wave

\[
\varphi(i) = \varphi_0 e^{i(ax - \omega t)},
\]

\[
\psi(i) = 0.
\]

If \( a \) is replaced by \( k(1-2\varepsilon)^{\frac{1}{2}} \) and the latter is approximated by the leading terms of its binomial expansion, \( \varphi^{(1)} \) takes the form:

\[
\varphi(i) = \varphi_0 e^{i k x} [1 - \varepsilon k \varepsilon i^{\frac{\pi}{2}} + 0(\varepsilon^2)] e^{-i\omega t}.
\]

Thus the first two order perturbations of \( \varphi(i) \) are

\[
\varphi(i)(0) = \varphi_0 \varepsilon e^{i(kx - \omega t)},
\]
and

\[ \varphi^{(1)} = k x \varphi_0 e^{i(kx - \omega t + \pi/2)}. \]

Consequently, the derived scattered waves are not accurate for large \( kx \) if the perturbation ends at the first order. However, results with a few orders of perturbations are good approximations at low frequencies (small \( k \)) and in the neighborhood of the origin (small \( x \)).

The method presumably can be extended to transient problems of diffraction of elastic waves. If the steady state response is obtained for all frequencies, the transient response is determinable by Fourier synthesis. Although the perturbation method is useful only at low frequencies, examples show that in many cases the information at high frequency range can be supplemented by the geometric ray theory of diffraction. By ray theory we mean here that the wavelength of the incident wave is so short (high frequency) that the scattering surface can be treated as a flat plane. Reflection of any type of incident wave by a plane is readily solvable. Joining the frequency responses at low and high frequencies obtained by these two methods respectively is sufficient for the calculation of the transient response.
CHAPTER II REFERENCES


0.3 Morse, Philip M., and Herman Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953.


CHAPTER II REFERENCES


Chapter III

CIRCULAR CYLINDER PROBLEMS

IN THIS AND THE FOLLOWING THREE CHAPTERS, the methods of wave function expansions and of integral transforms will be applied to the diffraction of elastic waves by inclusions with simple geometry. Among these geometries, the circular cylinder and sphere will be the most familiar.

Scattering of sound waves by a rigid or gaseous spherical obstacle was first discussed by Rayleigh in 1872, using the wave-function expansion method. The results were later used to determine the transmission of light through an atmosphere containing small particles in suspension, leading to the celebrated finding that the energy scattered varies inversely as the fourth power of the wavelength. This offers an explanation as to why the sky is blue, because the wavelength of blue is the longest of all visible light.

When the spherical obstacle is replaced by a circular cylinder, the wave pattern changes from an axially symmetric one to a two-dimensional one. The mathematical analysis can be carried out analogously. Scattering of sound with long wavelengths by a circular cylinder was discussed in Rayleigh's Theory of Sound, Vol. II, which first appeared in 1878. A systematic presentation of Rayleigh's work, and other related work, can be found in Chapter X of Lamb's Hydrodynamics. (0.1)

Sezawa can be credited with initiating a general treatment of the scattering of waves by a circular cylinder in an elastic solid.
In his paper published in 1927 he formulated, in terms of special wave functions, solutions for the scattering of an incident compressional wave (P wave) by circular and elliptical cylinders, as well as by spheres, though no detailed calculations were given for the stresses for the energy scattered. Recent revival of interest in the subject has led to numerous publications in the fields of acoustics (White, Ref. 0.3, and Kato, Ref. 0.4), in geophysics (Knopoff, Ref. 0.5), and in applied mechanics (Pao, Ref. 0.6, Pao and Mow, Ref. 0.7, and Mow and Mente, Ref. 0.8). Emphasis varies, with more attention given to the scattered energy in acoustics and geophysics, while in applied mechanics the emphasis is on stress concentrations. All papers quoted so far have dealt with the sinusoidal incident waves, i.e., with the steady-state response.

As for the transient response, i.e., the diffraction of a pulse by an obstacle, there have been only a few results published. The first significant contribution was made in 1962 by Baron and Matthews (Ref. 0.9), who calculated the stress response at the surface of a cylindrical cavity while a step compressional wave passes through the cavity. Displacement and velocity responses were reported in a subsequent paper by Baron and Parnes (Ref. 0.10). Other works where the transient problem is discussed are cited in Section 5 of this chapter.
1. BASIC EQUATIONS AND PLANE-WAVE REPRESENTATION

IN THIS CHAPTER and the two that follow, our discussion will focus mainly on elastic wave scattering in circular, elliptic, and parabolic cylinder coordinates. Before we begin the treatment of circular cylinder coordinates, it might be advantageous at this time to present some basic equations that will be common to all.

The basic equations for cylinder coordinates and for plane-wave representations are given here.

1.1. Equations in Cylindrical Coordinates

For circular, elliptic, and parabolic cylinder coordinates \((\xi, \eta, z)\) the scale factors \(h_\xi, h_\eta, h_z (i = 1, 2, 3)\) in (I-2.48) are of the form

\[
 ds^2 = (h_\xi d\xi)^2 + (h_\eta d\eta)^2 + dz^2, \tag{1.1}
\]

where \(\xi\) and \(\eta\) are related to the Cartesian coordinates \((x, y, z)\) by

\[
\xi + i\eta = f(x + iy), \quad z = z, \tag{1.2}
\]

with \(i = \sqrt{-1}\). The function \(f\) will be specified later for each coordinate system. Since the scale factors \(h_\xi\) and \(h_\eta\) are independent of \(z\) and \(h_z = 1\), we can set \(\omega(z) = 1\) in (I-2.61); Eqs. (I-2.60) through (I-2.63) then become

\[
\mathbf{u} = \mathbf{M} + \mathbf{N}; \tag{1.3}
\]

\[
\mathbf{l} = \nabla \phi, \tag{1.4}
\]

\[
\mathbf{M} = \nabla \times (e_2 \psi) = -e_2 \times \nabla \psi; \tag{1.4}
\]
\[ \mathbf{N} = \lambda \mathbf{\nabla} \left( \frac{\partial \mathbf{\chi}}{\partial z} \right) - \lambda \mathbf{\nabla} \cdot \mathbf{v}^2 \mathbf{\chi} = \rho \mathbf{\nabla} \times \mathbf{\nabla} \times (e_z \mathbf{\chi}), \]

\[ \sigma_p^2 \mathbf{\nabla}^2 \varphi = \mathbf{\ddot{\varphi}}, \]

\[ \sigma_p^2 \mathbf{\nabla}^2 (\psi, \chi) = (\ddot{\psi}, \ddot{\chi}). \]

In addition, we have the usual stress-displacement relation (I-2.29a):

\[ \mathbf{\sigma} = \lambda (\mathbf{\nabla} \cdot \mathbf{u}) \mathbf{I} + \mu (\mathbf{\nabla} \mathbf{u} + \mathbf{u} \mathbf{\nabla}). \]

The gradient operator \( \mathbf{\nabla} \) and the Laplacian \( \mathbf{\nabla}^2 \) in curvilinear coordinates are listed in (I-2.52). Although it is possible to express the stresses directly in terms of the potentials \( \varphi, \psi, \) and \( \chi \), we shall postpone this until the need arises.

1.2. Incident Plane Waves

Most elastic-wave diffraction studies are concerned with an incident plane wave. The plane wave problems are not only mathematically less complex, they are also a good approximation of many physically meaningful problems. Because of the importance of this type of incident wave, we shall recapitulate some of our earlier discussion on the representation of a plane wave by displacement potentials.

The displacement, \( \mathbf{u}^{(i)} \), of an incident plane wave may be defined as

\[ \mathbf{u}^{(i)} = A f(\mathbf{v} \cdot \mathbf{r} - ct), \]

where \( c \) can be either \( c_p \) or \( c_s \); \( A \) is a constant vector and \( \mathbf{v} \) a unit vector indicating the direction of propagation. The \( A, \mathbf{v}, \) and \( c \) are
related by (1-2.39) and the arbitrary function \( f \) shall be taken as a harmonic function for steady-state scattering. Since \( c_p > c_s \) for all real materials, the wave front which is defined by \( v \cdot r - ct = \text{constant} \) for the incident P wave is always separated from the one for the incident S wave.

In an infinite elastic medium, the propagation of a plane wave is disturbed by the insertion of an obstacle if the obstacle is composed of material different from its surroundings. In this and the next two chapters, the obstacle is taken to be a long, uniform cylinder with either a circular, elliptic, or parabolic cross section. We shall take the axis of the cylinder as the \( z \)-coordinate, and the wave normal \( v \) of a plane wave will, in general, be inclined to the \( z \)-axis. As shown in Fig. 1.1, the unit vector \( v \) makes an angle \( \phi_0 \) with the \( z \) (or \( z' \)) axis and its projection along the \( Ox' \) line is at an angle \( \theta_0 \) from the \( x \)-axis. It can then be represented as

\[
v = \sin \phi_0 e'_x + \cos \phi_0 e'_z = \sin \phi_0 (\cos \theta_0 e_x + \sin \theta_0 e_y) + \cos \phi_0 e_z
\]

where \( e_x, e_y, \) and \( e_z \) are the unit vectors along the Cartesian axes.

Using displacement potentials \( \varphi, \psi, \) and \( \chi, \) we can represent an incident plane P wave by

\[
\varphi(\vec{z}) = A_f \left[ \sin \phi_0 (x \cos \theta_0 + y \sin \theta_0) + \cos \phi_0 z - c_p t \right],

\psi(\vec{z}) = \chi(\vec{z}) = 0,
\]

(1.8)

where \( A \) is a constant. The corresponding displacement vector (see I-2.41) is simply
Fig. 1.1. Plane-Wave Geometry

\[ u(t) = v_{\psi}^{(t)} = Avf'(v \cdot r - c_\psi t). \]  

(1.9)

A special case of great interest is when \( \psi = \pi/2 \), i.e., when \( v \) is at right angles to the \( z \)-axis. The displacement is then independent of the \( z \)-coordinate,

\[ u(t) = A[\cos \theta_0 e_x + \sin \theta_0 e_y]f'(x \cos \theta_0 + y \sin \theta_0 - c_\psi t), \]

(1.10)

and the problem is of plane strain (see 1.3.8).

As mentioned in Chapter I, there is no ambiguity in the polarization of a plane \( P \) wave and the direction of its associated displacement.
This is not true, however, in the case of the shear wave. For plane shear waves, the displacement vector lies on the wave surface. This displacement can be resolved into two components, one on the \( \xi-\eta \) plane (\( \xi-\eta \) plane) along the line \( O'y' \), and the other perpendicular to the first and lying on the \( \xi'z' \) plane as shown in Fig. 1.1. According to this decomposition, the component along the \( y' \) axis is the displacement vector \( \mathbf{M} \) as defined in (1.3), and the other gives the displacement vector \( \mathbf{N} \). The incident shear waves which satisfy the above resolution may be specified by setting, in (1.4)

\[
\begin{align*}
\varphi(\mathbf{i}) &= 0, \\
\psi(\mathbf{i}) &= Bf(v \cdot r - c_g t), \\
\chi(\mathbf{i}) &= Cg(v \cdot r - c_g t),
\end{align*}
\]

(1.11)

where \( B \) and \( C \) are two constants to be specified and \( v \cdot r - c_g t = \sin \phi \).

\( (x \cos \theta_x + y \sin \theta_x) + \cos \phi \). \( z \).

The displacements due to \( \psi(\mathbf{i}) \) and \( \chi(\mathbf{i}) \), following (1.4), are

\[
\begin{align*}
\mathbf{M} &= (v \times \mathbf{e}_z)Bf'(v \cdot r - c_g t), \\
\mathbf{N} &= v \times (v \times \mathbf{e}_z)Cg''(v \cdot r - c_g t).
\end{align*}
\]

(1.12a)

(1.12b)

Because \( v \times \mathbf{e}_z = -\sin \phi \mathbf{e}_y \) (Fig. 1.1), it is evident that \( \mathbf{M} \) lies on the \( \xi-\eta \) plane and \( \mathbf{N} \) is perpendicular to \( \mathbf{M} \). The resultant displacement vector is

\[
\psi(\mathbf{i}) = \mathbf{M} + \mathbf{N} = -B \sin \phi \frac{\partial}{\partial \phi} e_y - \frac{\partial}{\partial \phi} (v \times \mathbf{e}_y) + C \sin \phi (v \times \mathbf{e}_y) g''.
\]

(1.13)
This is due to a superposition of two distinct shear waves each with a particular polarization.

If \( M \) and \( N \) are to be representing the two components of a single shear wave with the displacement vector making an angle \( \delta_0 \) with the \( O'y' \) axis, the two phase functions \( f' \) and \( g'' \) must be identical and coefficients \( B \) and \( C \) should be related by

\[
\tan \delta_0 = \frac{\xi C}{B}.
\]

The shear wave is then given by

\[
\psi(z) = A[\cos \delta_0 e'_y + \sin \delta_0 (v \times e'_y)]g''(v \cdot r - \alpha s t), \tag{1.14}
\]

where \( A \) is the magnitude of the displacement vector. Equations (1.13) and (1.14) show that any linear combination of the two potentials \( \psi(z) \) and \( \chi(z) \) in (1.11) yields a possible representation of shear waves.

In particular, when

\[
-B \sin \psi = A \cos \delta_0, \tag{1.15}
\]

\[
-\xi C \sin \psi = A \sin \delta_0,
\]

such a combination defines a shear wave with its displacement vector making an angle \( \delta_0 \) with \( O'y' \) axis. Conversely, a shear wave with arbitrary polarization can always be decomposed into two components. One lies along the \( e'_y \) axis, being derivable from the potential \( \psi(z) \). We shall call this the \( S_{\psi} \) component. The other lies along the line \( v \times e'_y \) and is derivable from the \( \chi(z) \) potential, hence the name \( S_{\chi} \) component. Scattering due to each component of the \( S \) wave can be treated independently.
The case of \( \phi_0 = 0 \) merits special attention. It represents a plane wave with phase function \( g''(z - c_s t) \) moving along the \( z \)-axis in an infinite medium. Equations (1.11) and (1.15) are still meaningful if the products \( B \sin \phi_0 \) and \( C \sin \phi_0 \) are understood to remain finite as \( \phi_0 \to 0 \). However, with the insertion of a prismatic cylinder as shown in Fig. 1.1, the propagation of such a plane wave may or may not be possible because the frequencies and wavelengths of the Fourier components of the phase function \( g''(v \cdot r - c_s t) \) must be so coordinated that the additional boundary conditions at the cylindrical surface are satisfied. It is a problem usually treated in the study of mechanical wave guides.

Another special case, \( \phi_0 = \pi/2 \), is of great importance in applications since the \( z \)-dependence is eliminated from the analysis and the problem is reduced to a two-dimensional one — plane strain or plane stress. If in addition, \( \delta_0 = \pi/2 \), \( u^{(i)} \) in (1.14) reduces to

\[
-u^{(i)} = A e^{S_\nu \eta'} (x \cos \theta_0 + y \sin \theta_0 - c_s t),
\]

(1.16)

which is actually the \( S_\nu \) component alone. To the cylinder, it is an incident SH wave, which was treated extensively in Chapter II. When \( \delta_0 = 0 \), only the \( S_\psi \) component remains and

\[
u^{(i)} = A(-\sin \theta_0 x + \cos \theta_0 y) g''(x \cos \theta_0 + y \sin \theta_0 - c_s t).
\]

(1.17)

These are what we call incident SV waves; they will be discussed again in the following sections.

Finally we note that in the preceding discussion, the \( x \)-\( y \) axes
are chosen to be the principal axes of the cross section of the cylinder, so that a simple coordinate transformation can be found in (1,2). Much of the complication arises from the fact that the projection of the wave normal ($O'x'$ line on Fig. 3.2 this chapter) onto the $x$-$y$ plane does not coincide with either axis. However, when the cylinder has a circular cross section, we can judiciously choose either the $x$ or $y$ axis to coincide with the projection of the wave normal because of symmetry about the $z$-axis of the cylinder. Thus the angle $\theta_o$ is either 0 or $\pi/2$ depending on choice, and the scattering problem is considerably simplified. The case of the circular cylinder is discussed first in this chapter, and elliptical and parabolic cylinders which do not possess axial symmetry will be treated in the ensuing chapters.

2. EQUATIONS IN CIRCULAR CYLINDER COORDINATES AND BESSEL FUNCTIONS

2.1 Circular Cylindrical Coordinates

For circular cylindrical coordinates, the transformation which relates $x$ and $y$ to $\xi$ and $\eta$ and the scale factors are as follows.

Transformation:

\[
x + iy = e^{\xi+i\eta},
\]

\[
x = e^{\xi} \cos \eta, \quad y = e^{\xi} \sin \eta, \quad z = z.
\]

The scale factors are $h_\xi = h_\eta = e^{\xi}$, and $h_z = 1$.

If in addition, we let $\xi = \ln r$, and $\eta = \theta$, then we have the familiar form of the transformation and scale factors. See Fig. (2.1).
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \] (2.2)

\[ h_r = 1, \quad h_\theta = r, \quad h_z = 1. \]

**Fig. 2.1. Circular Cylindrical Coordinates**

By substituting Eq. (2.2) into (1.2.52), we have the following equations for the circular cylinder coordinates

\[ \nabla f = r \frac{\partial F}{\partial r} + \frac{1}{r} \frac{\partial F}{\partial \theta} + z \frac{\partial F}{\partial z}, \]

\[ \nabla \cdot \mathbf{f} = \frac{\partial (r F_r)}{\partial r} + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}, \]

\[ \nabla \times \mathbf{f} = r \frac{1}{r} \left[ \frac{\partial F_z}{\partial \theta} - \frac{\partial (r F_\theta)}{\partial r} \right] + \frac{1}{r} \left[ \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial \theta} \right] + z \frac{1}{r} \left[ \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right]. \]

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{3}{r^2} \frac{\partial^2 f}{\partial z^2}. \]

The scalar wave equation, in expanded form, is
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial z^2}, \quad (2.4)
\]

which is common to \( \varphi, \psi, \) and \( \chi \) where \( c = c_\varphi \) for \( \varphi \) and \( c = c_\psi \) for \( \psi \) and \( \chi \).

The displacements \( u_r, u_\theta, \) and \( u_z \) are related to \( l, m, \) and \( n, \)

and consequently, to \( \varphi, \psi, \) and \( \chi \) as follows:

\[
l = v_\varphi = r \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} + \frac{\partial \varphi}{\partial z},
\]

\[
m = \nabla \times (e_\psi) = e_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} - e_z \frac{\partial \psi}{\partial r},
\]

\[
n = l \nabla \left( \frac{\partial \chi}{\partial z} \right) - \nabla^2 \chi,
\]

\[
= i \left[ e_r \left( \frac{\partial^2 \chi}{\partial r \partial z} \right) + e_\theta \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta \partial z} + e_z \frac{\partial^2 \chi}{\partial z^2} \right], \quad (2.5)
\]

and

\[
u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \chi}{\partial r \partial z},
\]

\[
u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta \partial z}, \quad (2.6)
\]

\[
u_z = \frac{\partial \varphi}{\partial z} - \left[ \frac{1}{r} \frac{\partial \varphi}{\partial r} \left( \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \right].
\]
The stresses are related to $\varphi$, $\psi$, and $\chi$ as follows:

\[ \sigma_{rr} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \left( \frac{\partial}{\partial \theta} \frac{\partial \psi}{\partial \theta} \right) + \frac{\lambda}{\bar{\alpha}^2} \frac{\partial^3 \chi}{\partial r^2 \partial \theta} \right] ; \quad (2.7a) \]

\[ \sigma_{\theta \theta} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{1}{r} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial^2 \varphi}{\partial \theta^2} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} \frac{\partial \psi}{\partial \theta} \right) - \frac{\lambda}{\bar{\alpha}^2} \frac{\partial^3 \psi}{\partial r \partial \theta^2} \right] ; \quad (2.7b) \]

\[ \sigma_{zz} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{r^2} \left( \frac{\partial^2 \varphi}{\partial \theta^2} \right) - \frac{\lambda}{\bar{\alpha}^2} \frac{\partial^3 \chi}{\partial z^3} \right] ; \quad (2.7c) \]

\[ \sigma_{r \theta} = \mu \left\{ 2 \left[ \frac{1}{r} \frac{\partial^2 \varphi}{\partial \theta \partial r} - \frac{1}{r^2} \frac{\partial \varphi}{\partial \theta} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right\} ; \quad (2.7d) \]

\[ \sigma_{rz} = \mu \left\{ 2 \frac{\partial^2 \varphi}{\partial r \partial z} + \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{\lambda}{r} \left[ 2 \frac{\partial^3 \chi}{\partial r \partial \theta^2} - \frac{\partial}{\partial r} \left( \nabla^2 \chi \right) \right] \right\} ; \quad (2.7e) \]

\[ \sigma_{\theta z} = \mu \left\{ 2 \frac{\partial^2 \varphi}{\partial \theta \partial z} + \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{\lambda}{r} \left[ 2 \frac{\partial^3 \chi}{\partial \theta \partial z^2} - \frac{\partial}{\partial \theta} \left( \nabla^2 \chi \right) \right] \right\} . \quad (2.7f) \]

2.2. *Circular Cylindrical Wave Functions*

In what follows the general solutions for wave equations in circular cylinder coordinates will be presented. The solutions given
are for the steady state case. However, as discussed in Chapters I and II, we can usually determine the transient solution if the steady-state solution is known. Therefore, no generality is lost in the presentation.

Let the function \( \varphi \) be of the form \( \varphi = \phi(r, \theta, z)e^{-i\omega t} \), then \( \phi(r, \theta, z) \) must satisfy the following Helmholtz equation:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\omega^2}{c^2} \phi = 0. \tag{2.8}
\]

Using the method of separation of variables with \( \phi = R(r) \omega(\theta)Z(z) \), Eq. (2.8) separates into

\[
x^2 R'' + xR' + (k^2 \omega^2 - \nu^2)R = 0,
\]

\[
\omega'' + \nu^2 \omega = 0, \tag{2.9}
\]

\[
z'' + \gamma^2 z = 0, \quad k^2 = \frac{\omega^2}{c^2} - \gamma^2 ,
\]

where \( \nu \) and \( \gamma \) are separation constants. The solutions for \( \omega \) and \( z \) are:

\[
\omega = e^{\pm i\nu \theta} \quad \text{or} \quad \begin{pmatrix} \sin \nu \theta \\ \cos \nu \theta \end{pmatrix},
\]

\[
z = e^{\pm i\gamma z} \quad \text{or} \quad \begin{pmatrix} \sin \gamma z \\ \cos \gamma z \end{pmatrix}. \tag{2.10}
\]

For most problems of interest, \( \omega \) must be single valued, i.e., \( \omega(\theta + 2\pi) = \omega(\theta) \), which requires \( \nu = n \), where \( n \) is an integer. The separation constant \( \gamma \) is related to the wave propagation in the \( z \)
direction, and in general it could be complex. Therefore, the field is not necessarily periodic along the z-axis.

The solution for $R(r)$ when $\nu = n$ can be expressed in terms either of Bessel functions of the first and second kind, $J_n(kr)$ and $Y_n(kr)$ respectively, or in terms of Hankel functions of the first and second kind, $H_n^{(1)}, H_n^{(2)}(kr)$. The choice of the radial function, i.e., $J_n(kr)$ etc., is dependent upon the physics of the problem.

For progressing waves, the appropriate solution is

$$\varphi_n(r, \theta, z, t) = [A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr)]e^{\pm i \nu \theta} e^{-i(\omega t + \gamma z)}.$$  \hspace{1cm} (2.11)

The solution above can be interpreted as follows. The sign of $\pm \gamma z$ in the argument of $\exp -i(\omega t + \gamma z)$ determines the direction of wave propagation in the z direction. $(\omega t + \gamma z)$ means that the wave is propagating in the negative z direction, and the converse is true if the argument is $(\omega t - \gamma z)$. $e^{-i \omega t}$ represents a wave diverging from the $r$-axis, and $e^{-i \omega t}$ is a representation of waves converging on the axis.

The last statement becomes apparent when one considers that

$$H_n^{(1)}, H_n^{(2)}(kr) = J_n(kr) \pm i Y_n(kr),$$

and when $kr \to \infty$

$$H_n^{(1)}, H_n^{(2)}(kr) \propto \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} + i(kr - \pi/4).$$

Thus the product of $H_n^{(1)}, H_n^{(2)}(kr)e^{-i \omega t}$ represents an inward or outward propagating circular, simple-harmonic wave. Also, as $kr \to \infty$, $k$ has the meaning of a wave number for a simple harmonic wave, except that
the wave front is circular.

The separation constants $\gamma$ and $k$ might be considered as propagation constants in $z$ and $r$ directions, although they are related by Eq. (2.8) as $k^2 = \omega^2/c^2 - \gamma^2$. They are not given a priori, rather they must be determined by the boundary conditions of the problem. For example, if a wave is propagating in the $x$-$y$ plane, then $\varphi$ will be independent of $z$ and $\gamma = 0$. It follows that $k = \omega/c$, which is the usual definition for the wave number.

For waves propagating symmetrically about the $z$ axis, $n = 0$ and Eq. (2.11) becomes

$$\varphi(r, \vartheta, z, t) = [A_0^+(kr) + B_0^-(kr)] e^{-i(\omega t + \gamma z)}. \tag{2.12}$$

To represent standing waves one uses $J_n(\gamma r)$ and $Y_n(\gamma r)$ as radial functions. Thus, the wave functions representing standing waves are

$$\varphi_n(r, \vartheta, z, t) = [A_n J_n(\gamma r) + B_n Y_n(\gamma r)] \begin{bmatrix} \sin \gamma \vartheta \\ \cos \gamma \vartheta \end{bmatrix} e^{-i\omega t}. \tag{2.13}$$

Here again, as described in the traveling wave case, the choice of the functions and the determination of the separation constants $\gamma$ and $k$ will depend on the boundary conditions of the problem.

2.3. Bessel Functions

In this subsection we will present some of the important relationships of Bessel functions pertinent to the elastic wave scattering problem.
Series Representation. The Bessel function of the first kind of order \( v \) is defined as

\[
J_v(z) = \frac{1}{2^v \pi} \sum_{r=0}^{\infty} \frac{(-1)^r v(2z)^{2r}}{r! \Gamma(v + r + 1)}
\]  

(2.14)

in which \( \Gamma \) is the gamma function. If \( v \) is an integer, \( n \), then the following relationships hold:

\[
J_{-n}(z) = (-1)^n J_n(z)
\]

\[
J_n(z) = \pm J_n(-z) \quad (+ \text{for } n \text{ even, } = \text{ for } n \text{ odd}).
\]  

(2.15)

The Bessel function of the second kind of nonintegral \( v \) is defined as:

\[
Y_v(z) = \frac{\cos \pi v J_v(z) - J_{-v}(z)}{\sin \pi v}.
\]  

(2.16)

When \( v \) is an integer \( n \), Eq. (2.16) takes the indeterminate form, \( 0/0 \).

The series for \( Y_{v=n}(z) \) is determined if the following definition for \( Y_n(z) \) is used.

\[
Y_n(z) = \left\{ \frac{3}{\pi} \left[ \cos \pi v J_v(z) - J_{-v}(z) \right] / \frac{3}{\pi} \sin \pi v \right\}_{v=n}.
\]  

(2.17)

The Bessel function of the second kind for integer order \( n \) is

\[
Y_n(z) = \frac{2}{\pi} \ln \left( \frac{z}{2} \right) J_n(z) - \frac{1}{\pi} \sum_{r=0}^{n-1} \frac{(n - r - 1)!}{r!} \left( \frac{2}{z} \right)^{n-2r}.
\]

\[
- \frac{1}{\pi} \sum_{r=0}^{n} \frac{(-1)^r (\frac{2}{z})^{n+r}}{r! (n+r)!} \left[ \psi(n + r + 1) + \psi(r + 1) \right].
\]  

(2.18)
where

$$\psi(n + r + 1) = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n + r}\right) - \gamma,$$

$$\psi(1) = -\gamma,$$

and $\gamma$ is Euler's constant as defined by

$$\gamma = \lim_{m \to \infty} \left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} - \ln m\right\} = 0.5772.$$

Also, when $\nu$ is an integer the following relationship holds

$$y_{-\nu}(z) = (-1)^{\nu} y_{\nu}(z).$$

The Hankel functions of the first and second kind (also known as Bessel functions of the third kind) for any order $\nu$ is defined as:

$$H_{\nu}^{(1,2)}(z) = J_{\nu}(z) \pm iY_{\nu}(z). \quad (2.19)$$

It follows that we may define $J_{\nu}$ and $Y_{\nu}$ as functions of $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$:

$$J_{\nu}(z) = \frac{i}{2} \left\{H_{\nu}^{(1)}(z) + H_{\nu}^{(2)}(z)\right\},$$

$$Y_{\nu}(z) = \frac{1}{2i} \left\{H_{\nu}^{(1)}(z) - H_{\nu}^{(2)}(z)\right\}. \quad (2.20)$$

There is an additional category of Bessel functions that are of interest, the modified Bessel functions. Consider the differential equation for $R(r)$. If $k$ is replaced by $ik$, then the governing differ-
The differential equation becomes

\[ r^2 f'' + rf' - (k^2 r^2 + v^2) f = 0. \]  

(2.21)

If we now replace \( r \) by \( i \zeta \) and \( dr \) by \( id\zeta \), we note that Eq. (2.21) takes the usual form

\[ \zeta^2 f'' + \zeta f' + (k^2 \zeta^2 - v^2) f = 0, \]

and it has the same solutions as previously described, except that the argument is now imaginary. It is usually desirable in applications to give the solution in real instead of in pure imaginary form. We note, for instance, that one solution is \( J_\nu (\zeta) \). We can write \( \zeta = ze^{\pm \frac{i\pi}{2}} \) and it follows, according to the definition in Eq. (2.14), that

\[ J_\nu (k\zeta) = J_\nu (ze^{\pm \frac{i\pi}{2}}) = \left( \frac{ze^{\mp (\pi i/2)}}{2} \right)^\nu \sum_{\nu=0}^{\infty} \frac{(\frac{z^2}{2})^{2r}}{r! \Gamma (\nu + r + 1)} \]

\[ = e^{\pm (\nu \pi i/2)} \sum_{\nu=0}^{\infty} \frac{(\frac{z^2}{2})^{\nu+2r}}{r! \Gamma (\nu + r + 1)} \]  

(2.22)

The sum in Eq. (2.22) represents the modified Bessel function of the first kind:

\[ I_\nu (z) = \sum_{\nu=0}^{\infty} \frac{(\frac{z^2}{2})^{\nu+2r}}{r! \Gamma (\nu + r + 1)}. \]  

(2.23)

Substituting Eq. (2.22) into (2.23) gives, for any order \( \nu \) and \( z \) positive, the following relationship between \( I_\nu \) and \( J_\nu \):
\[ I_v(z) = e^{i\frac{\pi}{2}v} J_v(ze^{\pm i\frac{\pi}{2}}). \] (2.24)

When \( z \) is regarded as a complex variable, the \( \pm \) signs in equation (2.24) are affixed according to the \( \arg z \). They are:

\[ I_v(z) = e^{-i\frac{\pi}{2}v} J_v(ze^{\pm i\frac{\pi}{2}}), \quad -\pi < \arg z \leq \frac{\pi}{2}. \] (2.25a)

\[ I_v(z) = e^{i\frac{\pi}{2}v} J_v(ze^{-\pm i\frac{\pi}{2}}), \quad \frac{\pi}{2} < \arg z \leq \pi. \] (2.25b)

If \( v = n \), and \( n \) is a positive integer, we have

\[ I_n(z) = (i)^{-n} J_n(zi). \]

Also we have the following relationships

\[ I_n(z) = I_{-n}(z), \] (2.26a)

\[ I_n(-z) = (-1)^n I_n(z). \] (2.26b)

The modified Bessel function of the second kind for \( v \neq n \) is defined in a similar way as for the ordinary Bessel function, as

\[ K_v(z) = \frac{1}{2\pi i} \left( I_{-v}(z) - i^{v} I_v(z) \right) / \sin v\pi. \] (2.27)

With this definition, the following useful relations are obtained

\[ K_v(z) = K_{-v}(z); \] (2.28a)

\[ K_v(z) = (\pi/2) e^{\pm i(\nu+1/2)\pi/2} I_{-\nu}(z). \] (2.28b)
\[ K_v(z e^{\pm \frac{i}{2} \pi}) = \pm \frac{i}{2} e^{\mp i \pi (v+1)} i^v H_v^{(2)},(1)(z); \] (2.28c)

in which \( H_v^{(2)},(1)(z) \) are the Hankel functions of the first and second kinds.

When \( v = n \), and \( n \) an integer, \( K_n \) is defined as

\[ K_n(z) = \left\{ \left(\frac{\pi}{2}\right) \left[ \frac{\partial}{\partial v} \left( I_{-v}(z) - I_v(z) \right) \right] / \frac{\partial}{\partial v} \sin \nu \pi \right\}_{\nu = n}. \] (2.29)

It follows that

\[ K_n(z) = (-1)^{n+1} \ln \left( \frac{\pi}{2} \right) I_n(z) + \sum_{r=0}^{n-1} \frac{(-1)^r (n - r - 1)!}{r!} \left( \frac{2}{z} \right)^{n-2r} \]

\[ + \sum_{r=0}^{\infty} \frac{(-1)^n}{2} \frac{(\frac{z}{2})^{n+2r}}{r! (n + r)!} \left[ \psi(n + r + 1) + \psi(r + 1) \right]. \] (2.30)

**Integral Representations**

There are occasions where it is more convenient to represent the various kinds of Bessel functions in integral form. The following lists of integral representations will be helpful in future discussions.

The Bessel function of the first kind is:

\[ J_v(z) = \frac{\left(\frac{z}{2}\right)^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^{\pi} e^{iz \cos \theta} \sin^{2v} \theta \, d\theta \]

\[ = \frac{2(\frac{z}{2})^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^{\pi/2} \cos (z \cos \theta) \sin^{2v} \theta \, d\theta, \quad \text{Re } v > -\frac{1}{2}; \] (2.31a)
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\[ J_\nu(z) = \frac{(i)^{-\nu}}{2\pi} \int_0^{2\pi} e^{iz} \cos \theta e^{i\nu \theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos (\nu \theta - z \sin \theta) \, d\theta. \]  

(2.31b)

The Bessel function of the second kind is:

\[ Y_\nu(z) = \frac{1}{\pi} \int_0^{\pi} \sin (z \sin \theta - \nu \theta) \, d\theta - \frac{1}{\pi} \int_0^{\pi} [e^{\nu t} + e^{-\nu t} \cos \nu r] e^{-z} \sinh t \, dt \]

\[ |\arg z| < \frac{\pi}{2}. \]  

(2.32)

Hankel functions of the first and second kind are:

\[ H^{(1)}_\nu(z) = \frac{e^{-i\nu \pi/2}}{\pi} \int_{-\pi/2+i\infty}^{\pi/2-i\infty} e^{iz} \cos \theta + i \nu \theta \, d\theta; \]

\[ H^{(2)}_\nu(z) = \frac{-e^{-i\nu \pi}}{\pi} \int_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{iz} \cos \theta + i \nu \theta \, d\theta, \quad |\arg z| < \frac{\pi}{2}. \]  

(2.33)

\[ \text{Fig. 2.2. Path of Integration for } H^{(1),(2)}(z) \]
The path of integration is shown in Fig. 2.2.

The modified Bessel functions, first and second kind, are:

\[ I_\nu = \frac{(\frac{2}{\pi z})^\nu}{\Gamma(\nu + \frac{1}{2}) \sqrt{\pi}} \int_0^\pi e^{\frac{z}{2} \cos \theta} \sin^{2\nu} \theta \, d\theta, \quad \text{Re } \nu > -\frac{1}{2}, \]

\[ = \frac{2(\frac{2}{\pi z})^\nu}{\Gamma(\nu + \frac{1}{2}) \sqrt{\pi}} \int_0^{\pi/2} \cosh (z \cos \theta) \sin^{2\nu} \theta \, d\theta, \quad \text{Re } \nu > -\frac{1}{2}, \]

\[ J_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \theta} \cos \nu \theta \, d\theta, \quad (2.34b) \]

and

\[ X_\nu(z) = \frac{\sqrt{\pi} (\frac{2}{z})^\nu}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty \cosh t \sinh^{2\nu} t \, dt, \quad \text{Re } \nu > -\frac{1}{2}, |\arg z| < \frac{\pi}{2}. \]

(2.35)

The integral representations for Bessel functions above are a very minor part of the many varieties available. On occasion we shall use additional ones. For a more detailed discussion of integral representations see McLachlan (2.1) and Watson (2.2).

Recurrence Formulas and Wronskians

The following formulas will prove to be quite helpful in the ensuing discussions.

Recurrence Formulas:

Although the recurrence formulas given below are written for \( J_\nu(a) \), they are also true for \( Y_\nu \) and \( H_\nu \) (\( \nu \) any order).
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\[ zJ'_v(z) = wJ_v(z) - zJ_{v+1}(z); \quad (2.36a) \]
\[ zJ'_v(z) = -wJ_v(z) + zJ_{v-1}(z); \quad (2.36b) \]
\[ 2J'_v(z) = J_{v-1}(z) - J_{v+1}(z); \quad (2.36c) \]
\[ \frac{2v}{z} J_v(z) = J_{v+1}(z) + J_{v-1}(z); \quad (2.36d) \]
\[ J_n''(z) = \frac{1}{z^2} \left[ (n^2 + n - z^2)J_n(z) - zJ_{n-1}(z) \right]. \quad (2.36e) \]

The recurrence formulas for \( I_v(z) \) and \( K_v(z) \) are as follows:

\[ zI'_v(z) = vI_v(z) + zI_{v+1}(z); \quad (2.37a) \]
\[ zI'_v(z) = -vI_v(z) + zI_{v-1}(z); \quad (2.37b) \]
\[ 2I'_v(z) = I_{v-1}(z) + I_{v+1}(z); \quad (2.37c) \]
\[ zK'_v(z) = vK_v(z) - zK_{v+1}(z); \quad (2.38a) \]
\[ zK'_v(z) = -vK_v(z) - zK_{v-1}(z); \quad (2.38b) \]
\[ \frac{2v}{z} K_v(z) = K_{v+1}(z) - K_{v-1}(z). \quad (2.38c) \]

**Wronskians**

The Wronskian determinant is used to determine the dependency of the solutions to differential equations. For a second-order differential equation, the Wronskian is defined as:
\[
W(y_1, y_2) = \begin{vmatrix}
y_1 & y_2 \\
y'_1 & y'_2
\end{vmatrix},
\]

where \(y_1, y_2, y'_1, \text{ and } y'_2\) are solutions and derivatives of the solutions of the second-order differential equation. If \(W\) is not zero, then the solutions are independent of each other. The purpose of presenting \(W\) here is that it is useful in simplifying the solution whenever it occurs in the solutions.

Wronskians involving \(J_\nu\) and \(Y_\nu\) are

\[
W(J_\nu(z), J_{-\nu}(z)) = -\frac{2 \sin \nu \pi}{\pi z}, \quad (z \neq 0),
\]  
(2.39a)

\[
W(J_\nu(z), Y_\nu(z)) = \frac{2}{\pi z}.
\]  
(2.39b)

Wronskians involving \(J_\nu, Y_\nu, \text{ and } H^{(1), (2)}\) are

\[
W(H^{(1)}_\nu(z), H^{(2)}_\nu(z)) = -\frac{4i}{\pi z},
\]  
(2.40a)

\[
W(J_\nu(z), H^{(1), (2)}_\nu(z)) = \pm \frac{2i}{\pi z}.
\]  
(2.40b)

\[
W(Y_\nu(z), H^{(1), (2)}_\nu(z)) = \mp \frac{2}{\pi z}.
\]  
(2.40c)

Wronskians involving \(I_\nu\) and \(K_\nu\) are

\[
W(I_\nu(z), I_{-\nu}(z)) = -\frac{2 \sin \nu \pi}{\pi z},
\]  
(2.41a)

\[
W(K_\nu(z), I_\nu(z)) = \frac{1}{z}.
\]  
(2.41b)
Behavior of the Bessel Functions (First and Second Kind) and Hankel Functions

Given above are some of the pertinent Bessel function formulas that will be used throughout this chapter. In what follows, the behavior of \( J_n, Y_n, I_n, \) and \( K_n \) will be discussed. Also, some of the asymptotic expressions will be given. Shown in Fig. (2.3a,b) are \( J_n(z) \) and \( Y_n(z) \) for \( n = 0, 1, 2, \) and \( 0 \leq z \leq 14.0 \). The behavior of \( J_0(z) \) resembles a cosine curve with slight damping. \( J_1(z) \) resembles that of the sine function. According to Eq. (2.15), \( J_0 \) is an even function while \( J_1(z) \) is an odd function. All \( J_n(0) = 0 \) for \( n \geq 1 \). When \( n = 0 \), \( J_0(0) = 1 \). The behavior of the Bessel function of the second kind is depicted in Fig. (2.3b). It is shown that as \( z \to 0 \) all \( Y_n(z) \to -\infty \). This is because of the logarithmic singularity shown in Eq. (2.18). Also, as a result of the logarithmic singularity,

![Fig. 2.3a. Bessel Function of the First Kind](image-url)
$Y_\eta(z)$ for $z < 0$ is complex. Therefore, $Y_\eta(z)$ is neither an even nor odd function of $z$. The behavior of the Hankel functions can be deduced from $J_\eta$ and $Y_\eta$. (See Ref. 2.3 for a complex plot.) The values of $J_\eta(z)$ and $Y_\eta(z)$, for most values of $z$, must be determined by the series given in Eqs. (2.14) and (2.18). There are, of course, many tables which give values of the Bessel functions for wide ranges of $\eta$'s and $z$'s. For many practical problems, however, we are interested only in certain ranges of $z$. For example, if only the early time response is desired, then we may focus our attention on the behavior of the function when $z$ is large, which means either high frequency or large $P$ (see Chapter III, Section 3, "Initial Value Theorem").
(\(p\) is the Laplace transform parameter.) Or, if only the long time solution is of interest, then we may have to examine the expression when \(z\) is very small, which may be considered as low frequency or small values of \(p\).

The asymptotic behavior of the Bessel functions depends on the order \(\nu\) and the argument \(z\). Given below are some of the more commonly used asymptotic expressions for \(J_\nu(z)\), \(Y_\nu(z)\), and \(R_\nu(z)\).

For \(|z| >> 1\), \(|z| >> |\nu|^2\), \(\nu\) real and \(-\frac{\pi}{2} \leq \text{arg} \ z \leq \frac{\pi}{2}\),

\[
J_\nu(z) \approx \frac{2}{\pi z} \left\{ \tau_\nu(z) \cos \psi - \xi_\nu(z) \sin \psi \right\}; \quad (2.42)
\]

\[
Y_\nu(z) \approx \frac{2}{\pi z} \left\{ \tau_\nu(z) \sin \psi + \xi_\nu(z) \cos \psi \right\}; \quad (2.43)
\]

\[
R_\nu^{(1,2)}(z) \approx \frac{2}{\pi z} e^{i\nu \psi} \left\{ \tau_\nu(z) \pm i \xi_\nu(z) \right\}, \quad (2.44)
\]

in which

\[
\tau_\nu(z) \approx 1 - \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)}{21(8z)^2} - \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)(4\nu^2 - 7^2)}{41(8z)^4} + R_1; \quad (2.45a)
\]

\[
\xi_\nu(z) \approx \frac{(4\nu^2 - 1^2)}{118z} - \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{31(8z)^3} + \ldots + R_2; \quad (2.45b)
\]

\[
\psi = (z - \frac{1}{4} - \frac{1}{2} \nu \pi); \quad (2.45c)
\]

where \(R_1\) and \(R_2\) are the remainder of the series.
When $\frac{1}{4}z^2 \ll \eta$, \( \text{Re} \, z > 0 \), and \( \eta > 0 \) is large, then the following asymptotic forms result

\[ J_n(z) \approx \frac{(\frac{\eta}{z})^n}{n!} \left\{ 1 - \frac{(\frac{\eta}{z})^2}{(n + 1)} + \frac{(\frac{\eta}{z})^4}{2!(n + 1)(n + 2)} \right\}; \quad (2.46) \]

\[ Y_n \approx -\left( \frac{2}{\pi} \right)^n \frac{(n - 1)!}{\eta} \left\{ 1 + \frac{(\frac{\eta}{z})^2}{(n - 1)} + \frac{(\frac{\eta}{z})^4}{2!(n - 1)(n - 2)} \right\}; \quad (2.47) \]

\[ g_n^{(1),(2)} \approx i\left( \frac{2}{\pi} \right)^n \frac{(n - 1)!}{\eta} \left\{ 1 + \frac{(\frac{\eta}{z})^2}{n - 1} + \frac{(\frac{\eta}{z})^4}{2!(n - 1)(n - 2)} \right\}. \quad (2.48) \]

The asymptotic formulas listed in Eqs. (2.42) through (2.48) are unsuitable when both the argument and order are large. Formulas with different restrictions are available and may be found in McLachlan and in Watson (Refs. 2.1 and 2.2).

There is another limiting case of interest, that is when \( z \to 0 \).

Examination of Eqs. (2.14) and (2.18) shows that when \( z \ll 1 \), the Bessel functions can be approximated by their leading terms.

For \( z \ll 1 \), \( \eta = 0 \)

\[ J_0(z) = 1 - \frac{z^2}{2} + O(z^4). \quad (2.49a) \]

For \( z \ll 1 \), \( \eta \geq 1 \)

\[ J_n(z) = \frac{(\frac{z}{\eta})^n}{\eta} \left\{ \frac{1}{n!} - \frac{(\frac{z}{\eta})^2}{1!(n + 1)} + O(z^4) \right\}. \quad (2.49b) \]
For \( n = 0, z \ll 1 \)

\[
\gamma_0 = \frac{2}{\pi} \ln z - (\frac{2}{3} z)^2 (1 - \frac{2}{\pi} \ln z) + O(z^4 \ln z). \tag{2.50a}
\]

For \( n \geq 1, z \ll 1 \)

\[
\gamma_n(z) = - \frac{1}{\pi} \left( \frac{2}{z} \right)^n \left((n - 1)! + \pi z^2 + O(z^4)\right). \tag{2.50b}
\]

In a similar way the behavior of Hankel function as \( z \to 0 \) can be deduced from the Bessel function:

\[
\begin{align*}
H_0^{(1) }, (2)(z) &= \pm \frac{2i}{\pi} \ln z - i \left( \frac{z}{2} \right)^2 (1 - \frac{2}{\pi} \ln z) + O(z^4 \ln z); \\
H_n^{(1) }, (2)(z) &= i \frac{2}{\pi} \left( \frac{2}{z} \right)^n \left((n - 1)! + \pi z^2 + O(z^4)\right).
\end{align*} \tag{2.51a,b}
\]

Fig. 2.4. Modified Bessel Function of First and Second Kind
Behavior of Modified Bessel Functions

Figure (2.4) shows the behavior of $I_n(z)$ and $K_n(z)$ for $n = 0, 1, 2$ and $0 \leq z \leq 3.5$. The function $I_n(z)$ is bounded for all $n$'s as $z \to 0$. The limit of $I_0(0) = 1$, and for $n \geq 1$ $I_n(0) = 0$. $K_n(z)$ tends to zero as $z \to \infty$ for all values of $n$. However, $K_0(z)$ and $I_n(z)$ become unbound as $z \to 0$ and $z \to \infty$, respectively. For a given $n$, $I_n(z)$ and $K_n(z)$ are functions of $z$, monotonically increasing and decreasing respectively. The limiting expressions for $I_n(z)$ and $K_n(z)$ for $z \ll 1$ are also dominated by the leading terms of their series representations. They are

$$I_n(z) \approx \left(\frac{e}{2}z\right)^n / \Gamma(n + 1); \quad (2.52a)$$

$$K_0 \approx - \ln z; \quad (2.52b)$$

$$K_n \approx \frac{1}{(\pi z)^{1/2}} \Gamma(n) \left(\frac{e}{2}z\right)^{-n}. \quad (2.52c)$$

The asymptotic expansions for $I_n(z)$ and $K_n(z)$ for large arguments are:

For $|z| \gg \epsilon$, $|z| \gg |v|^2$,

$$I_n(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{(4v^2 - 1^2)}{8z} + \frac{(4v^2 - 1^2)(4v^2 - 3^2)}{2!(8z)^2} + \ldots\right]$$

$$+ e^{-(v+\frac{1}{2})\pi i} \frac{e^{-z}}{\sqrt{2\pi z}} \left[1 + \frac{4v^2 - 1}{1!8z} + \ldots\right], \quad 0 \leq \arg z \leq \pi; \quad (2.53a)$$
\[ K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)}{21(8z)^2} + \ldots \right. \\
+ \left. \frac{(4\nu^2 - 1^2) \ldots (4\nu^2 - (2r - 3)^2)}{(r - 1)! (8z)^{r-1} \ldots} \right] - \pi < \arg z \leq \pi. \]

(2.53b)

For \( n \gg \frac{1}{4}z^2 \), \( \text{Re} z > 0 \), and \( n > 0 \) large enough

\[ I_n(z) = \frac{(\frac{2}{z})^n}{n!} \left\{ 1 + \frac{(\frac{3}{2}z)^2}{n + 1} + \frac{(\frac{3}{2}z)^4}{2!(n + 1)(n + 2)} \right\}; \quad (2.54a) \]

and

\[ K_n(z) = (\frac{2}{z})^n \frac{(n - 1)!}{2} \left\{ 1 - \frac{(\frac{3}{2}z)^2}{n - 1} + \frac{(\frac{3}{2}z)^4}{2!(n - 1)(n - 2)} \right\}. \quad (2.54b) \]

3. STEADY-STATE SOLUTION FOR CAVITY AND RIGID INCLUSIONS (PLANE-WAVE)

3.1. Incident P Wave

Let us consider an harmonically time-varying incident plane P wave propagating in the direction \( \nu = e_x \).

The incident wave can be represented as

\[ \varphi(i) = \varphi_0 e^{i(\alpha z - \omega t)}. \quad (3.1) \]

where \( \alpha = \omega/c_d \) is the compressional wave number, \( \varphi_0 \) is the amplitude, and \( \omega \) is the circular frequency.

As the waves impinge upon either a circular cylindrical cavity or a rigid inclusion, two waves are reflected from the boundary. They
are a P wave and an S wave $\psi$. The shear wave, $\chi$, is absent because of the plane-strain nature of the problem.

The two reflected waves are

$$
\varphi(r) = \sum_{n=0}^{\infty} A_n \eta_n^{(1)}(nr) \cos n\theta e^{-i\omega t},
$$

$$
\psi(r) = \sum_{n=0}^{\infty} B_n \eta_n^{(1)}(nr) \sin n\theta e^{-i\omega t},
$$

(3.2)

which represent waves diverging from the origin. $\beta = \omega/c_g$ is the shear wave number. The circular functions are chosen to satisfy the evenness and oddness of the displacements and stresses. $A_n$ and $B_n$ are coefficients of the expansions to be determined from the appropriate boundary conditions.

To determine the unknown coefficients, we first expand the incident wave in terms of the cylindrical wave function (see Chapter II, Section 1). It follows

$$
\varphi(i) = \varphi_0 \sum_{n=0}^{\infty} \epsilon_n(i) \eta_n^{(1)}(nr) \cos n\theta e^{-i\omega t},
$$

(3.3)

where

$$
\epsilon_n = \begin{cases} 
1, & n = 0, \\
2, & n \geq 1.
\end{cases}
$$

The total wave is the sum of the incident wave and the reflected waves.
\[ \varphi = \varphi^{(i)} + \varphi^{(r)}, \]  
\[ \psi = \psi^{(r)}, \]  
(3.4)

or

\[ \varphi = \sum_{n=0}^{\infty} \left[ c_n J_n(\sigma r) + A_n H_n^{(1)}(\sigma r) \right] \cos n\theta \ e^{-i\omega t}, \]  
(3.5)

\[ \psi = \sum_{n=0}^{\infty} B_n H_n^{(1)}(\sigma r) \sin n\theta \ e^{-i\omega t}. \]  
(3.6)

Cavity. First let us consider the cavity case. The boundary condition for a cavity of radius \( \alpha \) is a traction-free surface at \( r = \alpha \). It follows

\[ \sigma_{rr} = \sigma_{r\theta} = 0 \bigg|_{r=\alpha}, \]

where \( \sigma_{rr} \) and \( \sigma_{r\theta} \) are the stresses due to all the waves. In terms of the displacement potentials they are given as

\[ \sigma_{rr} = \frac{2\mu}{r} \sum_{n=0}^{\infty} \left( c_n J_n(\sigma) \varepsilon_{11}^{(1)} + A_n L_n^{(1)} + B_n L_n^{(3)} \right) \cos n\theta \ e^{-i\omega t}; \]  
(3.7)

\[ \sigma_{r\theta} = \frac{2\mu}{r} \sum_{n=0}^{\infty} \left( c_n J_n(\sigma) \varepsilon_{41}^{(1)} + A_n L_n^{(1)} + B_n L_n^{(3)} \right) \sin n\theta \ e^{-i\omega t}; \]  
(3.8)

\[ \sigma_{\theta\theta} = \frac{2\mu}{r} \sum_{n=0}^{\infty} \left( c_n J_n(\sigma) \varepsilon_{21}^{(1)} + A_n L_n^{(1)} + B_n L_n^{(3)} \right) \cos n\theta \ e^{-i\omega t}; \]  
(3.9)
where $\varepsilon_{11}^{(1)}$, $\varepsilon_{11}^{(3)}$ ... etc., are defined as a part of the contribution to the stresses due to the various waves (see the appendix to this Chapter). The superscripts denote which cylinder function $\zeta(z)$ is used (explained in the appendix). Thus, for example, the expressions for the radial stress are:

$$
\varepsilon_{11}^{(1)} = \left( n^2 + n - \frac{\beta^2}{2} \right) J_n(\alpha r) - \alpha n J_{n-1}(\alpha r); \quad (3.10a)
$$

$$
\varepsilon_{11}^{(3)} = \left( n^2 + n - \frac{\beta^2}{2} \right) J_n^{(1)}(\alpha r) - \alpha n J_{n-1}^{(1)}(\alpha r); \quad (3.10b)
$$

$$
\varepsilon_{12}^{(3)} = -n(n+1) J_n^{(1)}(\beta r) + \beta n J_{n-1}^{(1)}(\beta r). \quad (3.10c)
$$

The unknown coefficients $A_n$ and $B_n$ are determined by letting $r = \alpha$ in Eqs. (3.7) and (3.8) and by imposing the traction-free condition. It follows

$$
A_n = - \varepsilon_{n} J_n \zeta \text{ Im} \frac{\varepsilon_{11}^{(1)} \varepsilon_{12}^{(3)}}{\Delta_n}, \quad (3.11)
$$

$$
B_n = - \varepsilon_{n} J_n \zeta \text{ Im} \frac{\varepsilon_{11}^{(1)} \varepsilon_{11}^{(3)}}{\Delta_n}. \quad (3.12)
$$

where
\[ \Delta_n = \begin{bmatrix} E_{11}^{(3)} & E_{12}^{(3)} \\ E_{11}^{(3)} & E_{12}^{(3)} \end{bmatrix}, \]  

in which \( E_{11}^{(1)} \), etc., are the values of the function \( E_{11} \), etc., evaluated at \( r = \alpha \).

The stress field is determined once the coefficients \( A_n \) and \( B_n \) are known. Of particular interest are the dynamic stress concentration factors around the cavity. In the cavity case, the only stress at the boundary is the hoop stress, \( \sigma_{\theta \theta} \). Substituting Eqs. (3.11) and (3.12) into Eq. (3.9), and using the Wronskian relationship in Eq. (2.40b), establishes hoop stress at \( r = \alpha \) as

\[ \sigma_{\theta \theta} = \frac{\sigma_{\theta \theta}}{\sigma} \bigg|_{r=\alpha} = \frac{2}{\pi} \left( 1 - \frac{1}{\kappa} \right) \sum_{n=0}^{\infty} e^{i \pi n} \cos n \phi e^{-i \omega t}, \]  

where

\[ S_n = \left\{ (n^2 - 1)\beta a H_{n-1}(\beta a) - (n^3 - n + \frac{1}{2} \beta^2 a^2) H_n(\beta a) \right\} / \{ \alpha a H_{n-1}(\alpha a) \}
\]

\[ = \left[ (n^2 - 1)\beta a H_{n-1}(\beta a) - (n^3 - n + \frac{1}{2} \beta^2 a^2) H_n(\beta a) \right] - B_{1}^{(1)}(\alpha a)
\]

\[ = \left[ (n^3 - n + \frac{1}{2} \beta^2 a^2)\beta a H_{n-1}(\beta a) - (n^2 + n - \frac{1}{2} \beta^2 a^2) H_n(\beta a) \right]. \]

(3.15)

\( \sigma_o \), defined by \( \sigma_o = \mu \beta^2 \varphi_o \), denotes the stress intensity of the incident wave in the direction of propagation. \( \kappa^2 \), defined by

\[ \kappa^2 = 2(1-\nu)/(1-2\nu), \]

is the ratio of \( \beta^2/\alpha^2 \), or \( (\sigma_p/\sigma_p)^2 \), and it is a
function of the Poisson ratio of the medium. \( c_{q\theta}^* \) is dimensionless, and is considered as the Dynamic Stress Concentration Factor for the circular cylindrical cavity. The analogous static solution for this problem is Kirsch's static solution for biaxial loadings (see Fig. 3.1).

![Fig. 3.1. Biaxial Loading System](image)

The dynamic stress concentration factor shown in Eq. (3.14) is dependent upon the dimensionless wave number \( \alpha \) (\( \beta \) being related to \( \alpha \) by \( \kappa \alpha = 2\alpha \)), and upon the Poisson ratio \( \nu \). According to Chapter I, Section 4, the steady-state solution obtained in this manner may be considered the admittance of the system. In other words, it is the response of the system to a simple harmonic input of unit magnitude. Thus it may be used as the kernel in the Fourier inversion integral when we try to determine the transient response of the system. This will be discussed later.

Numerical evaluation of \( c_{q\theta}^* \) for a large range of \( \alpha \) and \( \nu \) is difficult because of the complexity of Eq. (3.14). In fact, it has been only recently that numerical results have become available. (0.6)

Equation (3.14) has the following complex form:
\[ \sigma_{\theta\theta}^* = (R + iI)e^{-i\omega t}, \]  

which may be interpreted as follows: The real part, \( R \), yields the stress at \( t = 0 \), and the imaginary part yields the stress at \( t = T/4 \) (see Fig. 3.2), where \( T \) is the period of the incident wave. As stated earlier, the value of \( \sigma_{\theta\theta}^* \) depends on Poisson's ratio \( \nu \) and the normalized wave number \( \alpha \). Since \( \alpha = \omega a/c_p \), the stresses will depend

![Incident wave at t = 0](image)

![Incident wave at t = T/4](image)

**Fig. 3.2. Position of the Incident Wave**

on the circular frequency of the incident wave and \( a/c_p \). \( a/c_p \) has the dimension of time, and it is the time it takes the P wave to travel the distance \( a \) which is the radius of the cavity. We define \( a/c_p \) as one-half of a transit time. A transit time is then the time required for the wave to propagate a distance equal to the diameter of the cavity.

It will be of interest to examine some of the limiting values of Eq. (3.14) before discussing the numerical results. The case we wish to examine is the limiting expression for \( \sigma_{\theta\theta}^* \) as \( \alpha \to 0 \) (which implies \( \omega \to 0 \)).

First let us examine the physical meaning of the incident fields as \( \alpha \to 0 \). From Eq. (3.1), we have
\[ \varphi(t) = \varphi_0 e^{i(ax-wt)} = \varphi_0 [\cos (ax - wt) + i \sin (ax - wt)]; \quad (3.17a) \]

\[ u_x(i) = i\alpha \varphi_0 e^{i(ax-wt)} = \alpha \varphi_0 [i \cos (ax - wt) - \sin (ax - wt)]; \quad (3.17b) \]

\[ \sigma(i) = (\lambda + 2\mu)\varphi_0^2 e^{i(ax-wt)}, \]

\[ = - (\lambda + 2\mu)\varphi_0^2 [\cos (ax - wt) + i \sin (ax - wt)]; \quad (3.17c) \]

\[ \sigma_{yy} = -\lambda \varphi_0^2 e^{i(ax-wt)} = -\lambda \varphi_0^2 [\cos (ax - wt) + i \sin (ax - wt)], \]

\[ = - \frac{\nu}{1 - \nu} (\lambda + 2\mu)\varphi_0^2 [\cos (ax - wt) + i \sin (ax - wt)], \quad (3.17d) \]

where the amplitude in Eq. (3.17c) was previously defined as \( \sigma_0 \). That is,

\[ \sigma_0 = - (\lambda + 2\mu)\varphi_0^2 = - \mu \varphi_0^2. \]

Maintaining the magnitude of the incident stress, \( \sigma_0 \), as a constant as \( \alpha \to 0 \), and using the approximations for sine and cosine — i.e., \( \sin ax \to ax, \cos ax \to 1 \) for \( ax \ll 1 \) — we obtain the following approximate expressions for the incident stresses and displacements:

\[ \sigma_{xx}(i) = \sigma_0 [1 + iax] \quad \sigma_0; \quad (3.18a) \]

\[ \sigma_{yy}(i) = \frac{-\nu}{1 - \nu} \sigma_0 [1 + iax] \quad \frac{-\nu}{1 - \nu} \sigma_0; \quad (3.18b) \]

\[ u_x(i) = \alpha \varphi_0 [-ax] \quad - \alpha \varphi_0 x. \quad (3.18c) \]

The equations above show that as \( \alpha \to 0 \) the imaginary part of the inci-
dent stresses vanish as expected, and the real part gives precisely
the biaxial loading shown in Fig. (3.1). With the foregoing anal-
ysis, we would then anticipate the dynamic solution in Eq. (3.14)
to degenerate into the Kirsch static solution as \( \alpha \to 0 \). To show this,
we substitute the small argument approximations in Eq. (2.51) for the
Hankel functions in Eq. (3.14). Retaining only the leading term, we
find

\[
S_0 \rightarrow \frac{\pi}{2i} ; \\
S_1 \rightarrow 0 ;
\]

and for \( n \geq 2 \)

\[
S_n \rightarrow \frac{\pi}{2i} \frac{(\frac{\alpha \kappa}{2})^{n-2}}{(\kappa^2 - 1)(n - 2)!} \begin{cases} \frac{\pi}{2i(k^2 - 1)} & n = 2, \\ 0 & n > 2. \end{cases}
\]

It follows that as \( \alpha \to 0 \), Eq. (3.14) becomes

\[
\sigma^*_{\theta \theta} = \frac{2}{\kappa^2} \left[ (\kappa^2 - 1) - 2 \cos 2\theta \right],
\]

which is the Kirsch static solution for the biaxial loading system
shown in Fig. 3.1. For values of \( \alpha \kappa \) different from zero, \( \sigma^*_{\theta \theta} \) must
be evaluated numerically from Eq. (3.14). Some numerical results
are shown in Figs. 3.3 through 3.7 (taken from Pao, Ref. 0.6).

Figure 3.3 shows both the real and imaginary parts of \( c^*_{\theta \theta} \) as
functions of the angle at three wave numbers: \( \alpha \kappa = 0, 0.2, 3.5. \)
Fig. 3.3. Distribution of the Dimensionless Stress \( \sigma_{\theta\theta}^* \) at \( r = a, \nu = 0.26; \) Solid Line \( t = 0, \) Dashed Lines \( t = \pi/4 \)

When \( \omega = 0, \) we have Kirsch's static distribution; \( \omega = 0.2 \) and 3.5 represent the low frequency and high frequency incident waves, respectively.

If we now compare the distribution of the real part of \( \sigma_{\theta\theta}^* \) at \( \omega = 0.2 \) with the static distribution, we note there is only a slight difference. This should not be surprising because at \( \omega = 0.2, \) the incident wavelength is \( 10\pi a (\lambda = (2\pi/\omega) a), \) which is long compared to the diameter of the cavity. As a result, the incident stress field for such a long wave is nearly uniform in the neighborhood of the cavity, or near the stress field of a static field. Hence, it might be expected that the resulting \( \sigma_{\theta\theta}^* \) might be closely approximated by the static solution.

At \( \omega = 3.5, \) however, the stress distribution is quite different from the low wave number distribution. The magnitude of \( \sigma_{\theta\theta}^* \) is
generally lower than for \( \alpha x = 0.2 \). What is of interest is that the peak is shifted toward the incident side of the cavity \( \theta = \pi \), and the presence of rather high negative stresses at \( \theta \approx 45^\circ \), and \( 135^\circ \). (Negative dimensionless stress means the real stresses have an opposite sign from the incident stress.) This negative stress field is quite contrary to the static solution. For example, in the case shown in Fig. 3.3 where the static stress distribution for \( \nu = 0.26 \), the minimum stress is approximately 0.03 at \( \theta = 0, \pi \), and nowhere on the circumference of the cavity does it become negative.

Figures 3.4 through 3.6b show the behavior of \( \sigma^{*}_{\theta \theta} \) as a function of wave number and Poisson ratio. It should be noted first that all curves of \( \sigma^{*}_{\theta \theta} \) degenerate to the static value as \( \alpha x \to 0 \). The maximum dynamic stress concentration factor around the cavity occurs at \( \theta = \pi/2 \) and at \( 0.25 < \alpha x < 0.30 \), which is a rather "low" wave number or low-frequency wave. The dynamic stress concentration factors are

![Diagram](image-url)

*Fig. 3.4. Dynamic Stress Concentration Factors (\( \sigma^{*}_{\theta \theta} \)); \( \theta = \pi/2 \), (A) \( t = 0 \), (B) \( t = T/4 \)*
higher than their corresponding static values. At high wave numbers, the value of $\sigma^*$ approaches an asymptote. For example, Fig. 3.4 shows $\text{Re } \sigma^*_{\theta\theta}$ approaches the value of unity. Physically, this means that when the frequency is very high, the circular boundary appears to be a plane boundary to the incident waves. Thus, at $\theta = \pi/2$, the value of the tangential stress should approach the value of the incident stress field, which implies $\sigma^*_{\theta\theta}$ should approach unity. A similar behavior is also observed in Figs. 3.5a and 3.5b. It is noted that $\text{Re } \sigma^*_{\theta\theta}$ at $\theta = \pi$ asymptotically approaches zero. This can be interpreted in the same manner as in the preceding discussion, i.e., that

![Graph](image1)

**Fig. 3.5a. Dynamic Stress Concentration Factors; $\theta = \pi, t = 0$**

![Graph](image2)

**Fig. 3.5b. Dynamic Stress Concentration Factors; $\theta = \pi, t = T/4$**
the incident side of the cavity appears to be a traction-free plane boundary to the incident wave. Hence, the value of the stresses at \( \theta = \pi \) should approach the value of the plane wave incident normally on the plane boundary, which mean all stresses vanish at the boundary.

There are several stationary values in Figs. 3.4 through 3.6b. The wave numbers at which these stationary values occur are related to the natural frequencies of the various modes of the cavity. The natural frequencies of the different modes are determined by the roots of the determinant shown in Eq. (3.13). The roots of Eq. (3.13) are
complex, with the real parts representing the natural frequencies, and the imaginary parts representing the rate at which the energy is being carried away from the vicinity of the cavity. We shall postpone further discussion here, since this point will be elucidated in the section on transient response.

The results shown in Figs. 3.4 through 3.6 can be presented in a slightly different manner, if we express $\sigma_{\theta\theta}$ as follows:

$$\sigma_{\theta\theta}^* = (R + iI)e^{-i\omega t} = |\sigma_{\theta\theta}^*|e^{-i(\omega t - \phi)},$$

where $\phi = \tan^{-1} I/R$ is the phase angle.

Shown in Fig. 3.7 is the behavior of $|\sigma_{\theta\theta}^*|$ at $\theta = \pi/2$ as a function of the wave number and the Poisson ratio. It represents the

---

Fig. 3.7. The Maximum Dynamic Stress Concentration Factors (Absolute Values of $\sigma_{\theta\theta}/\sigma_0$ at $\theta = \pi/2$)
maximum dynamic stress concentration under steady-state excitation. The time $t$ at which it occurs depends upon the phase angle $\phi$. Examination of the results in Fig. 3.7 shows that the maximum dynamic stress concentration factor is always 10% to 15% higher than the corresponding static value for any given Poisson ratio, $\nu$.

Rigid Inclusion. Consider now the case of a rigid inclusion. The expressions for the incident wave and reflected waves remain the same as in Eqs. (3.1) through (3.9). The essential difference between the rigid inclusion case and the cavity case arises from the boundary conditions (Refs. 0.7 and 3.1). The appropriate boundary condition for the rigid inclusion at $r = a$ is that the displacements of the surrounding medium must be the same as those of the inclusion, which translates as a rigid body. Let us denote the rigid body translation of the inclusion as $U$. For the present case, i.e., for an incident $P$ wave, the direction of $U$ must be in the direction of propagation of the incident wave. It follows that the boundary condition can be represented (at $r = a$) as

$$
u_r = u_r + u_r^{(r)} = U \cos \theta,$$

$$
u_\theta = u_\theta + u_\theta^{(r)} = -U \sin \theta,$$

(3.21)

where $u_r^{(r)}$, $u_r^{(r)}$ ... represent the displacements due to incident and reflected waves. At any radius $r$, we have
\[ u_r = r^{-1} \sum_{n=0}^{\infty} \left[ \varphi_n r^n w_n'(ar) + A_n r H_n^{(1)}(ar) + B_n H_n^{(1)}(br) \right] \cos n\theta e^{-i\omega t}; \]

\[ u_\theta = r^{-1} \sum_{n=0}^{\infty} \left[ \varphi_n r^n w_n'(ar) + A_n r H_n^{(1)}(ar) + B_n \beta r H_n^{(1)}(br) \right] \sin n\theta e^{-i\omega t}. \]

(3.22)

(3.23)

The boundary condition as expressed in Eq. (3.21) contains an additional unknown \( U \). However, \( U \) can be determined in terms of the stresses acting on the inclusion. If we write the equation of motion for the inclusion, we have

\[ (wa^2)\ddot{U} = \int_0^{2\pi} \left( \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta \right) |a \, d\theta \]

\[ \mid_{r=a} \]

(3.24)

Now using Eqs. (3.7) and (3.8) for \( \sigma_{rr} \) and \( \sigma_{r\theta} \), respectively, and the orthogonality condition in Eq. (3.24) — we shall drop the superscript (1) here for brevity — we have

\[ U = \eta a^{-1} [2i\varphi_{1}J_1(\alpha a) + A_1 H_1(\alpha a) + B_1 H_1(\beta a)], \]

(3.25)

where \( \eta = \rho / \rho_1 \) is the density ratio of the medium to the inclusion.

It is also clear that \( U \) is contributed solely by the \( n = 1 \) term, which is the translational mode.

Substituting Eqs. (3.22), (3.23), and (3.25) into Eq. (3.21), we have,
\[ A_n = -\frac{\Phi_0^2 \nu_n^{2n}}{\Delta_n} \left[ a \beta a^2 J_n(\alpha \alpha) H_n'(\beta \alpha) - n^2 R^J_n(\alpha \alpha) H_n(\beta \alpha) \right]; \quad (3.26) \]

\[ B_n = -\frac{\Phi_0^2 \nu_n^{2n}}{\Delta_n} \left( \frac{2i\nu_n}{n} \right); \quad (3.27) \]

\[ \Delta_n = a \beta a^2 H'_{n}(\alpha \alpha) H_{n}'(\beta \alpha) - n^2 R^H_n(\alpha \alpha) H_n(\beta \alpha); \quad (3.28) \]

and for \( n = 1 \),

\[ A_1 = \frac{2i\Phi_0}{\Delta_1} \left[ -4n J_1(\alpha \alpha) H_1(\beta \alpha) + (1 + n) J_1(\alpha \alpha) \beta a H_0(\beta \alpha) \right. \]

\[ + (1 + n) a \alpha J_0(\alpha \alpha) H_1(\beta \alpha) - a \beta a^2 J_0(\alpha \alpha) H_0(\beta \alpha) \left. \right]; \quad (3.29) \]

\[ B_1 = -\frac{2i\Phi_0}{\Delta_1} \left[ \frac{2i(1 - \nu_n)}{n} \right]; \quad (3.30) \]

\[ \Delta_1 = 4n H_1(\alpha \alpha) H_1(\beta \alpha) - (1 + n) \beta a H_0(\beta \alpha) H_1(\alpha \alpha) \]

\[ - (1 + n) a \alpha H_0(\alpha \alpha) H_1(\beta \alpha) + a \beta a^2 H_0(\alpha \alpha) H_0(\beta \alpha). \quad (3.31) \]

It is of interest here to note that when \( \eta = 0 \), that is when we have an infinitely dense inclusion, the rigid body motion vanishes.

This is the same as fixing the inclusion stationary in space. Furthermore, it can be shown easily that when \( \eta = 0 \), Eqs. (3.26), (3.27), and (3.28) apply to all \( n \), including \( n = 1 \).
The parameters of interest in this problem are $U$, and the stresses around the boundary of the inclusion. By substituting the coefficients $A_n$ and $B_n$ into Eqs. (3.7), (3.8), (3.9), and (3.26), the following quantities are obtained:

$$U^* = \frac{U}{i\omega a\varphi_0} = \frac{4i\eta [2H_1(\beta a) - \beta a H_0(\beta a)]}{\pi a a_{\Delta_1}} e^{-i\omega t};$$  \hspace{1cm} (3.32)$$

$$\sigma_{rr}^* |_{r=a} = \frac{\sigma_{rr}}{(-\mu \beta^2 \varphi_0)}.$$  \hspace{1cm} (3.33)$$

$$\sigma_{r\theta}^* |_{r=a} = \frac{\sigma_{r\theta}}{(-\mu \beta^2 \varphi_0)} = -\left(\frac{2}{\pi}\right) [2(1 - \eta) H_1(\beta a) \sin \theta/\Delta_1 + 2 \sum_{n=2}^\infty \frac{\beta \eta H_n'(\beta a)}{\Delta_n} \cos n\theta e^{-i\omega t}] e^{-i\omega t};$$  \hspace{1cm} (3.34)$$

and

$$\sigma_{\theta\theta}^* = \left(1 - \frac{2}{\kappa}\right) \sigma_{rr}^*.$$  

The denominator used to nondimensionalize the displacements and stresses is the amplitude of the parameters in the incident wave. $\Delta_1$ and $\Delta_n$ are defined by Eqs. (3.28) and (3.31), and $\kappa^2 = 2(1 - \nu)7(1 - 2\nu)$.
It is to be observed that the displacements and stresses are functions of wave numbers, of Poisson's ratio, and of the parameter \( \eta \) (the density ratio). It is also to be noted that the parameter \( \eta \) enters into the \( n = 1 \) term only in the series. There are two particular instances worth noting: it has already been mentioned that when \( \eta \) is zero, we have the equivalent of fixing the inclusion in space; when \( \eta = 1 \), that is, when the density of inclusion and medium are the same, then \( B_1 \) vanishes. What this implies is that even though we have a rigid inclusion, the shear wave of the \( n = 1 \) mode is not excited at the boundary of the inclusion. Furthermore, Eq. (3.32) for \( \eta = 1 \) reduces to

\[
U^* = \frac{-4\pi e^{-i\omega t}}{\pi a^2 \epsilon^2 H_2(\alpha x)} .
\]

(3.35)

Hence, \( U^* \) depends only upon the incident wave number.

A similar limiting process can be carried out on \( \sigma^*_{rr} \) and \( \sigma^*_{r\theta} \) for \( \alpha x \rightarrow 0 \), as in the cavity case. It can be shown that for any arbitrary values of \( \eta \neq 0 \), Eqs. (3.33) and (3.34) reduce to the static solution for a rigid inclusion. The static solutions for a rigid insert and biaxial loading of \( \sigma_o \) and \( \nu/(1 - \nu)\sigma_o \) — see Fig. 3.1 — are:

\[
\sigma^*_{rr} = 1 + \frac{2}{\kappa^2 + 1} \cos 2\theta ;
\]

(3.36a)

\[
\sigma^*_{r\theta} = \frac{2}{\kappa^2 + 1} \sin 2\theta ;
\]

(3.36b)
\[ \sigma_{\theta \theta}^* = \left(1 - \frac{2}{\kappa^2}\right) \sigma_{rr}^*. \]  

(3.36c)

The limiting case as \( \alpha \to 0 \), for \( \eta = 0 \), can be carried out in the exact manner for all terms of \( \kappa \neq 1 \). But special care must be exercised in the \( \eta = 1 \) term, since this is where \( \eta \) enters the expression. For \( \eta = 0 \) and \( \eta = 1 \) the terms in Eq. (3.33) and (3.34) are:

\[ \sigma_{rr}^*: \]
\[ \frac{-H_1(\beta \alpha) + \beta a H_0(\beta \alpha)}{\Delta_1}; \]

(3.37)

\[ \sigma_{\rho \rho}^*: \]
\[ \frac{H_1(\beta \alpha)}{\Delta_1}; \]

(3.38)

and

\[ \Delta_1 = -\beta a H_0(\beta \alpha) H_1(\alpha \alpha) - \alpha a H_0(\alpha \alpha) H_1(\beta \alpha) + \alpha \beta^2 a^2 H_0(\alpha \alpha) H_0(\beta \alpha). \]  

(3.39)

Now, using the first term of the approximation for the Hankel small-argument function, and carrying out the limiting process in Eqs. (3.37), (3.38), and (3.39), one finds:

\[ \sigma_{rr}^*: \]
\[ \frac{-H_1(\beta \alpha) + \beta a H_0(\beta \alpha)}{\Delta_1} \sim \frac{-1}{(1 + \kappa^2) \alpha}; \]

(3.40)
Thus the $n = 1$ term becomes dominant as $\alpha \eta \to 0$. In fact, it becomes singular as $\alpha \eta \to 0$. What this implies is that if we were to fix the inclusion stationary in space, the inclusion would have to exert infinite force on the medium in order to be in static equilibrium. This is what one would expect physically. Numerical results of Eqs. (3.32), (3.33), and (3.34) for $0 \leq \alpha \eta \leq 4.0$ are shown in Figs. 3.8, 3.9, and 3.10.

Figures 3.8a and 3.8b show the real and imaginary parts of $U^*$ as a function of $\alpha \eta$ and $\eta$ with $\nu = 0.20$. It is evident that the
Fig. 3.8b. Rigid Body Translation of the Inclusion, Imaginary Part of $U^*$

density ratio plays a major role in the behavior of $U^*$. For any arbitrary density, it can be shown that as $a\alpha \rightarrow \infty$, $U^* \rightarrow 0$ and as $a\alpha \rightarrow 0$, $U^* \rightarrow 1$. This is done as follows: when $a\alpha >> 1$ and $a\alpha \gg \pi^2$, we have (see Eq. 2.44):

$$H_n(a\alpha) \sim \left(\frac{2}{\pi a\alpha}\right)^{3/2} e^{i(a\alpha - \pi \pi/2 - \pi/4)}.$$

Substituting the above expression into Eq. (3.32) and collecting the dominant terms we have

$$U^* \sim 2 \left(\frac{2}{\pi}\right)^{3/2} \left(\frac{1}{a\alpha}\right)^{3/2} e^{-i(a\alpha + \pi \pi/2 - \pi/4)}$$

(3.42)

which shows that $U^* \rightarrow 0$ when $a\alpha \rightarrow \infty$ as $(a\alpha)^{-3/2}$.

The limiting case for $a\alpha \rightarrow 0$ is carried out by using the leading term for the small-argument Hankel function. One finds
\[ U^*(a_0) = 1 + \frac{a_0^2 a^2}{4n} [\kappa^2(n - 1) \ln (k a_0) + \ln a_0] \]
\[ + \frac{2\pi a_0^2 a^2}{8n} [(1 - \eta)\kappa^2 + (1 + \eta)] \]  

Thus, as \( a_0 \to 0 \), \( U^*(a_0) \to 1 \).

Shown in Figs. 3.9a and 3.9b are the polar distributions of \( \sigma^*_{rr} \).

The effects of density are quite noticeable on the stress distributions and their magnitudes. The magnitude of the radial stress is

![Diagram of normalized radial stresses](image)

**Fig. 3.9a. Distribution of Normalized Radial Stresses at \( r = a \)**

![Diagram of normalized radial stresses](image)

**Fig. 3.9b. Distribution of Normalized Radial Stresses at \( r = a \)**
always larger on the incident side of the inclusion. This observation is further elucidated in Fig. 3.10, which shows the behavior of $|\sigma^*|_{nn}$ as a function of the wave number. It is noted that $|\sigma^*|_{nn}$ approaches the value of 2 on the incident side of the inclusion when $\alpha \alpha > 1$.

![Fig. 3.10. Maximum Normalized Stress at $0$ and $\pi$ for Various $\eta$ and $\nu = 0.20$](image)

Again, this can be explained by pointing out that when the incident wavelength is very short, the curved boundary appears to be a plane to the incident wave — hence on the incident side of the inclusion we obtain radial stress magnitudes of 2, which are the same as for a plane wave incident on a plane rigid boundary. Also, as $\alpha \alpha$ becomes large, $|\sigma^*|_{nn}$ at $\theta = 0$ approaches zero, where $\theta = 0$ is the center of the shadow side of the inclusion. Therefore, as $\alpha \alpha$ becomes large the stress should approach zero. Similar observations of the "shadow-forming" waves have been observed in geometrical acoustics. For more
details see Morse, Ref. 3.2.

The magnitudes of \( \sigma^*_{rr} \) at \( \theta = \pi \) are all substantially higher than for the static value. It is noted that the denser the inclusion, the higher the stresses become at low wave number. The magnitudes of \( \sigma^*_{rr} \) on the shadow side are always lower than the static value for \( \alpha > 0.30 \). At small wave number and \( \eta = 0.10 \) there is a slight over-
shoot over the static value. Also, the wave number at which the over-
shoot occurs at \( \theta = \pi \) is coincident with the peak of \( u^* \) as shown in Fig. 3.8a. This is because the motion of the inclusion is related to the resultant stress acting on the inclusion.

3.2. Plane S Wave

The problem of the incident plane S wave can be solved in the same manner as the case of the plane P wave. We shall only highlight some of the differences between the two cases and present results of interest.

Consider an harmonically time-varying incident plane S wave, which can be represented as

\[
\psi(t) = \psi_o e^{i(2\pi - \omega t)}.
\]

Equation (3.44) can be expanded in polar coordinates as before:

\[
\psi(t) = \psi_o \sum_{n=0}^{\infty} \epsilon_{n,2}^m J_n(\alpha r) \cos n\theta e^{-i\omega t}.
\]

The essential difference between (3.45) and (3.3) is that the stresses \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) as derived from the potentials are now odd func-

tions of $\theta$ instead of even functions of $\theta$. The converse is true for $\sigma_{rr}$. For example:

$$
\sigma_{rr} = \psi \sum \left( \varepsilon_{12}^{(1)} + \varepsilon_{11}^{(3)} + \varepsilon_{12}^{(3)} \right) \sin n \theta e^{-i\omega t},
$$

where

$$
\varepsilon_{12}^{(1)} = n(n + 1)J_{n}(\beta r) - n\beta rJ_{n-1}(\beta r);
$$

$$
\varepsilon_{11}^{(3)} = (n^{2} + n - \beta^{2}r^{2}/2)H_{n}(\alpha r) - \alpha nH_{n-1}(\alpha r);
$$

$$
\varepsilon_{12}^{(3)} = -n[-(n + 1)H_{n}(\beta r) + \beta rH_{n-1}(\beta r)].
$$

The other stress components can be obtained similarly.

(a) Cavity Case. The traction free condition at $r = a$ is again imposed. Following the previous procedure, the dimensionless tangential stress at $r = a$ is:

$$
\sigma^{*}_{\theta\theta} = \frac{8}{\pi} \left( 1 - \frac{1}{\kappa^{2}} \right) \sum_{n=1}^{\infty} \frac{n^{2} - 1 - \frac{\beta^{2}a^{2}}{2}H_{n}(\beta a)}{\Delta_{n}} \sin n\theta e^{-i\omega t}.
$$

with

$$
\Delta_{n} = aaH_{n-1}(aa) \left[ (n^{2} - 1)\beta aH_{n-1}(\beta a) - (n^{2} - n + \frac{\beta^{2}a^{2}}{2})H_{n}(\beta a) \right] -
$$

$$
+H_{n}(aa) \left[ -(n^{2} - n + \frac{\beta^{2}a^{2}}{2})\beta aH_{n-1}(\beta a) + (n^{2} + n - \frac{\beta^{2}a^{2}}{2})\beta^{2}a^{2}H_{n}(\beta a) \right].
$$

$\Delta_{n}$ is the same as shown in Eq. (3.13).
(b) Rigid Inclusion. The motion of the rigid inclusion in the case of the incident shear wave is such that it both rotates and translates in the direction perpendicular to the direction of the shear waves' propagation. This becomes clear if we observe the following stress resultant and torque acting on the inclusion:

\[
F_x = \int_0^{2\pi} \left( \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta \right) a \, d\theta \bigg|_{r=r_i}; \quad (3.49a)
\]

\[
F_y = \int_0^{2\pi} \left( \sigma_{rr} \sin \theta + \sigma_{r\theta} \cos \theta \right) a \, d\theta \bigg|_{r=r_i}; \quad (3.49b)
\]

\[
T = \int_0^{2\pi} \sigma_{r\theta} a^2 \, d\theta \bigg|_{r=r_i}; \quad (3.49c)
\]

Using the fact that \( \sigma_{rr} \) is an odd function of \( \theta \), and \( \sigma_{r\theta} \) is an even function, we note that \( F_x = 0 \), \( F_y \) depends solely on the \( n = 1 \) term in \( \sigma_{rr} \) and \( \sigma_{r\theta} \), and \( T \) depends solely on the \( n = 0 \) term in \( \sigma_{r\theta} \). Furthermore, we may relate the acceleration of the inclusion to the stress resultant and the torque as:

\[
\left( \pi a^2 \rho_1 \right) \ddot{U}_y = F_y; \quad (3.50a)
\]

\[
\left( \frac{\pi \rho_1 a^4}{2} \right) \ddot{\varnothing} = T. \quad (3.50b)
\]

Thus, the rigid body translation \( U_y \) and rotation \( \varnothing \) of the inclusion can be determined by Eqs. (3.50a,b) in terms of the unknown coefficients \( A_n \) and \( B_n \). More precisely, the translations are
associated with the \( n = 1 \) term of the series, and the rotation with \( n = 0 \) term.

The boundary conditions at \( r = a \) are:

\[
\begin{align*}
\sigma_r &= U_y \sin \theta, \\
\sigma_\theta &= U_y \cos \theta + \alpha \theta.
\end{align*}
\]  

(3.51)

If the expressions for \( \sigma_r \) and \( \sigma_\theta \) are substituted into Eq. (3.51), and if the orthogonality condition is used, it is found that \( A_n \) and \( B_n \) can be determined similarly as in the P wave case. Substituting \( A_n \) and \( B_n \) into the stress formulas, and simplifying the expressions, we obtain the following dimensionless stresses:

\[
\sigma_{rr}^* = \frac{4}{\pi} \left\{ \frac{(1 - n)K_1(a)}{A_1} \sin \theta + \sum_{n=2}^{\infty} \frac{n-1}{n} \frac{K_n(a)}{A_n} \sin n\theta \right\} e^{-i\omega t} ;
\]

(3.52)

\[
\sigma_{\theta\theta}^* = \left( 1 - \frac{2}{\kappa^2} \right) \sigma_{rr}^* ;
\]

(3.53)

\[
\sigma_{r\theta}^* = \frac{2}{\pi} \left\{ -\frac{i \beta a^2}{8 a^3 \sqrt{1 - n^2}} \left[ (1 + n)K_1(a) - a\alpha K_0(a) \right] \cos \theta \\
- \frac{2}{A_1} \left[ (1 + n)K_1(a) - a\alpha K_0(a) \right] \cos \theta \\
- \frac{\sum_{n=2}^{\infty} \frac{n+1}{n} \frac{K_n(a) + a\alpha K_{n-1}(a)}{A_n} \cos n\theta}{e^{-i\omega t}} \right\} e^{-i\omega t} ;
\]

(3.54)

where
\[ \Delta_1 = -4nH_1(\alpha a)H_1(\beta a) + (1 + \eta)aaH_0(\alpha a)H_1(\beta a) + (1 + \eta)\beta aH_0(\beta a)H_1(\alpha a) \]
\[ - a^2H_0(\alpha a)H_0(\beta a); \]  
(3.55)

\[ \Delta_n = naaH_{n-1}(\alpha a)H_n(\beta a) + n\beta aH_{n-1}(\beta a)H_n(\alpha a) - a^2H_{n-1}(\alpha a)H_{n-1}(\beta a), \]  
(3.56)
in which \( \eta \) is the density ratio.

The dimensionless displacement and the rotation are found by normalizing with the displacement and rotation of the incident wave. One finds

\[ U_y^* = \frac{U_y}{(-\psi^\circ_0)} = \frac{8\eta n}{\beta^2 \nu} \left[ H_1(\alpha a) - \frac{\alpha a}{2} H_0(\alpha a) \right]; \]  
(3.57)

\[ \Theta^* = \frac{\Theta}{(-\beta^2 \psi^\circ_0)} = \frac{16\eta n}{\pi \beta a\left[ \beta^2 a^2 H_1(\beta a) + 8n\left( \frac{\beta a}{2} H_0(\beta a) - H_1(\beta a) \right) \right]}. \]  
(3.58)

As a point of interest, it may be noted that Eqs. (3.48), (3.52), and (3.54) will all degenerate into their corresponding static solutions by the same limiting process shown in the P wave case. (It may be of interest for the reader to carry out a limiting process for this case as an exercise.)

The static cavity solution for \( \sigma_{\theta \theta}^* \) is

\[ \sigma_{\theta \theta}^* = 4 \sin 2\theta. \]  
(3.59)

The static solution for the stresses around a rigid inclusion (see Goodier, Ref. 3.3) are:
\[ \sigma_{rr}^* = \frac{2\kappa^2}{\kappa^2 + 1} \sin 2\theta; \quad (3.60a) \]

\[ \sigma_{r\theta}^* = \frac{2\kappa^2}{\kappa^2 + 1} \cos 2\theta; \quad (3.60b) \]

\[ \sigma_{\theta\theta}^* = \left( 1 - \frac{2}{\kappa^2} \right) \sigma_{rr}^*. \quad (3.60c) \]

Results obtained in Ref. 0.8 are shown below. In Figs. 3.11a, b, and c the behavior of \(|\sigma_{\theta\theta}^*|\) is shown for the cavity case as a function of \(\theta\), the shear wave number \(\beta\alpha\), and Poisson's ratio \(\nu\).

Fig. 3.11a. Distribution of \(|\sigma_{\theta\theta}^*|\) for various \(\beta\alpha\) with \(\nu = 0.25\) (Cavity)
The stress distribution at $\beta \alpha = 0.10$ is almost identical to the static distribution. At higher wave numbers, however, multiple peaks appear along the circumference of the cavity. These multiple peaks are a result of the interference caused by the incident and reflected waves. The maximum of the maxima occurs at rather low wave numbers. As shown in Figs. 3.11b and 3.11c respectively, the maximum occurs at $\beta \alpha \sim 0.5$ for $\theta = \pi/4$, and at $\theta = 3\pi/4$, $\beta \alpha \sim 0.28$. Here again, the amplification due to dynamic effect is approximately 10% to 15% over the static value. It is dependent upon the Poisson ratio of the medium.

Figures 3.12 show the stress distribution of $|\sigma_{rr}^*|$ (Fig. 3.12a) and $|\sigma_{r\theta}^*|$ (Fig. 3.12b) as a function of polar angles, while Fig. 3.13 shows the variation of $|\sigma_{rr}^*|$ and $|\sigma_{r\theta}^*|$, at selected angles, as a function of the wave number $\beta \alpha$ and the density ratio $\eta$ of a rigid inclusion.
The shifting of stress concentration at high wave number toward the incident side of the inclusion is noted again (see Fig. 3.12).

Deemed in Fig. 3.13a is the behavior of $|\sigma_{rr}^*|$ at $\theta = \pi/4$, $3\pi/4$. It shows the pronounced effect of $\eta$ on the magnitude of $\sigma_{rr}^*$. The lower the density ratio, i.e., the denser the inclusion, the higher the overshoot becomes. When $\eta = 0$, that is when we fix the inclusion
Fig. 3.13a. $|\sigma_{rr}^*|$ Versus $Ba$ at $\theta = \pi/4, 3\pi/4$ for Various $n$

With $\nu = 0.25$ (Rigid Inclusion)

Fig. 3.13b. $|\sigma_{r\theta}^*|$ Versus $Ba$ at $\theta = \pi, \pi/2$ for Various $n$

With $\nu = 0.25$ (Rigid Inclusion)
in space, the stresses will increase without bound as \( \beta a \to 0 \).

An interesting observation can be made concerning the effects of \( \eta \) in \( |\sigma^*_r| \) and \( |\sigma^*_\theta| \). It is noted that there is only one notable peak in \( |\sigma^*_r| \) at \( \theta = 3\pi/4 \) and \( \eta = 0.1 \), while there are two distinct peaks for \( |\sigma^*_\theta| \) at \( \theta = \pi \) (see Fig. 3.13b). If we compare the wave

![Graph of |σ^*_r| vs. βa for various values of η](image1.png)

**Fig. 3.14a.** \(|\sigma^*_r| \text{ Versus } \beta a \text{ for Various Values of } \eta\)

*With \( \nu = 0.25 \) (Rigid Inclusion)*

![Graph of |U^*_y| vs. βa for various values of η](image2.png)

**Fig. 3.14b.** \(|U^*_y| \text{ Versus } \beta a \text{ for Various Values of } \eta\)

*With \( \nu = 0.25 \) (Rigid Inclusion)*
numbers at which these peaks occur with those shown in Figs. 3.14a and 3.14b for $|U_y^*|$ and $|\psi^*|$, we note that the peak at $8\alpha \sim 0.8$ corresponds to a $|\psi^*|$ peak, while the one at $8\alpha \sim 0.35$ corresponds to a $|U_y^*|$ peak. This correlation of the effects of rotation and translation of the inclusion on the stresses is further understood by the examination of Eqs. (3.52) and (3.54). Equation (3.52) shows that $\sigma_{rr}^*$ does not depend on the $\pi = 0$ term, which is related to the rotation of the inclusion. But, Eq. (3.54) shows that $\sigma_{rg}^*$ contains all terms of $\pi$ in the series. As a further illustration of the effects of the motion of the inclusion on the stresses, it is noted that $|\sigma_{rg}^*|$ at $\pi/2$ exhibits only one peak. This follows immediately from Eq. (3.54), because the $\pi = 1$ term vanishes at $\pi/2$. The wave number at which $|\sigma_{rg}^*|$ peaks at $\pi/2$ coincides almost exactly with that for $|\psi^*|$.

Summary. Presented above are two of the simplest types of elastic wave scattering problems for the dynamic stress concentration studies. It is noted that the dynamic effects, in general, increase the stress concentration factor for the cavity case by about 10% to 15% over the static value. For the rigid inclusion case, however, this does not apply, since the stress concentration factor will depend largely upon the density ratio between the inclusion and the medium.

For practical application, the maximum dynamic stress concentration factors for a cavity impinged upon by P waves and S waves are shown in Figs. 3.15 and 3.16 as functions of the Poisson ratio. For design purposes, these values may be used for structural members under simple harmonic loadings.
3.3. Oblique Incident Plane Waves

The preceding two subsections are concerned with normal incidence where the propagation direction is at a right angle to the axis of the circular cylinder, for which the plane strain or plane stress approximation can be applied. If the propagation vector is at an arbitrary angle with the cylinder axis, a full three-dimension treatment is required. The solution can, however, be formulated in an analogous manner.
Referring to Fig. 1.2, with the circular cylinder, we assume that a plane wave progresses obliquely to the z-axis at an angle \( \phi = \phi_p \) or \( \phi_s \). Since the circular cylinder has azimuthal symmetry, we can choose \( \theta = 0 \) in Fig. 1.2 so that the \( O'z' \) line coincides with the coordinate axis \( z \). As in the case of plane strain, incident P and S waves can be treated independently, each having a distinct angle of incidence, \( \phi_p \) or \( \phi_s \).

As in Eq. (1.8), for an incident plane harmonic P wave we let

\[
\begin{align*}
\varphi(t) &= ik_p (x \sin \phi_p + z \cos \phi_p - \sigma_p t), \\
\psi(t) &= x(t) = 0, \quad k_p \sigma_p = \omega.
\end{align*}
\]  

(3.61)

The corresponding displacements are

\[
\begin{align*}
u_x &= ik_p \sin \phi_p A e^{ik_p z}, \\
u_y &= 0, \\
u_z &= ik_p \cos \phi_p A e^{ik_p z},
\end{align*}
\]

with \( \Theta_p = x \sin \phi_p + z \cos \phi_p - \sigma_p t \). The displacement vector is obviously parallel to the propagation vector \( \omega \).

For an incident S wave, we assume

\[
\begin{align*}
\varphi(t) &= 0, \\
\psi(t) &= (ik_s) Be^{ik_s z}.
\end{align*}
\]  

(3.62)
$$\chi(\epsilon) = C e^{i k_{s} e_{g}}.$$ 

$$e_{g} = x \sin \phi_{g} + z \cos \phi_{g} - c_{g} \tau, \quad k_{s} e_{g} = \omega.$$ 

The actual displacement is, by (1.13),

$$u_{x}(\epsilon) = -kC \sin \phi_{g} \cos \phi_{g} (i k_{s} e_{g})^{2} e^{i k_{s} e_{g}} = -A \sin \delta_{o} \cos \phi_{g} e^{i k_{s} e_{g}},$$

$$u_{z}(\epsilon) = -B \sin \phi_{g} (i k_{s} e_{g})^{2} e^{i k_{s} e_{g}} = A \cos \delta_{o} e^{i k_{s} e_{g}},$$

$$u_{y}(\epsilon) = -kC \sin \phi_{g} (i k_{s} e_{g})^{2} e^{i k_{s} e_{g}} = A \sin \delta_{o} \sin \phi_{g} e^{i k_{s} e_{g}}.$$ 

As in (1.15), the $B$ and $C$ are related to $A$ so that the resultant displacement is a vector with magnitude $A$ at an angle $\delta_{o}$ from the $y$ axis:

$$B \sin \phi_{g} k_{s}^{2} = A \cos \delta_{o},$$

$$kC \sin \phi_{g} k_{s}^{2} = A \sin \delta_{o}. \tag{3.63}$$

For the potential $\varphi(\epsilon)$ in (3.61), the $z$-dependence is separated from the remaining space and time variables. We thus set

$$\alpha = \kappa_{p} \sin \phi_{p}, \quad \gamma = \kappa_{p} \cos \phi_{p},$$

and have

$$\varphi(\epsilon) = A e^{i(\gamma z - \omega t)} e^{i \omega \tau},$$

$$= A e^{i(\gamma z - \omega t)} \sum_{n} e^{i n J_{n}(\alpha r)} \cos n\theta. \tag{3.64}$$
Except for the factor $e^{i\gamma z}$ and the new radial wave number $\alpha$, with

$$
\alpha^2 = k_p^2 - \gamma^2 = \omega^2/c_p^2 - \gamma^2,
$$

and $\varphi^{(i)}$ is no different than the incident $P$ wave in the plane-strain case. In fact, (3.61) reduces to (3.1) when $\gamma = 0$ ($\phi_p = \pi/2$).

It is now obvious that we can assume the scattered waves as

$$
\varphi(r) = e^{i(\gamma z - \omega t)} \sum D_n H_n^{(1)}(ar) \cos \pi \theta; \quad (3.65a)
$$

$$
\psi(r) = e^{i(\gamma z - \omega t)} \sum E_n H_n^{(1)}(br) \sin \pi \theta; \quad (3.65b)
$$

$$
\chi(r) = e^{i(\gamma - \omega t)} \sum F_n H_n^{(1)}(cr) \cos \pi \theta; \quad (3.65c)
$$

with

$$
\beta^2 = \frac{\omega^2}{c_s^2} - \gamma^2 = \frac{\omega^2}{c_p^2} - \gamma^2 = \kappa^2(\kappa^2 - \cos \phi_p).
$$

The unknown coefficients $D_n$, $E_n$, and $F_n$ are fixed by the boundary conditions. For example, the appropriate conditions for a cylindrical cavity are

$$
\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0, \quad \text{at } r = a.
$$

By letting $\varphi = \varphi^{(i)} + \varphi^{(r)}$, $\psi = \psi^{(r)}$, and $\chi = \chi^{(r)}$, and substituting their values as given by (3.64) and (3.65) into (2.7), we can calculate explicitly the values for $D_n$, $E_n$, and $F_n$.

The case for an incident $S$ wave is treated in exactly the same way. The incident waves are represented by (3.62) and the exponential
function can again be expanded into a Fourier-Bessel series:

\[ e^{ik_\theta r} = e^{i\gamma z} \sum_n \epsilon_n \lambda J_n(\beta r) \cos \eta \theta, \]

where

\[ \beta = k_g \sin \phi, \quad \gamma = k_g \cos \phi, \]

and

\[ \beta^2 = k_g^2 - \gamma^2 = \omega^2/c_g^2 - \gamma^2. \]

Scattered waves are also representable by (3.65) with

\[ \alpha^2 = \omega^2/c_p^2 - \gamma^2 = k_g^2/k^2 - \gamma^2 = k_g^2(1/k^2 - \cos \phi_g). \]

However, since \( \kappa = c_p/c_\theta > 1 \), the wave number \( \alpha \) becomes imaginary whenever \( \cos \phi_g \geq 1/k^2 \). Thus, the Hankel function \( H_n^{(1)}(i\alpha r) \) changes to the nonoscillatory \( K_n^{(1)}(i\alpha r) \) as in (2.28b). The value \( \phi_g = \cos^{-1}(1/k^2) \) is the critical angle below which the scattered P wave is of the surface wave type which decays rapidly away from the surface of the cylinder. This critical angle is the same as the one for the total reflection of an SV wave by a plane surface (Ewing, Jardesky, and Press, p. 29, Ref. 3.4).

Scattering of P and S waves at oblique incidence was formulated by White (Ref. 0.2), but he gave no detailed calculations.
4. STABILITY RESPONSE OF AN "ELASTIC" INCLUSION

IN THIS SECTION we shall examine the scattering phenomenon and dynamic
stress concentration factors around a more general type of inclusion.
In other words, we shall consider the wave transmitted into the in-
clusion as well as the wave reflected back into the medium by the
inclusion. The cavity case and the rigid inclusion presented in the
preceding section may be considered as limiting cases of the general
elastic inclusion problem.

4.1. Solid Elastic Inclusion

Let us consider an elastic circular cylindrical inclusion of
infinite extent imbedded (welded contact) in an infinite elastic
medium. The elastic constants and the density of the medium are
denoted by \( \lambda_1, \mu_1, \) and \( \rho_1 \), and those of the inclusion by \( \lambda_2, \mu_2, \) and
\( \rho_2 \) (see Fig. 4.1). The incident wave is assumed to be a plane P wave
as in Section 3. If \( \lambda_1, \mu_1, \) and \( \rho_1 \) differ from \( \lambda_2, \mu_2, \) and \( \rho_2 \), waves
will be reflected from the boundary of the inclusion as well as re-
fractured into the inclusion. The reflected waves in this case, as in the cavity or rigid inclusion cases before, will be outward propagating waves. The refracted waves, however, being confined in the scatterer, are standing waves.

The displacement potentials for the reflected waves are the same as in Eq. (3.2):

\[ \varphi(r) = \sum_{n=0}^{\infty} A_n n_n^{(1)}(a_1 r) \cos n\theta e^{-i\omega t}, \]  

\[ \psi(r) = \sum_{n=0}^{\infty} B_n n_n^{(1)}(b_1 r) \sin n\theta e^{-i\omega t}, \]  

where \( a_1 \) and \( b_1 \) denote the compressional wave numbers and shear wave numbers, respectively, in the medium.

The displacement potentials of the refracted standing waves in the inclusion can be represented as:

\[ \varphi(f) = \sum_{n=0}^{\infty} C_n n_n^{(2)}(a_2 r) \cos n\theta e^{-i\omega t}, \]  

\[ \psi(f) = \sum_{n=0}^{\infty} D_n n_n^{(2)}(b_2 r) \sin n\theta e^{-i\omega t}, \]  

where \( a_2 \) and \( b_2 \) denote the compressional and shear wave numbers in the scatterer.

In medium 1, the resultant waves are then determined by superposing the incident and the reflected waves, while the refracted waves are the only ones in the scatterer. Thus we have
\[ \varphi_1 = \varphi(i) + \varphi(r); \]  
\[ \psi_1 = \psi(r); \]  
\[ \varphi_2 = \varphi(f); \]  
\[ \psi_2 = \psi(f). \] (4.5a)

There are four sets of expansion coefficients \( A_n, B_n, C_n, \) and \( D_n \) in Eq. (4.5) that are to be determined by the boundary conditions. The welded boundary conditions at \( r = \alpha \) in this instance are:

\[ \sigma_{rr1} = \sigma_{rr2}; \]  
\[ \sigma_{r\theta1} = \sigma_{r\theta2}; \]  
\[ \mu_{r1} = \mu_{r2}; \]  
\[ \mu_{\theta1} = \mu_{\theta2}; \] (4.6a)

where subscripts 1 and 2 denote the stresses and displacement in medium 1 and 2 due to the incident, reflected, and refracted waves. For example, \( \sigma_{rr1} \) will depend on \( \varphi_1 \) and \( \psi_1 \) or \( \varphi(i), \varphi(r), \) and \( \psi(r), \) while \( \sigma_{rr2} \) is contributed by \( \varphi(f) \) and \( \psi(f) \) only.

Using the equations in the appendix for the stresses and displacements, with the proper cylinder function substituted, the coefficients \( A_n, B_n, C_n, \) and \( D_n \) can be determined by four simultaneous equations. In the matrix form these equations are
\[
\begin{bmatrix}
E^{(3)}_{11}(\alpha_1) & E^{(3)}_{12}(\beta_1) & -\bar{\mu}E^{(1)}_{11}(\alpha_2) \\
E^{(3)}_{41}(\alpha_1) & E^{(3)}_{42}(\beta_1) & -\bar{\mu}E^{(1)}_{41}(\alpha_2) \\
E^{(3)}_{71}(\alpha_1) & E^{(3)}_{72}(\beta_1) & -E^{(1)}_{71}(\alpha_2) \\
E^{(3)}_{81}(\alpha_1) & E^{(3)}_{82}(\beta_1) & -E^{(1)}_{81}(\alpha_2)
\end{bmatrix}
\begin{bmatrix}
\bar{E}^{(1)}_{12}(\beta_2) \\
\bar{E}^{(1)}_{42}(\beta_2) \\
\bar{E}^{(1)}_{72}(\beta_2) \\
\bar{E}^{(1)}_{82}(\beta_2)
\end{bmatrix}
= -E_n^i 
\begin{bmatrix}
A_n \\
B_n \\
C_n \\
D_n
\end{bmatrix}
\begin{bmatrix}
E^{(1)}_{11}(\alpha_1) \\
E^{(1)}_{41}(\alpha_1) \\
E^{(1)}_{71}(\alpha_1) \\
E^{(1)}_{81}(\alpha_1)
\end{bmatrix},
\]

where \( E^{(3)}_{11}(\alpha_1) \) represents \( E^{(3)}_{11}(\alpha_1) \) evaluated at \( r = \alpha \), etc. \( \bar{\mu} \), defined by \( \bar{\mu} = \mu_2/\mu_1 \), is the ratio of the shear modulus of the inclusion to the shear modulus of the medium.

Special cases of the inclusion can be deduced from Eq. (4.7) by an appropriate limiting process. They will be discussed more in Chapter VI.

Although Eq. (4.7) is straightforward and does not present any mathematical complexity, the actual computations are tedious. In the following subsections we shall present two types of inclusions where the interaction between the inclusion and the medium are of interest.

### 4.2. Fluid Inclusion

This case is presented because it demonstrates clearly how the stresses in the medium are affected by the standing waves in the inclusion.

Consider a cavity in an elastic solid filled with an inviscid fluid, with only one wave refracted into the fluid. The boundary conditions at \( r = \alpha \) are:
\[ u_{r1} = u_{r2}; \quad (4.8a) \]
\[ \sigma_{rr1} = \sigma_{rr2}; \quad (4.8b) \]
\[ \sigma_{r\theta 1} = 0. \quad (4.8c) \]

The circumferential displacement \( u_\theta \) in the solid might be different from that in the fluid at the interface. Such a paradox in discontinuity is a result of the inviscid assumption for the fluid.

The displacement potentials for the medium remain the same as in Eqs. (4.1) and (4.2), while \( \psi (r) \) vanishes. Also, the inviscid assumption implies that \( u_2 = 0 \) and \( D_\eta = 0 \). The stresses in the fluid, then, are:

\[ \sigma_{rr2} = \sigma_{\theta \theta 2} = \lambda_2 r^2 \psi (r). \quad (4.9) \]

Using the boundary conditions in Eq. (4.9), the coefficients \( A_\eta, B_\eta, \) and \( C_\eta \) are determined as follows:

\[
\begin{bmatrix}
E_{11}'^{(3)}(\alpha_1) \\
E_{12}'^{(3)}(\beta_1) \\
E_{42}'^{(3)}(\beta_1) \\
E_{72}'^{(3)}(\beta_2)
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11}'^{(3)}(\alpha_1) \\
\varepsilon_{12}'^{(3)}(\alpha_2) \\
\varepsilon_{12}'^{(1)}(\alpha_1) \\
\varepsilon_{12}'^{(1)}(\alpha_2)
\end{bmatrix}
\begin{bmatrix}
A_\eta \\
B_\eta \\
C_\eta \\
D_\eta
\end{bmatrix}
= \begin{bmatrix}
E_{11}'^{(1)}(\alpha_1) \\
E_{41}'^{(1)}(\alpha_1) \\
E_{71}'^{(1)}(\alpha_1)
\end{bmatrix}. \quad (4.10)
\]

where

\[ \varepsilon_{11}'^{(1)}(\alpha_2) = \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right) E_1^2 J_\eta' \varepsilon_{12}'^{(1)}(\alpha_2). \quad (4.11) \]
The stresses in the medium are determined by substituting $A_n$ and $B_n$ into the following equations:

\[ \sigma_{rr} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left[ \psi_n \epsilon_{n} \epsilon_{11}^{(1)} \left(\alpha_1 r\right) + A_n \epsilon_{11}^{(3)} \left(\alpha_1 r\right) + B_n \epsilon_{12}^{(3)} \left(\beta_1 r\right) \right] \cos n\theta e^{-i\omega t} \; ; \tag{4.12a} \]

\[ \sigma_{\theta\theta} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left[ \psi_n \epsilon_{n} \epsilon_{21}^{(1)} \left(\alpha_1 r\right) + A_n \epsilon_{21}^{(3)} \left(\alpha_1 r\right) + B_n \epsilon_{22}^{(3)} \left(\beta_1 r\right) \right] \cos n\theta e^{-i\omega t} \; ; \tag{4.12b} \]

\[ \sigma_{r\theta} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left[ \psi_n \epsilon_{n} \epsilon_{41}^{(1)} \left(\alpha_1 r\right) + A_n \epsilon_{41}^{(3)} \left(\alpha_1 r\right) + B_n \epsilon_{42}^{(3)} \left(\beta_1 r\right) \right] \sin n\theta e^{-i\omega t} \; ; \tag{4.12c} \]

and the stresses in the fluid are determined by substituting $C_n$ into the following equation:

\[ \sigma_{rr}^2 = \sigma_{\theta\theta} = -\mu_1 r^{-2} \sum_{n=0}^{\infty} C_n \beta_1^2 r^2 \left(\frac{\rho_2}{\rho_1}\right) \epsilon_{n} \left(\alpha_2 r\right) \cos n\theta e^{-i\omega t} \; . \tag{4.13} \]

The stress concentration factors around the inclusion are determined by letting $r = a$ in Eqs. (4.12a) and (4.12b). Following a procedure similar to that shown in Section 3, the following expressions are obtained for $\sigma_{rr}^*$ and $\sigma_{\theta\theta}^*$: (4.1)

\[ \sigma_{rr}^* = \frac{\pi}{n} \sum_{n=0}^{\infty} \frac{\epsilon_{n} \epsilon_{n+1} \epsilon_{n} \left(\alpha_2 a\right) \beta_1^2 a^2}{A_n^2} \]

\[ \times \left\{ \beta_1 a H_{n-1} \left(\beta_1 a\right) - \left(n^2 + n - \frac{\beta_1^2 a^2}{2}\right) H_n \left(\beta_1 a\right) \right\} \cos n\theta e^{-i\omega t} \; . \tag{4.14} \]
\[ \sigma_{ee} = -4 \sum_{n=0}^{\infty} \frac{\epsilon_{n}}{\Delta_{n}} \left\{ \left[ \left( \frac{1}{\kappa} \right)^{2} - 1 \right] \left( \frac{n_{1}^{2}a^{2}}{2} (n-1) - n^{2}(n^{2}-1) - n \frac{\beta_{1}^{2}a^{4}}{4} \right) \right. \\
+ \frac{\beta_{1}^{2}a^{2}}{4} \eta \left( n(n+1) - \frac{\beta_{1}^{2}a^{2}}{2} \right) \right\} J_{n}(\alpha_{1}a) \tilde{H}_{n}(\beta_{1}a) \\
+ \left[ \left( \frac{1}{\kappa} - 1 \right) \left( \frac{n^{3} - n - \frac{\beta_{1}^{2}a^{2}}{2}}{2} \right) - \frac{n\beta_{1}^{2}a^{2}}{2} \right] \beta_{1} a \tilde{H}_{n}(\alpha_{1}a) \tilde{H}_{n-1}(\beta_{1}a) \\
+ \left( \frac{1}{\kappa} - 1 \right) \left( \frac{n^{3} - n + \frac{\beta_{1}^{2}a^{2}}{2}}{2} \right) \alpha_{1} \tilde{H}_{n-1}(\alpha_{1}a) \tilde{H}_{n}(\beta_{1}a) \\
+ \left( \frac{1}{\kappa} - 1 \right) \left( 1 - n^{2} \right) \alpha_{1} \beta_{1} \tilde{H}_{n-1}(\alpha_{1}a) \tilde{H}_{n}(\beta_{1}a) \right\} \cos \eta_{e} e^{-i\omega t}, \]

where \( \Delta_{n} \) is the determinant of the square matrix shown in Eq. (4.10).

Examination of Eqs. (4.14) and (4.15) shows explicitly that the important parameters governing the dynamic stress concentration factors are:

\[ \kappa^{2} = \frac{2(1-\nu)}{1-2\nu} = \frac{\beta_{1}^{2}}{a_{2}^{2}}; \]

and

\[ \eta = \rho_{2}/\rho_{1}. \]

In addition, if \( a_{1} \) (the incident wave) is considered as the primary variable we may express \( a_{2} = \gamma a_{1} \), where \( \gamma = \frac{\rho_{2}}{\rho_{1}} = \frac{\omega}{c_{1}}/\frac{c_{2}}{c_{1}} \), which is the ratio of the propagation velocity in the medium to the propagation velocity in the fluid. It may be concluded that the
stress concentration factor around a fluid inclusion is dependent on the incident wave number, \( \alpha_1 \alpha \), on the Poisson ratio of the medium, and on two dimensionless parameters \( \eta \) and \( \gamma \). Numerical results have been computed for a wide range of parameters and incident wave numbers. Figures 4.2a and 4.3 show some of the behavior of the stresses typical of the interaction problem.

![Graph showing radial stress as a function of wave number](image)

**Fig. 4.2a. Radial Stress as a Function of Wave Number**  
*For \( \gamma = 3.0, \eta = 1/2.7, \nu = 0.25*  

The behavior of \( |r_0| \) at \( \theta = 0, \pi/2 \), and \( \pi \) is shown in Fig. 4.2a as a function of wave number for \( \gamma = 3, \eta = 1/2.7 \) (a ratio that is typical of water over rock) and Poisson's ratio \( \nu = 0.25 \). The dominant features in Fig. 4.2a are the well-defined multiple peaks in the range of \( \alpha_1 \alpha \) from 0 to 2. These peaks occur at \( \alpha_1 \alpha = 0.60, 1.02, 1.27, \) and 1.77, and they resemble the familiar resonant phenomenon of an elastic system. As it turns out, the occurrence of these peaks is related to the natural vibratory modes of the fluid inclusion excited.
by the refracted standing waves.

This can be demonstrated readily if we consider the solution of a fluid inclusion in a rigid medium (a limiting case of an elastic medium). If the medium is rigid, then the boundary condition for the fluid inclusion at \( r = \alpha \) is \( u_r = 0 \). The displacement for the fluid inclusion at \( r = \alpha \) is

\[
 u_r = \sum_{n=0}^{\infty} C_n \alpha \, j''(\alpha \, \alpha) \cos n\theta \, e^{-i \omega t}. \tag{4.16}
\]

Thus, the admissible nontrivial values of \( \alpha \) are determined by the roots of

\[
 j''(\alpha \, \alpha) = 0. \tag{4.17}
\]

Table 4.1 shows the lowest two roots of Eq. (4.17) for \( n = 0 \) to 4.

<table>
<thead>
<tr>
<th>( n )</th>
<th>First Root</th>
<th>Second Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3.83171</td>
</tr>
<tr>
<td>1</td>
<td>1.84118</td>
<td>5.33144</td>
</tr>
<tr>
<td>2</td>
<td>3.05424</td>
<td>6.70613</td>
</tr>
<tr>
<td>3</td>
<td>4.20119</td>
<td>8.01524</td>
</tr>
<tr>
<td>4</td>
<td>5.31755</td>
<td>9.28240</td>
</tr>
</tbody>
</table>
For \( n = 0 \), the motion is symmetric about the origin so that waves have annular "ridges and furrows." The first unsymmetric mode comes from the roots for \( n = 1 \). The one nodal diameter is at \( \theta = \pm \pi/2 \). Additional symmetric and unsymmetric modes come, respectively, from the roots for even and odd values of \( n \). Guided by the fact that the medium is much "stiffer" than the inclusion, the roots in Table 4.1 are used, as an approximation, to determine whether there is any relation between the roots of \( J'_n(a_2a) \) and the critical values of \( a_1a \) in Fig. 4.2a. Table 4.1 is in terms of \( a_2a \). So, by dividing the values in Table 4.1 by 3, one obtains the roots of \( J'_n(a_2a) = 0 \) in terms of \( a_1a \), as shown in Table 4.2.

**Table 4.2**

<table>
<thead>
<tr>
<th>( n )</th>
<th>First Root</th>
<th>Second Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.28</td>
</tr>
<tr>
<td>1</td>
<td>0.61</td>
<td>1.78</td>
</tr>
<tr>
<td>2</td>
<td>1.02</td>
<td>2.24</td>
</tr>
<tr>
<td>3</td>
<td>1.40</td>
<td>2.67</td>
</tr>
<tr>
<td>4</td>
<td>1.77</td>
<td>3.09</td>
</tr>
</tbody>
</table>

It is noted that the first peak (\( a_1a = 0.60 \)) in Fig. 4.2a occurs very close to the first unsymmetric model (\( n = 1 \)). The second peak (\( a_1a = 1.02 \)) is almost exactly the value of the first root of the \( n = 2 \) term. The third peak (\( a_1a = 1.27 \)) is near the value of the second root of \( n = 0 \), while the fourth peak at \( a_1a = 1.75 \) occurs near the second root of the \( n = 1 \) term. Note in Fig. 4.2a that the
curve for $|\sigma_{rr}^*|$ at $\theta = \pi/2$ does not have a peak at $a_1a = 0.60$. This can be explained by the $n = 1$ root, because at $\theta = \pi/2$ the cosine of the angle for $n = 1$ is zero. Hence, there will be no peak at $\theta = \pi/2$ for $a_1a = 0.60$. The relationship between the peaks and natural frequencies of the inclusion becomes much clearer in Fig. 4.2b. Figure 4.2b shows an angular plot of $|\sigma_{rr}^*|$ for the four values of $a_1a$ at which the peaks occur. It is noted that at $a_1a = 0.60$ and 1.75, the behavior of $|\sigma_{rr}^*|$ is nearly that of $|\cos \theta|$. While at $a_1a = 1.05$,

![Graph](image)

Fig. 4.2b. Radial Stress as a Function of the Angle $\theta$
for $\gamma = 3.0$, $n = 1/2.7$, $\nu = 0.25$

it is nearly that of $|\cos 2\theta|$. At $a_1a = 1.25$ it is almost constant, with $|\cos 3\theta|$ superimposed on top of it.

It must be kept in mind that for a fluid inclusion at any frequency, $|\sigma_{rr}^*|$ includes contributions from all the different modes. However, at the critical frequencies or wave numbers, the modes corresponding to those frequencies will dominate.

The behavior of $|\sigma_{rr}^*|$ for different values of $\gamma$ at $\theta = \pi/2$ is
illustrated in Fig. 4.3. It is noted that there are sudden rises in magnitude at various values of $\alpha_1$. The sharp rise at $\alpha_1 = 1.05$ in the $\gamma = 3.0$ curve corresponds to the second peak shown in Fig. 4.2a.

![Graph](image)

**Fig. 4.3. Tangential Stress as a Function of Wave Number for $\theta = \pi/2$**

The peak for the $\gamma = 2$ curve is shifted to $\alpha_1 = 1.6$, which is almost 1.5 times the value of $\alpha_1$ for the $\gamma = 3.0$ curve. There are no peaks observed for the $\gamma = 1.0$ curve for the range of $\alpha_1$ computed. The curve of $\gamma = \infty$ is also of interest. It may be interpreted as the cavity case. The behavior of $|c_{\theta \theta}^*|$ for $\gamma = \infty$ is exactly that shown in Fig. 3.7.

As a point of interest, we noted that in Fig. 4.2a the values of $|c_{\theta \theta}^*|$ seemed to converge on the same value as $\alpha_1 \to 0$. It is to be recalled that $\alpha_1 \to 0$ represents the static case which should have a uniform pressure in the fluid. It can be shown, by the limiting process given in Section 3, that Eqs. (4.14) and (4.15) can be reduced
to the static solution. They are:

\[
\sigma^*_{rr} = \frac{\lambda_2}{\lambda_2 + \nu_1};
\]

and

\[
\sigma^*_{\theta\theta} = \frac{1}{1-\nu_1} \left[ \left( 1 - \frac{2\lambda_2(1-\nu_1)}{\lambda_2 + \nu_1} \right) - (2-4\nu_1) \cos 2\theta \right].
\]

For the case of \( \gamma = 3 \) with \( \eta = 1/2.7 \), the static value of \( \sigma^*_{rr} \) is 0.11, which is the value of \( |\sigma^*_{rr}| \) in Fig. 4.2a as \( a_1 \alpha \to 0 \). A similar observation can be made in the behavior of \( |\sigma^*_{\theta\theta}| \) as \( a_1 \alpha \to 0 \). For all cases of \( \gamma \), the dynamic solution converges on the corresponding static value.

4.3. Elastic Liners of Arbitrary Thickness

As a final example on the subject of steady-state response, we shall present the problem of an elastic liner of arbitrary thickness embedded in an elastic medium. There are no new conceptual difficulties in this problem, but the results are of practical interest -- they may be used, for example, to design underground tunnel linings, or plates with inside strengthening rings.

The incident wave is again assumed to be a plane P wave. There are still two reflected waves, as in all previous cases. However, there are now four refracted waves in the liner: two inward propagating waves and two outward propagating waves. Alternatively, we might think of them as standing waves excited by the incident wave.
If we consider the refracted waves as inward and outward propagating, then the total displacement potentials in the cylinder are:

\[
\psi^{(f)} = \sum_{n=0}^{\infty} \left[ C_n \tilde{H}^{(1)}_n (\alpha_2 r) + D_n \tilde{H}^{(2)}_n (\alpha_2 r) \right] \cos n\theta \, e^{-i\omega t}; \quad (4.18)
\]

and

\[
\psi^{(f)} = \sum_{n=0}^{\infty} \left[ M_n \tilde{H}^{(1)}_n (\beta_2 r) + N_n \tilde{H}^{(2)}_n (\beta_2 r) \right] \sin n\theta \, e^{-i\omega t}; \quad (4.19)
\]

where the \( \tilde{H}^{(1)}_n \) and \( \tilde{H}^{(2)}_n \) terms in the series represent the outward and inward propagating waves, respectively. There are six sets of unknown expansion coefficients: two for the reflected waves and four for the refracted waves. To solve for these unknowns, we have four equations of continuity at \( r = b \) and two traction-free conditions at \( r = a \) (see Fig. 4.4).

![Fig. 4.4. Elastic Cylinder](image-url)
Equations of continuity at \( r = b \) are:

\[
\begin{align*}
\sigma_{rr1} &= \sigma_{rr2}, \\
\sigma_{\theta\theta1} &= \sigma_{\theta\theta2}, \\
\sigma_{r\theta1} &= \sigma_{r\theta2}.
\end{align*}
\] (4.20)

Boundary conditions at \( r = a \) are:

\[
\begin{align*}
\sigma_{rr2} &= 0, \\
\sigma_{r\theta2} &= 0.
\end{align*}
\] (4.21)

The expressions for the stresses and displacements in the medium and cylinder with \( e^{-i\omega t} \) omitted are:

**Stresses in the medium:**

\[
\begin{align*}
\sigma_{rr1} &= 2\mu_1 r^{-2} \sum_{n=0}^{\infty} \left[ \varphi_n e_1^{(1)}(a_1 r) + A_n e_1^{(3)}(a_1 r) + B_n e_1^{(3)}(b_1 r) \right] \cos n\theta; \\
\sigma_{\theta\theta1} &= 2\mu_1 r^{-2} \sum_{n=0}^{\infty} \left[ \varphi_n e_2^{(1)}(a_1 r) + A_n e_2^{(3)}(a_1 r) + B_n e_2^{(3)}(b_1 r) \right] \cos n\theta; \\
\sigma_{r\theta1} &= 2\mu_1 r^{-2} \sum_{n=0}^{\infty} \left[ \varphi_n e_4^{(1)}(a_1 r) + A_n e_4^{(3)}(a_1 r) + B_n e_4^{(3)}(b_1 r) \right] \sin n\theta.
\end{align*}
\] (4.22a, 4.22b, 4.22c)
Displacements in the medium:

\[ u_{r1} = r^{-1} \sum_{n=0}^{\infty} \left[ \varphi_n e_n^{(1)}(\alpha_1 r) + A_n e_n^{(3)}(\alpha_1 r) + B_n e_n^{(3)}(\beta_1 r) \right] \cos n\theta; \]  
\[ (4.23a) \]

\[ u_{\theta 1} = r^{-1} \sum_{n=0}^{\infty} \left[ \varphi_n e_n^{(1)}(\alpha_1 r) + A_n e_n^{(3)}(\alpha_1 r) + B_n e_n^{(3)}(\beta_1 r) \right] \sin n\theta. \]  
\[ (4.23b) \]

Stresses in the cylinder:

\[ \sigma_{rr2} = 2\mu r^{-2} \sum_{n=2}^{\infty} \left[ C_n e_n^{(3)}(\alpha_2 r) + D_n e_n^{(4)}(\alpha_2 r) + N_n e_n^{(3)}(\beta_2 r) \right] \cos n\theta; \]  
\[ (4.24a) \]

\[ \sigma_{\theta 2} = 2\mu r^{-2} \sum_{n=0}^{\infty} \left[ C_n e_n^{(3)}(\alpha_2 r) + D_n e_n^{(4)}(\alpha_2 r) + N_n e_n^{(3)}(\beta_2 r) \right] \cos n\theta; \]  
\[ (4.24b) \]

\[ \sigma_{r\theta 2} = 2\mu r^{-2} \sum_{n=0}^{\infty} \left[ C_n e_n^{(3)}(\alpha_2 r) + D_n e_n^{(4)}(\alpha_2 r) + N_n e_n^{(3)}(\beta_2 r) \right] \sin n\theta. \]  
\[ (4.24c) \]

Displacements in the cylinder:

\[ u_{r2} = r^{-1} \sum_{n=0}^{\infty} \left[ C_n e_n^{(3)}(\alpha_2 r) + D_n e_n^{(4)}(\alpha_2 r) + N_n e_n^{(3)}(\beta_2 r) \right] \cos n\theta; \]  
\[ (4.25a) \]
\[ u_{\theta 1} = \sum_{n=0}^{\infty} \left[ C_n \epsilon_{11}^{(3)}(a_2r) + D_n \epsilon_{12}^{(4)}(a_2r) + M_n \epsilon_{82}^{(3)}(b_2r) 
+ N_n \epsilon_{82}^{(4)}(b_2r) \right] \sin n\theta, \]

where \( \epsilon_{11}^{(3)} \ldots \) are all defined in the appendix.

By letting \( r = a, b \) in the appropriate equations in Eqs. (4.22) through (4.25), and using the continuity and boundary conditions in Eqs. (4.20) and (4.21), the coefficients \( A_n, B_n, \ldots, N_n \) are determined by the following matrix:

\[
\begin{bmatrix}
\tilde{\mu}_{11}^{(3)}(a_1b) & \tilde{\mu}_{12}^{(3)}(a_1b) & E_{11}^{(3)}(a_1b) & E_{12}^{(3)}(a_1b) & E_{11}^{(4)}(a_2b) & E_{12}^{(4)}(a_2b) \\
\tilde{\mu}_{41}^{(3)}(a_1b) & \tilde{\mu}_{42}^{(3)}(a_1b) & E_{41}^{(3)}(a_1b) & E_{42}^{(3)}(a_1b) & E_{41}^{(4)}(a_2b) & E_{42}^{(4)}(a_2b) \\
E_{71}^{(3)}(a_1b) & E_{72}^{(3)}(a_2b) & E_{71}^{(3)}(a_2b) & E_{72}^{(3)}(a_2b) & E_{71}^{(4)}(a_2b) & E_{72}^{(4)}(a_2b) \\
E_{81}^{(3)}(a_1b) & E_{82}^{(3)}(a_2b) & E_{81}^{(3)}(a_2b) & E_{82}^{(3)}(a_2b) & E_{81}^{(4)}(b_2a) & E_{82}^{(4)}(b_2a) \\
0 & 0 & E_{11}^{(3)}(a_2b) & E_{11}^{(3)}(a_2b) & E_{12}^{(3)}(b_2a) & E_{12}^{(3)}(b_2a) \\
0 & 0 & E_{21}^{(3)}(a_2b) & E_{21}^{(3)}(a_2b) & E_{22}^{(3)}(b_2a) & E_{22}^{(3)}(b_2a)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n \\
C_n \\
D_n \\
M_n \\
N_n
\end{bmatrix}
\]

By adding the two sides together to get the following matrix:

\[
\begin{bmatrix}
\tilde{\mu}_{11}^{(1)}(a_1b) \\
\tilde{\mu}_{11}^{(1)}(a_1b) \\
\tilde{\mu}_{41}^{(1)}(a_1b) \\
E_{71}^{(1)}(a_1b) \\
E_{81}^{(1)}(a_1b)
\end{bmatrix} = -\psi \epsilon_i \tilde{v}_n
\]

(4.26)
It is no longer feasible to expand Eq. (4.26) as we have shown before; the numerical results are obtained through machine computation. The dimensionless parameters that affect the magnitudes of the stresses and displacements are:

\[
\tilde{\mu} = \frac{\mu_1}{\mu_2}
\]

the ratio of the shear moduli of the medium and the liner

\[
\gamma = \frac{c_{p1}}{c_{p2}}
\]

the ratio of the dilatational phase velocities

\[
\kappa_{1,2}^2 = \frac{2(1-\nu_{1,2})}{1-2\nu_{1,2}}
\]

the ratio of outer radius to inner radius of the liner

\[
\eta = \frac{b}{a}
\]

Four of these parameters are dependent upon the material properties of the medium and the liner, and one (\(\eta\)) upon the wall thickness of the cylinder.

Two sets of dimensionless parameters are used in Table 4.3 to

Table 4.3

Values of Dimensionless Parameters for the Cases Shown in Figs. 4.5a Through 4.10b

<table>
<thead>
<tr>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{\mu})</td>
<td>2.90</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1.50</td>
</tr>
<tr>
<td>(\nu_1)</td>
<td>0.25</td>
</tr>
<tr>
<td>(\nu_2)</td>
<td>0.20</td>
</tr>
<tr>
<td>(\eta)</td>
<td>1.05, 1.10, 1.20</td>
</tr>
</tbody>
</table>
show the behavior of the magnitude of the dimensionless stresses (or stress concentration factors) in the medium and in the cylinder, as functions of wave numbers. The values for Case I might be considered,
typically, as a concrete liner in granitic rock, while Case II might be considered as a steel liner in granitic rock.

The behavior of $\sigma^*_\theta\theta$ as a function of wave number and angle for Case I is shown in Figs. 4.5a and b and 4.6a and b. It should be noted that there is great similarity between the behavior of $\sigma^*_\theta\theta$ in this case and in the simple cavity case. However, the magnitudes

\begin{center}
\textbf{Fig. 4.6a.} Distribution of Dimensionless Tangential Stress $|\sigma^*_\theta\theta_2|$ at the Inner Side of the Liner $\eta = 1.1$
\end{center}

\begin{center}
\textbf{Fig. 4.6b.} Real and Imaginary Dimensionless Tangential Stresses at the Inner Side of the Liner vs Wave Numbers $\theta = \pi/2$
\end{center}
of the stresses have changed. Also, the stresses in the medium tend toward lower values as \( n \) increases at \( \theta = \pi/2 \) (Fig. 4.5a). This implies that as the thickness of the liner increases, we can reduce the maximum stress concentration factor in the medium. On the other hand, for Case I, we note that the stress concentration factor in the cylinder actually increases as \( n \) increases.

The dynamic stress concentration factor exceeds the static stress concentration factor by 10% to 15%, just as in Section 3. The wave number where the maximum occurs is approximately 0.30, which is about the same as in the cavity case.

The magnitude of the dimensionless stress distribution around the inside circumference of the cylinder, and the real and imaginary parts of \( \sigma_{\theta\theta}^{*} \), are shown in Fig. 4.6. At low wave numbers (\( \alpha = 0.20 \)), the angular distribution of \( |\sigma_{\theta\theta}^{*}| \) is approximately of the form \( \alpha + b \cos 2\theta \) (\( \alpha \) and \( b \) are two constants — see Savin, Ref. 4.3, p. 240). The shifting of the maximum stress toward the incident side of the liner is apparent in Fig. 4.6a. The behavior of the real and imaginary parts of \( \sigma_{\theta\theta}^{*} \) in Fig. 4.6b follows almost exactly the pattern in Fig. 3.4.

![Diagram](image-url)  
*Fig. 4.7a. Distribution of Normalized Dimensionless Tangential Stress \( |\sigma_{\theta\theta}^{*}| \) at \( r = a \) and \( n = 1.1 \) (Case II)*
Numerical values for Case II are shown in Figs. 4.7a and b and 4.8a and b. The differences between Cases I and II are primarily in the magnitude of the stresses, rather than in their general behavior. The maximum values of $|\sigma_{\theta \theta_2}^*|$, depending on the values of $\eta$, range
from 6.2 for $\eta = 1.2$ to over 8 for $\eta = 1.05$, while in Case I they range from 1 for $\eta = 1.05$ to 1.1 for $\eta = 1.2$. Thus it seems that the stiffer the liners, the higher the stresses in the liner. Another point of interest is that increasing $\eta$ produces opposite effects in the two cases. In Case I, an increase of $\eta$ causes $\sigma^*_{\theta_2}$ to increase. The converse is true in Case II. This point should be considered in actual design. Also, it is noted that the stiff liners are much more efficient in reducing the maximum stresses in the medium, as shown in Fig. 4.8a.

Other problems of the same geometry, but with an incident plane shear wave and with a rigid inclusion inside the liner, have been solved in Refs. 4.4 and 4.5. The similarity in the behavior of the stresses which was observed between the unlined cavity and the lined cavity under incident plane P waves was observed in the same Refs.
for their corresponding unlined cavity or simple rigid inclusion cases.

4.4. Parametric Study of the Lining Problem

The dynamic stresses of an elastic liner embedded in an elastic medium have been found to be dependent upon four parameters: \( \mu, v_{1,2}, \nu, \) and \( n \), in addition to the incident wave number \( \alpha z \). For practical applications, we shall present some numerical values for the maximum dynamic stress concentration factor in the medium \( K_1 \) and for the maximum dynamic stress concentration factor in the liner \( K_2 \), as functions of these parameters. The values of \( K_1 \) and \( K_2 \) are determined by selecting the maximum values of \( |\sigma_{\theta \theta 1}^*| \) and \( |\sigma_{\theta \theta 2}^*| \) at \( r = a, b \) for a given set of parameters in a \( |\sigma_{\theta \theta 1}^*| \) versus \( n \) plot. For example, Fig. 4.5a gives three maximum values of \( |\sigma_{\theta \theta 1}^*| \) at \( \alpha_1 = 0.3 \). They are 2.82, 2.70, and 2.59, for \( n = 1.05, 1.1 \) and 1.2, respectively.

\[\text{Fig. 4.5a. Maximum Medium Dynamic Stress Concentration Factor } K_1 \text{ versus Liner Thickness Parameter } n \text{ for Various } \mu \text{ and } \nu, (\nu_1 = \nu_2 = 0.25)\]
Figures 4.9 and 4.10 show the variation of $K_1$ and $K_2$ as a function of the parameters. As illustrated in Figs. 4.9a,b, the most effective way to lessen the stress in the medium is to have a stiff liner, i.e., with $\bar{\mu}$ small. Also, an increase in the thickness of the liner will always reduce the stress in the medium. However, the rate of decrease in stress decreases as $\eta$ increases. Figure 4.10 gives the values of $K_2$ as a function of $\eta$ and $\bar{\mu}$. An interesting point in Fig. 4.10a is that an increase in $\eta$ is not always accompanied by a reduction in $K_2$. It is observed that when $\bar{\mu} > 1$, $K_2$ increases as $\eta$ increases, and the converse is true when $\bar{\mu} < 1$. At $\bar{\mu} = 1$, the value of $K_2$ remains unchanged as $\eta$ varies. In fact when $\bar{\mu} = 1$, $K_2$ has the value of the unlined cavity. The effect of the shear moduli ratio on $K_2$ is shown in Fig. 4.10b. It is shown that very large values of $K_2$ will result in a very thin and "stiff" liner embedded in an elastic medium.
Fig. 4.10a. Maximum Linear Dynamic Stress Concentration Factor $K_2$ versus Linear Thickness Parameter $n$ for Various $\bar{\nu}$ and $\gamma$ ($\nu_1 = \nu_2 = 0.25$)

Fig. 4.10b. Maximum Linear Dynamic Stress Concentration Factor $K_2$ versus Ratios of Shear Moduli $\bar{\nu}$ for Various $n$, $\gamma = 0.5$ ($\nu_1 = \nu_2 = 0.25$)
To conclude this subsection we might add that various other Poisson ratios and \( \gamma \) were used in Ref. 4.6, but the most important parameters in effecting the values of \( K_1 \) and \( K_2 \) are those shown here in Figs. 4.9 and 4.10.

5. TRANSIENT RESPONSE OF A CYLINDRICAL CAVITY

5.1. Incident Compressional Wave

The response of various types of cylindrical inclusions subjected to periodic excitations were examined in detail in the previous sections. In many practical problems, however, the incident disturbances are more often than not aperiodic. Therefore, we shall consider the transient behavior of a cylindrical cavity when it is subjected to an aperiodic disturbance.

Consider first a unit step P wave propagating in the positive \( z \) direction. In accordance with the plane strain theory the stress field can be represented as

\[
\sigma_{xx} = H(\hat{t}),
\]

\[
\sigma_{yy} = \frac{\nu}{1-\nu} H(\hat{t}),
\]

where \( \hat{t} = t - (x + a/c_P) \), and \( t \) is zero when the wave has just arrived at \( x = -a \). The \( x,y \) coordinate system used in Eq. (5.1) is that shown in Fig. 4.1 — i.e., the origin of the \( x-y \) coordinates coincides with the center of the cylinder.

There are several methods available for solving the transient
problem. The Laplace transform method was used in Chapter II, Section 3, to solve the problem of a cavity subjected to an SH wave. However, we shall now use the Fourier synthesis technique as outlined in Chapter I, Section 4.

It was stated in Chapter I, Section 4, that the transient response of a linear system \( g(x, t) \), due to an input \( f(t) \), is determined by:

\[
g(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x, \omega) F(\omega) e^{-i\omega t} d\omega, \tag{5.2}
\]

where \( \chi(x, \omega) \) is defined in that Section as the admittance function of the system, \( F(\omega) \) is the Fourier transform of the function \( f(t) \), and \( \hat{t} \) is the local time taken to be zero at the wave front. If we are interested in finding the transient stress behavior at the boundary of the cavity, then Eq. (3.14) is precisely the \( \chi(x, \omega) \) that we need for the problem. To proceed with the problem, we first take the Fourier transform of \( f(\hat{t}) \) and substitute it together with Eq. (3.14) into Eq. (5.2). We then have the formal expression for the transient behavior of \( \sigma_{\theta\theta} \) at the boundary of the cavity.

The Fourier transform of the unit step input along the boundary of the cavity is:

\[
F(\zeta) = \frac{i}{\sqrt{2\pi}} \left( \frac{\sigma_p}{a^2} \right), \quad \text{Im} \; \zeta > 0,
\]

where

\[
\text{Re} \; \zeta = \frac{\omega a}{c_p} = \alpha a, \quad \text{and} \; \zeta = \xi + i\eta).
\]
The dimensionless stress \( \tilde{\sigma}_{\theta\theta}(t) \) due to the unit step input is determined by using Eq. (3.14), and by assuming that the summation and integration are interchangeable, as:

\[
\tilde{\sigma}_{\theta\theta}(t) = \frac{2\xi}{n^2} \left( \frac{\kappa^2 - 1}{\kappa^2} \right)^n \sum_{n=0}^{\infty} \varepsilon_n^{1+n+1} \int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{S_n e^{-i(t\zeta)}}{\zeta} \, d\zeta \cos n\theta, \quad (5.3)
\]

where \( \tilde{t} \) denotes the dimensionless time, \( \tilde{t} = \frac{t}{\alpha} \).

In this representation the stress \( \tilde{\sigma}_{\theta\theta}(\tilde{t}) \) as shown in Eq. (5.3) can be viewed as a superposition of the stresses due to the response of each mode \( \kappa_n \).

It is sufficient to say then that the integral for each \( \kappa \) yields the time history of the stress associated with that particular mode \( \kappa \).

Let us now examine the integral in Eq. (5.3). Here again we shall replace the integral along \( -\infty+i\gamma + \infty+i\gamma \) by an appropriate contour integral. The choice of an appropriate contour \( C \) is dictated by the singularities of the integrand. The singularities in this case are a simple pole at the origin due to the incident wave, a branch point at the origin due to the logarithmic singularity in the Hankel function, and simple poles at the roots of the following transcendental equation:

\[
\xi H_{n-1}(\zeta)[(\kappa^2 - 1)\kappa H_{\kappa-1}(\kappa\xi) - (\kappa^3 - \kappa + \frac{1}{2}\kappa^2 \xi^2)H_{\kappa}(\kappa\xi)]
\]

\[\quad - H_{\kappa}(\zeta)[(\kappa^3 - \kappa + \frac{1}{2}\kappa^2 \xi^2)\kappa H_{\kappa-1}(\kappa\xi) - (\kappa^2 + \kappa - \frac{1}{2}\kappa^2 \xi^2)\kappa^2 \xi^2 H_{\kappa}(\kappa\xi)] = 0, \quad (5.4)
\]

which is the denominator of \( S_n(\zeta) \) (see Eq. 3.14).
A convenient contour $C$ is shown in Fig. 5.1 for $\tau > 0$.

The contour integration gives

$$\int_{C} \frac{S_n(\xi)}{\xi} e^{-i\tau \xi} d\xi = 2\pi i \sum_{k=1}^{m} R_{n,k}.$$  \((5.5)\)

where $R_{n,k}$ denotes the residues. It follows

$$\int_{-R+iY}^{R+iY} \frac{S_n(\xi)}{\xi} e^{-i\tau \xi} d\xi = \int_{C} \frac{S_n(\xi)}{\xi} e^{-i\tau \xi} d\xi + 2\pi i \sum_{k=1}^{m} R_{n,k}.$$  \((5.6)\)

In the limit as $R \to \infty$, the limit of the integral on the left-hand side of Eq. (5.6) is the integral desired in Eq. (5.3).

Let us now examine the right-hand side of Eq. (5.6). It is noted that in the third and fourth quadrants, the imaginary part of $\xi$ is always negative. Thus, when $\tau > 0$, the integrand vanishes as $R \to \infty$.

Then according to Jordan's lemma, the integrals along $AB$ and $DE$ vanish.
also.

The term $R_{n,k}$ is the residue of the $n^{th}$ mode and $k^{th}$ pole in $S_n(\xi)$. For the case of a simple pole, the values of the residue are determined by expressing:

$$
\frac{S_n(\xi)}{\xi} e^{-i\xi \tau} = \frac{K_n(\xi)}{\xi D_n(\xi)} e^{-i\xi \tau}.
$$

(5.7)

It follows

$$
R_{n,k} = \frac{K_n(\xi_n,k)}{D_n(\xi_n,k)} e^{-i\xi_n,k}. \tag{5.8}
$$

where $\xi_n,k$ denotes the location of the pole.

The other parts of the RHS of Eq. (5.6) that remain to be examined are two branch integrals and the integral around the origin.

The integral around the origin yields the residue due to the simple pole at the origin, which yields the "long time" solution due to a unit step input. To illustrate we shall make use of the limiting analysis that was presented in Section 3. It was shown in Eq. (3.19) that as $n\tau \rightarrow 0$, the steady-state solution is reduced to the static solution shown in Eq. (3.20). From the theorem of residue, the residue from the simple pole at the origin is given as:

$$
R_n = \lim_{\tau \rightarrow 0} S_n(\tau) e^{-i\xi \tau}. \tag{5.9}
$$

To determine the value of the residue, we can now use the same limiting process as we have used in Section 3 to determine the value
of \( S_n(\zeta) \) as \( \zeta \to 0 \). Since the expression for \( \lim S_n(\zeta) \) reduces to the corresponding terms in Kirsch's static solution, it might be expected that the residue of the pole at the origin gives the long time solution for \( \tilde{\sigma}_{\theta\theta} \). In fact, by using Eqs. (3.19) and (5.3) we find the following expression for the stress due to the residue at the origin:

\[
\tilde{\sigma}_{\theta\theta} = \frac{2}{\pi^2} \left[ (\pi^2 - 1) - 2 \cos 2\theta \right] H(\hat{\eta}),
\]

where \( H \) is the unit step function. This is exactly the static solution one would anticipate after the initial response has died down.

The branch integrals remain to be examined. Along the branch \( B-C \) the argument of \( \zeta \) is \( -\pi/2 \), and the argument of \( \zeta \) along \( D-E \) is \( 3\pi/2 \). Thus \( \zeta \) is pure imaginary on either branch. In addition, since \( \eta_n^{(1)}(z) \) is defined only in the range of \( -\pi < \arg \zeta \leq \pi \), an analytic continuation of \( \eta_n^{(1)}(\zeta) \) for \( \arg \zeta = 3\pi/2 \) is required.

Let us express \( \zeta = \rho e^{-\pi i/2} \) and \( \zeta = \rho e^{3\pi i/2} \) as follows:

\[
\begin{align*}
\rho e^{-\pi i/2} &= \rho e^{\pi i/2} e^{-\pi i}, \\
\rho e^{3\pi i/2} &= \rho e^{\pi i/2} e^{3\pi i}.
\end{align*}
\]

A formula of analytic continuation of \( \eta_n^{(1)}(z) \) which will enable us to determine the values along the two branches is:

\[
\begin{aligned}
\eta_n^{(1)}(ze^{-im\pi}) &= -(m-1)(-1)^{m-1} \eta_n^{(1)}(z) - e^{im\pi(m-1)}(m+1)\eta_n^{(2)}(z).
\end{aligned}
\]

Now, using Eqs. (5.11) and (5.12) we find:
arg $z = - \pi i/2,$

$$H_n^{(1)}(pe^\pi/2, e^{-\pi i/2}) = 2(-1)^n H_n^{(1)}(pe^{\pi i/2}) + (-1)^n H_n^{(2)}(pe^{\pi i/2})$$
$$= (-1)^n [H_n^{(1)}(pe^{\pi i/2}) + 2J_n(pe^{\pi i/2})].$$

(5.13)

Similarly, when $\arg \zeta = 3\pi/2,$ $H_n^{(1)}(z)$ is

$$H_n^{(1)}(pe^{3\pi i/2}, e^{\pi i}) = (-1)^n [H_n^{(1)}(pe^{\pi i/2}) - 2J_n(pe^{\pi i/2})].$$

(5.14)

It is to be noted that the arguments of the Hankel and Bessel functions in Eqs. (5.12) through (5.14) are pure imaginary numbers. If, however, we desire to use functions of real arguments, then we may use the relationship shown in Eqs. (2.25a) and (2.28b), and obtain the following expression for $H_n^{(1)}(z)$ at $\arg z = -(\pi/2), (3\pi/2)$ in terms of modified Bessel functions of first and second kind:

arg $z = - \pi i/2,$

$$H_n^{(1)}(pe^{-\pi i/2}) = -\frac{2(i)^{n+1}}{\pi} [K_n(\rho) + i\pi(-1)^n J_n(\rho)];$$

(5.15)

arg $z = 3\pi/2,$

$$H_n^{(1)}(pe^{3\pi i/2}) = -\frac{2(i)^{n+1}}{\pi} [K_n(\rho) - i\pi(-1)^n J_n(\rho)];$$

(5.16)

where $\rho > 0$.

With the proper expression for $H_n^{(1)}(z)$ along the two branches determined, the branch integrals along $BC$ and $DE$ can now be evaluated.
Without going into any further details of the lengthy mathematical substitution, we may summarize the contribution made by the three parts of the inversion integrals as follows:

(1) The "long time" solution is determined by the residue of the pole at the origin, shown in Eq. (5.10);

(2) The residues of the simple poles located by the roots of the transcendental equation shown in Eq. (5.4) represent free oscillations of the cavity as excited by the incident step pulse, as in Eq. (5.8);

(3) The branch integrals contribute to the "early time" response. Examination of the integrand shows that along the paths BC and DE the exponential factor in $\exp \left(-\frac{\tau}{\varepsilon}\right)$ for $\tau > 0$ is always negative. This implies then that the greatest contribution by the branch integrals is when $\tau$ is near zero.

Numerical calculations in this problem can be described as tedious at best. They are done primarily on large computers. First of all, the roots of Eq. (5.4) have to be determined so that the residues can be determined, and secondly, the branch integral must also be numerically integrated.

Shown in Fig. 5.2 are the locations of the roots of Eq. (5.4) in a complex plane for $\nu = 0 \ldots 5$ for $\nu = 0.25$. It appears that the number of poles associated with each order $\nu$ are a function of $n$. Furthermore, they always occur in pairs, located in the third and fourth quadrants of the complex plane. The number of poles and their locations for each order $n$ for $|\zeta| < 5$ are shown in Table 5.1.
From the table it is noted that the number of pairs of poles for each \( n \) are as follows: \( n = 0,1 \), 1 pair of poles; \( n = 2,3 \), 3 pairs of poles; and \( n = 4,5 \), 5 pairs of poles. The physical interpretation for the increase in number of pairs of poles is, of course, that
there are more nodes associated with higher modes than the lower modes.

The significance of the location \((\zeta_{n,k})\) of the pole — see Eq. (5.8) — is that the real part represents the frequency of the free oscillation, while the imaginary part represents the radiation damping. The node of a particular mode with high imaginary values will be damped out quickly, and those \((\zeta_{n,k})\) with high real value represent high frequencies.

If one is interested in the long time solution then one needs to study the contribution made by residues of the poles closest to the origin (since they represent low frequency and low damping). On the other hand, if one is interested in the early time history, then one must scrutinize the residues associated with the poles located away from the imaginary axis, since they represent the high-frequency response of the system. Unfortunately, computation for roots of higher order \(n\) and large value \(\hat{\tau}\) becomes extremely time-consuming as well as inaccurate, even with present-day large computers. The values of the roots shown in Table 5.1 are believed to be more complete and accurate than those given in Refs. 0.9, 5.1, 5.2, and 5.3.

In order to illustrate some of the preceding discussions, we have depicted in Figs. 5.3 and 5.4 the contribution made by the branch integrals to the stresses, and the stresses due to the different modes. Shown in Fig. 5.3 are results of branch integrals for \(n = 0 \ldots 5\) modes. The greatest contributions made by the branch integrals are those from the \(n = 0, 2\) mode at \(\hat{\tau} = 0\). The value decreases as \(\hat{\tau}\) increases as noted before. Figure 5.4 illustrates the stress at \(\theta = \pi/2\) due to various modes (both residue and branch inte-
Fig. 5.3. Contribution to $\hat{\sigma}_{2\theta}$ at $\theta = 0$ From Branch Integral of $n = 0, 1, 2, \ldots, 5$

Fig. 5.4. Contribution to $\hat{\sigma}_{zz}$ at $\theta = \pi/2$ From Various Modes and the Pole at Origin

gral are included), and due to the pole at the origin. We note that the dominant terms are those from the $n = 0, 2$ modes and that due to the pole at the origin. It may be of interest here to note also that the solution due to the pole at origin consists of two terms, i.e., $n = 0$ and $n = 2$ as shown in Eq. (5.10). Thus it appears that the dominant modes are $n = 0, 2$ both for the static loading case and for the dynamic loading case.

Shown in Fig. 5.5 are the stresses at three angular positions on the boundary of the cavity as a function of $\tau$. The first arrival of the wave at the cavity is at $\theta = \pi$ for $\tau = 0$. The time of arrival at $\theta = \pi/2$ is $\tau = 1$. The determination of time of arrival on the
Fig. 5.5. Transient Behavior of $\tilde{c}_{\theta\theta}$ ($\tilde{c}_{\theta\theta}$ vs $\tau$)

at $\theta = 0$, $\pi/2$, $\pi$

illuminating side (incident side) of the cavity is clear; the same, however, cannot be said for points on the shadow side of the cavity, as in Fig. 5.6. Points in the shadow zone, according to Fermat’s principle, can receive signals via waves traveling along the surface, $r = a$. Therefore, the earliest signal that can arrive at $\theta = 0$ is $\tau = 1 + \pi/2$, which is the time required for a P wave to travel a distance equal to a radius and a quarter of the circumference of the cavity. (For detailed discussions of arrival time of other types of incident waves see Ref. 5.4, and the book by Freidelander, Ref. 5.5.) Here we have used geometrical reasoning to ex-
plain the various arrival times because it is known that the solution as obtained in this section is inadequate to provide detailed information near the wave front. This is due to truncation of the series and to the limit range of argument used in the calculation. Methods similar to geometrical optics are available (Refs. 5.5 and 5.6) to determine the early time history; however, these are found to be inadequate for predicting the late time response (Ref. 5.7). Insofar as maximum dynamic stress concentration for a cavity is concerned, it is a rather late time phenomenon — i.e., the maximum stresses occur at 3 or 4 transit times after onset of the wave. For this reason, almost all literature that deals with the determination of maximum dynamic stress concentration has used one form or other of
the series expansion or integral transform methods. See Refs. 5.8, 5.9, and 5.10.

The value of the maximum dynamic stress concentration factor, for the case computed \((\lambda = \mu, \nu = 1/4)\), is, at \(\theta = \pi/2\), 2.94. It occurs at \(\tau \approx 6\), or 3 transit times. Comparing this with the static value of 2.67 it is noted there is an overshoot of 10%. It is of interest to note here that in both the transient and steady state cases we found the maximum dynamic stress concentration factors 10% above their corresponding static values.

Figure 5.5 also provides the information that there appears to be some tension, albeit small, at \(\theta = 0, \pi\) for \(\tau > 7\). The appearance of tension at these locations, as well as some fairly large compressive stress at \(\theta = 0\), is different from the static solution and may warrant some consideration in design practice.

We have illustrated an example of transient elastic P wave scattering and the dynamic stress concentration factor around a simple cavity. In what follows we shall present some additional results from recent literature on the incident shear wave, as well as on Rayleigh-type wave effects.

5.2. Incident Shear Wave

The method outline for the P wave case can be applied to the unit step incident shear wave case also. The only difference between this case and the P wave lies in the incident stress field, and in the admittance function in the Fourier integral. The incident shear wave with unit step in stress can be represented by
\[ \sigma_{\theta\theta}^{(\hat{t})} = H \left( t - \frac{x + a}{c_g} \right). \]  \hspace{1cm} (5.17)

Following the same procedure as in the P wave case, but using Eq. (3.48) as the admittance function, we obtain the following expression for the transient hoop stress around the cavity:

\[ \tilde{\sigma}_{\theta\theta}(\hat{t}) = \frac{\Delta}{\pi^2} \left( \frac{n^2 - 1}{n^2} \right) \]

\[ \times \sum_{n=1}^{\infty} i^{n+1} \int_{-\infty}^{\infty} \frac{n \left( n^2 - 1 - \frac{n^2 - 2}{2} \right) \Theta_n(\xi)}{\zeta \Delta_n(\xi)} e^{-i\xi \hat{t}} d\xi \sin n\theta, \]  \hspace{1cm} (5.18)

where \( \Delta_n(\xi) \) is given in Eq. (5.4).

The singularities of the integrand of Eq. (5.18) are precisely those described for the P wave case, therefore the path of integration of Eq. (5.18) will be the same as in the P wave case in Fig. 5.1.

Shown in Fig. 5.7 is the hoop stress \( \tilde{\sigma}_{\theta\theta}(\hat{t}) \) obtained in Ref. 5.1 at \( \theta = \pi/4 \) from the incident side. We note here also that the dynamic

![Fig. 5.7. Dynamic Stress Concentration Factor as Given by Ref. 5.1 at \( \theta = 3\pi/4 \) for Incident Shear Wave](image)
stress concentration factor is somewhat higher in the same problems than their corresponding static value.

5.3. Effects of Rayleigh Wave

Dynamic stress concentration factors determined in the preceding pages do not show the effects of Rayleigh-type surface wave propagation around the circumference of the cavity. The existence of the Rayleigh-type surface wave near the surface of either a convex or concave cylindrical surface was postulated by Viktorov in 1957 (Ref. 5.11). The effects of the surface wave on dynamic stress concentration was not studied, however, until 1964 (Miklowitz, Ref. 5.12).

The formal solution to the Rayleigh wave effects was based on the technique given by Freidlander (see Ref. II-3.2 and Chapter II, Section 3). In this technique one first represents function $\phi$ as

$$\phi(r, \theta, t) = \sum_{n=-\infty}^{\infty} \varphi(r, \theta + 2n\pi, t).$$

This solution gives $\phi$, not only in the physical plane $r \geq a$, $-\pi \leq \theta \leq \pi$, but also on a Riemann surface, the sheets of which are given by

$(2n - 1)\pi < \theta < (2n + 1)\pi$, and $n = 0, \pm 1, \pm 2, \ldots$  
The physical significance in the wave sum representation, as mentioned in Chapter II, Section 3, is that the effects at any given $\theta$ can be considered as the sum of $\varphi$ that have overlapped each other at given $\theta$ as $\varphi$ winds around the cavity. In other words, $n = 0$ will imply the first time around, and $n = 1$ that the waves have traversed once around already.
and are on the second round, etc. The number of \( n \) which one must
sum over is thus related to the time, which after the onset of the
wave can be determined by geometric reasoning.

Next, both the bilateral Laplace transform with respect to time
and the Fourier transform with respect to \( \theta \) were applied to \( \varphi \). They are

\[
\tilde{\varphi} = \int_{-\infty}^{\infty} \varphi(r, \theta, t) e^{-pt} \, dt, \quad \varphi = \frac{1}{2\pi} \int_{B^2} \varphi(r, \theta, p) e^{pt} \, d\varphi,
\]

(5.19)

\[
\tilde{\varphi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(r, \theta, t) e^{i\nu \theta} \, d\theta, \quad \varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(r, \nu, t) e^{-i\nu \theta} \, d\nu.
\]

Application of the transforms to the wave equation as well as to
the incident wave of the form \( \psi^{(i)} = \varphi_0 p(t - \frac{r}{a}) \) give the following
expressions for the incident and scattered waves:

\[
\varphi^{(i)} = 2\pi \varphi_0(p) \int_{|\nu|} (k_p r);
\]

(5.20)

\[
\varphi^{(s)} = A_\nu(p, \nu) K_\nu(k_p r);
\]

(5.21)

\[
\varphi^{(s)} = B_\nu(p, \nu) K_\nu(k_g r);
\]

(5.22)

where \( k_p = p/\sigma_p \), and \( k_g = p/\sigma_g \).

The coefficients \( A_\nu \) and \( B_\nu \) are determined by substituting Eqs.
(5.20) and (5.21) in the stress displacement potential relationship
and the stress free condition at \( r = a \). This is accomplished in a
manner similar to that in Section 2 — hence
\[ A_v(p, v) = \frac{2\pi \varphi_\varphi(p)}{\Delta(v, p)} \left\{ \left[ (2v^2 + k_g^2\alpha^2)I_v(k_p\alpha) - 2k_p\alpha I'_v(k_p\alpha) \right] \right. \]
\[ \times \left[ (2v^2 + k_g^2\alpha^2)K_v(k_g\alpha) - 2k_g\alpha K'_v(k_g\alpha) \right] \]
\[ - 4v^2[K_v(k_p\alpha) - k_p\alpha I'_v(k_p\alpha)][K_v(k_g\alpha) - k_g\alpha K'_v(k_g\alpha)] \varepsilon \right\}; \]

\[ B_v(p, v) = \frac{2\pi \varphi_\varphi(p) (2i\nu)(2v^2 + k_g^2\alpha^2 - 2)}{\Delta(v, p)} ; \]  

(5.24)

and

\[ \Delta(v, p) = - \left[ (2v^2 + k_g^2\alpha^2)K_v(k_g\alpha) - 2k_g\alpha K'_v(k_g\alpha) \right] \]
\[ \times \left[ (2v^2 + k_g^2\alpha^2)K_v(k_g\alpha) - 2k_g\alpha K'_v(k_g\alpha) \right] \]
\[ + 4v^2[K_v(k_p\alpha) - k_p\alpha I'_v(k_p\alpha)][K_v(k_g\alpha) - k_g\alpha K'_v(k_g\alpha)] \varepsilon . \]

(5.25)

The transformed stress \( \ddot{\sigma}_{\theta \theta} \) at \( r = \alpha \) is obtained by superposing the stresses due to incident and scattered waves.

\[ \ddot{\sigma}_{\theta \theta} = \frac{\mu}{a^2} \left\{ 2\pi \varphi_\varphi(p) [(k_g^2\alpha^2 - 2v^2 - k_g^2\alpha^2)I_v(k_p\alpha) + 2k_p\alpha I'_v(k_p\alpha)] \right. \]
\[ + A_v(v, p)(k_g^2\alpha^2 - 2v^2 - k_g^2\alpha^2)K_v(k_p\alpha) + 2k_p\alpha K'_v(k_p\alpha) \]
\[ - 2i\nu B(v, p)[K_v(k_g\alpha) - k_g\alpha K'_v(k_g\alpha)] \varepsilon \right\} . \]  

(5.26)

The stress is obtained through the inverse transform, i.e.,
\[ c_{\theta \theta}(r, \theta, t) = \frac{1}{4\pi i} \int_{\Gamma} e^{\nu \zeta} \left[ \sum_{n=-\infty}^{\infty} \zeta n \phi_{\nu \phi}^n \right] d\zeta. \]  

(5.27)

The double inversion integral was investigated in detail by Peck (Ref. 5.13) and in the work by Miklowitz mentioned earlier. The inversion with respect to \( \nu \) was found by the residue theorem, as the only singularities in \( \phi_{\nu \phi}(\nu, p) \) as a function of \( \nu \) are zeros of \( \Delta(\nu, p) \). It was through the examination for the zeros of \( \Delta(\nu, p) \) that the following results were found. First by letting \( p = -i\omega \), and using the relationship

\[ K_{\nu}(-i\omega) = \frac{1}{\pi i} e^{\frac{i\nu\pi}{2}} \frac{\nu i}{\nu} H_{\nu}^{(1)}(\omega), \]

the zeros of \( \Delta(\nu, p) \) are given by the following transcendental equation:

\[
\Delta(\nu, -i\omega) = \frac{1}{2} a^2 b^2 \left( \frac{\nu}{2} \right)^2 e^{\nu \pi i} \left\{ H_{\nu+2}(\alpha \omega)H_{\nu-2}(\beta \omega) + H_{\nu-2}(\alpha \omega)H_{\nu+2}(\beta \omega) \right. \\
- \left( \frac{\beta^2}{\alpha^2} - 1 \right) H_{\nu}(\alpha \omega) \left[ H_{\nu+2}(\beta \omega) + H_{\nu-2}(\beta \omega) \right] \right\} = 0, \quad (5.28)
\]

where \( a = \omega/c_p \) and \( \beta = \omega/c_s \). Equation (5.28) is the frequency equation first obtained by Viktorov for a surface wave propagating on a concave cylindrical surface.

In particular, when \( |\nu| \) is large, and with vanishingly small imaginary part and large \( \omega \), there was a limiting root corresponding to the Rayleigh surface wave. It was shown further in Ref. 5.14 that when \( \alpha \omega \gg 1 \), the roots of Eq. (5.28) can be split into three parts:
A Dilatational Part:

\[ \nu_{\alpha_j}(\alpha a) = \alpha a + (2)^{-1/3}e^{-2\pi i/3} a_j(\alpha a)^{1/3} + O(\alpha a), \quad j = 1, 2, 3, \ldots; \]

A Shear Part:

\[ \nu_{\beta_j}(\beta a) = \beta a + (2)^{-1/3}e^{-2\pi i/3} a_j(\beta a)^{1/3} + O(\beta a), \quad j = 1, 2, 3, \ldots; \]

and A Single Root:

\[ \nu_R(\alpha a) = \frac{\alpha}{c_0} a, \]

where \( a_j \) are the roots of the Airy function \( A_{\nu} \), and \( c_0 \) is the Rayleigh wave velocity for a plane surface. When \( 0 < \alpha \ll 1 \), the roots split into two groups,

\[ \nu_\alpha \sim -1, \]

\[ \nu_\beta \sim \nu_R \sim 1. \]

For Rayleigh wave effects, when \( \theta + 2m \) is taken to be large, the important effect was shown to be contributed primarily by the roots associated with the Rayleigh wave. Numerical results obtained by

---

*The Airy function was used for Hankel function approximation, i.e., \( |\nu| \to = |\arg \nu| < \pi/2 \),

\[ H_{\nu}(\nu \pm) \sim 2e^{-(1/3)\pi i} \nu^{-1/3} \left( \frac{4\nu}{1 - z^2} \right)^{2/3} A_{\nu}(\nu^{2/3} e^{(2/3)\pi i} \xi), \]

where \( \frac{2}{3} \xi^{3/2} = \ln \frac{1 + \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} - \sqrt{1 - \xi^2} \).
Miklowitz — Fig. 5.7 — show the behavior of dimensionless $\sigma_{\theta\theta}/\sigma_1$, contributed by the Rayleigh wave, due to a step input, $E(t)$. Poisson's ratio of the medium is 0.25.

The results shown in Fig. 5.7 indicate there is a tensile stress with an amplitude of $\approx 0.4$ associated with the Rayleigh surface wave which propagates around the cavity at large $\tau = ta/\sigma_p$. The number of circuits which the circumferential wave has traversed in Fig. 5.8 is $n = 6$, which implies it is valid after approximately $\tau \approx 38$.

![Graph showing the relationship between $\sigma_{\theta\theta}/\sigma_1$ and $\theta$ with annotations for $\theta + (2n-1)\pi/2$ and $\theta + [(2n+1)-\frac{1}{2}]\pi$]

It may be of interest here to compare the Rayleigh wave contribution to the solution obtained earlier in Fig. 5.5. We note in Fig. 5.5 that the dynamic solution approaches Kirsch's static solution after $\tau \approx 10$. Thus, in terms of the maximum dynamic stress concentration factor, the Rayleigh wave contribution is comparatively small.
However, as pointed out by Miklowitz, because of the tensile nature of
the stress associated with the Rayleigh wave, and the fact that for
v = 0.25 Kirsch's solution yields zero stresses at \( \theta = 0, \pi \), the exis-
tence of this tensile stress would be important, especially at \( \theta = 0, \pi \)
where the magnitude is \( \approx 0.8 \) due to positive and negative \( \theta \) propagating
waves. This method has also been effectively applied to evaluate par-
ticle velocities in the shadow region (5.13) and sound scattering by
cylinders (5.14).

6. SCATTERING OF FLEXURAL WAVES

In Chapter I, subsection 3.4, the classical theory of plate motion in
flexure is discussed. Equation (I-3.31) governs the transverse dis-
placement \( \omega \) of a plate in bending,

\[
D \nabla^2 \omega^2 + 2b \rho \frac{\partial^2 \omega}{\partial t^2} = q,
\]

(6.1)

whereas the bending moments \( M_{i,j} \) and shear forces \( Q_i \) can be calculated
from Eqs. (I-3.29) and (I-3.30). For \( q = 0 \) and harmonic wave motion,
solutions of (6.1) are given by

\[
\omega(x,y,t) = [\hat{W}_1(x,y) + \hat{W}_2(x,y)]e^{-i\omega t}.
\]

where \( \hat{W}_1, \hat{W}_2 \) satisfy the Helmholtz and Modified Helmholtz equations
respectively,

\[
(\gamma^2 + \gamma^2)\hat{W}_1 = 0, \quad \gamma^2 = \omega \sqrt{\frac{2b}{D}},
\]

\[
(\gamma^2 - \gamma^2)\hat{W}_2 = 0.
\]
$W_1$ represents the part of the flexural wave that travels with the speed $c_f = \gamma (D/2b_0)^{1/2}$ and $W_2$ represents the part attenuating as it progresses.

When harmonic forces are applied perpendicularly to a thin plate, both parts of the flexural wave are excited. If on the other hand the scatterer is at a large distance from the load, the attenuation will reduce the $W_2$ to a negligible amount and only $W_1$ needs to be considered as the incident flexural wave. For a plate with a circular inclusion, the incident wave along the $x$-axis is represented by Fig. 6.1.

![Fig. 6.1. Geometry of Plate Subjected to Bending](image)

$$w(i) = W_1 = w_0 e^{i(\gamma x - \omega t)}$$

$$= w_0 \sum_{n=0}^{\infty} \epsilon_n e^{i n J_n(\gamma r)} \cos n \delta e^{-i \omega t}.$$  \hspace{1cm} (6.2)

The corresponding moments and shears in Cartesian coordinates are

$$M_{xx} = w_0 DY^2 e^{i(\gamma x - \omega t)},$$

$$M_{yy} = w_0 D^2 e^{i(\gamma y - \omega t)},$$

$$M_{xy} = 0.$$  \hspace{1cm} (6.3)
\[ Q_x = \omega_0 D \gamma^3 e^{i(\gamma x - \omega t + \pi/2)}. \]

\[ Q_y = 0. \]

In the limit of zero frequency \((\gamma \to 0)\), the excitation is equivalent to the application of moments \(M_{max} = M_0\) and \(M_{max} = \omega M_0\) at the four sides of a plate, where \(\omega_0\) is so chosen that the product \(M_0^2 = \omega_0 D\gamma^2\) remains finite.

Since in polar coordinates, solutions for \(w_1\) are of the form \(H_n^{(1)\gamma} + i\gamma e^{i\pi\theta} \), and those for \(w_2\) are \(i\gamma e^{i\pi\theta} \), we take the following expression to represent the flexural waves scattered by a circular inclusion:

\[
\omega(s) = \omega_1(s) + \omega_2(s) = \omega_0 \sum_{n=0}^{\infty} \epsilon_n i^n [A_n H_n^{(1)\gamma} + B_n - H_n^{(1)\gamma}] \cos n\theta e^{-i\omega t}.
\]

The unknown constants \(A_n\) and \(B_n\) are determined from the boundary conditions at the interface of the inclusion. The total wave in the plate is then given, with the omission of the time factor, by

\[
\omega = \omega(i) + \omega(s) = \omega_0 \sum_{n=0}^{\infty} \epsilon_n i^n [J_n^{(1)\gamma} + A_n H_n^{(1)\gamma} + B_n - H_n^{(1)\gamma}] \cos n\theta. \tag{6.4}
\]

Inside an elastic inclusion, the refracted waves are standing waves which can be represented by
$$\omega_1 = - \omega_0 \sum_{n=0}^{\infty} e_n e^n [c_n J_n (\gamma_1 r) + D_n J_n (\delta \gamma_1 r)] \cos n\theta,$$

(6.5)

where the subscript 1 signifies the corresponding physical quantities pertaining to the inclusion, e.g. $\gamma_1 = (2\omega^2 \rho_{11}/D_1) \hat{\delta}$. The unknown coefficients are again determined by the conditions at the interface.

In polar coordinates, the bending moments and shears are given by

$$M_{rr} = - D [\nu^2 \omega + (1-\nu) \frac{\partial^2 \omega}{\partial \theta^2}],$$

$$M_{\theta\theta} = - D [\nu^2 \omega - (1-\nu) \frac{\partial^2 \omega}{\partial r^2}],$$

$$M_{r\theta} = D (1-\nu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \omega}{\partial \theta} \right),$$

$$Q_r = - D \frac{\partial}{\partial r} (\nu^2 \omega),$$

$$Q_\theta = - D \frac{\partial}{\partial \theta} (\nu^2 \omega),$$

$$V_r = Q_r - \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta},$$

$$v^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$
\[ M_{rr} = -D\omega r^{-2} \sum_{n=0}^{\infty} \varepsilon_n i^n [M_n(\gamma r) + A_n \bar{M}_n(\gamma r) + B_n \bar{M}_n(i\gamma r)] \cos n\phi; \]

\[ M_{\theta\theta} = -D\omega r^{-2} \sum_{n=0}^{\infty} \varepsilon_n i^n [N_n(\gamma r) + A_n \bar{M}_n(\gamma r) + B_n \bar{M}_n(i\gamma r)] \cos n\phi, \quad (6.7) \]

\[ V_r = -D\omega r^{-3} \sum_{n=0}^{\infty} \varepsilon_n i^n [L_n(\gamma r) + A_n \bar{M}_n(\gamma r) + B_n \bar{M}_n(i\gamma r)] \cos n\phi, \]

where

\[ \Sigma_n(z) = -(z^2 + (1-\nu)n^2)zH_n^{(1)}(z) + (1-\nu)n^2H_n^{(1)}(z), \]

\[ \bar{M}_n(z) = -(1-\nu)zH_n^{(1)}(z) + [(1-\nu)n^2 - z^2]H_n^{(1)}(z), \]

\[ \bar{\eta}_n(z) = (1-\nu)zH_n^{(1)}(z) - [(1-\nu)n^2 + \nu z^2]H_n^{(1)}(z), \]

and

\[ L_n(z) = \text{Re}[\Sigma_n(z)], \]

\[ M_n(z) = \text{Re}[\bar{M}_n(z)], \]

\[ N_n(z) = \text{Re}[\bar{\eta}_n(z)]. \]

Moments and shears inside the inclusion can be obtained in an analogous manner.

The usual boundary constraints for a plate are:

(i) Clamped edge;

(ii) Simply-supported edge;

(iii) Free edge;
(iv) Edge adjoint to another elastic plate;
(v) Edge adjoint to another plate with infinite rigidity.

For the flexural waves scattered by a circular inclusion, solution for the case of a free edge (a circular cavity) was given by Kung (Ref. 6.1) with detailed calculation of moment and shear forces as a function of frequencies. About the same time, Konenkov (Ref. 6.2) presented the solutions for all cases except (v) but without giving numerical results. Case (v), a rigid-moveable inclusion, was investigated by Chou (Ref. 6.3). We shall list the results of cases (i), (ii), and (iii) without giving derivation, followed by a detailed discussion of the solution for a rigid-moveable inclusion. To abbreviate, the symbol \( \zeta \) is used to denote \( \gamma r \) at \( r = a \).

1. Clamped circular edge or the edge of a rigid-fixed inclusion: at the surface \( r = a \),

\[
\omega = 0, \quad \frac{\partial \omega}{\partial r} = 0;
\]

\[
A_n = [iJ_n(\zeta)K_n^{(1)}(i\zeta) - J_n'(\zeta)K_n^{(1)}(i\zeta)]/\Delta_l, \tag{6.8}
\]

\[
B_n = 2i/\pi \zeta \Delta_l,
\]

\[
\Delta_l = iK_n^{(1)}(\zeta)K_n^{(1)}(i\zeta) - K_n^{(1)}(i\zeta)K_n^{(1)}(\zeta), \quad \zeta = \gamma a.
\]

2. Circular hole with the edge simply supported:

\[
\omega = 0, \quad M_{rr} = -D[\nu v^2 \omega + (1-\nu) \frac{\partial^2 \omega}{\partial r^2}] = 0, \quad \text{at } r = a
\]
\[
A_n = \frac{B_n^{(1)}(iz)M_n(z) - \mathcal{M}_n(iz)J_n(z)}{\Delta_2},
\]
\[
B_n = \frac{\mathcal{M}_n(z)J_n(z) - B_n^{(1)}(z)M_n(z)}{\Delta_2},
\]
\[
\Delta_2 = B_n^{(1)}(iz)M_n(iz) - \mathcal{M}_n(iz)H_n^{(1)}(iz).
\]

(3) **Circular cavity (free edge):**

\[
M_{rr} = 0, \quad V_r = Q_r - \frac{1}{r} \frac{3M_{r\theta}}{\partial \theta} = 0, \quad \text{at } r = a,
\]

\[
A_n = \frac{[M_n(z)S_n(iz) - L_n(z)M_n(iz)]}{\Delta_3},
\]
\[
B_n = \frac{[L_n(z)M_n(z) - M_n(z)S_n(z)]}{\Delta_3},
\]
\[
\Delta_3 = S_n(z)M_n(iz) - L_n(iz)M_n(z).
\]

At the surface of the cavity, the nonvanishing moment as normalized by \(M_o = \omega_o D_y^2\) is

\[
(M_{r\theta}/M_o)_{r=a} = \sum_{n=0}^{\infty} \epsilon_n i^n S_n \cos n\theta,
\]

where

\[
S_n = \frac{4(1-v^2)}{\pi \zeta^2 \Delta_n} \left[ (n^2+n)(n^2-n+\frac{\xi^2}{1-v})K_n(\zeta) + \frac{\xi^2}{1-v} \zeta K_{n-1}(\zeta) \right],
\]

\[
\zeta^2 \Delta_n = 4n^2(n+1)\xi^2 B_n^{(1)}(\zeta)K_n(\zeta) - 2\xi B_n^{(1)}(\zeta)K_{n-1}(\zeta).
\]
SCATTERING OF FLEXURAL WAVES

\[-(n^4 - n^2(1 - \nu) - 2(n^2 + n)\zeta^2 + \zeta^4/(1 - \nu))H_n^{(1)}(\zeta)K_{n-1}(\zeta)\]

\[+ [(n^2 - n^4)(1 - \nu) - 2(n^2 + n)\zeta^2 - \zeta^4/(1 - \nu)]K_n(\zeta)K_n^{(1)}(\zeta),\]

and the Hankel functions with imaginary argument have been replaced by the modified Bessel functions of the second kind, \(K_n(\zeta)\), according to (2.28).

As the frequency \(\omega\) of the incident wave approaches zero, we find, with \(\eta \to 0\),

\[S_0 \to 1 + \nu,\]

\[S_1 \to 0,\]

\[S_2 \to (1 - \nu^2)/(3 + \nu),\]

\[S_n \to 0, \quad n > 2.\]

Thus

\[(M_{rr}/M_0)_{\eta = 0} \to 1 + \nu - 2[(1 - \nu^2)/(3 + \nu)] \cos 2\Theta.\]

This is in agreement with the static solution of a plate with a circular hole subjected to bending moments \(M_0\) and \(\nu M_0\) at two perpendicular directions (Goodier, Ref. 6.4). A summary of moment concentration in a plate under statical bending can be found in the textbook by Timoshenko and Woinowsky-Krieger (Ref. 6.5).
(4) Circular elastic inclusion: At the interface, the following four conditions of continuity must be satisfied,

\[ \omega = \omega_1, \]
\[ \frac{\partial \omega}{\partial r} = \frac{\partial \omega_1}{\partial r}, \]
\[ M_{rr} = (M_{rr})_1, \]
\[ V_r = (V_r)_1. \]

They give rise to the four simultaneous equations for the unknown coefficients \( A_n, B_n, C_n, \) and \( D_n: \)

\[
\begin{bmatrix}
H_n^{(1)}(\zeta) & H_n^{(1)}(i\zeta) & J_n(\zeta_1) & J_n(i\zeta_1) \\
\xi H_n^{(1)}(\zeta) & i\xi H_n^{(1)}(i\zeta) & \zeta J_n'(\zeta_1) & i\zeta J_n'(i\zeta_1) \\
m_n(\zeta) & m_n(i\zeta) & pM_n(\zeta_1) & pM_n(i\zeta_1) \\
\Sigma_n(\zeta) & \Sigma_n(i\zeta) & pL_n(\zeta_1) & pL_n(i\zeta_1)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n \\
C_n \\
D_n
\end{bmatrix}
= \begin{bmatrix}
-J_n(\zeta) \\
-\zeta J_n'(\zeta) \\
-M_n(\zeta) \\
-L_n(\zeta)
\end{bmatrix}
\]

(6.12)

with \( p = D_1/D_2, \quad \zeta = \gamma a, \quad \zeta_1 = \gamma_1 a, \quad \eta = \rho_1/\rho. \)

(5) Circular rigid-moveable inclusion: Solutions for the scattering of flexural waves by a rigid inclusion in a plate presumably
can be deduced from the previous case. One way is to set the coefficients $C_n$ and $D_n$ in (6.12) equal to zero. This implies not only that no wave is transmitted to the insert, but also that the inclusion is constrained from moving ($\omega \rightarrow 0$), which is equivalent to having the edge of the circular inclusion clamped in space as in Case (1). Another approach is to calculate the coefficients $A_n$, $B_n$ from the matrix equation (6.12) in the limit $D_1/D \rightarrow \infty$. This is then the solution for a rigid-moveable inclusion because, although the insert is so rigid as to have no elastic deformation, it is still allowed to translate or rotate as a rigid body.

Consider the third and fourth columns of the $4 \times 4$ matrix in (6.12), and recall that

$$\zeta_1 = \gamma_1^a = \left( \frac{p_1 D}{\rho D_1} \right)^{\frac{1}{2}} \gamma_2,$$

(6.13)

$$\frac{1}{p} = \frac{D}{D_1} = \frac{\mu (1-\nu)}{\mu_1 (1-\nu)},$$

$\eta = p_1/p$ being a finite constant. A rigid inclusion means that $\mu/\mu_1 \rightarrow 0$ and $\nu_1/\nu \rightarrow 0$. Thus as $\mu/\mu_1 = \epsilon \rightarrow 0$, $1/p$ approaches zero in the order of $\epsilon$ and $\zeta_1$ approaches zero in the order of $\epsilon^2$. Replacing the Bessel functions with small arguments by their limiting values (2.49), we have for the third column, as $\epsilon \rightarrow 0$,

$$E_{31} = J_n(\zeta_1) + \frac{1}{n!} \left( \frac{\zeta_1}{2} \right)^n + O(\epsilon^{n/4}), \quad n \geq 2,$$
\[ E_{32} = \zeta_1^\nu (\zeta_1) + \frac{1}{(n-1)!} \left( \frac{\zeta_1}{2} \right)^n + O(\epsilon^{n/4}), \quad n \geq 2, \]

\[ E_{33} = \rho^2 \left( \frac{1-\nu_1}{(n-2)!} \right) \left( \frac{\zeta_1}{2} \right)^n + O(\epsilon^{n/4-1}), \quad n \geq 2, \]

\[ E_{34} = \rho \left( \frac{1-\nu_1}{(n-2)!} \right) \left( \frac{\zeta_1}{2} \right)^n + O(\epsilon^{n/4-1}), \quad n \geq 2, \]

where \( O(\epsilon^m) \) denotes "the order of \( \epsilon^m \)" and \( \epsilon \) is an infinitesimal quantity. It is seen that the elements \( E_{31} \) and \( E_{32} \) are of smaller order than \( E_{33} \) and \( E_{34} \); the same is true for \( E_{41} \) and \( E_{42} \) when compared with \( E_{43} \) and \( E_{44} \) in the fourth column. For the evaluation of coefficients \( A_n, B_n \) \( (n \geq 2) \), we can replace the elements \( E_{31}, E_{32}, E_{41}, \) and \( E_{42} \) by zeros and obtain the following matrix equation:

\[
\begin{bmatrix}
H_n^{(1)}(\zeta) & H_n^{(1)}(i\zeta) \\
\zeta H_n^{(1)'}(\zeta) & i\zeta H_n^{(1)'}(i\zeta)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix} =
\begin{bmatrix}
-\nu_1(\zeta) \\
-\nu_1'(\zeta)
\end{bmatrix}, \quad n \geq 2. \quad (6.14a)
\]

The \( A_n \) and \( B_n \) thus obtained are the same as those in (6.8).

For \( n = 0 \) or 1, the above approach fails because \( E_{31} \) may not have the same order as \( E_{32} \) and both of them may not be much smaller than \( E_{33} \) and \( E_{34} \). For instance, when \( n = 0 \),

\[ E_{31} = J_0(\zeta_1) + 1 + O(\epsilon^0), \]

\[ E_{32} = \zeta_1 J_0'(\zeta_1) - \frac{1}{2}(\zeta_1)^2 + O(\epsilon^2), \]
\[ E_{33} = pM_0(\xi_1) + \frac{3}{2}p(\xi_1)^2 + O(\varepsilon^{-\frac{1}{2}}), \]
\[ E_{34} = pL_0(\xi_1) + \frac{1}{2}p(\xi_1)^4 + O(\varepsilon^0). \]

Hence the complete $4 \times 4$ matrix equation (6.12) is needed to evaluate $A_0, B_0, \text{ and } A_1, B_1$. By carefully taking the limiting value of each element in the third and fourth columns where a two-term or three-term series expansion for Bessel functions is usually required, we find

\[ A_0 = [-i\xi J_0(\xi)H_1^{(1)}(i\xi) + iJ_1(\xi)H_0^{(1)}(i\xi) + 4\pi H_1^{(1)}(i\xi)J_1(\xi)]/\Delta_0, \]
\[ B_0 = [-\xi H_0^{(1)}(\xi)J_1(\xi) + iJ_0(\xi)H_1^{(1)}(\xi)]/\Delta_0, \]  
(6.14b)

\[ \Delta_0 = i\xi H_0^{(1)}(\xi)H_1^{(1)}(i\xi) - iH_0^{(1)}(i\xi)H_1^{(1)}(\xi) - 4\pi B_1^{(1)}(i\xi)J_1(\xi); \]

\[ A_1 = \left\{ \frac{8\pi [2J_1(\xi) - iJ_0(\xi)] [2H_1^{(1)}(i\xi) - i\xi E_0^{(1)}(i\xi)]}{\Delta_1} + \xi^3 [J_0(\xi)H_1^{(1)}(i\xi) - iH_0^{(1)}(i\xi) - iH_0^{(1)}(i\xi)J_1(\xi)] \right\}/\Delta_1, \]
\[ B_1 = \xi^2 [\xi J_0(\xi)H_0^{(1)}(\xi) - iH_1^{(1)}(\xi)J_0(\xi)]/\Delta_1, \]  
(6.14c)

\[ \Delta_1 = 8\pi [2H_1^{(1)}(\xi) - iE_1^{(1)}(\xi)] [2B_1^{(1)}(i\xi) - i\xi E_0^{(1)}(i\xi)] + \xi^3 [H_0^{(1)}(\xi)H_1^{(1)}(i\xi) - iH_0^{(1)}(i\xi)H_1^{(1)}(\xi)]. \]
The equations above, together with (6.14a) for \( A_n \) and \( B_n \) when \( n \geq 2 \), complete the solution for scattering by a rigid-moveable inclusion.

At the boundary of the circular inclusion \( r = a \), the moment \( M_{rr} \) and resultant shear \( V_r \) can be expressed as

\[
\frac{M_{rr}}{M_0} \bigg|_{r=a} = \sum_{n=0}^{\infty} \epsilon_n^r n S_n \cos n\theta,
\]

\[
\frac{V_r}{V_0} \bigg|_{r=a} = \sum_{n=0}^{\infty} \epsilon_n^r n T_n \cos n\theta,
\]

with

\[
S_0 = \frac{2i}{\pi \delta_0} \left\{ 2\zeta K_0(\zeta) + 4nK_1(\zeta) \right\},
\]

\[ T_0 = -\frac{4i}{\pi \delta_0}, \]

\[ \delta_0 = \zeta K_1(\zeta) R_0^{(1)}(\zeta) - \zeta K_0(\zeta) R_1^{(1)}(\zeta) - 4nH_1^{(1)}(\zeta) K_1(\zeta); \]

\[
S_1 = \left( -2i / \pi \delta_1 \right) \left\{ 8n \left[ 2K_1(\zeta) + \zeta K_0(\zeta) \right] + 2i^2 K_1(\zeta) \right\},
\]

\[
T_1 = \left( -2i / \pi \delta_1 \right) \left\{ 8n \left[ 2K_1(\zeta) + \zeta K_0(\zeta) \right] - 2i^2 \left[ K_1(\zeta) + \zeta K_0(\zeta) \right] \right\},
\]

\[
\delta_1 = 8n \left[ 2K_1(\zeta) + \zeta K_0(\zeta) \right] \left[ 2H_1^{(1)}(\zeta) - \zeta H_0^{(1)}(\zeta) \right]
\]

\[ - \zeta \left[ \zeta H_0^{(1)}(\zeta) K_1(\zeta) + \zeta K_0(\zeta) H_1^{(1)}(\zeta) \right]; \]

\[ S_n = \frac{4i}{\pi \delta_n}, \quad n \geq 2, \]

\[ T_n = \left( -4i / \pi \delta_n \right) \left[ nK_1(\zeta) + \zeta K_{n-1}(\zeta) \right], \quad n \geq 2, \]
\[ \Delta_n = c H_n^{(1)}(\zeta) K_{n-1}(\zeta) + c H_{n-1}^{(1)}(\zeta) K_n(\zeta), \quad n \geq 2. \]

In the equations above, \( \zeta = \gamma a \), and \( n = \rho_1 / \rho \); \( M^*_n = \nu^*_o \gamma^2 D \) and \( V^*_o = \nu^*_o \gamma^3 D \) are the maximum values of moment and shear in the plate without an inclusion (Eq. 6.3).

As the frequency \( \omega \to 0 \), \( \zeta \to 0 \). In the zero frequency limit, we find in (6.15)

\[ S_0 \to 1, \quad S_1 \to 0, \quad S_2 \to -1 \quad \text{and} \quad S_n \to 0, \quad (n > 2), \]
\[ T_0 \to 0, \quad T_1 \to 0, \quad T_2 \to 2, \quad \text{and} \quad T_n \to 0, \quad (n > 2), \]

and

\[
\left( \frac{M^*_r}{M^*} \right)_{r=a} = 1 + 2 \cos 2\theta, \\
\left( \frac{V^*_r}{V^*_o} \right)_{r=a} = -4 \cos 2\theta.
\]

These two values agree with those for a plate with a rigid inclusion subjected to static bending moments \( M^*_0 \) and \( \nu^*_o M^*_o \) at two perpendicular directions. For such a biaxial loading, the static solution (see Goland, Ref. 6.6) is

\[ \omega = -\frac{M^*_o}{2D} \left[ a^2 \ln (a/r) + \frac{1}{2} \left( r^2 - a^2 \right) + \frac{1}{2} \left( r^2 - 2a^2 + \frac{a^4}{r^2} \right) \cos 2\theta \right], \]

\[ M^*_r = \frac{1}{2} M^*_o \left\{ (1+\nu) + (1-\nu) \frac{a^2}{r^2} + \left[ 1 - \nu + 4\nu \frac{a^2}{r^2} + 3(1-\nu) \frac{a^4}{r^4} \right] \cos 2\theta \right\}, \]

\[ Q^*_r = -4M^*_o \left( a^2 / r^3 \right) \cos 2\theta. \]
Without recourse to the solution of an elastic insert, the solution for a rigid-moveable inclusion can be derived directly from Eq. (6.4), and the kinetic boundary conditions at \( r = a \) (see Chou, Ref. 6.3). As shown in Fig. 6.2, the inclusion has a translation as well as a rotation during the passage of a plane flexural wave. Let \( \dot{w} \) be the displacement of the centroid along the \( z \)-axis, and \( \phi \) the rotation of the rigid insert about \( y \)-axis. From Newton's equation of motion for a rigid body

\[
m_1 \ddot{w} = \int_0^{2\pi} [V_r]_{r=a} \alpha \, d\theta,
\]

\[
I_1 \ddot{\phi} = \int_0^{2\pi} [M_{r\theta} \cos \theta - V_r \cos \theta]_{r=a} \alpha \, d\theta,
\]

where \( m_1 = 2\pi \rho_1 ba^2 \) is the mass of the insert and \( I_1 = 2\pi \rho_1 ba^4 / 4 \) is the moment of inertia about the \( y \)-axis. The boundary conditions for a rigid-moveable inclusion are, at \( r = a \):

\[\text{Fig. 6.2. Motion of a Rigid Inclusion in an Elastic Plate}\]
\[ \omega = \omega - a \phi \cos \theta, \]

\[ \frac{\partial \omega}{\partial r} = - \phi \cos \theta, \]

which are quite different from those in Case (i). Only when \( r/L \rightarrow \infty \) does the above solution reduce to the case of a fixed-rigid insert.

Solutions obtained from the boundary conditions Eq. (6.17) are the same as those given in (6.14).

As mentioned before, values of the stress concentration factors, or more appropriately, the moment concentration factors, are known for Cases (iii) and (v). Fig. 6.3 shows the variation of \( M_{\theta \theta}/M_0 \) at the boundary of a circular cavity \((\theta = \pi/2)\) as a function of the normalized incident wave number \( \gamma \alpha \) (Kung, Ref. 6:1). It is seen that for all frequencies considered, the dynamic stress concentration factor is less than the static one \((\gamma \alpha = 0)\). For a rigid-moveable inclusion, the results are shown in Fig. 6.4 for \( M_{rr}/M_0 \) at \( r = \alpha \) and \( \theta = 0 \).
Aside from the references cited so far, there is the paper by Kyukin and Sergeev (6.7) who discuss the scattering of flexural waves by a row of circular inclusions. Chambers (6.8) discusses the scattering of flexural waves by a semi-infinite strip which can be clamped, simply-supported, or free from moment and shear. The geometry of a semi-infinite strip can be treated as the limiting case of a parabolic cylinder which is discussed in Chapter V.

From the steady state solutions, stress concentration factors due to an impulsive force can usually be calculated by applying the Fourier synthesis (see Chapter I, Section 4) provided the steady state solution for the entire frequency range (0 < \(\omega\) < \(\infty\)) is known. For the flexural wave, however, solutions based on the classical theory are in good agreement with experiments only for low frequencies or long wavelengths (compared with the thickness of the plate); thus no solutions derived from Eq. (6.1) are valid for high frequencies.

A theory which covers a wider range of frequencies has been proposed by Mindlin (6.9). Based on his theory, the scattering of flexural waves by a circular cavity in a plate is investigated by
Pao and Chao, (6.10) and by a rigid circular inclusion by Lu. (6.11) For the case of a circular cavity, the moment concentration factors \( M_{86}/M_0 \) as a function of the incident wave frequency \( \omega \) are shown in Fig. 6.5. The frequency has been normalized by \( \omega_0 = \pi a_b/2b \), which is the fundamental frequency of the simple thickness shear mode of a thick plate in vibration.

![Graph](image)

**Fig. 6.5. Absolute Values of Moment Concentration Factors at the Edge of a Circular Cavity (Mindlin Theory) \((\theta = \pi/2)\) \((\nu = 0.30)\)**

Since \( \omega_0 \) has a very large value even for a moderately thick plate, the frequency range covered in Fig. 6.5 is much wider than that in Fig. 6.3. Only when \( \alpha/2b \) (the ratio of the radius of the cavity to the thickness of the plate) approaches zero, and when the ratio \( \omega/\omega_0 \) is small, do the results based on the classical theory agree with those derived from Mindlin's theory.

To end this section we would like to caution the use of the
moment or shear concentration factor in calculating stresses or strains in a plate subjected to bending. In dynamic theory, the shear and the moment are respectively the average of shearing stress and the weighted average of normal stress across the thickness of the plate. For example,

\[ Q_x = \int_{-b}^{b} \sigma_{xz} \, dz, \]

\[ M_{xx} = \int_{-b}^{b} \sigma_{xx} \, dz. \]

Even using Mindlin's theory, which takes into account the shear and rotatory inertia effects, the actual concentration of stress at a point may differ from the moment concentration across a section passing through that point, especially at high frequencies.

7. EFFECTS OF INCIDENT WAVE'S CURVATURE

IN ALL PREVIOUS SECTIONS in this chapter, the incident waves have always been assumed to be plane waves. In this section we shall examine some of the effects on the dynamic stress concentration factors when the incident wave is of non-planar nature.

7.1. Cylindrical Wave

Consider, first, the case of a plane wave, it may be considered as waves generated by a line source located at infinity. Let us suppose that a harmonic dilatational line source is now located at \( \vec{0} \).
which is at a distance \( r_0 \) from 0 — see Fig. 7.1. We also have a cavity of radius \( a \) with its axis coinciding with 0. Thus the effects of the incident wave's curvature on the dynamic stress concentration can be examined by observing the effects of \( r_0 \) — i.e., the distance between the source and the axis of the cavity — on the stresses.

The harmonic waves generated by a dilatational line source located at \( O \) can be represented by

\[
\varphi(z) = \varphi_0 E_0^{(1)}(\alpha r)e^{-i\omega t}.
\]  

(7.1)

In this representation, we have a cylindrical wave propagating outward from the source at \( O \) in the \((\bar{r}, \bar{\theta})\) coordinates. Only the 0th order Hankel function is required because of the axial symmetric nature of the source.

As the incident cylindrical wave impinges on the cavity, we have the same two reflected waves as in the plane incident wave case. They are:
\[ \varphi(r) = \sum_{n=0}^{\infty} A_n H_n^{(1)}(ar) \cos n\theta e^{-i\omega t}, \]

(7.2)

\[ \psi(r) = \sum_{n=0}^{\infty} B_n H_n^{(1)}(br) \sin n\theta e^{-i\omega t}, \]

where \( A_n, B_n, \alpha, \) and \( \beta \) are the familiar coefficients of expansion and wave numbers. Here we note that the reflected waves are expressed as waves propagating outward from axis 0, in the \((r, \theta)\) coordinates.

In the method of wave function expansion, what we did in the plane wave case was first to expand the incident wave in the \((r, \theta)\) coordinates. Then, using the boundary condition together with the orthogonality condition provided by the sinusoidal functions, we determined the unknown coefficients, \( A_n \) and \( B_n \), for each order of \( n \). We wish, therefore, to transform first the incident wave in \((\overline{r}, \overline{\theta})\) into the \((r, \theta)\) coordinates. Then, we impose the appropriate boundary conditions to determine the coefficients of expansion. Proceeding with the transformation, we have, from Fig. 7.1, the following relationships between \((\overline{r}, \overline{\theta})\) and \((r, \theta)\):

\[ \overline{r} \sin \Omega = r_O \sin \overline{\theta}, \]

(7.3)

\[ \overline{r} \cos \Omega = r - r_O \cos \overline{\theta}. \]

We also have the following integral representation of \( H_n^{(1)}(ar)e^{in\theta} \)

\[ H_n^{(1)}(ar)e^{in\theta} = \frac{1}{\pi} \int_{C_1} e^{iar \cos \varphi + i\overline{n}(\varphi + \overline{\theta} - \pi/2)} d\varphi, \]

(7.4)
where $C_1$ is the contour shown in Fig. 2.1. (We are interested only in the $v = 0$ case at present, but since the transformation of the general case is just as readily derived, and just as useful, we have decided to present the transformation of $H_v(\alpha r)e^{i\nu \theta}$ into $(r, \theta)$ coordinates.)

Noting the periodic properties of the integrand, Eq. (7.4) is equivalent to

$$H_n^{(1)}(\alpha r)e^{i\nu \theta} = \frac{1}{\pi} \int_{C_1} e^{i\alpha r} \cos (\varphi + \Omega) + i\nu(\varphi + \theta + \Omega - \pi/2) d\varphi.$$  

(7.5)

From Eq. (7.3) and Fig. 7.1 we have also the following relationship:

$$\overline{r} \cos (\varphi + \Omega) = r \cos \varphi + r_0 \cos (\theta + \varphi),$$  

(7.6)

$$\Omega + \theta = \theta.$$

Substituting Eq. (7.6) into (7.5) results in

$$H_n^{(1)}(\alpha r)e^{i\nu \theta} = \frac{1}{\pi} \int_{C_1} e^{i\alpha r} \cos \varphi + i\nu_0 \cos (\theta + \varphi) + i\nu(\varphi + \theta - \pi/2) d\varphi.$$  

(7.7)

If $r \geq r_0$, and noting that

$$i\nu_0 \cos (\theta + \varphi) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-i\pi m/2} J_m(\nu_0) e^{im(\theta + \varphi)},$$

(see Eq. II-1.14), and by exchanging order of integration and summation, we obtained
\[ H_n(\mathbf{r}) e^{i\omega t} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} (-1)^m J_m(\alpha \rho) \int_{C_1} e^{i\alpha \rho \cos \varphi + i(n+m)\varphi} d\varphi \]

\[ = \sum_{m=-\infty}^{\infty} (-1)^m J_m(\alpha \rho) H_{n+m}(\alpha \rho) e^{i(n+m)\omega}. \quad (7.8) \]

If now \( r < r_o \), Eq. (7.8) fails to converge and it is replaced by

\[ H_n(\mathbf{r}) e^{i\omega t} = \sum_{m=-\infty}^{\infty} (-1)^m J_m(\alpha \rho) H_{n+m}(\alpha \rho) e^{i(n+m)\omega}. \quad (7.9) \]

Equations (7.8) and (7.9) are special cases of the well-known addition formula for the Hankel function.

We may now use the above relationship to expand Eq. (7.1) into \((r, \theta)\) coordinates. The appropriate equation to use, for the purpose of determining \( A_n \) and \( B_n \), is Eq. (7.9), because at \( r = a \), \( r < r_o \).

Thus

\[ \varphi(t) = \varphi_0 H_0(\alpha \rho) e^{-i\omega t} = \varphi_0 \sum_{n=0}^{\infty} (-1)^n \epsilon_n H_n(\alpha \rho) J_n(\alpha \rho) \cos n\theta e^{-i\omega t}. \quad (7.10) \]

Applying the traction-free boundary condition at \( r = a \), as one would have done in the plane wave case, the unknown coefficients of expansion \( A_n \) and \( B_n \) are determined for each \( n \). Without belaboring the details again (reader may refer to Section 3 for details), the stress at the boundary of the cavity is as follows:

\[ \sigma_{\theta \theta}(a, \theta) = \frac{-4i}{\pi} \beta^2 \mu \varphi_0 \left( 1 - \frac{1}{\lambda^2} \right) \sum_{n=0}^{\infty} (-1)^n \epsilon_n H_n(\alpha \rho) S_n \cos n\theta e^{-i\omega t}, \quad (7.11) \]
where \( S_n \) is defined in Eq. (3.15). It may be of interest here to note
the similarity between Eqs. (7.11) and (3.14).

The dynamic stress concentration factor is determined by norm-
alizing \( c_{\theta\theta}(\alpha, \theta, t) \) by the magnitude of the radial stress \( (c_{rr}^{(i)}) \) of
the incident wave at the same point in a medium with no opening. This
definition is consistent with our previous definition, except that
now the normalizing factor \( (c_{rr}^{(i)}) \) is a function of the distance \( \bar{r} \)
from \( \bar{0} \). The incident radial stress in terms of incident wave poten-
tial is

\[
c_{rr}^{(i)} = \frac{\alpha^2}{\omega} \varphi_B \left[ H_2(\alpha \bar{r}) + (1 - \kappa^2) H_0(\alpha \bar{r}) \right] e^{-i \omega t}.
\]

The dimensionless tangential stress \( c_{\theta\theta}^* \) is

\[
c_{\theta\theta}^* = \frac{4 \alpha}{\pi} \left[ H_0(\alpha \bar{r}) - \frac{1}{\kappa^2 - 1} H_2(\alpha \bar{r}) \right]^{-1} \sum_{n=0}^{\infty} (-1)^n \kappa_n \varphi_{n} \varphi_0 S_n \cos n \theta e^{-i \omega t},
\]

where \( \bar{r} \) is the distance from \( \bar{0} \) to any point on the surface of the
cavity. At any angle \( \theta \) it is given (see Fig. 7.1) as

\[
\bar{r} = \alpha^2 + \bar{r}_0^2 + 2 \alpha \bar{r}_0 \cos \theta.
\]

Since Eq. (7.13) can be used to compute stress for any value of
\( \bar{r}_0 \), it can be used to study the effects of the incident wave curva-
ture. Before presenting numerical results, we shall show several
limiting cases that are of interest. As was stated at the beginning
of this section, a plane wave can be considered as a wave generated
by a line source at infinity. To show this, let us first fix $a$ as finite but let $r_o \to \infty$ in Eq. (7.13).

From the geometry of this problem, we see that as $r_o$ becomes large, the distance $r$ can be approximated by $r_o$ without much error. And since $a$ is finite and $r_o$ is large we may use the asymptotic expansion for $H_n(z)$ in Eq. (7.13). The leading term of the asymptotic expansion for $H_n(z)$ (see Eq. 2.44) is

$$
\lim_{z \to \infty} H_n^{(1)}(z) = \left( \frac{2}{\pi a} \right)^{\frac{1}{2}} \frac{1}{e^{i(\pi/4 - \frac{1}{2} \pi n)}}.
$$

Substitution of the above equation into (7.13) produces

$$
\lim_{r_o \to \infty} \sigma_{00} = \frac{4i}{\pi} \left( 1 - \frac{1}{4a^2} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{e^{i(\pi/4 - \frac{1}{2} \pi n)}} e^{-i\pi n/2} \left( \frac{2}{\pi a r_o} \right) e^{-i\pi n/2} \times S_n \cos \eta e^{-i\omega t}
$$

which is exactly the expression in Eq. (3.14) for a plane incident wave. If we let $a \to 0$ in Eq. (7.15), we obtain the static solution as in Eq. (3.19).

There is another limiting case of considerable interest, i.e., the reduction of Eq. (7.13) to the static case. For this case, we shall fix $r_o$ as finite but let $a \to 0$. The limit of $S_n$ as $a \to 0$ has already been examined in Eqs. (3.19a) and (3.19c), so will not be...
repeated here. The other terms to be examined in Eq. (7.13), as \( \alpha \to 0 \), are:

\[
\lim_{\alpha \to 0} H_0(\alpha r) \propto \frac{2}{\pi} \ln(\alpha r),
\]

\[
\lim_{\alpha \to 0} H_\alpha(\alpha r) \propto -\frac{i(n-1)!}{\pi} \frac{2^n}{(\alpha r)^n}.
\]

It follows

\[
\lim_{\alpha \to 0} c_{66}^* \propto \frac{8}{(\alpha^2-1)\ln(\alpha r) + 2(\alpha r)^2} \sum_{n=2}^{\infty} (-1)^{n+1}(n-1) \left(\frac{\alpha r}{\alpha r_0}\right)^{n-2} \cos n\theta
\]

\[
= 4 \left(\frac{r}{r_0}\right)^2 \sum_{n=2}^{\infty} (-1)^{n+1}(n-1) \left(\frac{\alpha}{\alpha r_0}\right)^{n-2} \cos n\theta.
\]  \( (7.16) \)

Now using the relationship between \( \vec{r}, r_0, \alpha, \) and \( \cos \theta \) in Eq. (7.14), (7.16) can be reduced to

\[
c_{66}^* = -4 \left[ 1 - \frac{2 \sin^2 \theta}{1 + (\alpha/r_0)^2} \right].
\]  \( (7.17) \)

Equation (7.17) is in fact the solution of the static stress concentration factor around a cavity due to a line of dilatation located at distance \( r_0 \) from its axis. We may take an additional limit on Eq. (7.17) by letting \( r_0 \to \infty \), and obtain

\[
c_{66}^* = -4 \cos 2\theta,
\]

\( \alpha \to 0, \)
The solution to degenerate into a pure shear solution becomes clear if one notices the stress field due to a line of dilatation is 
\[ \sigma_{rr} = \varphi o / r^2 \sigma_{66} = \varphi o / r^2 . \]  
We have shown two limiting processes for Eq. (7.13) which give two different static solutions for physically different problems. From the foregoing discussion, the importance of the order of the limiting procedure becomes apparent.

The stress concentration factors for the static and dynamic line sources are given in Ref. 7.1, and presented in Figs. 7.2 through 7.6. The values of parameters used are \( \nu = 0.25 \), 0 \( \leq \alpha \leq 4.0 \), and \( r_o / \alpha = 2, 5, 10, 20, \infty \). Figure 7.2 depicts the effects of \( r_o / \alpha \) on the static stress concentration factors. Here we note that there are three maxima, all with a value of 4. Two maxima occur at \( \theta = 0, \pi \).
irrespective of the value of \( r_0/a \). The third one occurs at \( \theta = \pi/2 \) when \( r_0/a \to \infty \). However, the point where the maximum stress occurs is a function of \( r_0/a \). The angular position on the cavity where this maximum occurs is given by

\[
\theta = \pi - \cos^{-1}(a/r_0),
\]

which defines the point of tangency on the line drawn from the source to the cavity.

Dynamic stress concentration factors at two wave numbers \( \alpha a = 0.10, 1.0 \), are used to illustrate the effects of dynamic loading. Figure 7.3 presents the results at \( \alpha a = 0.10 \). Here, the effects both of the dynamics and of the incident wave's curvature are quite apparent. At

\[ r_0/a = 2, \] the results quite closely resemble that of the static loading at the same \( r_0/a \) value. However, at \( r_0/a = 10, 20 \), the dynamic stresses are quite different from their corresponding static cases.
In fact, if we compare the results for \( \frac{r_o}{a} = 20 \) with the plane wave case shown in Fig. 3.3, we will note the similarity. At wave number \( \alpha a = 1.0 \), as shown in Fig. 7.4, there is little resemblance between the static distribution and the dynamic distribution. The maximum value tends to shift toward the incident side of the cavity as \( \frac{r_o}{a} \) decreases. Figs. 7.5 and 7.6 illustrate the behavior of \( \left| \varepsilon_{gg}^* \right| \) at two fixed angular positions, \( \theta = 0, \pi \), on the cavity, respectively.

Here we note for \( \frac{r_o}{a} \geq 5 \), and \( \alpha a > 0.20 \), at \( \theta = \pi/2 \) --- see Fig. 7.5 --- that the plane wave solution and the cylindrical wave solution have become almost the same. At \( \theta = \pi \), Fig. 7.6, the two solutions are approximately the same at \( \alpha a \geq 0.7 \). The large differences between the plane wave solution and the present solution lie in the range when \( r_o/a < 5 \) and \( \alpha a \leq 0.20 \) or \( \alpha a \leq 0.7 \) at \( \theta = \pi/2, \pi \) respectively. The point of particular interest is at \( \theta = \pi \), because in the plane wave analysis, the stress is always very low, never exceeding a value \( \sim 0.4 \) at \( \alpha a \approx 0.5 \); but in the cylindrical wave case, the stress concentration can be as high as 5, which is an order of magnitude higher than the
Fig. 7.5. $|\sigma_{\theta\theta}^*|$ Versus $\alpha \alpha$ ($\theta = \pi/2$)

Fig. 7.6. $|\sigma_{\theta\theta}^*|$ Versus $\alpha \alpha$ ($\theta = \pi$)
plane wave prediction. This obviously warrants consideration in design.

7.2. Waves Due to a Concentrated Force

A dilatational line source, due to its symmetricity, may be considered as a special case of a system of forces acting at a point. In what follows we shall discuss a method of determining dynamic stress concentration factors due to waves generated by a harmonically varying single force acting at a point. It was shown earlier that only the P waves were generated by a dilatational source. In the case of a concentrated single force, both the dilatational wave and shear wave will be generated. Therefore, our first step will be to derive the waves generated by a concentrated force.

A detailed study of the waves generated by a concentrated force was presented by Stoke (1.17) and by Lamb (7.2). To facilitate our present discussion we shall highlight some of Lamb's derivations, though without going into all the details of his study, which is beyond the scope of the present book.

Consider a concentrated force $F_0 e^{-i\omega t}$ acting in an arbitrary direction of angle $\theta$ from the $x$ axis at $O$ — see Fig. 7.7. To deter-

![Fig. 7.7. Geometry of a Concentrated Force Problem](image-url)
mine the magnitude of the waves generated by this force, we shall first resolve \( F \) into its \( \overline{z} \) and \( \overline{y} \) components and determine the waves generated by these components. Then by methods of superposition we can determine the total wave generated by \( \overline{F}e^{-\omega t} \). Let us now proceed to determine the waves generated by \( F_{\overline{z}} \).

Following Lamb, we shall construct the solution due to a concentrated force \( F_{\overline{z}}e^{-\omega t} \) by the method of Fourier synthesis. Let us first assume \( F_{\overline{z}} \) is of the form \( F_{\overline{z}}(\overline{y}) = \chi e^{i\overline{z}y} \exp(-i\omega t) \), acting at \( \overline{z} = 0 \). The solutions for \( \varphi \) and \( \psi \) which will satisfy the imposed \( F_{\overline{z}} \) at \( \overline{z} = 0 \) as well as the finiteness and radiation conditions are of the form, for \( \overline{z} > 0 \):

\[
\begin{align*}
\varphi^+ &= A_1 e^{\overline{a}x} e^{i\overline{y}y} e^{-i\omega t}, \\
\psi^+ &= B_1 e^{i\overline{y}y} e^{-i\omega t}, \\
\varphi^- &= A_2 e^{\overline{a}x} e^{i\overline{y}y} e^{-i\omega t}, \\
\psi^- &= B_2 e^{i\overline{y}y} e^{-i\omega t}.
\end{align*}
\]

(7.18)

(7.19)

where \( \overline{a} = (\xi^2 - \alpha^2)^{1/2}, \overline{b} = (\xi^2 - \beta^2)^{1/2} \) are either real positive numbers for \( \xi > \alpha, \beta \), or negative imaginary numbers when \( \xi < \alpha, \beta \). The above choices are made so that the finiteness and radiation conditions are satisfied. \( \alpha, \beta \) are the usual compressional and shear wave numbers. \( A_1, B_1 \) and \( A_2, B_2 \) are the unknown coefficients for the dilatational and shear potentials for the \( \overline{z} > 0 \) and \( \overline{z} < 0 \) regions respectively.

The unknown coefficients are determined by the following condi-
tions at \( \Xi = 0 \): a) the normal stress \( \sigma_{xx} \) will be discontinuous across the plane; b) the shearing stresses and the displacements must be continuous. Substituting the displacement potentials into the potential-displacement and potential-stress relationships, and using the conditions at \( \Xi = 0 \), we have the following set of equations for the determination of \( A_1, B_1, A_2, \) and \( B_2 \):

From the conditions

\[
[a_{xx}]_0^+ - [a_{xx}]_0^- = x e^{i\xi y}, \tag{7.20}
\]

\[
[a_{xy}]_0^+ - [a_{xy}]_0^- = 0; \tag{7.21}
\]

we have

\[
(2\xi^2 - \beta^2)(A_1 - A_2) - 2i\xi\beta(B_1 + B_2) = \frac{X}{\mu} \tag{7.22}
\]

\[-2i\xi\beta(A_1 + A_2) - (2\xi^2 - \beta^2)(B_1 - B_2) = 0.
\]

And the condition of continuity in displacements yields two additional equations,

\[-\bar{\beta}(A_1 + A_2) + i\xi(B_1 - B_2) = 0, \tag{7.23}
\]

and

\[-i\xi(A_1 - A_2) + \bar{\beta}(B_1 + B_2) = 0.
\]

Hence

\[A_1 = -A_2, \quad B_1 = B_2,\]
\[ A_1 = -\frac{X}{2k^2 \mu}, \quad B_1 = \frac{i\xi X}{2k^2 \mu}. \]

It follows that the solution for \( \varphi, \psi \), which satisfy the applied load \( F_\pi = Xe^{i\xi y} \) at \( \pi = 0 \) are, for \( \pi > 0 \):

\[ \varphi^+ = -\frac{X}{2k^2 \mu} e^{-\pi\xi y + i\omega t}, \]
\[ \psi^+ = \frac{i\xi y}{2k^2 \mu} e^{-\pi\xi y + i\omega t}. \] \hspace{1cm} (7.24)

where \( \mu \) is the shear modulus.

In a similar manner we can derive the dilatational potential and distortional potential for an applied load in the y direction of the form: \( F_\pi = Y e^{i\pi x} \). The solutions are:

\[ \varphi^+ = -\frac{Y}{2k^2 \mu} e^{-\pi y + i\omega t}, \]
\[ \psi^+ = \frac{i\pi y}{2k^2 \mu} e^{-\pi y + i\omega t}. \] \hspace{1cm} (7.25)

Having determined the displacement potentials for the spatially sinusoidal loadings, we may now proceed to use the Fourier synthesis technique to derive the solution for a concentrated force. To achieve this, we use the same concepts used previously in synthesizing a periodic loading in time from harmonic time-varying forces (see Chapter I, Section 4). Thus we can view Eqs. (7.24) and (7.25) as the product of the Fourier transform of an arbitrary loading function and the admittance function due to a spatially sinusoidal loading. Hence
we interpret

\[ X(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\eta) e^{-i\xi \eta} \, d\eta, \]

\[ X(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\xi) e^{i\xi \eta} \, d\xi. \]

Then, according to Eq. (I-4.38), the displacement potentials for an arbitrary \( X(\eta) \) for \( \overline{\eta} > 0 \) become:

\[ \varphi(x, y, t) = \frac{e^{-i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\xi) e^{\frac{-i\xi x + i\eta y}{2\beta^2 \mu}} \, d\xi, \tag{7.26} \]

\[ \psi(x, y, t) = \frac{e^{-i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\xi) \frac{i\xi e^{-\frac{i\xi x + i\eta y}{2\beta^2 \mu}}}{2\beta^2 \mu} \, d\xi, \]

or

\[ \varphi(x, y, t) = \frac{e^{-i\omega t}}{2\sqrt{2\pi} \, \beta^2 \mu} \int_{-\infty}^{\infty} X(\xi) e^{\frac{-i\xi x + i\eta y}{\beta^2 \mu}} \, d\xi, \tag{7.27} \]

\[ \psi(x, y, t) = \frac{e^{-i\omega t}}{2\sqrt{2\pi} \, \beta^2 \mu} \int_{-\infty}^{\infty} X(\xi) \frac{e^{-i\xi x + i\eta y}}{\beta} \, d\xi. \]

For the concentrated force acting at \( \overline{0} \) in the \( \overline{\eta} \) direction, it can be represented by a Dirac-delta function as \( X(\eta) = X \delta(\eta) \), and its Fourier transform is \( X_0 / \sqrt{2\pi} \). Hence the displacement potentials due to a concentrated force acting in the \( \overline{\eta} \) direction become:
EFFECTS OF INCIDENT WAVE'S CURVATURE

\[
\phi(\bar{x}, y, t) = \frac{X_o e^{-i\omega t}}{4\pi\beta^2} x \int_{-\infty}^{\infty} \frac{e^{-\xi + i\xi y}}{\alpha} d\xi,
\]

\[
(7.28)
\]

\[
\psi(\bar{x}, y, t) = \frac{X_o e^{-i\omega t}}{4\pi\beta^2} y \int_{-\infty}^{\infty} \frac{e^{-\xi + i\xi y}}{\alpha} d\xi.
\]

Similarly, the displacement potentials due to a concentrated force acting in the \( y \) direction at \( \bar{0} \) are:

\[
\phi(\bar{x}, \bar{y}, t) = \frac{X_o e^{-i\omega t}}{4\pi\beta^2} \frac{\partial}{\partial \bar{y}} \int_{-\infty}^{\infty} \frac{e^{-\xi y + i\bar{y} x}}{\alpha} d\xi,
\]

\[
(4.29)
\]

\[
\psi(\bar{x}, \bar{y}, t) = \frac{X_o e^{-i\omega t}}{4\pi\beta^2} \frac{\partial}{\partial \bar{y}} \int_{-\infty}^{\infty} \frac{e^{-\xi y + i\bar{y} x}}{\beta} d\xi.
\]

We notice here that all four integrals in Eqs. (7.28) and (7.29) are the same type. These integrals are a form of the integral representation of the Hankel function of 0th order of the first or second kind. The equation given below is the integral representation of the Hankel functions.

\[
\int_{-\infty}^{\infty} e^{-i\xi y} \frac{d\xi}{\alpha} = \int_{-\infty}^{\infty} e^{\pm i\xi y - \sqrt{\xi^2 - \alpha^2}} \frac{d\xi}{\sqrt{\xi^2 - \alpha^2}} = \mp \pi i H_0(1, 2)(\sqrt{x^2 + y^2}),
\]

with the conditions that \( \xi, \alpha, \bar{x} \) are real and positive, and \( \sqrt{\xi^2 - \alpha^2} \) is positive for \( \xi > \alpha \) and negative imaginary for \( 0 < \xi < \alpha \). These conditions we note satisfy precisely those finiteness and radiation conditions stated earlier. Noting further that \( \bar{x} = \bar{r} \cos \bar{\theta}, \bar{y} = \bar{r} \sin \bar{\theta}, \)

\( X_o = F \cos \Theta, Y_o = F \sin \Theta, \) we finally arrive at the following expres-
sions for $\varphi$ and $\psi$ due to $\overline{F}$ acting at $\overline{O}$:

\[
\varphi(\overline{r},\overline{\theta},t) = \frac{-iF}{4\rho \omega C_a} \left[ \cos \overline{u} \cos \overline{v} + \sin \overline{u} \sin \overline{v} \right] H_1(\alpha r) e^{-i\omega t}; \quad (7.30)
\]

\[
\psi(\overline{r},\overline{\theta},t) = \frac{-iF}{4\rho \omega C_B} \left[ \cos \overline{u} \sin \overline{v} - \sin \overline{u} \cos \overline{v} \right] H_1(\beta r) e^{-i\omega t}. \quad (7.31)
\]

Thus it is seen that both shear waves and compressional waves are generated by a concentrated force, while in the dilatational line source problem only the dilatational waves are generated.

To study the dynamic stress concentration factor due to the incident waves around a cavity, shown in Eqs. (7.30) and (7.31), we shall first expand the incident waves into $(r, \theta)$ coordinates.

Here again, the addition formulae derived in Eq. (7.9) will be used to expand $\varphi$ and $\psi$ into $(r, \theta)$ coordinates. Without going into details, it can be shown that, due to a concentrated force at $\overline{O}$ expressed in the $(r, \theta)$ coordinate, the incident waves are:

\[
\varphi(\overline{r}) = \frac{-iF}{4\rho \omega C_a} \sum_{n=-\infty}^{\infty} (-1)^n \cos [(n+1)\theta - \theta] J_n(\alpha r) \overline{H}_{n+1}(\alpha \overline{r}) e^{-i\omega t}; \quad (7.32)
\]

\[
\psi(\overline{r}) = \frac{-iF}{4\rho \omega C_B} \sum_{n=-\infty}^{\infty} (-1)^n \sin [(n+1)\theta - \theta] J_n(\beta r) \overline{H}_{n+1}(\beta \overline{r}) e^{-i\omega t}. \quad (7.33)
\]

Determination of the dynamic stress concentration factor around the cavity can be carried out in the manner shown in earlier sections. The only added difficulty is in algebra, not in theory. References 7.3 (1967) has investigated the problem for two types of cylindrical discontinuities: rigid inclusions, and cavities due to a concentrated
force. Some numerical results are given in terms of "energy concentration factor."

8. TRANSIENT INTERACTION PROBLEM OF SHELL AND ELASTIC MEDIUM

THE INTERACTION of an acoustic wave and an elastic cylindrical shell has been a subject of extensive study in recent years (see for example Refs. 8.1 through 8.5). Studies of the transient interaction between elastic wave and elastic shell have appeared only recently; for example, in Refs. 0.10, 8.6, 8.7, and 5.10.

In the studies of elastic wave interaction problems, the methods used for the acoustic wave were generally adopted for the elastic wave problem. For example, Baron and Parnes, Ref. 0.10, used an approach, the modal expansion technique, similar to that used in Baron's earlier work, Ref. 8.5. Yoshihara, et al., Ref. 8.6, used an expansion similar to that in Haywood's investigation, Ref. 8.3. Although similarity in the methods themselves are observed, the actual solution to the elastic wave, shell interaction problem is governed by a pair of coupled integral equations, as compared to a single integral equation in the acoustic case.

8.1. Integral Equation Method

Let us consider a thin elastic cylindrical shell bound to an elastic medium subject to a unit stress incident P wave. The approach is to use the displacements from the cavity solution (see Section 5) as the influence coefficient to obtain the corresponding quantities for the diffraction of the P wave by a lined cavity. The governing equa-
tions of motion in terms of the radial and tangential displacements \( \omega \) and \( \nu \) of a shell compatible with the plane-strain problem of the elastic medium can be expressed as:

\[
- m \ddot{\omega} + \frac{EA}{(1-\nu^2)\alpha} \left( \frac{\partial \nu}{\partial \theta} - \dot{\omega} \right) - \frac{EI}{(1-\nu^2)\alpha} \left( \frac{\partial^4 \nu}{\partial \theta^4} + \dot{\omega} \right) = 0,
\]

\[
- m \ddot{\nu} + \frac{E}{1-\nu} \left( \frac{\partial^2 \nu}{\partial \theta^2} - \frac{\partial \omega}{\partial \theta} \right) = 0,
\]

where \( m \) is the mass of the shell per unit area, \( A \) the cross-sectional area of the shell, \( \alpha \) the radius of the shell, and \( I \) the moment of inertia. We now express the radial and tangential displacements \( \omega \) and \( \nu \) of the shell in terms of the generalized coordinates \( q_n \) and \( \bar{q}_n \):

\[
\omega(\theta, t) = \bar{q}_0(t) + \sum_{n=1}^{\infty} \left[ q_n(t) + \bar{q}_n(t) \right] \cos n\theta; \quad (8.3)
\]

\[
\nu(\theta, t) = \sum_{n=1}^{\infty} \left[ \frac{q_n(t)}{d_n} - \frac{\bar{q}_n(t)}{d_n} \right] \sin n\theta, \quad (8.4)
\]

where \( q_n(t) \) are the generalized coordinates for primarily inextensional motion and the coordinates \( \bar{q}_0(t) \) and \( \bar{q}_n(t) \) refer primarily to extensional shell motion. The coefficient \( d_n \) is:

\[
d_n = \frac{n^2-1}{2n} - \frac{(1-n^2)^2}{2n} \frac{1}{\alpha^2 A} + \frac{1}{2} \left[ \left( \frac{n^2+1}{n} \right)^2 + \frac{2(1-n^2)^3}{n^2} \frac{I}{\alpha^2 A} + \frac{(1-n^2)^4}{n^2} \frac{r^2}{\alpha^2 A^2} \right]^\frac{1}{2}.
\]

(8.5)

For shells of dimensions \( \alpha^2 A \gg I \), and \( n \ll 5 \), there is little coupling
between bending and extensional effects. For such cases \( d_n^2 = \kappa_n \), and the frequencies of vibration in vacuo, \( \omega_n \), \( \omega_n^* \) for Eqs. (8.1) and (8.2) are:

\[
\frac{\omega_n^2}{\omega_n} = \frac{EI}{(1-\nu^2)\kappa_n^2m_n^2(\kappa_n^2+1)} \tag{8.6a}
\]

\[
\frac{\omega_n^2}{\omega_n^*} = \frac{EA(\kappa_n^2+1)}{(1-\nu^2)m_n^2} \tag{8.6b}
\]

where \( \omega_n \) and \( \omega_n^* \) denote natural frequencies for inextensional and extensional modes, respectively. Using Eq. (8.6), the equations of motion for the shell may be written in terms of the generalized coordinates \( q_n \) and \( \bar{q}_n \) as:

\[
\ddot{q}_n + \omega_n^2 q_n = \frac{Q_n}{m_n \kappa_n} \quad n = 1, 2, 3, \ldots \tag{8.7}
\]

\[
\ddot{\bar{q}}_n + \omega_n^2 \bar{q}_n = \frac{\bar{Q}_n}{m_n \kappa_n} \quad n = 0, 1, 2, \ldots \tag{8.8}
\]

where \( Q_n \) and \( \bar{Q}_n \) are the generalized forces due to the applied radial and tangential forces on the shell in the \( n \)th mode, and \( m_n \) and \( \bar{m}_n \) are the generalized masses:

\[
m_n = m\left(1 + \frac{1}{d_n^2}\right) \quad \bar{m}_n = m(1 + d_n^2) \tag{8.9}
\]

(Detail derivation of the equations of motion in terms of the generalized coordinates can be found in Appendix I of Ref. 8.5.)
The generalized forces due to applied traction can be computed once the applied traction is specified. It is through the applied traction that one determines the structure-medium interaction.

Let us assume now that due to the interaction between the shell and the surrounding medium there exist radial and tangential forces at the boundary that can be represented by

\[ R(\theta, t) = \sum_{n=0}^{\infty} R_n(t) \cos n\theta, \quad (8.10) \]

\[ \Theta(\theta, t) = \sum_{n=0}^{\infty} \Theta_n(t) \sin n\theta. \]

However, the form of \( R_n(t) \) and \( \Theta_n(t) \) as yet has not been specified. The generalized forces \( Q_n \) and \( \overline{Q}_n \) can now be expressed in terms of these unknown applied tractions as:

\[ Q_n = a \int_0^{2\pi} \left[ -R_n(t) \cos^2 n\theta + \frac{\Theta_n(t)}{d_n} \sin^2 n\theta \right] d\theta = -\pi a \left[ \frac{R_n(t)}{d_n} - \frac{1}{d_n} \Theta_n(t) \right], \quad (8.11) \]

and

\[ \overline{Q}_n = a \int_0^{2\pi} \left[ -R_n(t) \cos^2 n\theta - \Theta_n(t) \sin^2 n\theta \right] d\theta = -\pi a \left[ R_n(t) + d_n \Theta_n(t) \right]. \]

Substituting (8.11) into the equations of motion in (8.7) and (8.8), the following ensue:
\[
\ddot{q}_n + \omega_n^2 q_n = -\frac{\left[R_n(t) - \frac{1}{d_n} \theta_n(t)\right]}{m_n},
\]
\[
\ddot{q}_n + \omega_n^2 q_n = -\frac{\left[R_n(t) + \frac{1}{d_n} \theta_n(t)\right]}{m_n},
\]

or the unknown force applied at the boundary as a result of the interaction is now expressible in terms of the generalized coordinates as:

\[
R_n(t) = -m \left[\ddot{q}_n + \ddot{\theta}_n + \omega_n^2 q_n + \omega_n^2 \dot{\theta}_n\right];
\]

\[
\theta_n(t) = \frac{m}{d_n} \left[\ddot{q}_n + \omega_n^2 q_n - d_n^2 (\ddot{q}_n + \omega_n^2 \dot{q}_n)\right].
\]

To determine unknown tractions at the boundary of the shell and the medium, we shall impose the condition of compatibility at \( r = a \). That is, the radial and tangential displacement at \( r = a \) for the shell and the medium must be the same. Hence at \( r = a \),

\[
\omega(\theta) = \omega_r(m),
\]

\[
\nu(\theta) = \omega_\theta(m),
\]

where \( \omega(\theta) \) and \( \nu(\theta) \) are the shell's displacements and \( \omega_r(m) \) and \( \omega_\theta(m) \) are the displacements of the medium; and

\[
\omega_r(m) = \omega_r^{(1)} + \omega_r^{(2)};
\]

\[
\omega_\theta(m) = \omega_\theta^{(1)} + \omega_\theta^{(2)};
\]
in which \( u_r^{(1)} \), \( u_\theta^{(1)} \) denote that portion of the displacement due to the incident step wave without the shell, and \( u_r^{(2)} \) and \( u_\theta^{(2)} \) are the contributions due to unknown boundary tractions \( R(t) \) and \( \Theta(t) \).

The displacements for the cavity case can be similarly determined by the method shown in Section 5; they are given in Ref. 5.2. They can be expressed as

\[
\begin{align*}
    u_r^{(1)} &= \sum_{n=0}^\infty F_n(t) \cos n\theta; \quad (8.15a) \\
    u_\theta^{(1)} &= \sum_{n=1}^\infty G_n(t) \sin n\theta. \quad (8.15b)
\end{align*}
\]

The displacements due to \( R(t) \) and \( \Theta(t) \) are obtained, first by assuming a Heaviside unit step function acting at the boundary, then by means of the Duhamel integral. The displacements due to \( R(t) \) and \( \Theta(t) \) are found to be

\[
\begin{align*}
    u_r^{(2)} &= u_r^{(2)'} + u_r^{(2)''}, \\
    u_\theta^{(2)} &= u_\theta^{(2)'} + u_\theta^{(2)''},
\end{align*}
\]

where \( u_r^{(2)'} \), \( u_\theta^{(2)'} \), and \( u_r^{(2)''} \), \( u_\theta^{(2)''} \) denote the displacements produced by \( R(t) \) and \( \Theta(t) \) respectively, and are given by

\[
    u_r^{(2)'} = \sum_{n=0}^\infty \left( R_n(0)\tilde{u}_r^{(n)}(t) + \int_0^t \frac{dR_n(t)}{dt} \tilde{u}_r^{(n)}(t - \tau) d\tau \right) \cos n\theta,
\]
\[ u_2^{(1)} = \sum_{n=0}^{\infty} \left( R_n(0) \tilde{u}_{\theta n}(t) + \int_0^t \frac{\partial R_n(\tau)}{\partial \tau} \tilde{u}_{\theta n}(t - \tau) d\tau \right) \sin n\theta, \]
\[ u_2^{(2)} = \sum_{n=0}^{\infty} \left( \omega(0) \tilde{u}_{\theta n}(t) + \int_0^t \frac{\partial \omega_n(\tau)}{\partial \tau} \tilde{u}_{\theta n}(t - \tau) d\tau \right) \end{aligned} \]
\[ u_2^{(2)} = \sum_{n=0}^{\infty} \left( \Theta(0) \tilde{u}_{\theta n}(t) + \int_0^t \frac{\partial \Theta_n(\tau)}{\partial \tau} \tilde{u}_{\theta n}(t - \tau) d\tau \right) \sin n\theta, \]

in which \( \tilde{u}_{\theta n}, \tilde{u}_{\theta n}, \tilde{u}_{\theta n}, \tilde{u}_{\theta n} \) are the indicial responses obtained by assuming the boundary traction as a step function; hence the responses are also known.

Using Eqs. (8.16), (8.15), (8.3), and (8.4), and the relationship between \( R_n(t), \theta_n(t), \) and \( q_n(t), \bar{q}_n(t) \) in Eq. (8.15), together with initial conditions \( q_n(0) = \bar{q}_n(0) = \bar{q}_n(0) = \bar{q}_n(0) = 0, \) results in the following sets of coupled integral equations for the determination of \( q_n \) and \( \bar{q}_n: \)

\[ n = 0, \]
\[ F_0(t) = \int_0^t \left[ q_n(0) + \omega_n^2 \bar{q}_n(t) \right] \tilde{u}_{\theta n}(t - \tau) d\tau - m[q_n(0)] \tilde{u}_{\theta n}(t) + \bar{q}_n(t); \]
\[ \begin{aligned} \end{aligned} \]
\[ n \neq 0, \]
\[ F_n(t) = \int_0^t m[q_n(t) + \omega_n^2 \bar{q}_n(t) + \omega_n^2 \bar{q}_n(t)] \tilde{u}_{\theta n}(t - \tau) d\tau \]
\[ + \int_0^t m \frac{d}{dn} \left[ q_n(t) + \omega_n^2 \bar{q}_n(t) - \frac{d}{dn} (q_n(t) + \omega_n^2 \bar{q}_n(t)) \right] \tilde{u}_{\theta n}(t - \tau) d\tau \]
\[ - m[q_n(0) + \bar{q}_n(0)] \tilde{u}_{\theta n}(t) + m \frac{d}{dn} [q_n(0) - \frac{d}{dn} q_n(0)] \tilde{u}_{\theta n}(t) \]
\[ + q_n(t) + \bar{q}_n(t) = 0; \]
\[ \begin{aligned} \end{aligned} \]
\[ g_n(t) = \int_0^t m[\dddot{q}_n(\tau) + \dddot{q}_n(\tau) + \omega_n^2 q_n(\tau) + \frac{2}{\omega_n^2} \ddot{q}_n(\tau)] \ddot{u}_{\theta n} (t - \tau) d\tau \]

\[ + \int_0^t \frac{m}{d_n} [\dddot{q}_n(\tau) + \omega_n^2 q_n(\tau) - \frac{1}{d_n} (q_n(\tau) + \frac{\omega_n^2}{d_n} \ddot{q}_n(\tau))] \ddot{u}_{\theta n} (t - \tau) d\tau \]

\[ - m[\dddot{q}_n(0) + \dddot{q}_n(0)] \ddot{u}_{\theta n} (t) + \frac{m}{d_n} [\dddot{q}_n(0) - \frac{\omega_n^2}{d_n} \ddot{q}_n(0)] \ddot{u}_{\theta n} (t) \]

\[ - \frac{q_n(t)}{d_n} + d_n \dddot{q}_n(t) = 0. \] (8.19)

Once the generalized coordinates are determined, the stresses in the shell are given in terms of the generalized coordinates as:

\[ \sigma_{\theta \theta} = - \frac{E}{(1-\nu^2) \alpha} \sum_{n=0}^{\infty} (n^2 + 1) \dddot{q}_n \cos n\theta; \] (6.20a)

\[ \sigma_B = - \frac{E d}{a^2} \sum_{n=0}^{\infty} [q_n(t) + \dddot{q}_n(t)] \cos n\theta. \] (8.20b)

where \( \sigma_B \) is the flexure stress due to the change in curvature in the shell, and \( d \) is the distance from the neutral axis of the shell to its extreme fiber. \( \sigma_{\theta \theta} \) is the hoop stress due to extensional modes only.

The approach above was first derived by Baron and Parnes (Ref. 0.10). Numerical results of Eqs. (8.17) through (8.19) were obtained by numerical method using a step integration in time technique.

Typical results for two steel shells \( (E = 30 \times 10^6 \text{ psi}, \rho = 0.283 \text{ lb/in}^3, h/a = 0.0192; h/a = 0.0476) \) embedded in a rock-type earth material \((8.9 \times 10^6 \text{ psi}, \rho = 0.0967 \text{ lb/in}^3)\) are shown in Figs. 8.1 through 8.5. Throughout the calculation, only \( n = 0,1,2 \) modes are used.
Thus, the early time solution, as pointed out earlier, remains questionable.

Fig. 8.1. Radial Displacement Component; $n = 0$

Fig. 8.2. Radial Displacement Component; $n = 1$
Fig. 8.3. Radial Displacement Component; $n = 2$

Fig. 8.4. Hoop Stress (Extensional), $\sigma_{\theta \theta}$ Liner 1, $h/a = 0.019$
Shown in Figs. 8.4 and 8.5 are the hoop stresses of the two liners due to an incident step wave, i.e., the stresses defined by Eq. (8.20a). The flexure stress \( c_{\theta \theta} \) defined by (8.20b) is proportional to \( \mu_{\text{m}} \), and therefore can be computed from Figs. 8.1, 8.2, and 8.3.

Shown are the stresses at two positions in the liner, \( \theta = 0, \pi/2 \).

The predominant modes are the \( n = 0, 2 \) modes, while the \( n = 1 \) mode contributes to early time behavior and is quickly damped out.

Some information worth noting here would be the relative magnitude of the flexure stress to hoop stress, the comparison between the dynamic solution and the static solution (the asymptote), and comparison of the maximum value of transient solution to the steady-state solution.

First we note for shell (1) \((h/a = 0.019)\) the maximum hoop
stresses due to $n = 0, 2$ modes are 4.79 and 4.69 respectively, while the maximum flexure stresses $\sigma_\theta$ due to the same two modes are computed by using values given in Figs. 8.1 and 8.3, and in Eq. (8.20b). They are 0.043 and 0.140. Thus flexure stress for a very thin shell is relatively unimportant compared with hoop stress. On the other hand, for shell (2) ($h/a = 0.0476$), we find the flexure stress is now 0.55 and 1.70 as compared with $\sigma_\theta = 4.79$ and 3.42 for $n = 0, 2$. Thus flexure stress is quite important in the total stress buildup in shell (2).

Second, we note the similarity in pattern of behavior between the shell medium interaction solution and the transient cavity solution. We find that the maximum value for the dynamic solution is always higher than the static value by approximately 10% to 20%.

Third, if we now compare the maximum stress from the transient analysis with that from the corresponding steady-state solution, as in Figs. 4.8b through 4.10a, we note that the amount of overshoot above the corresponding static solution for both cases is almost the same. This phenomenon was also observed previously for the cavity case.

We have shown a technique for solving structure-medium interaction problems. Needless to say, it is complex and numerically tedious. In what follows, we shall present some approximate methods for solving the transient problem.
8.2. Approximate Techniques for Solving Transient Structure-Medium Interaction Problems

There exist several other expositions on transient interaction problems. In Refs. 8.1 and 8.7 the Fourier expansion technique of the incident wave, coupled with the shell equations expressed in the generalized coordinates, is used in a way similar to that in the example in subsection 8.1. However, in Ref. 5.10 a train of pulses from steady-state components is used, where each pulse represents the time history of the transient stress in the incident wave. Obtaining the numerical results desired in each of these approaches, however, requires a great deal of effort, primarily by electronic computer.

The Fourier expansion technique should, theoretically, give an exact solution. In practice, however, solutions are often truncated after a few terms because of the time required on a computer. Therefore, the accuracy of the solution, especially for a short time, is often found to be inadequate. Nevertheless, an overall insight into the general behavior of structure-medium interaction problems is provided.

The Method of Substitute Kernel. Methods are available which avoid much of the tedious computation. One such method, referred to as the "substitute kernel method" by Carrier (Ref. 8.8), has demonstrated its usefulness in a number of problems. The method, though, is motivated heuristically, and is without rigorous mathematical proof; results obtained to date compare with other known results, indicating its validity.

Let us consider the transient cavity response problem presented
in Section 5. We note that the dynamic response due to an aperiodic arbitrary input \( f(t) \) can be determined as:

\[
g(x^*_t, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x^*_t, \omega) F(\omega) e^{-i\omega t} \, d\omega, \tag{8.21}
\]

where \( \chi(x^*_t, \omega) \) is the admittance function, and \( F(\omega) \) is the Fourier transform of \( f(t) \). It was shown further in Section 5 that \( \chi(x^*_t, \omega) \) for the dynamic stress concentration problem is precisely that obtained in Section 3 for the harmonic time-varying incident wave. Much of the difficulties in obtaining an analytic solution of Eq. (8.21) stem from the fact that \( \chi(x^*_t, \omega) \) is a very complicated function of \( \omega \).

The complexity of the function is evident in Eq. (3.14). For any other type of obstruction, i.e., a liner, a thickwall cylinder, etc., the complexity multiples, and any explicit analysis such as that carried out in Section 5 becomes almost impossible. The advantage of the "substitute kernel method" is to replace a very complex algebraic kernel, in this case \( \chi(x^*_t, \omega) \), by a suitable simple algebraic function which will facilitate inversion.

The attempt to find a suitable replacement for \( \chi(x^*_t, \omega) \) in Eq. (8.21) is aided by noting that \( g(x^*_t, t) \) through the convolution theorem, can be written as

\[
g(x^*_t, t) = \int_{-\infty}^{\infty} g^*_\delta(x^*_t, \tau) F(\tau) \, d\tau, \tag{8.22}
\]

where \( g^*_\delta(x^*_t, \tau) \) is the impulse response or the Fourier inversion of \( \chi(x^*_t, \omega) \) — see Section 4 of Chapter I. The operation implied by
Eq. (8.22) is sketched in Fig. 8.6. It is apparent that for each
time \( t \), one must calculate the area under the curve associated with
the product of \( g_\delta(x_{i+}, t-\tau) \) and \( F(\tau) \). If we are to replace the curve
\( g_\delta(x_{i+}, t-\tau) \) by a suitable approximation -- which is equivalent to

![Graphical representation of synthesis integral](image)

*Fig. 8.6. Graphical representation of synthesis integral*

replacing \( \chi(x_{i+}, \omega) \) in Eq. (8.21) -- \( g_\delta(x_{i+}, t-\tau) \) should have in common
with its replacement, say \( g^*(x_{i+}, t-\tau) \), any gross feature which charac-
terizes the results of the integration. In Carrier's paper, Ref.
8.8, it has been shown that the important gross properties of \( g^* \) are
the area under \( g^* \), the first moment of \( g^* \) (which measures the lop-
sidedness of \( g \)), and any particular resonances that \( g \) may have as
depicted by large responses of \( \chi(x_{i+}, \omega) \) at particular frequencies \( \omega \).

Thus we first require

\[
\int_{-\infty}^{\infty} g^* \, d\tau = \int_{-\infty}^{\infty} g(\tau) \, d\tau; \tag{8.23}
\]

\[
\int_{-\infty}^{\infty} \tau g^* \, d\tau = \int_{-\infty}^{\infty} \tau g(\tau) \, d\tau. \tag{8.24}
\]
According to the moment theorems given in Eq. (I-4.18),

\[ \int_{-\infty}^{\infty} g(\tau) \, d\tau = \chi(x,0) \, , \]

\[ \int_{-\infty}^{\infty} \tau g(\tau) \, d\tau = i \frac{d\chi(x,0)}{d\omega} \, . \]

Then Eqs. (8.23) and (8.24) imply

\[ \chi^*(x,0) = \chi(x,0) \, ; \quad (8.25) \]

\[ \frac{d\chi^*(x,0)}{d\omega} = \frac{d\chi(x,0)}{d\omega} \, . \quad (8.26) \]

where \( \chi^* \) denotes the Fourier transform of \( g^* \). In other words, the replacement admittance function \( \chi^* \), and its first derivative should have the same value as the admittance function \( \chi \) and \( \partial\chi/\partial\omega \) at \( \omega = 0 \). The other gross features that \( \chi^* \) must possess, besides the behavior at \( \omega = 0 \), would be a particular resonance-like behavior at any particular frequency that \( \chi \) possesses, and the approaching of \( \chi^* \) to \( \chi \) as \( \omega \to \infty \). The first requirement implies that \( \chi^* \) will reproduce the major characteristic of \( \chi \), while the latter implies that the singular behavior of \( g^* \) is similar to \( g \) at \( t = \tau \).

The following illustration exemplifies the process. To illustrate the scope of the idea, we shall use the same problem presented in Section 5 and then present additional results on a thickwall cylinder embedded in an elastic medium.

As we show in Section 5, the admittance function \( \chi \) for the stresses at the boundary — see Eq. (3.14) — is
\[
\chi(\theta, \alpha, \omega) = \frac{4}{\pi} \left( \frac{\kappa^2 - 1}{\kappa^2} \right) \sum_{n=0}^{\infty} \epsilon_n \epsilon_n^{n+1} \cos n\theta.
\]

In particular, the behavior of \( \chi \) at \( \theta = \pi/2 \) as depicted in Fig. 3.4 may be approximated by the simple algebraic expression

\[
\chi^*(\pi/2, \zeta) = C_1 + \frac{C_2 + iC_3}{\zeta - \zeta_0} + \frac{C_2 - iC_3}{\zeta + \overline{\zeta}_0},
\]

where \( \Re \zeta = \omega_\alpha / \sigma_\alpha \) and \( C_1, C_2, \) and \( C_3 \) are constants that must be chosen to satisfy the criteria given in Eqs. (8.25) and (8.26) and to satisfy the condition as \( \omega \to \infty. \)

\( \zeta_0 \) and \( \overline{\zeta}_0 \) represent a particular pair of poles which may represent the predominant resonant behavior in \( \chi \). This becomes clear if we recall from our earlier discussion that the real part of the pole represents the frequency and the imaginary part represents the radiation damping. Keeping this in mind, we note from Fig. 3.4 that the first major resonance in \( \Re \sigma_{66} \) occurs near \( \alpha = 0.28. \) This observation, coupled with the discussion of the physical significance of the real and imaginary parts of the pole, lead us to choose a pair of poles close to the origin. This pair of poles will yield results that are most persistent in time.

Using the values of \( \chi(\pi/2, 0) \), \( (\partial \chi / \partial (\omega \alpha)) \bigg|_{\omega \alpha = 0} \) and \( \chi(\pi/2, \omega \alpha) \) for \( \omega \alpha \) large, and \( \zeta_0 = 0.286 - i 0.279 \) (the closest pair of poles to the origin in Table 5.1), to determine \( C_1, C_2, \) and \( C_3, \) produces the results shown in Fig. 8.7 for \( \chi^* \). It is evident from this that \( \chi^* \) is in excellent agreement with \( \chi \) in this case.
Fig. 8.7. Fit of Frequency Response of Cylindrical Cavity

If we assume now that the incident wave is again a step function, then the Fourier inversion

$$
\bar{\sigma}_{\theta\theta}(\pi/2, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \chi^*(\omega) e^{-i\omega t} \, d\omega
$$

is immediate. It is

$$
\bar{\sigma}_{\theta\theta}(\theta, t) = R(t)C_1 + \frac{2R(t)}{\xi_0^2 + \eta_0^2} \left\{ \eta_0 \xi_0^\tau \left[ C_3(\eta_0 \cos \xi_0^\tau + \xi_0 \sin \xi_0^\tau) \right] \right.

- \left. C_2(\eta_0 \sin \xi_0^\tau - \xi_0 \cos \xi_0^\tau) - (C_3 \eta_0 + C_2 \xi_0) \right\}. \quad (8.28)
$$

Here $\tau$ is the usual dimensionless time -- i.e., $\tau = (C_o / a)t$ -- $\xi_0$ is the real part of $\xi_0$, and $\eta_0$ denotes the imaginary part of $\xi_0$. The results of Eq. (8.28) and the result shown in Fig. 5.5 for $\theta = \pi/2$ are shown in Fig. 8.8. The results of the approximate method as given by Eq. (8.28) (from Ref. 8.9), and results obtained in Section 5,
were very close indeed.

Fig. 8.8. Comparison Between Results Obtained by Approximate Method and by Series Expansion Method for a Cylindrical Cavity

Additional results using the replacement kernel in Eq. (8.27) were obtained in Ref. 8.9 for the thickwall cylinder embedded in an elastic medium. The difficulties in this case, as pointed out in Ref. 8.9, were that we do not know the value of the pair of poles closest to the origin. This difficulty may be bypassed, however, by choosing a set of values of $\zeta_0$ (guided by the bump in $\text{Re } \bar{\sigma}_{66}$) such that one obtains a good fit for the admittance function which we intend to replace.

In particular, the thickwall cylinder case shown in Fig. 4.7b was examined. Results obtained by the approximate technique as given in Ref. 8.9 and those determined in Ref. 0.10 are shown in Fig. 8.9 for a ratio of outer to inner radii of 1.05. Here too, the approxi-
mations were shown to be in good agreement with those obtained by the more refined analysis presented earlier in this section.

![Graph showing comparison of results](image)

**Fig. 6.9. Comparison Between Results Obtained by Approximate Method and by Integral Equation Method**

The simplicity of the method was demonstrated by the two foregoing examples. It was shown that the method works well both qualitatively and quantitatively. Unfortunately, it is a method without rigor, and the choice of the form of $\chi^*$ to a large extent remains ad hoc. Nevertheless, the method shows promise of being a powerful tool for solving otherwise intractable problems.

The Trapezoidal Approximation. The foregoing technique involves some quite intricate reasoning as to the choice of the function $\chi^*$.

There is a more direct technique recently applied by Stovel in his study of the transient response of a cavity (Ref. 8.10). The method was developed originally by V. Solodovnikov (See Ref. 4.1, Chapter 1). Let us
consider the Fourier inversion integral expressed in Eq. (8.21). It has been observed in Chapter I, Section 4, that for a causal function \( f(t) = 0 \) for \( t < 0 \), the response \( g(x_i, t) \) can be expressed alternatively in terms of sine and cosine transforms. If the input is an impulse function, then

\[
g_\delta(x_i, t) = \frac{2}{\pi} \int_0^\infty R(\omega) \cos \omega t \, d\omega = \frac{2}{\pi} \int_0^\infty I(\omega) \sin \omega t \, d\omega,
\]

(8.29)

where \( R(\omega) \) and \( I(\omega) \) are the real and imaginary parts of the frequency response:

\[
\chi(x_i, \omega) = R(\omega) + iI(\omega),
\]

(8.30)

and \( g_\delta(x_i, t) \) is the impulse response. Also, we have shown earlier in Chapter I, that if \( g_\delta(x_i, t) \) is known, then the response to a unit step function \( H(t) \) is determined by a simple quadrature, i.e.,

\[
g_H(x_i, t) = \int_0^t g_\delta(x_i, \tau) \, d\tau.
\]

Using the real part of \( \chi(x_i, \omega) \), we obtain

\[
g_H(x_i, t) = \int_0^t g_\delta(x_i, \tau) \, d\tau = \frac{2}{\pi} \int_0^t \int_0^\infty R(\omega) \cos \omega \tau \, d\tau
\]

\[
= \frac{2}{\pi} \int_0^\infty \frac{R(\omega)}{\omega} \sin \omega t \, d\omega.
\]

(8.31)

Equation (8.31) will now be used to determine the transient re-
response of the cavity. Direct integration of Eq. (8.31) is just as difficult as the procedure discussed in Section 5. However, we have on hand the numerical results of $R(\omega)$ as depicted in Fig. 8.7. We can, in general, approximate a function such as $R(\omega)$ by a sum of simple functions. These simple functions may take a form of a triangle, a rectangle, or a trapezoid, as sketched in Fig. 8.10. Noting that the triangle and rectangle are special cases of the trapezoid,

![Diagram showing $R_1(\omega)$, $R_2(\omega)$, and $R_3(\omega)$ functions.](image)

Fig. 8.10. Some Simple Functions for the Approximations

we try for an approximation of $R(\omega)$ by dividing the function graphically into trapezoids and expressing the approximate solution as the sum of responses to the trapezoidal functions. Figure 8.11 exemplifies the decomposition of $R(\omega)$. The four trapezoids denoted by $R_1(\omega)$ which are used for the approximation are shown in Fig. 8.12.
Fig. 8.11. Approximation of $R(aa)$ by Four Trapeoids

Fig. 8.12. Functions of $R_i(aa)$

Hence we may consider

$$R(aa) = \sum_{i=0}^{4} R_i(aa).$$
Each $R_i(x)$ can be typically represented by the following function. Denoting $x = \xi$, we have

$$R_i(\xi) = \begin{cases} R_{oi}, & 0 < \xi < \xi_{ti}, \\ R_{oi}\left(\frac{\xi_{bi} - \xi}{\xi_{bi} - \xi_{ti}}\right), & \xi_{ti} < \xi < \xi_{bi}, \\ 0 & \xi_{bi} < \xi, \end{cases}$$

(8.32)

where the parameters $R_{oi}$, $\xi_{ti}$, and $\xi_{bi}$ are as shown in Fig. 8.12.

Using this approximation, the response becomes

$$g_n(x, \tau) = \frac{2}{\pi} \int_0^\infty \frac{R_i(\xi) \sin \xi \tau}{\xi} d\xi,$$

(8.33)

in which $\tau = (E_a/a)\tau$. The response for a typical trapezoidal function

$$g_{bi}(x, \tau) = \frac{2}{\pi} \int_0^\infty \frac{R_i(\xi) \sin \xi \tau}{\xi} d\xi$$

$$- \frac{2}{\pi} \left[ \int_0^{\xi_{ti}} \frac{R_{oi} \sin \xi \tau}{\xi} d\xi + \int_{\xi_{ti}}^{\xi_{bi}} \frac{R_{oi}}{\xi_{bi} - \xi_{ti}} \frac{\xi_{bi} - \xi}{\xi} \sin \xi \tau d\xi \right]$$

$$= \frac{2}{\pi} \left\{ \text{Si}(\xi_{ti} \tau) + \frac{\xi_{bi}}{\xi_{bi} - \xi_{ti}} \left[ \text{Si}(\xi_{bi} \tau) - \text{Si}(\xi_{ti} \tau) \right] \right.$$  

$$+ \frac{1}{\xi_{bi} - \xi_{ti}} \left[ \frac{\cos \xi_{bi} \tau - \cos \xi_{ti} \tau}{\tau} \right] \right\},$$

(8.34)

where
\[ Si(\xi t) = \int_0^{\xi t} \frac{\sin \xi t}{\xi} d\xi \]

is the sine integral and it is a well-tabulated function. The advantage of the method is evident by the results shown in Eq. (8.34). The otherwise complex response is now a sum of simple responses. The results obtained in Ref. 8.10, using the trapezoidal approximation, are shown in Fig. 8.13. These results and results from Chapter 3, Section 5 are almost identical.

![Graph showing comparison between results obtained by trapezoidal approximation and series expansion method.]

**Fig. 8.13.** Comparison Between Results Obtained by Trapezoidal Approximation and by Series Expansion Method

The question of the accuracy of an approximate method is important. Here, we may assess the accuracy of the method as follows. Consider \( R(\xi) \) as the actual frequency response curve, and \( R_1(\xi) \) as the approximate curve. We call the difference between the two curves
$R^*(\xi)$, so that

$$R^*(\xi) = R(\xi) - R_1(\xi).$$

Then the error for the step response due to the approximation is

$$E^*(\tau) = \frac{2}{\pi} \int_0^\infty \frac{R^*(\xi)}{\xi} \sin \xi \tau \, d\xi,$$

$$E^*(\tau) = \frac{2}{\pi} \int_{0^+}^\infty \frac{|R(\xi) - R_1(\xi)|}{\xi} \, d\xi. \quad (8.35)$$

Thus the error incurred by the approximation can be bounded by the integral shown in Eq. (8.35). Reference 8.10 has shown that with a reasonable approximation such as that shown in Fig. 8.11 the maximum error is less than 0.12. Considering the simplicity of the technique, this is well within the expected accuracy.
Some Formulas in Cylindrical Coordinates

Displacements -- Displacement potential relationship:

\[ u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \zeta \frac{\partial^2 \chi}{\partial r \partial z}, \]

\[ u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r} + \frac{\zeta}{r} \frac{\partial^2 \chi}{\partial \theta \partial z}, \]

\[ u_z = \frac{\partial \varphi}{\partial z} - \zeta \left[ \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} \left( r \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \right], \]

where \( \zeta \) is a scalar factor having the dimension length, used for the sole purpose of giving \( \chi \) the same dimension as \( \psi \).

Strains -- Displacement relationship:

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \]

\[ \varepsilon_{r\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \]

\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \]

\[ \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial r \partial \theta} - \frac{u_\theta}{r} \right), \]

\[ \varepsilon_{r z} = \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \]

\[ \varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_z}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right). \]
Stress -- Displacement potential relationships:

\[
\sigma_{rr} = \lambda \nu^2 \psi + 2\mu \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \left( \frac{\partial \psi}{\partial r} \right) + \nu \left( \frac{\partial^2 \psi}{\partial r^2} \right) \right],
\]

\[
\sigma_{\theta \theta} = \lambda \nu^2 \psi + 2\mu \left[ \frac{1}{r} \left( \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{1}{r} \left( \frac{\partial^2 \psi}{\partial r \partial \theta} \right) + \frac{1}{r} \left( \frac{\partial^2 \psi}{\partial \theta^2} \right) \right],
\]

\[
\sigma_{zz} = \lambda \nu^2 \psi + 2\mu \left[ \frac{\partial^2 \psi}{\partial z^2} + \nu \left( \frac{\partial^2 \psi}{\partial r^2} \right) \right],
\]

\[
\sigma_{r \theta} = \mu \left[ 2 \left( \frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + 2\nu \left( \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \right) \right],
\]

\[
\sigma_{r z} = \mu \left[ 2 \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right] + 2\nu \left( 2 \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \right),
\]

\[
\sigma_{\theta z} = \mu \left[ 2 \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right] + 2\nu \left( 2 \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \right).
\]

Components of stresses due to cylindrical wave functions

Displacement potentials:

\[
\varphi = \xi_n^{(i)}(\alpha r) \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i\gamma z} e^{-i\omega t},
\]

\[
\psi = \xi_n^{(i)}(\beta r) \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i\gamma z} e^{-i\omega t},
\]

\[
\chi = \xi_n^{(i)}(\beta r) \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i\gamma z} e^{-i\omega t},
\]

where \(\xi_n^{(i)}\) denotes cylinder function. \((i) = 1, 2, 3, 4\) are used to
identify the different Bessel functions, with $\zeta_n^{(1)} = J_n$, $\zeta_n^{(2)} = Y_n$, $\zeta_n^{(3)} = H_n^{(1)}$, $\zeta_n^{(4)} = H_n^{(2)}$; and

$$a^2 = p^2 - \gamma_p^2, \quad b^2 = s^2 - \gamma_s^2,$$

$$p^2 = \left(\omega/c_p\right)^2 \quad \text{... compressional wave numbers},$$

$$s^2 = \left(\omega/c_s\right)^2 \quad \text{... shear wave numbers}.$$

Stresses — Cylindrical wave function relationships ($e^{-i \omega t}$ omitted),

$\sigma_{rr}$ due to:

$$\varphi: \quad \lambda \nabla^2 \varphi + 2 \mu \frac{\partial^2 \varphi}{\partial r^2} = \left(\frac{2\mu}{r^2}\right) \zeta_{11}^{(i)} \begin{cases} \cos \eta \theta \\ \sin \eta \theta \end{cases} \ e^{\pm i \gamma_r z},$$

$$\zeta_{11}^{(i)} = (n^2 + n - s^2 r^2/2 + \gamma_p^2 r^2) \zeta_n^{(i)}(ar) \ - \ ar \zeta_{n-1}^{(i)}(ar).$$

$$\psi: \quad \frac{2\mu}{r^2} \left[ r \frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{\partial \psi}{\partial \theta} \right] = \left(\frac{2\mu}{r^2}\right) \zeta_{12}^{(i)} \begin{cases} \sin \eta \theta \\ \cos \eta \theta \end{cases} \ e^{\pm i \gamma_s z},$$

$$\zeta_{12}^{(i)} = - \eta \left[ - (n+1) \zeta_n^{(i)}(br) + \beta r \zeta_{n-1}^{(i)}(br) \right].$$

$$\chi: \quad 2 \mu \frac{\partial^3 \chi}{\partial r^2 \partial z} = \frac{2\mu}{r} \zeta_{13}^{(i)} \begin{cases} \cos \eta \theta \\ \sin \eta \theta \end{cases} \ e^{\pm i \gamma_s z},$$

$$\zeta_{13}^{(i)} = \ell (\pm i \gamma_s) \left[ (n^2 + n - s^2 r^2) \zeta_n^{(i)}(br) \ - \ \beta r \zeta_{n-1}^{(i)}(br) \right].$$

$\sigma_{\theta \theta}$ due to:

$$\varphi: \quad \lambda \nabla^2 \varphi + 2 \mu \left( \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right) = \frac{2\mu}{r} \zeta_{21}^{(i)} \begin{cases} \cos \eta \theta \\ \sin \eta \theta \end{cases} \ e^{\pm i \gamma_r z},$$
\[ \varepsilon_{21}^{(i)} = - \left( n^2 + n + \frac{s^2}{2} r^2 - \frac{p^2}{2} r^2 \right) G_n^{(i)}(ar) + ar G_{n-1}^{(i)}(ar). \]

\[ \psi : \quad \frac{2 \mu}{r^2} \left[ \frac{\partial^2}{\partial \theta^2} - r \frac{\partial^2}{\partial r \partial \theta} \right] = \frac{2u}{r^2} \psi^{(i)} \left\{ \begin{array}{l} \sin n \theta \\ \cos n \theta \end{array} \right\} e^{\pm i \gamma \theta^2}, \]

\[ \varepsilon_{22}^{(i)} = \mp n \left[ (n + 1) G_n^{(i)}(br) - br G_{n-1}^{(i)}(br) \right]. \]

\[ \chi : \quad \frac{2 \mu}{r^2} \left[ r \frac{\partial^2}{\partial r \partial z} + \frac{1}{\partial z^2} \right] = \frac{2 \mu}{r^2} \varepsilon_{23}^{(i)} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{\pm i \gamma \theta^2}, \]

\[ \varepsilon_{23}^{(i)} = \pm (\pm i \gamma \theta) \left[ - (n^2 + n) G_n^{(i)}(br) + br G_{n-1}^{(i)}(br) \right]. \]

\[ \sigma_{zz} \text{ due to:} \]

\[ \varphi : \quad \lambda \nabla^2 \psi + 2u \frac{\partial^2 \psi}{\partial z^2} = \frac{2 \mu}{r^2} \varepsilon_{31}^{(i)} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{\pm i \gamma \theta^2}, \]

\[ \varepsilon_{31}^{(i)} = \left( \pm 2r^2 - \frac{s^2}{2} \right) G_n^{(i)}(ar). \]

\[ \psi : \quad \text{None.} \]

\[ \chi : \quad - \frac{2 \mu}{r^2} \left[ \frac{\partial^2}{\partial r \partial z} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} \right] = \frac{2 \mu}{r^2} \varepsilon_{33}^{(i)} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{\pm i \gamma \theta^2}, \]

\[ \varepsilon_{33}^{(i)} = \pm (\pm i \gamma \theta) s^2 r^2 G_n^{(i)}(br). \]

\[ \sigma_{\theta \theta} \text{ due to:} \]

\[ \varphi : \quad \frac{2 \mu}{r^2} \left[ r \frac{\partial^2 \varphi}{\partial r \partial \theta} - \frac{p \varphi}{\partial \theta} \right] = \frac{2 \mu}{r^2} \varepsilon_{41}^{(i)} \left\{ \begin{array}{l} \sin n \theta \\ \cos n \theta \end{array} \right\} e^{\pm i \gamma \theta^2}, \]

\[ \varepsilon_{41}^{(i)} = \mp n \left[ - (n + 1) G_n^{(i)}(ar) + ar G_{n-1}^{(i)}(ar) \right]. \]
\[ \psi: \quad \frac{\mu}{r^2} \left[ \frac{3^2 \psi}{\partial \theta^2} - r^3 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] = \frac{2\mu}{r^2} \varepsilon_{42} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{42} = - \left( n^2 + n - \frac{2}{3} \right) \varepsilon_n^{(i)}(\beta r) + \beta r \varepsilon_{n-1}^{(i)}(\beta r). \]

\[ \chi: \quad \mu \left[ r^2 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} \right) + \frac{\partial^3 \chi}{\partial \theta \partial \theta \partial \theta} \right] = \frac{2\mu}{r^2} \varepsilon_{43} \left\{ \begin{array}{l} \sin n \theta \\ \cos n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{43} = \varepsilon(\pm i \gamma \varphi)(z) \left[ - (n + 1) \varepsilon_n^{(i)}(\beta r) + \beta r \varepsilon_{n-1}^{(i)}(\beta r) \right]. \]

\( \sigma_{rz} \) due to:

\( \varphi: \quad 2\mu \frac{\partial^2 \varphi}{\partial r \partial \theta} = \frac{2\mu}{r^2} \varepsilon_{51} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{51} = \varepsilon(\pm i \gamma \varphi)(r) \left[ - n \varepsilon_n^{(i)}(\alpha r) + \alpha r \varepsilon_{n-1}^{(i)}(\alpha r) \right]. \]

\( \psi: \quad \mu \frac{\partial^2 \psi}{\partial \theta^2} = \frac{2\mu}{r} \varepsilon_{52} \left\{ \begin{array}{l} \sin n \theta \\ \cos n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{52} = \frac{\varepsilon(\pm i \gamma \varphi)(\alpha r) \varepsilon_n^{(i)}(\beta r)}{2}. \]

\( \chi: \quad \mu \left[ \frac{2^3 \chi}{3 \partial \theta^2} - \frac{2}{2 \partial r} \nabla^2 \chi \right] = \frac{2\mu}{r^2} \varepsilon_{53} \left\{ \begin{array}{l} \cos n \theta \\ \sin n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{53} = \frac{\varepsilon(\pm i \gamma \varphi)(\alpha r) \varepsilon_n^{(i)}(\beta r)}{2r} \left[ - n \varepsilon_n^{(i)}(\beta r) + \beta r \varepsilon_{n-1}^{(i)}(\beta r) \right]. \]

\( \sigma_{gz} \) due to:

\( \varphi: \quad \frac{2\mu}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} = \frac{2\mu}{r^2} \varepsilon_{61} \left\{ \begin{array}{l} \sin n \theta \\ \cos n \theta \end{array} \right\} e^{i \gamma \varphi z}, \]

\[ \varepsilon_{61} = \varepsilon(\pm i \gamma \varphi)(r) \left[ - n \varepsilon_n^{(i)}(\alpha r) + \alpha r \varepsilon_{n-1}^{(i)}(\alpha r) \right]. \]
\( \varepsilon_{61}^{(i)} = (\pm i \gamma_p r)(\pm n) \varepsilon_{n}^{(i)}(ar). \)

\[ \psi: \quad -\mu \frac{\partial^2 \psi}{\partial r^2} = \frac{2\mu}{r^2} \varepsilon_{62}^{(i)} \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i \gamma_\theta^2 z}, \]

\[ \varepsilon_{62}^{(i)} = \frac{\mp i \gamma_\theta^2 r}{2} \left[ n \varepsilon_{n}^{(i)}(\beta r) - \beta r \varepsilon_{n-1}^{(i)}(\beta r) \right]. \]

\[ x: \quad \frac{\mu k}{r} \frac{\partial}{\partial \theta} \left[ 2 \frac{\partial^2 \chi}{\partial \theta^2} - \nabla^2 \chi \right] = \frac{2\mu}{r^2} \varepsilon_{63}^{(i)} \begin{cases} \sin n\theta \\ \cos n\theta \end{cases} e^{\pm i \gamma_\theta^2 z}, \]

\[ \varepsilon_{63}^{(i)} = \frac{\pm i \gamma_\theta^2}{2r} (\pm n)(\beta^2 r^2 - \gamma_\theta^2) \varepsilon_{n}^{(i)}(\beta r). \]

Displacement — Cylindrical wave function relationship:

\( u_r \) due to:

\[ \begin{align*}
\varphi: & \quad \frac{\partial \varphi}{\partial r} = \frac{1}{r} \varepsilon_{71}^{(i)} \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i \gamma_\theta^2 z}, \\
\varepsilon_{71}^{(i)} & = \left[ ar \varepsilon_{n-1}^{(i)}(ar) - n \varepsilon_{n}^{(i)}(ar) \right]. \\
\psi: & \quad \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \varepsilon_{72}^{(i)} \begin{cases} \sin n\theta \\ \cos n\theta \end{cases} e^{\pm i \gamma_\theta^2 z}, \\
\varepsilon_{72}^{(i)} & = \pm n \varepsilon_{n}^{(i)}(\beta r). \\
x: & \quad \frac{\partial^2 x}{\partial r \partial z} = \frac{\pm i \gamma_\theta^2}{r} \varepsilon_{73}^{(i)} \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} e^{\pm i \gamma_\theta^2 z}, \\
\varepsilon_{73}^{(i)} & = \pm i \gamma_\theta^2 \left[ \beta r \varepsilon_{n-1}^{(i)}(\beta r) - n \varepsilon_{n}^{(i)}(\beta r) \right].
\end{align*} \]
\( u_\theta \) due to:

\[
\psi: \quad \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{1}{r^2} \mathcal{E}_{81} \left\{ \begin{array}{c} \sin \pi \theta \\ \cos \pi \theta \end{array} \right\} e^{i \gamma_p z}.
\]

\[
\mathcal{E}_{81} = i \pi \mathcal{E}_{n} (ar).
\]

\[
\psi: \quad -\frac{\partial \Phi}{\partial r} = \frac{1}{r^2} \mathcal{E}_{82} \left\{ \begin{array}{c} \cos \pi \theta \\ \sin \pi \theta \end{array} \right\} e^{i \gamma_p z}.
\]

\[
\mathcal{E}_{82} = - \left[ \pi r \mathcal{E}_{n-1} (\pi r) - 2 \pi \mathcal{E}_{n} (\pi r) \right].
\]

\[
\psi: \quad \frac{\partial^2 \chi}{\partial \theta^2} = \frac{1}{r^2} \mathcal{E}_{83} \left\{ \begin{array}{c} \sin \pi \theta \\ \cos \pi \theta \end{array} \right\} e^{i \gamma_p z}.
\]

\[
\mathcal{E}_{83} = \left( i \pi \right) \left( -i \gamma_p \right) \mathcal{E}_{n} (ar).
\]

\( u_z \) due to:

\[
\psi: \quad \frac{\partial \Phi}{\partial z} = \mathcal{E}_{91} \left\{ \begin{array}{c} \cos \pi \theta \\ \sin \pi \theta \end{array} \right\} e^{i \gamma_p z}.
\]

\[
\mathcal{E}_{91} = \left( -i \gamma_p \right) \mathcal{E}_{n} (ar).
\]

\[
\psi: \quad \text{None.}
\]

\[
\psi: \quad -i \left[ \frac{\partial^2 \chi}{\partial \theta^2} - \frac{\partial \chi}{\partial z} \right] = \frac{1}{r} \mathcal{E}_{93} \left\{ \begin{array}{c} \cos \pi \theta \\ \sin \pi \theta \end{array} \right\} e^{i \gamma_p z}.
\]

\[
\mathcal{E}_{93} = \beta^2 r \mathcal{E}_{n} (ar).
\]
DIFFRACTION AND STRESS CONCENTRATIONS

CHAPTER III REFERENCES


Chapter IV

ELLIPtic CYLINDER PROBLEMS

BASICALLY, THE DIFFRACTION OF WAVES by an elliptic cylinder is not much different from the diffraction caused by a circular cylinder, especially when the eccentricity of the elliptical cross section is small. But in mathematical analysis, because of the geometry of the scatterer, an entirely different wave function is used, involving products of Mathieu functions; the angular Mathieu function contains the wave number in its argument. Thus at the boundary of an ellipse, the wave functions which are the sum of the P wave and S wave parts do not form an orthogonal set, and the boundary conditions cannot be satisfied exactly by using the wave-function expansion method. This imposes a serious difficulty in calculating stresses and displacements near the scatterer.

We shall, however, carry out the analysis as far as we can in a manner analogous to that in the previous chapter, and indicate how to get around this particular difficulty. The literature concerning the diffraction of elastic waves is much less abundant for elliptic cylinders than for circular cylinders or spheres. The first paper we can trace is by Sezawa, in 1927,\(^{(0.1)}\) in which the solution for the scattering of a P wave was formulated in terms of Mathieu functions.
In 1961, Harumi discussed the scattering of both P and S waves, and calculated the energy distribution-in-angle of the wave scattered by a rigid ribbon, which is treated as a limiting case (infinite eccentricity) of a general ellipse. In the meantime, the diffraction of an acoustic or electromagnetic wave by an elliptical obstacle has been treated extensively. The formal solution in terms of Mathieu functions can be found in the books by McLachlan (Ref. 0.3, p. 358), and by Morse and Feshbach (Ref. 0.4, p. 1428).

1. EQUATIONS IN ELLIPTIC COORDINATES AND MATHIEU FUNCTIONS

As in equation III-1.2, elliptical cylinder coordinates \((\xi, \eta, z)\) are defined by the transformation \(\xi + i\eta = \cosh^{-1} [(x + iy)/a]\) and \(z = z\).

Explicitly, we have

\[
x = a \cosh \xi \cos \eta, \quad 0 < \xi < \infty,
\]

\[
y = a \sinh \xi \sin \eta, \quad 0 < \eta < 2\pi,
\]

\[
z = z, \quad -\infty < z < \infty.
\]

The scale factors \(h_i^2\) \((i = \xi, \eta, z)\) in (III-1.1)

\[
da^2 = (h_\xi d\xi)^2 + (h_\eta d\eta)^2 + (h_z dz)^2
\]

are given by

\[
h_\xi^2 = h_\eta^2 = a^2 j^2, \quad h_z = 1;
\]

\[
j^2 = \cosh^2 \xi - \cos^2 \eta = \sinh^2 \xi + \sin^2 \eta
\]

\[
= \frac{1}{2} (\cosh 2\xi - \cos 2\eta).
\]
Elimination of $\eta$ from the first two equations of (1.1) results in the equation $(x/\alpha \cosh \xi)^2 + (y/\alpha \sinh \xi)^2 = 1$, which defines a family of ellipses on a plane $z = \text{constant}$. The ellipses have focal lengths $2\alpha$ and eccentricity $e = 1/cosh \xi$, as shown in Fig. 1.1. Similarly, elimination of $\xi$ gives rise to a family of hyperbolas defined by $(x/\alpha \cos \eta)^2 - (y/\alpha \sin \eta)^2 = 1$. Analogous to circular polar coordinates $r$ and $\theta$, we call $\xi$ and $\eta$ the radial and angular coordinates, respectively. They are related by

$$r = \alpha (\cosh^2 \xi - \sin^2 \eta)^{\frac{1}{2}} = \alpha (\sinh^2 \xi + \cos^2 \eta)^{\frac{1}{2}},$$

$$\theta = \tan^{-1} (\tanh \xi \tan \eta).$$

Fig. 1.1. Elliptic Coordinate System -- A Family of Confocal Ellipses (Focal Length $2\alpha$) Intersecting Orthogonally with Another Family of Hyperbolas

The ellipses have major axes $2\alpha \cosh \xi$ and minor axes $2\alpha \sinh \xi$. 
At the extremities of the major axis, \( \eta = 0 \) or \( \pi (y = 0) \) and \( x = \pm a \cosh \xi = \pm p \). At the extremities of the minor axis, \( \eta = \pi/2 \) or \( 3\pi/2 (x = 0) \) and \( y = \pm a \sinh \xi \). The eccentricity of the ellipses is given by \( e = 1/\cosh \xi = a/p \). As \( e \to 1 \), \( \xi \to 0 \) while \( a \to p \) -- thus a long ellipse degenerates to a line segment of length \( 2a \), which is also the distance between two focal points. If \( e \to 0 \), \( \xi \to \infty \), and \( p \) is assumed to remain constant, the foci coalesce \( (a \to 0) \) at the origin, and \( (a \cosh \xi) \) and \( (a \sinh \xi) \to 0 \). Then the ellipse tends to a circle of radius \( r \). In the meantime, the equation of hyperbola becomes \( x^2/cos^2 \eta - y^2/sin^2 \eta = a^2 = 0 \), or \( y/x = \tan \eta \). The confocal hyperbolas degenerate to radii of the circle, making angles \( \theta \) with the \( x \)-axis. Table 1.1 lists the relationships among the various parameters for an ellipse.

Table 1.1

<table>
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<th>( \xi )</th>
<th>( e )</th>
<th>minor axis/major axis</th>
</tr>
</thead>
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</tr>
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<td>0.9640</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
11. Mathieu Functions

In elliptic cylinder coordinates, the steady state wave equation — see Eq. 1.40 — takes the form

\[ \nabla^2 \phi + \kappa^2 \phi = \frac{1}{\alpha^2} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right) + \frac{\partial^2 \phi}{\partial z^2} + \kappa^2 \phi = 0. \] (1.4)

By separation of variables \( \phi(\xi, \eta, z) = X(\xi)Y(\eta)Z(z) \), we find that \( X, Y, \) and \( Z \) satisfy the following equations respectively

\[
\begin{align*}
X''(\xi) + \left( b - \frac{1}{2} a^2 (k^2 - \gamma^2) \cosh 2\xi \right) X(\xi) &= 0, \\
Y''(\eta) + \left( b - \frac{1}{2} a^2 (k^2 - \gamma^2) \cos 2\eta \right) Y(\eta) &= 0, \\
Z''(z) + \gamma^2 Z(z) &= 0,
\end{align*}
\] (1.5)

the \( b \) and \( \gamma^2 \) are separation constants. The third of the equations above obviously has solutions \( e^{\pm \gamma z} \), thus for \( \gamma^2 > 0 \), the separation constant \( \gamma \) has the meaning of wave number along the \( z \)-axis. The second is a Mathieu equation, which can be reduced easily to the canonical form (see Refs. 1.1, 1.2, and 1.3)

\[ \frac{d^2 Y}{d\eta^2} + \left( b - 2q \cos 2\eta \right) Y = 0, \quad q = \frac{1}{4} a^2 (k^2 - \gamma^2). \] (1.6)

The first is known as a modified Mathieu equation; it can be reduced to a Mathieu equation by setting \( \xi = t \eta \).

Generally, with \( q > 0 \), the coefficients \( b \) and \( q \) in (1.6) may have any value. Thus the solutions \( Y(\eta) \) may be unbounded or bounded as \( \eta \) increases, or they may be periodic with period \( \pi \) or \( 2\pi \), depending
on the values of \( b \) and \( q \). For the study of diffraction of waves by an elliptic cylinder, the surface \( \xi = \xi_0 \) (constant) is the boundary interior of the infinite space. In order to have a unique and single-valued solution, the function \( Y(\eta) \) should return to the same value as \( \eta \) increases by \( 2\pi \). Thus only periodic solutions of (1.6) with period \( \pi \) or \( 2\pi \) in \( \eta \) are of interest.

Solutions with period \( \pi \) can be expressed in terms of series \( \cos 2\pi r \) and \( \sin 2\pi r \) with \( r \) being integers. We write

\[
Y(\eta) = \sum_{r=0}^{\infty} [A_{2r} \cos 2\pi r + B_{2r} \sin 2\pi r]. \tag{1.7a}
\]

Substituting the series into (1.6), and collecting the coefficients of cosine and sine functions, we obtain two sets of homogeneous equations, one for \( A_{2r} \) and the other for \( B_{2r} \), as given below:

Coefficients of \( \cos 2\pi r \):

\[
\begin{align*}
(b - 4)A_0 - qA_2 &= 0, \\
(b - 4)A_2 - q(A_4 + 2A_0) &= 0, \\
(b - 4r^2)A_{2r} - q(A_{2r+2} + A_{2r-2}) &= 0, \quad r \geq 2.
\end{align*}
\]

Coefficients of \( \sin 2\pi r \):

\[
\begin{align*}
(b - 4)B_2 - qB_4 &= 0, \quad B_0 = 0, \\
(b - 4r^2)B_{2r} - q(B_{2r+2} + B_{2r-2}) &= 0, \quad r \geq 2.
\end{align*}
\]

In order to have nontrivial \( A_{2r} \) or \( B_{2r} \), their coefficients in the equations above, which form an infinite matrix, must satisfy a secular
equation. Through various approximations, each secular equation may be solved for \( b \) in terms of \( q \) and the order \( r \). We designate the roots of the secular equations by \( b = a_{2m}(q) \) \((m = 0,1,2,\ldots)\) for the cosine series and \( b = b_{2m}(q) \) \((m = 1,2,3,\ldots)\) for the sine series. \( a_{2m} \) and \( b_{2m} \) are known as the characteristic numbers of the Mathieu equations.

In other words only when \( b = a_{2m}(q) \) can equation (1.6) have the solution \( \sum A_{2r}^{(2m)} \cos 2r_\eta \). Similarly \( b = b_{2m}(q) \) is the necessary condition for the existence of the other solution \( \sum B_{2r}^{(2m)} \sin 2r_\eta \). The characteristic numbers are usually expressed in power series of \( q \) or in continuous fractions involving \( q \), and they are tabulated in Refs. 1, 4.

![Graph showing characteristic numbers](image)

*Fig. 1.2. Characteristic Numbers \( a_r, b_r, r = 0,1(1)5 \) (from Handbook of Mathematical Functions, Abramowitz and Stegun, eds.)*
and 1.5. Some of the results are shown in Fig. 1.2.

Once the characteristic numbers are determined, the set of homogeneous equations for \( A^{(2m)}_r \) will fix the ratios among them for a given \( a_{2m} \). If in addition, a normalization formula for these coefficients is adopted, then the values for all \( A^{(2m)}_r \) are completely determined. The same is true for coefficients \( B^{(2m)}_r \). Currently, there are two normalization procedures used in tabulating Mathieu functions; one is to assign a constant value, usually unity, for \( Y(\eta) \) or its derivative \( Y'(\eta) \) at \( \eta = 0 \); the other is to base upon the orthogonality conditions for \( Y(\eta) \) -- see Eq. 1.11 -- and to assign a constant value for \( \int Y^2(\eta) \, d\eta \). We shall follow the second scheme. Solutions with period \( 2\pi \) are also expressible in series:

\[
Y(\eta) = \sum_{r=0}^{\infty} [A_{2r+1} \cos (2r + 1)\eta + B_{2r+1} \sin (2r + 1)\eta]. \tag{1.7b}
\]

Again there exists a set of characteristic numbers \( a_{2m+1}(q) \) and \( b_{2m+1}(q) \) (Fig. 1.2), and for each characteristic number the coefficients \( A^{(2m+1)}_{2r+1} \) or \( B^{(2m+1)}_{2r+1} \) can be calculated.

The periodic solutions of the Mathieu equation, when \( b \) in (1.6) equals the characteristic numbers \( a_{m} \) or \( b_{m} \) mentioned above, are called Mathieu functions, which are designated by \( \phi e_m(\eta, q) \) and \( \phi s_m(\eta, q) \) (these stand for cosine-elliptic and sine-elliptic respectively), with \( m = 0, 1, 2, \ldots \). Each of them can be divided further according to their periodicity. The four types of Mathieu functions are:
\[ ae_{2m}(n,q) = \sum_{r=0}^{\infty} A_{2r}^{(2m)}(q) \cos 2rn, \quad \text{when } b = a_{2m}, \]

\[ se_{2m+2}(n,q) = \sum_{r=0}^{\infty} B_{2r}^{(2m+2)}(q) \sin 2rn, \quad b = b_{2m+2}, \]

(1.8a)

\[ ae_{2m+1}(n,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}(q) \cos (2r + 1)n, \quad b = a_{2m+1}, \]

\[ se_{2m+1}(n,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)}(q) \sin (2r + 1)n, \quad b = b_{2m+1}. \]

For \( n = 0 \) or \( \pi /2 \), the Mathieu functions have the following values:

\[ ae_{m}(0,q) = \sum_{r=0}^{\infty} A_{r}^{(m)}(q) \neq 0, \quad se_{m}(0,q) = 0, \]

\[ ae_{2m}(\pi/2,q) = \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)}(q) \neq 0, \]

(1.8b)

\[ ae_{2m+1}(\pi/2,q) = 0, \quad se_{2m}(\pi/2,q) = 0, \]

\[ se_{2m+1}(\pi/2,q) = \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2m+1)}(q) \neq 0. \]

The values for the coefficients \( A_{r}^{(m)}(q) \) and \( B_{r}^{(m)}(q) \) for lower orders \( m \) are tabulated in Refs. 1.4 and 1.5. As an illustration, the Mathieu functions of the lower integral orders are shown in Figs. 1.3 and 1.4.
Fig. 1.3a. Even Periodic Mathieu Function of Order Zero

Fig. 1.3b. Even Periodic Mathieu Function of Order One

Fig. 1.4a. Odd Periodic Mathieu Function of Order Zero
Fig. 1.4b. Odd Periodic Mathieu Function of Order One

The functions $ae_{2m}$ and $ae_{2m+2}$ ($m = 0, 1, 2, \ldots$) have period $\pi$, and there is a constant term in the series $ae_{2m}$, which is a function of $q$. As a consequence, $ae_0$ is never zero. The other two functions $ae_{2m+1}$ and $se_{2m+1}$ have a period $2\pi$. All functions of order $m$ have $m$ real zeros in $0 < \eta < \frac{1}{2}\pi$. The functions $ae_m(\eta, q)$ are even about $\eta = 0$, whereas $se_m(\eta, q)$ are odd. From the definition (1.7), it is easy to show that

$$ae_m(p\pi + \eta) = ae_m(p\pi - \eta), \quad p = \text{integers},$$

$$ae_m(p\pi + \eta) = (-1)^m ae_m(p\pi - \eta), \quad m$$

$$ae_m(\eta + \pi) = (-1)^m ae_m(\eta).$$

Thus it is sufficient to give tabular values in the interval $0 \leq \eta \leq \frac{1}{2}\pi$. 
The even and odd Mathieu functions defined in (1.8) satisfy the following orthogonality conditions ($\nu, m$ positive integers):

\[
\int_0^{2\pi} a_{\nu m}(\eta, \varrho) a_{\nu n}(\eta, \varrho) \, d\eta = 0, \quad \nu \neq m, \quad (1.10)
\]

\[
\int_0^{2\pi} a_{\nu m}(\eta, \varrho) a_{\nu n}(\eta, \varrho) \, d\eta = 0, \quad \nu \neq m.
\]

When $\nu = m$ = integers, the values of the first two integrals can be calculated by substituting (1.8) into (1.10). They are obviously functions of the values of the coefficients $A_{\nu r}^{(m)}$ and $B_{\nu r}^{(m)}$ which in turn depend on how they are normalized. The Mathieu functions shown in Figs. 1.3 and 1.4 are based on the following normalization formulas (Refs. 1.2 and 1.6)

\[
2[a_{0}^{(2m)}]^2 + \sum_{r=1}^{\infty} [a_{2r}^{(2m)}]^2 = \sum_{r=0}^{\infty} [a_{2r+1}^{(2m+1)}]^2
\]

\[
= \sum_{r=0}^{\infty} [b_{2r+1}^{(2m+1)}]^2 = \sum_{r=0}^{\infty} [b_{2r+2}^{(2m+2)}]^2 = 1, \quad (1.11)
\]

which are stipulated by requiring that

\[
\int_0^{2\pi} a_{\nu m}^2(\eta, \varrho) \, d\eta = \int_0^{2\pi} a_{\nu n}^2(\eta, \varrho) \, d\eta = \pi, \quad (1.12)
\]

for real values of $\varrho$ (Ref. 1.7). We shall adhere to this normalization
in this book. The other commonly used normalization formulas are 
\( a_m(0,q) = \Gamma A_r^{(m)} = 1, \) and \( a_m'(0,q) = \Gamma r B_r^{(m)} = 1, \) as in Refs. 1.4 and 1.8.

So far our discussion is restricted to \( q = \frac{1}{4} \alpha^2 (k^2 - \gamma^2) > 0. \)
For the case of \( q < 0, \) we set \( q = -q_1 \) with \( q_1 > 0, \) and Eq. (1.5b) changes to

\[ Y'' + (b + q_1 \cos 2\eta)Y = 0, \quad q_1 = \frac{1}{4} \alpha^2 (\gamma^2 - k^2) > 0. \]

If the independent variable is also changed to \( \eta_1 = \eta \pm \pi/2, \) the equation above can be reduced to canonical form because \( q_1 \cos 2\eta = -q_1 \cos 2\eta_1. \) With \( -q \) in (1.6), its solution is simply

\[ ce_{2m+p}(\eta, q) = (-1)^m ce_{2m+p}(\frac{\pi}{2} - \eta, q), \quad p = 0, 1, \]

\[ se_{2m+p}(\eta, q) = (-1)^m se_{2m+p}(\frac{\pi}{2} - \eta, q), \quad p = 1, 2. \]

The factor \((-1)^m\) appears because the coefficients \( A_r^{(m)} \) and \( B_r^{(m)} \) in the series for Mathieu functions depend on \( q. \)

In summary, the Mathieu equation (1.6) has periodic solutions \( ce_m(\eta, q) \) or \( se_m(\eta, q) \) according to \( b = a_m(q) \) or \( b_m(q) \), the characteristic numbers, respectively, as given in (1.8). The former is known as the even Mathieu function and the latter the odd Mathieu function. Even order Mathieu functions \( (m = 0, 2, 4, \ldots) \) have period \( \pi \) and odd order ones have period \( 2\pi. \) For a given value of \( b, \) either \( a_m \) or \( b_m, \) the second independent solution of (1.6) is non-periodic and will not be considered here.
Numerical values for $\alpha_m$ and $\beta_m$, the coefficients $A_m$ and $B_m$, and the functions $\omega_m(\eta)$ and $\sigma_m(\eta)$, were tabulated by Ince (1932) with the coefficients being normalized by Eq. (1.12). Since this table is not readily accessible, we reproduce a condensed version in Tables 1.2 and 1.3. A more extensive table was prepared later by the Computation Laboratory of the National Bureau of Standards in 1951 (Ref. 1.4), in which the $\alpha_m$, $\beta_m$ and $A_m$, $B_m$ are listed. The latter are normalized by the conditions $\omega_m(0,q) = \sigma_m(0,q) = 1$, which were first used in the table prepared by Stratton et al. (Ref. 1.5). The functions $\omega_m(\eta,q)$ and $\sigma_m(\eta,q)$ are tabulated by Wiltse and King (Ref. 1.9).

Table 1.2
Even Periodic Mathieu Functions

<table>
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<tr>
<th>$q$</th>
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| \( a_2 \) |       |     |     |     |     |     |     |
| \( n \) |       |     |     |     |     |     |     |
| 0      | 4.3713 | 5.1727 | 6.0452 | 6.8291 | 7.4491 | 7.8701 |
| 15     | 1.0860 | 1.0488 | 0.9637 | 0.8558 | 0.7353 | 0.6120 |
| 30     | 0.9793 | 1.0038 | 0.9577 | 0.8850 | 0.7946 | 0.6946 |
| 45     | 0.7294 | 0.8382 | 0.8891 | 0.9097 | 0.9065 | 0.8821 |
| 60     | 0.2986 | 0.4963 | 0.5875 | 0.7573 | 0.9586 | 0.9403 |
| 75     | -0.2137 | -0.0065 | 0.1457 | 0.2783 | 0.4043 | 0.5242 |
| 90     | -0.8157 | -0.7038 | -0.6721 | -0.6863 | -0.7245 | -0.7721 |

| \( a_3 \) |       |     |     |     |     |     |     |
| \( n \) |       |     |     |     |     |     |     |
| 15     | 1.0672 | 1.1283 | 1.1586 | 1.1502 | 1.1112 | 1.0515 |
| 30     | 0.8172 | 0.9237 | 1.0015 | 1.0397 | 1.0444 | 0.8418 |
| 45     | 0.1734 | 0.3577 | 0.5272 | 0.6631 | 0.7654 | 0.8418 |
| 60     | -0.5550 | -0.3689 | -0.1713 | 0.0120 | 0.1720 | 0.3126 |
| 75     | -0.9487 | -0.8560 | -0.7363 | -0.6158 | -0.5096 | -0.4188 |
| 90     | -0.7242 | -0.7165 | -0.6902 | -0.6622 | -0.6455 | -0.6439 |

| \( a_4 \) |       |     |     |     |     |     |     |
| \( n \) |       |     |     |     |     |     |     |
| 0      | 16.0338 | 16.1412 | 16.3387 | 16.6498 | 17.0966 | 17.6888 |
| 15     | 1.0351 | 1.0744 | 1.1168 | 1.1586 | 1.1933 | 1.2141 |
| 30     | 0.5747 | 0.6548 | 0.7396 | 0.8256 | 0.9065 | 0.9747 |
| 45     | -0.4012 | -0.2881 | -0.1606 | -0.0209 | 0.1256 | 0.2705 |
| 60     | -0.9890 | -0.9553 | -0.8975 | -0.8150 | -0.7098 | -0.5884 |
| 75     | -0.5846 | -0.6546 | -0.7086 | -0.7488 | -0.7620 | -0.7620 |
| 90     | 0.4299 | 0.3630 | 0.2981 | 0.2346 | 0.1732 | 0.1155 |

\[ \sigma_1(n,q) \]

\[ \sigma_2(n,q) \]

\[ \sigma_3(n,q) \]

\[ \sigma_4(n,q) \]
### Table 1.3

Odd Periodic Mathieu Functions

<table>
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<th>q</th>
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<th>4</th>
<th>5</th>
<th>6</th>
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<td></td>
<td>(a_{1}(n, q))</td>
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</tr>
<tr>
<td>n₀</td>
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<td>0.0000</td>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>15</td>
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<td>0.0417</td>
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<td>0.1362</td>
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<td>0.4865</td>
<td>0.4348</td>
<td>0.3913</td>
<td>0.3544</td>
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</table>

|    | \(a_{2}(n, q)\) |      |      |      |      |      |
| n₀ |      |      |      |      |      |      |
| 0  | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 15 | 0.4292 | 0.3631 | 0.3038 | 0.2523 | 0.2088 | 0.1726 |
| 30 | 0.7915 | 0.7148 | 0.6297 | 0.5691 | 0.5046 | 0.4470 |
| 45 | 0.9940 | 0.9768 | 0.9504 | 0.9175 | 0.8807 | 0.8420 |
| 60 | 0.9346 | 0.9943 | 1.0436 | 1.0825 | 1.1171 | 1.1328 |
| 75 | 0.5725 | 0.6438 | 0.7114 | 0.7739 | 0.8308 | 0.8822 |
| 90 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

|    | \(a_{3}(n, q)\) |      |      |      |      |      |
| n₀ |      |      |      |      |      |      |
| 0  | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 15 | 0.6712 | 0.6246 | 0.5732 | 0.5203 | 0.4681 | 0.4180 |
| 30 | 1.0149 | 1.0049 | 0.9786 | 0.9418 | 0.8981 | 0.8501 |
| 45 | 0.8219 | 0.9071 | 0.9703 | 1.0171 | 1.0506 | 1.0729 |
| 60 | 0.1502 | 0.2787 | 0.3905 | 0.4901 | 0.5801 | 0.6621 |
| 75 | -0.6113 | -0.5332 | -0.4697 | -0.4166 | -0.3704 | -0.3288 |
| 90 | -0.9462 | -0.9115 | -0.8935 | -0.8879 | -0.8911 | -0.8999 |

|    | \(a_{4}(n, q)\) |      |      |      |      |      |
| n₀ |      |      |      |      |      |      |
| 0  | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 15 | 0.8543 | 0.8355 | 0.8098 | 0.7781 | 0.7415 | 0.7016 |
| 30 | 0.9327 | 0.9870 | 1.0276 | 1.0549 | 1.0697 | 1.0740 |
| 45 | 0.1326 | 0.2610 | 0.3820 | 0.4936 | 0.5950 | 0.6863 |
| 60 | -0.7894 | -0.7064 | -0.6203 | -0.5340 | -0.4492 | -0.3669 |
| 75 | -0.8714 | -0.8722 | -0.8703 | -0.8678 | -0.8658 | -0.8653 |
| 90 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
1.2. Modified Mathieu Functions

As mentioned before, the first of Eq. (1.5) is a modified Mathieu equation. With \( q = \frac{1}{4} a^2 (k^2 - \gamma^2) \), it takes the form

\[
\frac{d^2 X}{d \xi^2} - (b - 2q \cosh 2\xi) X = 0. \tag{1.13}
\]

Since this equation is derivable from (1.6) by writing \( \xi = \pm \eta \), it follows that (1.13) has solutions like (1.7) with \( \eta \) replaced by \( \pm i \xi \).

The solutions, in series of hyperbolic functions, are no longer periodic, which is permissible for the geometry shown in Fig. 1.1. Denoting the solutions of (1.13) by \( C_m(\xi, q) \) and \( S_m(\xi, q) \), we have, when \( b = a_m \) or \( b_m \):

\[
C_m(\xi, q) = se(i \xi, q) = \sum_{r=0}^{\infty} A_r^{(m)} \cosh r \xi, \quad (b = a_m), \tag{1.14}
\]

\[
S_m(\xi, q) = (-i)se(i \xi, q) = \sum_{r=1}^{\infty} E_r^{(m)} \sinh r \xi, \quad (b = b_m).
\]

They are called modified Mathieu functions.

For wave propagation problems it is convenient to express the solutions of (1.13) in terms of a series of Bessel functions like

\[
X(\xi) = \sum_{r=0}^{\infty} (-1)^r C_{2r} J_{2r}(2\sqrt{q} \cosh \xi). \tag{1.15}
\]

The coefficients \( C_{2r} \) can be determined in exactly the same manner as \( A_{2r} \) in (1.7a). Surprisingly, the recursion formulas for \( C_{2r} \) are the same as those for \( A_{2r} \). Hence the characteristic numbers for (1.13)
are the same as those for (1.6) and $c_{2r}^{(2m)}$ is proportional to $A^{(2m)}_{2r}$.

We write $c_{2r}^{(2m)} = K A_{2r}^{(2m)}$ and fix the multiplier $K$ by equating (1.15) to the first series of (1.14) at $\xi = \pi i/2$. Listed below are the four types of modified Mathieu functions in terms of series of Bessel functions:

$$
Ce_{2m}(\xi, q) = \frac{\alpha_{2m}(\frac{\pi}{2}, q)}{A_0^{(2m)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} J_{2r}(2\sqrt{q} \cosh \xi), \quad b = a_{2m},
$$

$$
Ce_{2m+1}(\xi, q) = -\frac{\alpha_{2m+1}(\frac{\pi}{2}, q)}{\sqrt{q} A_1^{(2m+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2m+1)} J_{2r+1}(2\sqrt{q} \cosh \xi), \quad b = a_{2m+1},
$$

$$
Se_{2m+2}(\xi, q) = -\frac{s_{2m+2}(\frac{\pi}{2}, q)}{q B_2^{(2m+2)}} \tanh \xi \times \sum_{r=0}^{\infty} (-1)^r (2r+2)^{2m+2} B_{2r+1}^{(2m+2)} J_{2r+1}(2\sqrt{q} \cosh \xi), \quad b = b_{2m+2},
$$

$$
Se_{2m+1}(\xi, q) = \frac{s_{2m+1}(\frac{\pi}{2}, q)}{\sqrt{q} B_1^{(2m+1)}} \tanh \xi \times \sum_{r=0}^{\infty} (-1)^r (2r+1)^{2m+1} B_{2r+1}^{(2m+1)} J_{2r+1}(2\sqrt{q} \cosh \xi), \quad b = b_{2m+1},
$$

where $\alpha_m(x, q) = [\tilde{\alpha}_m(x, q)/dn]_{x=\pi/2}$, etc. Because $\cosh \xi$ is even in $\xi$, the function $Ce_m(\xi, q)$ is an even function. The other solution $Se_m(\xi, q)$ is an odd function because of the factor $\tanh \xi$.

The modified Mathieu functions in (1.16) are a solution of (1.13) for $b = a_m$ or $b_m$. A second independent solution is found by expanding
$X(\xi, q)$ into a series of Bessel functions of the second kind $Y_n(2\sqrt{q} \cosh \xi)$. The coefficient of each term of the series is identical to that of $J_n(2\sqrt{q} \cosh \xi)$. If we denote the second independent solution by $E\psi_m(\xi, q)$ or $G\psi_m(\xi, q)$ (see p. 159 of Ref. 1.2), the complete solution is given by

$$X(\xi, q) = aE\psi_m(\xi, q) + bE\psi_m(\xi, q), \quad (b = a_m),$$

and

$$X(\xi, q) = aG\psi_m(\xi, q) + bG\psi_m(\xi, q), \quad (b = b_m),$$

where $a$ and $b$ are two arbitrary constants.

Recall that for circular cylinder coordinates, we use Hankel functions $H_n^{(1)}, H_n^{(2)}(kr) = J_n(kr) \pm iY_n(kr)$ to represent the scattered waves because as $kr \to \infty$, $H_n^{(1)}, H_n^{(2)}(kr) \to (kr)^{-\frac{1}{2}}e^{\pm ikr}$, thus displaying the desired asymptotic behavior for a diverging or converging wave. The same can be said for wave functions in elliptical cylinder coordinates. In other words, to study travelling waves, we choose $C\psi_m(\xi, q) \pm iE\psi_m(\xi, q)$ for $b = a_m$, or $S\psi_m(\xi, q) \pm iG\psi_m(\xi, q)$ for $b = b_m$, as the two independent solutions.

The discussion above shows how the complete solution of the modified Mathieu equation is derived. The functions $C\psi$ and $F\psi$ or $S\psi$ and $G\psi$ indeed can be used with $ce$ or $se$ in the previous subsection to construct elliptical wave functions (see Chapter 9 of Ref. 1.2).

However, in Eq. (1.16) the use of two different numerical factors in the series to define the even and odd order functions causes considerable inconvenience in wave propagation analysis. Following Meixner, (1.3)
and Schäfke, (1.3) we adopt the following symbols and definitions for
the modified Mathieu functions \((\xi > 0)\) (1.6):

For \(b = a_m\), \(m = 0, 1, 2, \ldots\), \((p = 0 \text{ or } 1)\),

\[
M_m^{(j)}(\xi, q) = \left[ae_{2m+p}(0, q)\right]^{-1} \sum_{r=0}^{\infty} (-1)^{r+m} (2m+1) (2r+1) \frac{\partial^r}{\partial \xi^r} \varphi^{(j)}(2\sqrt{q} \cosh \xi);
\]

\[
(1.17)
\]

For \(b = b_m\), \(m = 0, 1, 2, \ldots\), \((p = 1 \text{ or } 2)\),

\[
M_m^{(j)}(\xi, q) = [ae_{2m+p}(0, q)]^{-1} \tanh \xi \sum_{r=0}^{\infty} (-1)^{r+m} (2m+1) \frac{\partial^r}{\partial \xi^r} \varphi^{(j)}(2\sqrt{q} \cosh \xi);
\]

In the above, \(j = 1, 2, 3, \text{ or } 4\) and \(\varphi_m^{(j)}(z)\) are the circular cylinder
functions of the \(j\)th kind:

\[
\begin{align*}
\varphi_m^{(1)}(z) &= J_m(z), & \varphi_m^{(2)}(z) &= Y_m(z), \\
\varphi_m^{(3)}(z) &= H_m^{(1)}(z), & \varphi_m^{(4)}(z) &= H_m^{(2)}(z).
\end{align*}
\]

\[(1.18)\]

For \(b = a_m\), we can either take the modified Mathieu functions of
the first kind \(M_m^{(1)}(\xi, q)\) and of the second kind \(M_m^{(2)}(\xi, q)\) as the two
independent solutions of Eq. (1.13), or take the pair \(M_m^{(3)}(\xi, q)\) and
\(M_m^{(4)}(\xi, q)\). For \(b = b_m\), the two independent solutions are either
\(M_m^{(1)}\) and \(M_m^{(2)}\) or the remaining pair. It follows from the definition
of Hankel functions that
\[ M_\nu^{(3,4)}(\xi, q) = M_\nu^{(1)}(\xi, q) \pm iM_\nu^{(2)}(\xi, q), \]
\[ \text{(1.19)} \]

Also, by comparing (1.17) with (1.16) we find that
\[ M_\nu^{(1)}(\xi, q) = \frac{(-1)^{m} \varphi^{(2m)}}{\varphi^{'2m}(\frac{\pi}{2}, q) \varphi^{'2m}(0, q)} \bar{c}_{\nu,2m}(\xi, q), \]
\[ \text{(1.20)} \]
\[ M_\nu^{(1)}(\xi, q) = \frac{(-1)^{m+1} \varphi^{(2m+1)}}{\varphi^{'2m+1}(\frac{\pi}{2}, q) \varphi^{'2m+1}(0, q)} \bar{c}_{\nu,2m+1}(\xi, q), \]
\[ M_\nu^{(1)}(\xi, q) = \frac{(-1)^{m} \varphi^{(2m+2)}}{\varphi^{'2m+2}(0, q) \varphi^{'2m+2}(\frac{\pi}{2}, q)} \bar{c}_{\nu,2m+2}(\xi, q), \]
\[ M_\nu^{(1)}(\xi, q) = \frac{(-1)^{m} \varphi^{(2m+3)}}{\varphi^{'2m+3}(0, q) \varphi^{'2m+3}(\frac{\pi}{2}, q)} \bar{c}_{\nu,2m+3}(\xi, q). \]

They differ only by a constant multiplier.

In the sequel, only the notations \( M_\nu^{(j)} \) and \( M_\nu^{(j)} \) will be used for the modified Mathieu functions. Since they are associated with the radial coordinates \( \xi \), they are also known as Radical Mathieu Functions. Other symbols used in literature for Mathieu functions and their relations are listed in "Table 20.10, Comparative Notations" in the Handbook of Mathematical Functions (Ref. 1.6). Graphs of radial Mathieu functions as normalized by equations similar to (1.12) are shown in Figs. 1.5 and 1.6.
Fig. 1.5. Radial Mathieu Functions of the First and Second Kinds and Their Derivatives (from Ref. 1.10, Blanch and Clemm, Tables Relating to Radial Mathieu Functions, Vol. 2, 1963).

Although the series (1.18) for $M_m^{(j)}$ and $M_m^{(j)}$ are uniformly convergent for $\xi > 1$ and are convergent even at $\xi = 1$, the convergence is rather slow even for moderate values of $\xi$. For numerical calculation and also for evaluating the derivatives of $M_m^{(j)}$ and $M_m^{(j)}$ at $\xi = 0$, it is convenient to represent them by the series of products of Bessel functions as given below (see Chapter 13 of Ref. 1.2).

$$M_m^{(j)}(\xi, q) = \sum_{r=0}^{\infty} (-1)^{r+m} A_{2r+p}^{(2m+p)} \left[ J_{r-s}(u_1) Z_{r+p+s}(u_2) + J_{r+p+s}(u_1) Z_{r-s}(u_2) \right] / A_{2(p+s)}^{(2m+p)} \xi_{2s+p}^s.$$
where \( \epsilon_{2s+p} = 2 \) if \( 2s+p = 0 \), \( \epsilon_{2s+p} = 1 \), otherwise,

\[
M_s^{(j)}(x, q, \epsilon) = \sum_{r=0}^{\infty} \frac{(-1)^{r+m}}{B_{2r+p}(2m+p)} \left( \frac{1}{2r-s}(u_1)Z_{r-p-s}(u_2) \right) \left( \frac{1}{2r+p}(u_2) \right),
\]

where

\[
u_1 = \sqrt{q} e^{-\xi}, \quad \nu_2 = \sqrt{q} e^{+\xi}.
\]

The choice of \( s \) in the above is quite arbitrary so long as \( A_{2s+p} \).
\( p = 0, 1 \) and \( E_{28+p} (p = 1, 2) \) are not zero. Series so defined converge uniformly for all values of \( \xi \) when \( q \neq 0 \). Thus, the values of derivatives of \( M_0 \) and \( M_{q} \) can be calculated by differentiating the series termwise.

Since \( \xi = 0 \) defines a strip of width \( 2a \) (Fig. 1.1) on the \( x-z \) plane, the values for the radial Mathieu functions at \( \xi = 0 \) are of particular interest. Of the four quantities, \( M_0^{(1)} \), \( M_0^{(1)} \), \( M_0^{(2)} \), and \( M_0^{(2)} \), and their derivatives with respect to \( \xi \), we note that

\[
M_0^{(1)} (0, q) = 0, \quad \left[ dM_0^{(1)} (\xi, q) / d\xi \right]_{\xi=0} = 0, \quad (1.22)
\]

and the rest all have nonvanishing values at \( \xi = 0 \).

Extensive numerical tables for radial Mathieu functions and their derivatives were prepared recently by Blanch and Clemm in two volumes of Tables Relating to the Radial Mathieu Functions (Ref. 1.10).

### 1.3. Wave Functions in Elliptical Coordinates

With the solutions of each of the equations in (1.5) obtained, the wave function \( \psi (\xi, \eta, z) \) which satisfies Eq. (1.4) can be constructed by combining the angular and radial Mathieu functions according to

\[
\psi_m (\xi, \eta, z) = X_m (\xi) Y_m (\eta) e^{i \gamma z}.
\]

Since we consider only waves in a region either interior or exterior to a complete ellipse, all wave functions must be periodic in \( \eta \), with a period \( 2a \) or its fractions. This limits the choice of solutions for \( Y_m (\eta) \) to angular Mathieu functions \( cs_m (\eta, q) \) and \( se_m (\eta, q) \) with integral order \( m \). None of the solutions for \( X_m (\xi) \) is periodic in real \( \xi \). Which of the four radial...
Mathieu functions $M_m^{(j)}(\xi, q)$ and $M_m^{(j)}(\xi, q)$, ($j = 1, 2, 3, 4$), or their combinations thereof, should be used for wave problems depending on their asymptotic behavior and the region of interest.

To have a meaningful combination, the separation constant $b$ in (1.5) for $X(\xi)$ and $Y(\eta)$ must assume the same characteristic number $a_m$ or $b_m$ which, in turn, depends on the value of $q = \frac{1}{4} a^2 (k^2 - \gamma^2)^2$. The order of the characteristic number fixes the order of wave function $\varphi_m$. Various product pairs of $X_m(\xi)$ and $Y_m(\eta)$ are listed in Table 1.4.

<table>
<thead>
<tr>
<th>$b = a_m$</th>
<th>$b = b_m$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_m^{(1)}(\xi, q) c e_m(\eta, q)$</td>
<td>$M_m^{(1)}(\xi, q) s e_m(\eta, q)$</td>
<td>Standing wave</td>
</tr>
<tr>
<td>$M_m^{(2)}(\xi, q) c e_m(\eta, q)$</td>
<td>$M_m^{(2)}(\xi, q) s e_m(\eta, q)$</td>
<td>Standing wave in a region excluding $\xi = 0$</td>
</tr>
<tr>
<td>$M_m^{(3)}(\xi, q) s e_m(\eta, q)$</td>
<td>$M_m^{(3)}(\xi, q) s e_m(\eta, q)$</td>
<td>Diverging wave**</td>
</tr>
<tr>
<td>$M_m^{(4)}(\xi, q) c e_m(\eta, q)$</td>
<td>$M_m^{(4)}(\xi, q) s e_m(\eta, q)$</td>
<td>Converging wave**</td>
</tr>
</tbody>
</table>

* Period $\pi$ or $2\pi$ in $\eta$ according to $m$ is even or odd integers. The factor $e^{\pm i \gamma \eta}$ is omitted in the table.

** When combined with the time factor $e^{-i\omega t}$. 

In Table 1.4 functions $a_{m}^{*}$ and $b_{m}^{*}$ are defined in (1.8), with the coefficients $A_{r}^{(m)}(q)$ and $B_{r}^{(m)}(q)$ normalized by formula (1.11). Various properties of symmetry for $a_{m}^{*}(n,q)$ and $b_{m}^{*}(n,q)$ are listed in Eq. (1.9). As a quick guide, their symmetry or antisymmetry about the major and minor axes of an ellipse are the same as for the degenerate forms $\cos mn$ and $\sin mn$ about $n = 0$ and $\pm n$, respectively. The modified Mathieu functions $M_{c}^{(j)}_{m}$ and $M_{e}^{(j)}_{m}$ are defined in (1.17) or in (1.21).

For waves in a region which contains the inter focal line ($\xi = 0$), unless there is a distribution of source function at $\xi = 0$, the wave function $\varphi_{m}(\xi,n,z)$ and its gradient must be continuous across the inter focal line (see Fig. 1.1). Continuity of $\varphi_{m}$ requires that

\begin{equation}
(a) \quad \varphi_{m}(0,n,z) = \varphi_{m}(0,-n,z),
\end{equation}

and continuity of the gradient of $\varphi_{m}$ demands that

\begin{equation}
(b) \quad \frac{\partial \varphi_{m}(\xi,n,z)}{\partial \xi}
\bigg|_{\xi=0} = -\frac{\partial \varphi_{m}(\xi,-n,z)}{\partial \xi}
\bigg|_{\xi=0}.
\end{equation}

Since $a_{m}^{*}(n) = a_{m}^{*}(-n)$ and $M_{c}^{(1)}_{m}(0)$ is a nonzero constant, the product $M_{c}^{(1)}_{m}(\xi)a_{m}^{*}(n)$ satisfies condition (a). Furthermore, as seen in (1.22),

\begin{equation}
M_{c}^{(1)}_{m}(0,q) = 0,
\end{equation}

so (b) is satisfied.

For the product $M_{c}^{(1)}_{m}(\xi)a_{m}^{*}(n)$, condition (a) is satisfied
because $M_s(1)_m(0) = 0$. So is the second condition because $se_m(\eta) =
- se_m(-\eta)$ and $[\partial M_2(1)_m(\xi) / \partial \xi]_{\xi=0} \neq 0$.

As for the product $M_2(2)_m(\xi) se_m(\eta)$, condition (a) is satisfied
but not condition (b) because $\partial M_2(2)_m(0) / \partial \xi \neq 0$. Similarly, the pro-
duct $M_s(2)_m(\xi) se_m(\eta)$ does not satisfy condition (a). Hence, these
two products cannot be used for a region that contains the interfocal
line $\xi = 0$. However, they are valid wave functions whenever the line
$\xi = 0$ is excluded from the region of concern, such as for the exterior
of an ellipse, or when there is a distribution of source at $\xi = 0$ as
in the case of radiation from a thin strip at $\xi = 0$.

For waves diverging from an elliptical boundary $\xi = \xi_0$, the wave
function to be used is $M_2(3)_m$ or $M_s(3)_m$. Their asymptotic values for
large $\xi$ are given by the following formula (see Ref. 1.6):

$$
\begin{align*}
M_2(3)_m(\xi, q) & \quad \longrightarrow \quad \frac{1}{\xi(2\sqrt{q} \cosh \xi - \frac{\pi}{4} + \frac{m\pi}{2})} e^{i(2\sqrt{q} \cosh \xi - \frac{\pi}{4} + \frac{m\pi}{2})} \\
(-1)^m M_s(3)_m(\xi, q) & \quad \xi \to (\pi\sqrt{q} \cosh \xi)^{\frac{1}{2}}
\end{align*}
$$

(1.24)

Both functions vanish at $\xi = \infty$. Also, for large $\xi$, (1.3) yields
$cosh \xi + r/a$, and the exponential factor of (1.24) when combined with
the time factor has the form $e^{i(x\theta - \omega t)}$, which indeed represents a
radially diverging wave.

The asymptotic formulas for $M_2(4)_m$ and $M_s(4)_m$ are

$$
\begin{align*}
M_2(4)_m(\xi, q) & \quad \longrightarrow \quad \frac{1}{\xi(2\sqrt{q} \cosh \xi - \frac{\pi}{4} + \frac{m\pi}{2})} e^{i(2\sqrt{q} \cosh \xi - \frac{\pi}{4} + \frac{m\pi}{2})} \\
(-1)^m M_s(4)_m(\xi, q) & \quad \xi \to (\pi\sqrt{q} \cosh \xi)^{\frac{1}{2}}
\end{align*}
$$

(1.25)
They are used for converging waves when combined with \(e^{-i\omega t}\). This is expected, as \(M_m^{(4)}\) and \(M_m^{(4)}\) are expressed in series of Hankel functions of the second kind, and each term when multiplied by \(e^{-i\omega t}\) represents a converging wave.

Finally, we examine the limiting case when the fundamental ellipse tends to a circle. It was mentioned before (Eq. 1.3) that as the eccentricity \(e \to 0\) \((\xi \to \infty)\), the focal length \(a \to 0\) but the product \(\alpha \cosh \xi = \rho\), a nonzero constant, so that the ellipse tends to a circle of radius \(r = \rho\). In the meantime, the confocal hyperbola degenerates to radii of the circle with \(n \to \theta\). Since \(q = \frac{1}{4\alpha^2}(k^2 - \gamma^2)\), \(q\) tends to zero except when \(k \to \infty\).

As \(q \to 0\), the coefficients \(A_n^{(m)}\) and \(B_n^{(m)}\) can still be calculated from the recursion formulas in Eq. 1.7, together with the power series expressions for the characteristic numbers \(a_m\) and \(b_m\). It is found that for \(q = 0\), all coefficients vanish except for \(r = m\), i.e., \(A_m^{(m)}\) and \(B_m^{(m)}\). From the normalization formula (1.11), we then have

\[
\begin{align*}
\psi_0(n,0) &= A_0^{(0)} = 1/\sqrt{2}, \\
\psi_m(n,0) &= A_m^{(m)} \cos mn = \cos m\theta, \\
\psi_m(n,0) &= B_m^{(m)} \sin mn = \sin m\theta.
\end{align*}
\]

(1.26)

This can be verified easily because the Mathieu equation (1.6) has solutions \(Y(n) = \cos \sqrt{\varphi} n\), \(\sin \sqrt{\varphi} n\) when \(q = 0\), and Fig. 1.2 shows that the characteristic numbers \(b = a_m\) and \(b_m\) all have values \(n^2\) if \(q = 0\).

Limiting values for modified Mathieu functions for \(e = 0\) can be
found from the limiting values of the series in (1.17). The coefficients $A_{2m+p}^{(2m+p)}$ and $B_{2m+p}^{(2m+p)}$ have the same values as those in (1.8).

The argument of the Bessel and Hankel functions is $2\sqrt{q} \cosh \xi$ approaches $\frac{r\sqrt{k^2 - \gamma^2}}{2}$. Hence

$$M_a^{(j)}(\xi, 0) = \frac{[e_2m+p(0,0)]^{-1} - A_{2m+p}^{(2m+p)}e_{2m+p}^{(j)}(\xi, 0)}{r\sqrt{k^2 - \gamma^2}},$$

$$p = 0 \text{ or } 1,$$

$$M_b^{(j)}(\xi, 0) = \frac{[e'_2m+p(0,0)]^{-1} - (2m + p)B_{2m+p}^{(2m+p)}e_{2m+p}^{(j)}(\xi, 0)}{r\sqrt{k^2 - \gamma^2}},$$

$$p = 1 \text{ or } 2.$$

Because

$$e_2m+p(0,0) \rightarrow A_{2m+p}^{(2m+p)} \quad \text{and} \quad e'_2m+p(0,0) \rightarrow (2m + p)B_{2m+p}^{(2m+p)},$$

we have

$$M_a^{(j)}(\xi, 0) = M_b^{(j)}(\xi, 0) + Z_2m+p^{(j)}(r\sqrt{k^2 - \gamma^2}). \quad (1.27)$$

The results above can also be verified by first approximating the modified Mathieu equation (1.13) for large $\xi$ and small $q$ by

$$x'' - (b - qe^{2\xi})x = 0, \quad q = \frac{1}{2}ae^2(k^2 - \gamma^2).$$

Putting $r = \frac{1}{2}ae^\xi$ changes the equation above to

$$r^2 \frac{d^2x}{dr^2} + r \frac{dx}{dr} + [(k^2 - \gamma^2)r^2 - b]x = 0.$$
From (1.26) and (1.27), it follows that when \( a \to 0 \) and \( \xi \to \infty \),

\[
M_0^{(j)}(\xi, q) \to 0, q \to 0, \frac{1}{\sqrt{2}} 2^{(j)}(r\sqrt{k^2 - \gamma^2}),
\]

\[
M_m^{(j)}(\xi, q) e_m(n, q) \to 2^{(j)}(r\sqrt{k^2 - \gamma^2}) m \cos \theta, \quad m > 0, \tag{1.28}
\]

\[
M_m^{(j)}(\xi, q) e_m(n, q) \to 2^{(j)}(r\sqrt{k^2 - \gamma^2}) m \sin \theta, \quad m > 0.
\]

The elliptic wave functions indeed reduce to circular wave functions (Section 2 of Chapter III), as the fundamental ellipse tends to a circle.

1.4. Expansion of Plane Waves in Elliptical Wave Functions

Central to the analysis of diffraction of waves by an elliptical cylinder is the expressing of the incident wave in terms of wave functions in elliptical coordinates. This depends on an important integral relation of Mathieu functions. We shall first establish this integral relation and then use it to expand a plane wave function into a series of elliptical wave functions.

Consider a plane wave with an unspecified amplitude \( f(\theta) \), with its wave normal making an angle \( \theta \) with the \( x \)-axis:

\[
f(\theta) e^{j[k(x \cos \theta + y \sin \theta) - \omega t]} = f(\theta) e^{j[k + \omega t]},
\]

\[
\omega = \cosh \xi \cos n \cos \theta + \sinh \xi \sin n \sin \theta.
\]
Then the integral

\[ u(x,y) = \int_0^{2\pi} f(\theta) e^{ik(x \cos \theta + y \sin \theta)} d\theta, \]  

(1.30)

represents the superposition of plane waves propagating in all directions. The integral (1.30) obviously satisfies the two-dimensional steady state wave equation

\[ (\nabla^2 + k^2)u(x,y) = 0. \]

In plane elliptical coordinates \((\xi, \eta, 0)\), the integral takes the form

\[ u(\xi, \eta) = \int_0^{2\pi} e^{iak\xi} f(\theta) d\theta, \]  

(1.31)

and it satisfies the equation \((\nabla^2 + k^2)u(\xi, \eta) = 0\), or

\[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{1}{2} k^2 a^2 \cos 2\eta \right) u = -\left( \frac{a^2}{\partial \xi^2} + \frac{1}{2} k^2 a^2 \cosh 2\xi \right) u. \]

The equation above is just the two-dimensional version of (1.4).

The kernel \(e^{iak\xi}\) in (1.31) also satisfies the two-dimensional wave equation. Since it is congruent in \(\eta\) and in \(\theta\), the following equality holds:

\[ \left( \frac{\partial^2}{\partial \eta^2} - \frac{1}{2} k^2 a^2 \cos 2\eta \right) e^{iak\xi} = \left( \frac{a^2}{\partial \eta^2} + \frac{1}{2} k^2 a^2 \cos 2\theta \right) e^{iak\xi}. \]

Differentiating both sides of (1.31) with respect to \(\eta\), we obtain, by utilizing the above equality,
\[
\frac{3^2 u(\xi, n)}{\zeta^2_n} = \int_0^{2\pi} f(\theta) \frac{3^2}{\zeta^2_n} \left( e^{ik\omega} \right) d\theta
\]

\[
= \int_0^{2\pi} f(\theta) \left\{ \frac{3^2}{\zeta^2_n} + \frac{1}{2} k^2 a^2 (\cos 2\eta - \cos 2\theta) \right\} e^{ik\omega} d\theta
\]

\[
= \frac{1}{2} k^2 a^2 \cos 2\eta u(\xi, n) + \int_0^{2\pi} f(\theta) \left( \frac{3^2}{\zeta^2_n} - \frac{1}{2} k^2 a^2 \cos 2\theta \right) e^{ik\omega} d\theta.
\]

Integration by parts changes the first term of the last integral to

\[
\int_0^{2\pi} f(\theta) \frac{3^2}{\zeta^2_n} \left( e^{ik\omega} \right) d\theta = \left[ f(\theta) \frac{3^2}{\zeta^2_n} e^{ik\omega} - e^{ik\omega} \frac{df}{d\theta} \right]_0^{2\pi}
\]

\[
+ \int_0^{2\pi} e^{ik\omega} \frac{df}{d\theta} d\theta.
\]

Thus, if \( f(\theta) \) is periodic in \( \theta \) with period \( 2\pi \) or \( \pi \),

\[
\left( \frac{3^2}{\zeta^2_n} - \frac{1}{2} k^2 a^2 \cos 2\eta \right) u = \int_0^{2\pi} e^{ik\omega} \left( \frac{3^2}{\zeta^2_n} - \frac{1}{2} k^2 a^2 \cos 2\theta \right) f(\theta) d\theta.
\]

Adding \( bu(\xi, n) \) to both sides of this equation, where \( b \) is the separation constant in (1.5), and then letting \( u(\xi, n) = X(\xi)Y(\eta) \), we obtain

\[
X(\xi) \left[ \frac{d^2 Y(\eta)}{d\eta^2} + (b - \frac{1}{2} k^2 a^2 \cos 2\eta) Y(\eta) \right] = -\int_0^{2\pi} e^{ik\omega} \left[ \frac{d^2 f(\theta)}{d\theta^2} + (b - \frac{1}{2} k^2 a^2 \cos 2\theta) f(\theta) \right] d\theta.
\]

If \( Y(\eta) \) satisfies the Mathieu equation (1.5) (with \( \gamma = 0 \), the relation
above shows that $X(\xi)$ must be a modified Mathieu function, satisfying
Eq. (1.5), and that $f$ is proportional to a Mathieu function. Writing
$f(\theta) = \rho Y(\theta)$ in (1.31), where $\rho$ is a constant, we finally arrive at
the important integral relation for Mathieu functions:

$$X(\xi)Y(\eta) = \rho \int_0^{2\pi} e^{i\kappa \xi \theta} y(\theta) \, d\theta.$$  

(1.32)

In the equation above, $Y(\eta)$ stands for any one of the Mathieu functions
and $X(\xi)$ for the corresponding modified Mathieu function.

The constant $\rho$ can be fixed by evaluating the integral on the
right-hand side of (1.32) for large $\xi$. Because

$$\omega = \frac{i}{2} e^{\xi} \cos(\eta - \theta) + \frac{i}{2} e^{-\xi} \cos(\eta + \theta),$$

for large $\xi$ only the first term becomes important and the integral

$$\int_0^{2\pi} e^{i\kappa \xi \theta} \cos(\eta - \theta) y(\theta) \, d\theta,$$

can be evaluated by the method of stationary phase for large $\xi$. It
has an asymptotic value:

$$\left[\frac{\pi}{\kappa \xi} \right]^{\frac{1}{2}} e^{i\kappa \xi \left(\frac{\pi}{2} - \frac{\eta}{4}\right)} y(\eta) \rho$$

$0 < \eta < 2\pi$.

Using the asymptotic value of $X(\xi)$, and comparing both sides of (1.32),
we can then determine the value for $\rho$, which of course depends on the
choice of the functions $Y(\eta)$ and $X(\xi)$. The following are the results
for $X(\xi) = M_{2^*}^{(1)}(\xi)$ and $M_{2^*}^{(1)}(\xi)$ (see Section 2.68 of Ref. 1.38).
Chapt. 10 of Ref. 1.2):

\[
\int_0^{2\pi} e^{i k \omega} se_m(\theta, q) \, d\theta = 2\pi i m e_m(n, q) M_m^{(1)}(\xi, q),
\]

(1.33)

\[
\int_0^{2\pi} e^{i k \omega} se_m(\theta, q) \, d\theta = 2\pi i m e_m(n, q) M_m^{(1)}(\xi, q),
\]

with \( q = (\frac{1}{2} k a)^2 \) and \( \omega = \cosh \xi \cos n \cos \theta + \sinh \xi \sin n \sin \theta \).

A kinematic interpretation of the important integral relation (1.32) is worth noting. As listed in Table 1.4, the products \( X(\xi)Y(n) \) represent waves when expressed in elliptical coordinates. Since \( \xi = \xi_0 \) defines a family of confocal ellipses, the function \( X(\xi)Y(n)e^{+i \omega t} \) can loosely be called "elliptical waves" even though the curve \( \xi = \xi_0 \) may not be the real wave front. With this understanding, Eq. (1.32) shows that "elliptical waves" are synthesized from plane waves moving in all directions, with their amplitudes being proportional to the Mathieu function \( Y(\theta) \).

Just like the Fourier expansion of a periodic function into a series of circular functions (cosine and sine functions), a function of period \( \pi \) or \( 2\pi \) can be expanded into a series of Mathieu functions \( se_m \) and \( se_m \). The plane wave function

\[
e^{i k (x \cos \theta + y \sin \theta)} = e^{i k \omega},
\]

which was introduced in (1.29), is periodic in the variable \( \theta \). It then has the following expansion:
\[ e^{ik\omega} = \sum_{m=0}^{\infty} C_m^{e} e_m(\theta, q) + \sum_{m=1}^{\infty} D_m^{s} e_m(\theta, q). \]

The coefficients \( C_m \) and \( D_m \) are determined in the usual manner by applying the orthogonality conditions (1.10). With the normalization condition (1.12), we find:

\[ C_m = \frac{1}{\pi} \int_0^{2\pi} e^{ik\omega} e_m(\theta, q) \, d\theta, \]

\[ D_m = \frac{1}{\pi} \int_0^{2\pi} e^{ik\omega} e_m(\theta, q) \, d\theta. \]

These two integrals have already been evaluated in (1.33). The final result for the plane wave expansion is

\[ e^{ika} (\cosh \xi \cos \eta \cos \theta + \sinh \xi \sin \eta \sin \theta) \]

\[ = 2 \sum_{m=0}^{\infty} i^m C_m(\eta, q) M_\omega(1)(\xi, q)e_m(\theta, q) + 2 \sum_{m=1}^{\infty} i^m D_m(\eta, q) M_\omega(1)(\xi, q)e_m(\theta, q). \]  

(1.34)

Two cases are of special interest: when \( \theta = 0 \), the wave normal is parallel to the major axis of the ellipse; when \( \theta = \pi/2 \), the wave normal is parallel to the minor axis. In view of (1.8a), we have, with \( \theta = 0 \), the plane wave travelling parallel to the z-axis (Fig. 1.1):

\[ e^{ika} \cosh \xi \cos \eta = 2 \sum_{m=0}^{\infty} i^m C_m(0, q) M_\omega(1)(\xi, q)e_m(\eta, q); \]  

(1.35)
with \( \theta = \pi/2 \), the plane wave travelling parallel to the \( y \)-axis:

\[
\epsilon \, i \omega \sin \xi \sin \eta = 2 \sum_{m=0}^{\infty} (-1)^m se_{2m}(\tfrac{\pi}{2}, q) Ms_{2m}^{(1)}(\xi, q) se_{2m}(\eta, q) \\
+ 2 \sum_{m=0}^{\infty} (-1)^m is_{2m+1}(\tfrac{\pi}{2}, q) Ms_{2m+1}^{(1)}(\xi, q) se_{2m+1}(\eta, q). 
\]

(1.36)

Like the interpretation given to Eq. (1.33), we may regard the series in (1.34), with the time factor \( e^{-i \omega t} \) restored, as a synthesis of "elliptical waves." When their amplitudes and phases are properly coordinated, the net result is a plane wave.

1.5. Elasticity Equations in Elliptic Cylinder Coordinates

With the coordinate transformation (1.1) and the scalar factors (1.2) defined at the beginning of this Section, we have, for elliptic cylindrical coordinates, the following vector formulas:

\[
J^2 = \cosh^2 \xi - \cos^2 \eta = \frac{1}{2} (\cosh 2\xi - \cos 2\eta),
\]

\[
\nabla J = \frac{1}{\alpha^2 J} \left( e_\xi \frac{\partial J}{\partial \xi} + e_\eta \frac{\partial J}{\partial \eta} \right) + e_z \frac{\partial J}{\partial z},
\]

\[
\nabla \cdot J = \frac{1}{\alpha^2 J^2} \left( \frac{\partial J}{\partial \xi} \frac{\partial J}{\partial \xi} + \frac{\partial J}{\partial \eta} \frac{\partial J}{\partial \eta} \right) + \frac{\partial J}{\partial z} \frac{\partial J}{\partial z},
\]

\[
\nabla \times J = e_\xi \left( \frac{1}{\alpha^2} \frac{\partial J}{\partial \eta} - \frac{\partial J}{\partial z} \right) + e_\eta \left( \frac{\partial J}{\partial \xi} - \frac{1}{\alpha^2} \frac{\partial J}{\partial z} \right) + e_z \frac{1}{\alpha^2 J^2} \left[ \frac{\partial (J \alpha^2 F)}{\partial \xi} - \frac{\partial (J \alpha^2 F)}{\partial \eta} \right],
\]

\[
\nabla^2 J = \frac{1}{\alpha^2 J^2} \left( \frac{\partial^2 J}{\partial \xi^2} + \frac{\partial^2 J}{\partial \eta^2} \right) + \frac{\partial^2 J}{\partial z^2}.
\]
As in the case of circular cylindrical coordinates, we let the displacement vector \( \mathbf{u} \) be decomposed into three parts,

\[
\mathbf{u} = \mathbf{L} + \mathbf{M} + \mathbf{N},
\]

with

\[
\mathbf{L} = \nabla \phi, \quad \mathbf{M} = \nabla \times (\mathbf{e}_\phi \psi), \quad \mathbf{N} = \mathbf{e}_z \nabla^2 \chi.
\]

and each of the three displacement potentials \( \phi, \psi, \) and \( \chi \) satisfies the scalar wave equation

\[
\nabla^2 (\phi, \psi, \chi) = \left[ \frac{1}{a^2 \eta^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + \frac{\partial^2}{\partial z^2} \right] (\phi, \psi, \chi) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\phi, \psi, \chi).
\]

(1.39)

The velocity \( c = c_p \) for \( \phi \), and \( c_s \) for \( \psi \) and \( \chi \).

Without further explanation, we list all the pertinent equations.

**Displacement-Displacement Potential Relation:**

\[
\begin{align*}
\xi &= \frac{1}{a^2} \left( 3 \phi + \frac{\partial \psi}{\partial \eta} + \xi \frac{\partial^2 \chi}{\partial \xi \partial z} \right), \\
\eta &= \frac{1}{a^2} \left( 3 \phi - \frac{\partial \psi}{\partial \xi} + \xi \frac{\partial^2 \chi}{\partial \eta \partial z} \right), \\
z &= \frac{3 \phi}{a^2} + \xi \left( \frac{\partial^2 \chi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial \eta \partial t} \right).
\end{align*}
\]

(1.40)

**Strain-Displacement Relation:**

\[
\epsilon_{\xi \xi} = \frac{1}{a^2} \left( \frac{\partial \xi}{\partial \xi} + \frac{\sin 2\eta}{2z^2} \xi \right),
\]

\[
\epsilon_{\eta \eta} = \frac{1}{a^2} \left( \frac{\partial \eta}{\partial \eta} + \frac{\sin 2\eta}{2z^2} \eta \right),
\]

\[
\epsilon_{z z} = \frac{1}{a^2} \left( \frac{\partial z}{\partial z} + \frac{\sin 2\eta}{2z^2} z \right).
\]
\[ \varepsilon_{\eta \eta} = \frac{1}{a\omega} \left( \frac{3u_\eta}{\eta} + \frac{\sinh 2\xi}{2\eta^2} \frac{u_\xi}{\xi} \right), \]
\[ \varepsilon_{zn} = \frac{3u_z}{3z}, \]
\[ \varepsilon_{\xi z} = \frac{1}{2a\omega^3} \left[ J^2 \left( \frac{3u_\xi}{\xi} + \frac{3u_\eta}{\eta} \right) - \frac{1}{2} \sin 2\eta \frac{u_\xi}{\xi} - \frac{1}{2} \sinh 2\xi \frac{u_\eta}{\eta} \right], \]
\[ \varepsilon_{\xi \eta} = \frac{1}{2} \left( \frac{3u_\eta}{\eta} + \frac{3u_z}{3z} \right), \]
\[ \varepsilon_{\eta z} = \frac{1}{2} \left( \frac{3u_\eta}{\eta} + \frac{3u_z}{3z} \right). \]

**Stress-Displacement Relation:**

\[ \sigma_{\xi \xi} = \frac{(\lambda + 2\mu)}{a\omega} \frac{3u_\xi}{\xi} + \frac{\lambda}{a\omega} \frac{3u_\eta}{\eta} + \frac{\lambda}{a\omega} \frac{3u_z}{3z} + \frac{\lambda}{2a\omega^3} \frac{\sinh 2\xi}{\xi} u_\xi + \frac{\lambda}{2a\omega^3} \frac{\sin 2\eta}{\eta} u_\eta, \]
\[ \sigma_{\eta \eta} = \frac{\lambda}{a\omega} \frac{3u_\xi}{\xi} + \frac{(\lambda + 2\mu)}{a\omega} \frac{3u_\eta}{\eta} + \frac{\lambda}{a\omega} \frac{3u_z}{3z} + \frac{\lambda}{2a\omega^3} \frac{\sinh 2\xi}{\xi} u_\xi + \frac{\lambda}{2a\omega^3} \frac{\sin 2\eta}{\eta} u_\eta, \]
\[ \sigma_{zz} = \frac{\lambda}{a\omega} \frac{3u_\xi}{\xi} + \frac{\lambda}{a\omega} \frac{3u_\eta}{\eta} + \frac{(\lambda + 2\mu)}{a\omega} \frac{3u_z}{3z} + \frac{\lambda}{2a\omega^3} \frac{\sinh 2\xi}{\xi} u_\xi + \frac{\lambda}{2a\omega^3} \frac{\sin 2\eta}{\eta} u_\eta, \]
\[ \sigma_{\xi \eta} = \frac{\mu}{a\omega} \left[ \left( \frac{3u_\xi}{\xi} + \frac{3u_\eta}{\eta} \right) - \frac{1}{2\omega} \sin 2\eta u_\xi - \frac{1}{2\omega} \sinh 2\xi u_\eta \right], \]
\[ \sigma_{\xi z} = \mu \left( \frac{3u_\xi}{\xi} + \frac{3u_z}{3z} \right), \]
\[ \sigma_{\eta z} = \mu \left( \frac{3u_\eta}{\eta} + \frac{3u_z}{3z} \right). \]
Stress-Displacement Potential Relation:

\[
\sigma_{\xi\xi} = \mu \left( \frac{1}{\mu} \frac{\partial^2}{\partial \xi^2} + \frac{2}{\alpha^2} \frac{\partial^2}{\partial \eta^2} + \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} + \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \varphi \\
+ \frac{\mu}{\alpha^2} \left( \frac{\partial^2}{\partial \xi^2} - \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \psi \\
+ \frac{2\mu}{\alpha^2} \left( \frac{\partial^2}{\partial \xi^2} - \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} + \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \chi,
\]

\[
\sigma_{\eta\eta} = \mu \left( \frac{1}{\mu} \frac{\partial^2}{\partial \eta^2} + \frac{2}{\alpha^2} \frac{\partial^2}{\partial \xi^2} + \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \varphi \\
+ \frac{\mu}{\alpha^2} \left( -\frac{\partial^2}{\partial \xi^2} + \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} + \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} \right) \psi \\
+ \frac{2\mu}{\alpha^2} \left( \frac{\partial^2}{\partial \eta^2} + \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \chi,
\]

\[
\sigma_{zz} = \mu \left( \frac{1}{\mu} \frac{\partial^2}{\partial z^2} + 2 \frac{\partial^2}{\partial \xi^2} \right) \varphi - 2\mu \frac{\partial}{\partial z} \nabla^2 \chi,
\]

\[
\sigma_{\xi\eta} = \mu \left[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \varphi \\
+ \left( \frac{\partial^2}{\partial \eta^2} + \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \psi \\
+ \mu \frac{\partial}{\partial z} \left( \frac{\partial^2}{\partial \xi^2} - \frac{\sinh 2\xi}{\alpha^2} \frac{\partial}{\partial \xi} - \frac{\sin 2\eta}{\alpha^2} \frac{\partial}{\partial \eta} \right) \chi \right],
\]

\[
\sigma_{\xi z} = \mu \left[ 2 \frac{\partial^2 \varphi}{\partial \xi \partial z} + \frac{\partial^2 \psi}{\partial \eta \partial z} + \mu \frac{\partial}{\partial \xi} \left( \frac{\partial^2 \varphi}{\partial \xi^2} - \frac{1}{\alpha^2} \frac{\partial^2 \psi}{\partial \xi^2} \right) \chi \right].
\]
\[ c_{\eta z} = \frac{\nu}{\alpha} \left[ 2 \frac{\partial^2 \varphi}{\partial \eta \partial z} - \frac{\partial^2 \psi}{\partial \eta \partial z} + \frac{1}{\alpha} \left( 2 \frac{\partial^2}{\partial z^2} - \frac{1}{c_g^2} \frac{\partial^2}{\partial t^2} \right) \lambda \right] \]

in which

\[ v_1^2 = \frac{1}{\alpha^2 \nu^2} \left( \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \right) = \frac{1}{c_g^2} \frac{\partial^2}{\partial z^2} - \frac{3^2}{\partial \eta^2} . \]

2. DIFFRACTION OF SH WAVE

To illustrate the application of angular and radial Mathieu functions to the diffraction of elastic waves, we begin with the simplest problem — diffraction of an SH wave by an elliptical cylinder. The mathematics involved in dealing with an elliptical cylinder are similar to those used in the case of the circular cylinder, discussed in Section 1 of Chapter II. Here the radial Mathieu functions \( M_0^{(j)} \) and \( M_0^{(j)} \) are used instead of Bessel functions, and the trigonometric functions in circular coordinates are replaced by the angular Mathieu functions \( ce_m \) and \( se_m \).

For the same geometric boundary, the analogous problem of the scattering of electric waves was investigated in 1897 by Rayleigh, \(^{(2.1)}\) using Cartesian coordinates, and in 1908 by Sieger, \(^{(2.2)}\) who also contributed a great deal to the elliptic wave functions. The problem of sound waves was dealt with in 1938 by Morse and Rubenstein, \(^{(2.3)}\) who first presented detailed numerical results for diffraction by a slit (degenerate ellipse). Subsequent publications were reviewed by Bouwkamp \(^{(2.4)}\) and by Jones (Chapter 8 of Ref. 2.5). Additional numerical results for scattered wave-energy densities at low and medium
frequency ranges were reported recently by Barakat. (2.6) No numerical results have yet been reported for stress concentration factors based on the solutions shown in this Section.

As in subsections 1.1, 1.2, and 1.3 of Chapter II, we denote the only nonzero displacement component of the incident wave by \( u_n(z) \), but unlike the case of the circular cylinder, the direction of the incident wave must be specified relative to the major or minor axis of the ellipse. Let the wave normal make an angle \( \theta_0 \) with the \( x \)-axis (Fig. 2.1). The incident plane wave is represented by \((\omega = k\varepsilon_0)\)

![Diagram](image)

**Fig. 2.1. Incident Plane Wave on an Elliptic Cylinder**
\[ 
u_z(x, y, t) = u_0 e^{i k (x \cos \theta_c + y \sin \theta_c) - i \omega t}, \]

\[ = u_0 e^{ika (\cosh \xi \cos \eta \cos \theta_c + \sinh \xi \sin \eta \sin \theta_c) - i \omega t}, \]

(2.1)

\[ u_x^{(i)} = u_y^{(i)} = 0. \]

The exponential function as shown in (1.34) can be expanded into a series of Mathieu functions with \( q = \frac{1}{2} k^2 a^2 \),

\[ u_z^{(i)} = 2u_0 \sum_{m=0}^{\infty} i^m \left[ \cos_{m} (\theta_c, q) M_{m}^{(1)} (\xi, q) \cos_{m} (\eta, q) \right. \]

\[ + \left. \sin_{m}(\theta_c, q) M_{m}^{(1)} (\xi, q) \sin_{m} (\eta, q) \right] e^{-i \omega t}. \]

(2.2)

In the sequel, the time factor \( e^{-i \omega t} \) will be omitted.

Associated with the incident wave, the stresses can be calculated from (1.42):

\[ \sigma_{xz} = \frac{\mu}{\omega} \frac{\partial u_z}{\partial \xi}, \]

(2.3)

\[ \sigma_{nz} = \frac{\mu}{\omega} \frac{\partial u_z}{\partial \eta}. \]

All other stress components vanish.

The waves scattered by the elliptical cylinder, with hitherto unspecified boundary conditions, are constructed from the solution of the two-dimensional steady-state wave equation

\[ (\nu^2 + k^2) u_z(\xi, \eta) = 0. \]
From Table 1.4, the scattered waves are taken as

\[
    u_z(r) = \sum_{m=0}^{\infty} \left[ B_m se_m(\theta) M_{c}^{(3)}(\xi) ce_m(\eta) + C_m se_m(\theta) M_{s}^{(3)}(\xi) se_m(\eta) \right],
\]

in which \( B_m \) and \( C_m \) are the unknown coefficients and \( ce_m(\theta) \) and \( se_m(\xi) \) are constant factors inserted to match the series expansion of incident waves. In (2.4) and the sequel, the parameter \( q = (\frac{1}{2}k\alpha)^2 \) is omitted from the argument of all Mathieu functions. The complete wave outside the elliptical cylinder is then

\[
    u_z(\xi, \eta, t) = u_z(i) + u_z(r).
\]

The unknown coefficients \( B_m \) and \( C_m \) are fixed by the boundary conditions at the surface \( \xi = \xi_o \). The results are given below for three types of diffracting cylinders.

2.1. Rigid-Fixed Cylinder

Appropriate boundary conditions for a rigid elliptic cylinder are

\[
    u_z = 0, \quad \text{at} \ \xi = \xi_o.
\]

Combining (2.2) and (2.4) according to (2.5), and applying the orthogonality conditions for \( ce_m(\eta) \) and \( se_m(\eta) \) and the above boundary conditions, we obtain

\[
    B_m = -2u_o \frac{M_{c}^{(1)}(\xi_o)/M_{c}^{(3)}(\xi_o)}{M_{s}^{(3)}(\xi_o)}.\]
\[ C_m = -2u_o e^{im\xi_o} \frac{\mu^{(1)}(\xi_o)}{\mu^{(3)}(\xi_o)}. \]

The final answer is:

\[
\begin{align*}
\varepsilon_2(\xi, \eta) &= 2u_o \sum_{m=0}^{\infty} e^{im\xi_o} \left\{ \cos m(\eta) \left[ \frac{\mu^{(1)}(\xi) - \mu^{(3)}(\xi)}{\mu^{(1)}(\xi_o) - \mu^{(3)}(\xi_o)} \right] \cos m(\eta) \\
+ \sin m(\eta) \left[ \frac{\mu^{(1)}(\xi) - \mu^{(3)}(\xi)}{\mu^{(1)}(\xi_o) - \mu^{(3)}(\xi_o)} \right] \sin m(\eta) \right\}. \quad (2.7)
\end{align*}
\]

As the boundary \( \xi = \xi_o \) tends to a circle of radius \( a \), the Mathieu functions degenerate to circular functions according to (1.26), and the modified Mathieu functions to Bessel and Hankel functions according to (1.27). Thus when \( q \to 0 \),

\[
\cos m(\eta) \to \frac{1}{2},
\]

\[
\begin{align*}
\cos m(\eta) &\to \cos m(\eta) = \cos m(\eta) \\
\sin m(\eta) &\to \sin m(\eta) = \sin m(\eta) \\
\mu^{(1)}(\xi) &\to J_m(kr) \\
\mu^{(3)}(\xi) &\to H_m^{(1)}(kr), \quad m \geq 0,
\end{align*}
\]

and

\[
\begin{align*}
u_2(\xi, \eta) + u_o \sum_{m=0}^{\infty} e^{im\xi_o} \left[ J_m(kr) - \frac{J_m(ka)}{B_m^{(1)}(ka)} H_m^{(1)}(kr) \right] \cos m(\theta - \theta_o),
\end{align*}
\]

which agrees with the solution in Section 1 of Chapter II.

When \( \xi_o = 0 \), the elliptic cylinder degenerates to a thin strip
of width 2a. Since $Ma_m^{(1)}(0) = 0$ -- Eq. (1.22) -- the solution above reduces to a simpler form

$$u_z(\xi, \eta) = u_0 e^{i k (x \cos \theta_o + y \sin \theta_o)}$$

$$- 2u_0 \sum_{m=0}^{\infty} i^m c_m(\theta_o) \frac{Ma_m^{(1)}(0)}{Mo_m^{(3)}(0)} Ma_m^{(3)}(\xi) se_m(\eta). \quad (2.9)$$

This can be simplified further when $\theta_o = \pi/2$ because $se_{2m+1}(\pi/2) = 0$, thus only even order terms are left in the series.

2.2. Cavity

On a stress-free boundary $\xi = \xi_o$, all three stress components $\sigma_{\xi\xi}$, $\sigma_{\xi\eta}$, and $\sigma_{\xi\zeta}$ must vanish. For the case of an SH wave, the only boundary condition needed is $\sigma_{\xi\zeta}(\xi_o, \eta) = 0$, or, from (2.3),

$$\left[ \frac{\partial u_z}{\partial \xi} \right]_{\xi = \xi_o} = 0.$$

The solution can be found easily, as

$$u_z(\xi, \eta) = 2u_0 \sum_{m=0}^{\infty} i^m \left\{ c_m(\theta_o) \left[ \frac{Ma_m^{(1)}(\xi)}{Mo_m^{(3)}(\xi)} - \frac{Ma_m^{(1)}(\xi_o)}{Mo_m^{(3)}(\xi_o)} \right] se_m(\eta) \right. \left. + ce_m(\theta_o) \left[ \frac{Ma_m^{(1)}(\xi)}{Mo_m^{(3)}(\xi)} - \frac{Ma_m^{(1)}(\xi_o)}{Mo_m^{(3)}(\xi_o)} \right] se_m(\eta) \right\}. \quad (2.10)$$

If $\xi_o = 0$, $Ma_m^{(1)}(0) = 0$, and we have
\[ u_n(\xi, \eta) = u_o e^{ik(x \cos \theta_o + y \sin \theta_o)} - 2u_o \sum_{m=0}^{\infty} i^m s_e_m(\theta_o) \frac{M_{\theta_m}^{(1)}}{M_{\theta_m}^{(3)}}(0) M_{\theta_m}^{(3)}(\xi) s_e_m(\eta). \] (2.11)

For \( \theta_o = 0 \), the incident wave is unimpeded by the crack at \( y = 0 \).

This is substantiated by the vanishing of the scattered wave part in the above solution on account of \( s_e_m(0) = 0 \). For \( \theta_o = \pi/2 \), only odd order terms remain in the series because \( s_e_m(\pi/2) = 0 \).

With Eq. (1.8a), both solutions (2.7) and (2.10) can be simplified if the incident angle \( \theta_o \) is 0 or \( \pi/2 \). For the wave parallel to the major axis (\( x \)-axis) of the ellipse, only the series with even Mathieu functions \( s_e_m(\eta) \) remains. This is because \( s_e_m(\eta) \) is antisymmetric about the \( x \)-axis, which is incompatible with the symmetry about the same axis for the incident wave. If the wave is propagating along the minor axis (\( y \)-axis), the \( m \) in (2.7) and (2.10) are even integers for the \( s_e_m(\eta) \) series, and odd for the \( s_e_m(\eta) \) series -- see Eqs. (1.35) and (1.36).

2.3. Elastic Inclusion

So far, the examples cited are very much like the corresponding circular cylinder problems. However, there is a noticeable difference in the angular functions. For circular wave functions, the angular functions are \( \sin m\theta \), \( \cos m\theta \), where \( m \) is just an integer. For elliptic wave functions, however, the angular functions are \( s_e_m(n, q) \) and \( s_e_m(n, q) \) which, in addition to the order \( m \), are dependent on \( q = (\hat{\omega} k a)^2 \).

Since \( k = \omega/c_s \), the angular function depends also on the wave velocity.
\( \sigma_z \). Mathematically this creates a difficulty whenever two wave functions with different wave speeds are to be superimposed in order to satisfy the boundary conditions.

Consider now an elastic elliptic cylinder with shear wave velocity \( c_z \), surrounded by an infinite medium with velocity \( c_{z1} \). Assume the two surfaces are bounded such that both displacement and tractions are continuous across the interface. We use subscript 1 to indicate all physical quantities pertinent to the surrounding matrix, and subscript 2 for the elliptical inclusion. Due to an incident plane wave \( u_{z1}^{(i)} \) as given by (2.1), the total wave in the matrix is

\[
u_{z1} = u_{z1}^{(i)}(\xi, \eta; q_1) + u_{z1}^{(r)}(\xi, \eta; q_1),
\]

where the scattered wave \( u_{z1}^{(r)} \) is still given by (2.4) with \( q = q_1 \).

Inside the cylinder, the motion is a standing wave. From Table 1.4, we find

\[
u_{z2}(\xi, \eta; q_2) = \sum_{m=0}^{\infty} \left[ -D_m \mathcal{C}_m(\theta_0, q_2) \mathcal{M}_m(1)(\xi, \eta; q_2) \mathcal{E}_m(\eta, q_2) \\
- E_m \mathcal{E}_m(\theta_0, q_2) \mathcal{M}_m(1)(\xi, \eta; q_2) \mathcal{E}_m(\eta, q_2) \right] .
\]

(2.12)

The unknown coefficients \( B_m, D_m, E_m \) are fixed by the continuity condition at the interface \( \xi = \xi_0 \):

\[
u_{z1}(\xi_0, \eta; q_1) = \nu_{z2}(\xi_0, \eta; q_2),
\]

(2.13)

\[
\sigma_{z1}(\xi_0, \eta; q_1) = \sigma_{z2}(\xi_0, \eta; q_2).
\]
For the case of the incident wave parallel to the major axis, 

\[ \theta_0 = 0, \text{ Eq. (2.2) becomes} \]

\[ u_z(t) = 2\mu_0 \sum_{m=0}^{\infty} i^m c_\sigma_m (0,q_1) M \sigma_m (l) (\xi,q_1) c_\sigma_m (n,q_1). \]

Since the motion is symmetrical about \( n = 0, \eta \) axis, the coefficients 

\[ C_m = \Sigma_m = 0, \]

and

\[ u_z^,(r) = \sum_{m=0}^{\infty} B_m M \sigma_m (3) (\xi,q_1) c_\sigma_m (n,q_1), \]

\[ u_z = -\sum_{m=0}^{\infty} D M \sigma_m (1) (\xi,q_2) c_\sigma_m (n,q_2). \]

Substitution in the boundary conditions leads to two equations:

\[ \sum_{m=0}^{\infty} \left[ B_m M \sigma_m (3) (\xi_0,q_1) c_\sigma_m (n,q_1) + D M \sigma_m (1) (\xi_0,q_2) c_\sigma_m (n,q_2) \right] \]

\[ = -2\mu_0 \sum_{m=0}^{\infty} i^m c_\sigma_m (0,q_1) M \sigma_m (l) (\xi_0,q_1) c_\sigma_m (n,q_1), \]

\[ \text{and} \]

\[ \sum_{m=0}^{\infty} \left[ B_m M \sigma_m (3) \prime (\xi_0,q_1) c_\sigma_m (n,q_1) + \mu_1 D M \sigma_m (1) \prime (\xi_0,q_2) c_\sigma_m (n,q_2) \right] \]

\[ = -2\mu_0 \sum_{m=0}^{\infty} i^m c_\sigma_m (0,q_1) M \sigma_m (l) (\xi_0,q_1) c_\sigma_m (n,q_1). \]

(2.14)

Since \( c_\sigma_m (n,q_1) \) and \( c_\sigma_m (n,q_2) \) are not orthogonal to each other, there is no way to determine exactly the unknown coefficients \( B_m \) and \( D_m \) as in the corresponding case of the circular elastic inclusion.

Several approximate methods can be employed, however. One is to
use the orthogonality condition for one of these two angular functions.

Multiplying both equations in (2.14) by $ae_n(n, q_1)$, and integrating
from 0 to $2\pi$, we find

$$
B_n^{(3)}(q_1) + \sum_{m=0}^{\infty} D_{mn} M_{m1}^{(1)}(q_2) = -2\mu_0 i^n a_n(0, q_1) M_{n}^{(1)}(q_1),
$$

$$
B_n^{(3)}' (q_1) + \frac{\nu_2}{\mu_1} \sum_{m=0}^{\infty} D_{mn} M_{m1}^{(1)}'(q_2) = -2\mu_0 i^n a_n(0, q_1) M_{n}^{(1)}'(q_1),
$$

where

$$
p_{mn} = \frac{1}{\pi} \int_0^{2\pi} a_n(n, q_1) a_n(n, q_2) \, dn.
$$

Elimination of $B_n$ from the two equations above leads to a system of an
infinite number of algebraic equations for $D_m$:

$$
\sum_{m=0}^{\infty} D_{mn} D_m = b_n, \quad n = 0, 1, \ldots, \quad (2.15)
$$

with

$$
d_{mn} = p_{mn} \left[ \frac{M_{m1}^{(1)}(q_2)}{M_{n}^{(3)}(q_1)} \right] - \frac{\nu_2}{\mu_1} \left[ \frac{M_{m1}^{(1)}'(q_2)}{M_{n}^{(3)}'(q_1)} \right],
$$

$$
b_n = -2\mu_0 i^n a_n(0, q_1) \left[ \frac{M_{n}^{(1)}(q_1)}{M_{n}^{(3)}(q_1)} - \frac{M_{n}^{(1)}'(q_1)}{M_{n}^{(3)}'(q_1)} \right].
$$

The first few coefficients $D_0, D_1, \ldots$ etc. can be determined approxi-
mately by truncating the infinite system to a finite number of equa-
tions. Once the $D_m$ are known, the $B_n$ can be calculated from either of the previous equations.

A second method is to replace $\cos_m(q_1)$ and $\cos_m(q_2)$ in (2.14) by their series representations (1.8), and then to apply the orthogonality conditions to $\sin m\xi_0$ and $\cos m\xi_0$. This leads to two systems of an infinite number of equations for both coefficients $B_m$ and $D_m$:

$$
\sum_{m=0}^{\infty} (b_{r_m} B_m + d_{r_m} D_m) = a_r, \quad r = 0, 1, 2, \ldots, \\
\sum_{m=0}^{\infty} (b'_{r_m} B_m + d'_{r_m} D_m) = a'_r,
$$

(2.16)

with

$$
b_{r_m} = A^{(m)}_r(q_1)M_s^{(3)}(0, q_1),
$$

$$
b'_{r_m} = A^{(m)}_r(q_1)M_s^{(3)'}(0, q_1),
$$

$$
d_{r_m} = A^{(m)}_r(q_2)M_s^{(1)}(0, q_2),
$$

$$
d'_{r_m} = \frac{\nu_2}{\nu_1} A^{(m)}_r(q_2)M_s^{(1)'}(0, q_2),
$$

$$
a_r = -2u_0 \sum_{m=0}^{\infty} i^m \cos_m(0, q_1)M_s^{(1)}(0, q_1)A^{(m)}_r(q_1),
$$

$$
a'_r = -2u_0 \sum_{m=0}^{\infty} i^m \cos_m(0, q_1)M_s^{(1)'}(0, q_1)A^{(m)}_r(q_1).
$$

Again, only approximate values for the first few coefficients $B_0$, $B_1$, 

...; \( D_0, D_1, \ldots \) can be found by truncating the double infinite system of equations into a finite system.

3. SCATTERING OF P AND SV WAVES

3.1. Application of Method of Wave Function Expansions to an Elliptic Cylinder

Next in complexity to SH problems is the scattering of P or SV waves at normal incidence — a problem of plane strain. Let an incident P and SV wave be represented by

\[
\varphi(i) = \varphi_0 e^{i \alpha(x \cos \theta_o + y \sin \theta_o - c_p t)},
\]

\[
\psi(i) = \psi_0 e^{i \beta(x \cos \theta_o + y \sin \theta_o - c_s t)},
\]

\[
\alpha c_p = \beta c_s = \omega.
\]

The waves are propagating at an angle \( \theta_o \) with the major axis of the ellipse \( \xi = \xi_o \) as shown in Fig. 2.1. From (1.34), they can be expanded into a series of elliptic wave functions

\[
\varphi(i) = 2\varphi_o \sum_{n=0}^{\infty} i^n \{ \cos \vartheta_0 (\theta_o, \xi) M_{2n}^{(1)} (\xi, \xi) \cos \theta_0 (n, \xi) \}
\]

\[
+ 2\psi_o \sum_{n=0}^{\infty} i^n \{ \cos \vartheta_0 (\theta_o, \xi) M_{2n}^{(1)} (\xi, \xi) \cos \theta_0 (n, \xi) \},
\]

\[
\psi(i) = 2\psi_o \sum_{n=0}^{\infty} i^n \{ \cos \vartheta_0 (\theta_o, \xi) M_{2n}^{(1)} (\xi, \xi) \cos \theta_0 (n, \xi) \}
\]

\[
+ 2\psi_o \sum_{n=0}^{\infty} i^n \{ \cos \vartheta_0 (\theta_o, \xi) M_{2n}^{(1)} (\xi, \xi) \cos \theta_0 (n, \xi) \},
\]
where

\[ q_p = \frac{1}{2} a \beta^2 \frac{a^2}{4\alpha^2}, \quad q_g = \frac{1}{2} a \beta^2 \frac{a^2}{4\beta^2}. \]  \hspace{1cm} (3.3)

In the series of (3.2) it is understood that \( s e_0(\eta) = 0 \), and that the
time factor \( e^{-i\omega t} \) has been omitted.

The scattered P and SV waves are represented by

\[ \varphi^{(r)} = \sum_{n=0}^{\infty} \left[ B_n M_n(3) (\xi, q_p) ce_n(\eta, q_p) + C_n M_n(3) (\xi, q_p) se_n(\eta, q_p) \right], \]

\[ \psi^{(r)} = \sum_{n=0}^{\infty} \left[ D_n M_n(3) (\xi, q_g) ce_n(\eta, q_g) + E_n M_n(3) (\xi, q_g) se_n(\eta, q_g) \right], \]

in which the unknown coefficients \( B, C, D, \) and \( E \) are to be determined
by the boundary conditions. The total potentials are then

\[ \varphi = \varphi^{(i)} + \varphi^{(r)}, \]

\[ \psi = \psi^{(i)} + \psi^{(r)}, \]  \hspace{1cm} (3.5)

and the corresponding displacements and stresses of the total field
can be calculated from (1.40) and (1.42).

Unlike the circular functions where the derivative of a sine
function is a cosine function and vice versa, the derivative of the
elliptic sine function \( se_{n}(\eta) \) does not equal the elliptic cosine
function \( ce_{n}(\eta) \). In fact, the first derivatives of either \( se_{n}(\eta) \) or
\( ce_{n}(\eta) \) cannot be expressed in terms of the functions themselves, and
their values must be evaluated independently. The same is true for
the modified Mathieu functions. Thus in the process of substituting
(3.5) into (1.40) or (1.46), little can be done to simplify the expres-
sions through combination and elimination. The final results are long
and cumbersome. For instance, the displacements due to an incident P
wave \( \psi_i = 0 \) and scattered waves are respectively:

\[
\begin{align*}
\psi_i^{(i)} &= (2\varphi_i/\omega) \sum_{n=0}^{\infty} t^n [c_n(\theta_{o', q_p}) M_0^{(1)}(\xi, q_p) c_n(n, q_p) \\
&\quad + s_n(\theta_{o', q_p}) M_0^{(1)}(\xi, q_p) s_n(n, q_p)], \\
&\quad + (2\varphi_i/\omega) \sum_{n=0}^{\infty} t^n [c_n(\theta_{o', q_p}) M_0^{(1)}(\xi, q_p) c_n(n, q_p) \\
&\quad + s_n(\theta_{o', q_p}) M_0^{(1)}(\xi, q_p) s_n(n, q_p)],
\end{align*}
\]

(3.6)

\[
\begin{align*}
\psi_i^{(r)} &= (\omega)^{-1} \sum_{n=0}^{\infty} [E_n M_0^{(3)}(\xi, q_p) c_n(n, q_p) + C_n M_0^{(3)}(\xi, q_p) s_n(n, q_p) \\
&\quad + D_n M_0^{(3)}(\xi, q_p) c_n(n, q_p) + E_n M_0^{(3)}(\xi, q_p) s_n(n, q_p)], \\
&\quad + (\omega)^{-1} \sum_{n=0}^{\infty} [E_n M_0^{(3)}(\xi, q_p) c_n(n, q_p) + C_n M_0^{(3)}(\xi, q_p) s_n(n, q_p) \\
&\quad - D_n M_0^{(3)}(\xi, q_p) c_n(n, q_p) + E_n M_0^{(3)}(\xi, q_p) s_n(n, q_p)],
\end{align*}
\]

(3.7)

where prime indicates differentiation with respect to \( \xi \) or \( n \). The
total displacement is the sum of these two! Worse yet, the \( c_n(n, q_p) \)
and \( c_n(n, q_g) \) are not orthogonal to each other because \( q_p \neq q_g \), a case
encountered also in (2.14). The functions \( c_n(n, q_p) \) and \( c_n(n, q_p) \) are
not orthogonal either.

Lack of the orthogonality conditions for the angular functions rules out any possibility of determining the unknown coefficients exactly from the prescribed boundary conditions. We turn now to an approximate evaluation of them for the case of a fixed-rigid elliptic cylinder. The method is to replace the Mathieu functions \( se_{n}(\eta) \), and \( se_{n}(\eta) \) by their cosine and sine series. Orthogonality conditions are then applied to the trigonometric functions, which leads to a system of an infinite number of equations for the coefficients \( B_{n} \) and \( E_{n} \), and another set for \( C_{n} \) and \( D_{n} \). Approximate values for these coefficients can be calculated by truncating the infinite system to a finite number of equations.

At the surface \( \xi = \xi_{o} \), the displacement components \( u_{\eta} \) and \( u_{\xi} \) must vanish for a fixed-rigid insert. Substitution of (3.6) and (3.7) into the following boundary conditions

\[
\begin{align*}
    u_{\xi} &= u_{\xi}^{(r)} + u_{\xi}^{(r)} = 0, \quad u_{\eta} = u_{\eta}^{(r)} + u_{\eta}^{(r)} = 0, \quad \text{at } \xi = \xi_{o}, \quad (3.8)
\end{align*}
\]

results in two equations, containing the unknown coefficients \( B_{n} \), \( C_{n} \), \( D_{n} \), and \( E_{n} \), and the angular Mathieu functions \( se_{n}(\eta) \) and \( se_{n}(\eta) \) and their derivatives. The latter will be replaced by their series definition (1.8) which can be written compactly as

\[
\begin{align*}
    se_{n}(\eta,q) &= \sum_{r=0}^{\infty} A_{r}^{(n)}(q) \cos rn \quad \text{when } b = a_{n}, \\
    se_{n}(\eta,q) &= \sum_{r=0}^{\infty} B_{r}^{(n)}(q) \sin rn \quad \text{when } b = b_{n},
\end{align*}
\]

(3.9)
with $se_0(n,q) = 0$. Their derivatives are

$$se_n'(n,q) = - \sum_{r=0}^{\infty} rA_r(n)(q) \sin rn,$$

$$se_n'(n,q) = \sum_{r=0}^{\infty} rB_r(n)(q) \cos rn. \tag{3.10}$$

The coefficients $A_r(n)$ and $B_r(n)$ which depend on the parameter $q(q = q_p$ or $q_s)$ are normalized by (1.11):

$$\sum_{r} [A_r(n)]^2 = \sum_{r} [B_r(n)]^2 = 1.$$

The two resulting equations are:

$$\sum_{n=0}^{\infty} \left[ B_n \sum_{r=0}^{\infty} (b_{rn} \cos rn) + C_n \sum_{r=0}^{\infty} (c_{rn} \sin rn) + D_n \sum_{r=0}^{\infty} (d_{rn} \sin rn) + E_n \sum_{r=0}^{\infty} (e_{rn} \cos rn) \right] = -2\varphi_0 \sum_{r=0}^{\infty} (U_r \cos rn + V_r \sin rn), \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \left[ B_n \sum_{r=0}^{\infty} (b_{rn} \sin rn) + C_n \sum_{r=0}^{\infty} (c_{rn} \cos rn) + D_n \sum_{r=0}^{\infty} (d_{rn} \cos rn) + E_n \sum_{r=0}^{\infty} (e_{rn} \sin rn) \right] = -2\varphi_0 \sum_{r=0}^{\infty} (U_r \sin rn + V_r \cos rn),$$

where

*The parameter $q_p$ or $q_s$ is dropped from all Mathieu functions and from the constants $A_r(n)$ and $B_r(n)$.
\[
\begin{align*}
\begin{cases}
    b_{m}^{(n)}(q) = -rM_{m}^{(3)}(x_{0})A_{r}^{(n)}, \\
    b_{m}^{(n)}(q) = M_{m}^{(3)}(x_{0})A_{r}^{(n)}, \\
    c_{m}^{(n)}(q) = rM_{m}^{(3)}(x_{0})B_{r}^{(n)}, \\
    c_{m}^{(n)}(q) = M_{m}^{(3)}(x_{0})B_{r}^{(n)}.
\end{cases}
\end{align*}
\] (3.12)

\[
\begin{align*}
\begin{cases}
    d_{m}^{(n)}(q) = -rM_{m}^{(3)}(x_{0})A_{r}^{(n)}, \\
    d_{m}^{(n)}(q) = M_{m}^{(3)}(x_{0})A_{r}^{(n)}, \\
    \epsilon_{m}^{(n)}(q) = rM_{m}^{(3)}(x_{0})B_{r}^{(n)}, \\
    \epsilon_{m}^{(n)}(q) = -M_{m}^{(3)}(x_{0})B_{r}^{(n)}.
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
    U_{r}(q) = -r \sum_{n} [i^{n}c_{e_{n}}(x_{0})M_{m}^{(1)}(x_{0})A_{r}^{(n)}], \\
    U_{r}'(q) = \sum_{n} [i^{n}c_{e_{n}}(x_{0})M_{m}^{(1)}(x_{0})A_{r}^{(n)}], \\
    V_{r}(q) = r \sum_{n} [i^{n}se_{e_{n}}(x_{0})M_{m}^{(1)}(x_{0})B_{r}^{(n)}], \\
    V_{r}'(q) = \sum_{n} [i^{n}se_{e_{n}}(x_{0})M_{m}^{(1)}(x_{0})B_{r}^{(n)}].
\end{cases}
\end{align*}
\] (3.13)

It is now a simple matter to apply the orthogonality conditions to (3.11) for \(\cos mn\) and \(\sin mn\). The results are:
\[
\begin{align*}
\sum_{n=0}^{\infty} \left( b_{m}^{r} E_{n} + e_{m}^{r} F_{n} \right) &= -2\psi_{r}^{(s)} U_{r}, \\
\sum_{n=0}^{\infty} \left( b_{m}^{r} P_{n} + e_{m}^{r} E_{n} \right) &= -2\psi_{r}^{(s)} V_{r}, \\
\sum_{n=0}^{\infty} \left( a_{m}^{r} C_{n} + d_{m}^{r} D_{n} \right) &= -2\psi_{r}^{(s)} V_{r},
\end{align*}
\]

which are similar to (2.16) for the scattering of SH waves by an elastic cylinder.

So far our calculation pertains to an incident P wave. For an incident S wave as defined by \(\psi^{(i)}\) in (3.1) and (3.2), the displacements are

\[
\begin{align*}
\psi_{r}^{(i)} &= \left(2\psi_{r}/a'\right) \sum_{n=0}^{\infty} \left[ e_{n}^{(s)}(n,q_{s})M_{n}^{(1)}(n,q_{s})e_{n}^{(s)}(n,q_{s}) \right. \\
&\quad \quad \left. + e_{n}^{(s)}(n,q_{s})M_{n}^{(1)}(n,q_{s})e_{n}^{(s)}(n,q_{s}) \right], \\
\psi_{\theta}^{(i)} &= \left(2\psi_{r}/a'\right) \sum_{n=0}^{\infty} \left[ -e_{n}^{(s)}(n,q_{s})M_{n}^{(1)}(n,q_{s})e_{n}^{(s)}(n,q_{s}) \right. \\
&\quad \quad \left. - e_{n}^{(s)}(n,q_{s})M_{n}^{(1)}(n,q_{s})e_{n}^{(s)}(n,q_{s}) \right].
\end{align*}
\]

The scattered waves can still be represented by (3.7). Hence, (3.14) is valid for determining the unknown coefficients provided the \(\psi_{r}\) on the right-hand side is replaced by \(\psi_{r}^{(s)}\), and the \(U_{r}', V_{r}'\), and \(V_{r}\).
take the following values:

$$U_r(q_r) = - \sum_n \left[ i^n s e_n(\theta_o) M_n^{(1)}(\xi_o) B_{1r}^{(n)} \right],$$

$$U'_r(q_r) = r \sum_n \left[ i^n s e_n(\theta_o) M_n^{(1)}(\xi_o) B_{1r}^{(n)} \right],$$

$$V_r(q_r) = - \sum_n \left[ i^n c e_n(\theta_o) M_n^{(1)}(\xi_o) A_{1r}^{(n)} \right],$$

$$V'_r(q_r) = - r \sum_n \left[ i^n c e_n(\theta_o) M_n^{(1)}(\xi_o) A_{1r}^{(n)} \right].$$ (3.17)

The above can be compared with (3.13) which is for an incident P wave.

The solutions for a rigid-fixed elliptical cylinder as outlined above are due to Harumi (Ref. 3.1),* but the use of Mathieu functions for scattering of elastic waves by an elliptical cylinder was proposed as early as 1927 by Sezawa. (3.2) Harumi evaluated the unknown coefficients for the case of a rigid ribbon (\( \xi = \xi_o = 0 \)) by approximating (3.14) with a system of a finite number of equations with 20 unknowns. He was able to calculate the distribution in angle of displacements and the scattering cross sections in far field. In a subsequent paper he formulated solutions for a stress-free cavity and then discussed the numerical results of scattering energy for a

---

*The notation for Mathieu functions as used by Harumi follows that in Morse and Feshbach's book. Except for a difference of a constant multiplier, the conversion is as follows:

<table>
<thead>
<tr>
<th>Ref:</th>
<th>( Se_m(\eta) )</th>
<th>( Sc_m(\eta) )</th>
<th>( He_m^{(2)}(\xi) )</th>
<th>( He_m^{(2)}(\xi) )</th>
<th>( De_m^{(m)} )</th>
<th>( De_m^{(m)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text:</td>
<td>( ce_m(\eta) )</td>
<td>( se_m(\eta) )</td>
<td>( Mc_m^{(4)}(\xi) )</td>
<td>( Ms_m^{(4)}(\xi) )</td>
<td>( A_{1r}^{(m)} )</td>
<td>( B_{1r}^{(m)} )</td>
</tr>
</tbody>
</table>
"cavity-ribbon" \( (\xi_0 = 0) \). (3.3)

After setting up the equations of (3.14), there still remains the major task of solving them. As can be seen from Eqs. (3.12) and (3.13) the coefficients \( b_m, b_m', \) etc., depend on the values of Mathieu functions, modified Mathieu functions, and their first order derivatives. Only very recently have tables pertaining to these functions become available, and only for a limited range of the order \( n \), parameter \( q \), and the variable \( \xi \) or \( \eta \). The computational task is simplified if only special cases are considered, such as normal incidence to the major axis \( (\theta_0 = \pi/2) \), parallel incidences \( (\theta_0 = 0) \), and, in particular, an ellipse degenerating into a line with \( \xi_0 = 0 \). The reader is referred to the original work by Harumi for further details of computations.

3.2. Application of Method of Perturbation to a Rigid Ribbon

Because the angular Mathieu functions \( ce_m(n, q_p) \) and \( ce_n(n, q_s) \) are not orthogonal to each other, the coefficients \( B_n, \ldots, F_n \) in the series solutions (3.5) cannot be determined exactly even for the simple case of scattering by a rigid-fixed ribbon with \( \xi_0 = 0 \) in (3.8). For problems of stress concentrations, interest in studying scattering by a rigid ribbon or a line crack lies in the determination of stress intensification factors at the edges of the discontinuity \( \xi_0 = 0 \). The application of the method of perturbation of wave numbers (Section 4, Chapter II) can supply an approximate answer (Ref. 3.4).

Consider a rigid strip of width \( 2\alpha \), clamped along the \( x \)-axis. Its edges are then at \( y = 0 \) and \( x = \pm \alpha \) \( (\xi = 0, \eta = 0 \) and \( \pi) \). A plane compressional wave is propagating perpendicularly to the ribbon and
is represented by Fig. 2.1, with \( \theta_o = \pi/2 \).

\[
\begin{align*}
\hat{u}_x(i) &= 0, \\
\hat{u}_y(i) &= A e^{iay}.
\end{align*}
\] (3.18)

The boundary conditions are

\[
\hat{u} = \hat{u}(i) + \hat{u}(r) = 0, \quad \text{at } y = 0, \quad |x| < a
\] (3.19)

and \( \hat{u} \) remains finite at the edges \( x = \pm a, y = 0 \), which is known as the "edge condition."

Since the boundary conditions are prescribed in terms of displacements, we shall apply Eqs. (4.15) through (4.18) of Chapter II. Of the Cartesian components of the 0th order displacement vector, each satisfies a scalar wave equation

\[
(\nabla^2 + \kappa^2) (\hat{u}_x^{(0)}, \hat{u}_y^{(0)}) = 0,
\]

where \( \kappa^2 = \frac{1}{2}(\alpha^2 + \beta^2) \). Because the components \( \hat{u}_x^{(0)} \) and \( \hat{u}_y^{(0)} \) are also uncoupled in the boundary conditions, each can be solved independently as in the case of scattering of an SH wave by a ribbon. The result, which is derivable from Eq. (2.9), with \( \theta_o = \pi/2 \), is

\[
\begin{align*}
\hat{u}_x^{(0)}(\xi, \eta) &= 0, \\
\hat{u}_y^{(0)}(\xi, \eta) &= A e^{iky} - 2A \sum_{n=0}^{\infty} B_n(q) M_{2n}^{(3)}(\xi, \eta) e_{2n}(\eta, q),
\end{align*}
\] (3.20)

where

\[
B_n(q) = (-1)^n e_{2n}(\pi/2, q) M_{2n}^{(1)}(0, q) / M_{2n}^{(3)}(0, q),
\]
\[ q = k^2 \alpha^2 / 4, \]

and

\[ y = a \sinh \xi \sin \eta. \]

In Cartesian components, the first order particular solution (Eq. II-4.17) has the following forms:

\[
\begin{align*}
\frac{u_1^{(1)}}{x_p} &= -x \frac{\partial u_0^{(0)}}{\partial x} + \frac{\partial u_0^{(0)}}{\partial y} - 2y \frac{\partial u_0^{(0)}}{\partial x}, \\
\frac{u_1^{(1)}}{y_p} &= y \frac{\partial u_0^{(0)}}{\partial x} - \frac{\partial u_0^{(0)}}{\partial y} - 2y \frac{\partial u_0^{(0)}}{\partial x}.
\end{align*}
\]

(3.21)

With \( u_p^{(1)} = u_c^{(1)} + u_x^{(1)} \) and \( u_x^{(0)} = 0 \), the first order solution is then given by

\[
\begin{align*}
\frac{u_1^{(1)}}{x} &= -2y \frac{\partial u_y^{(0)}}{\partial x} + u_x^{(1)}, \\
\frac{u_1^{(1)}}{y} &= x \frac{\partial u_y^{(0)}}{\partial x} - y \frac{\partial u_y^{(0)}}{\partial y} + u_y^{(1)},
\end{align*}
\]

(3.22)

where the complementary solutions \( u_x^{(1)} \) and \( u_y^{(1)} \) must be chosen to satisfy the first order boundary conditions

\[
\begin{align*}
u_1^{(1)} &= 0, \quad \text{at } |x| < a, \quad y = 0, \quad (3.23)
\end{align*}
\]

and the edge conditions at \( x = \pm a, \) \( y = 0. \)

Since \( 2y \frac{\partial u_0^{(0)}}{\partial x} \) vanishes on \( y = 0 \) and remains finite at \( x = \pm a \) as \( y \to 0 \), we have it simply that \( u_x^{(1)} = 0. \) The first two terms of
\( u_y^{(1)} \) also vanish on \( y = 0, \ |x| < a \), but \( x^2 u_y^{(0)}/2x \) will be singular at the edges as shown below. Thus \( u_y^{(1)} \) must be chosen to satisfy the boundary condition (3.23) and to eliminate singularities at the edges.

We proceed by introducing polar coordinates, locally at each edge, with \((\rho_+, \gamma_+)\) originating at \( x = a, y = 0 (\xi = \eta = 0) \) and \((\rho_-, \gamma_-)\) originating at \( x = -a, y = 0 (\xi = 0, \eta = \pi) \) (Fig. 3.1). Thus, at the two edges we have

\[ x = \pm a + \rho_\pm \cos \gamma_\pm, \quad y = \rho_\pm \sin \gamma_\pm, \]

where

\[ -\pi < \gamma_+ < \pi, \quad 0 < \gamma_- < 2\pi. \]

Near the edges, the elliptic coordinates are transformed to the local polar coordinates as

![Fig. 3.1. Motion of a Rigid Ribbon and the Local Coordinates \( \rho_\pm, \gamma_\pm \) at the Edges of the Ribbon](image)
\[ \xi + (2p_+/a)^{\frac{1}{2}} \begin{cases} \cos \left( \gamma_+/2 \right), \\ \sin \left( \gamma_-/2 \right), \end{cases} \quad \eta \rightarrow \begin{cases} (2p_+/a)^{\frac{1}{2}} \sin \gamma_+/2, \\ -(2p_-/a)^{\frac{1}{2}} \cos \gamma_-/2. \end{cases} \]

Carrying out the differentiation of \( u^{(0)}_y \) in (3.22) according to

\[ x \frac{3}{3x} = a \cosh \xi \cos \eta \left[ \frac{3\xi}{3\xi} + \frac{3\eta}{3\eta} \right], \]

and noting that \( ce^{(0)}_{2n}(0) = ce^{(1)}_{2n}(0) = 0 \) and \( ce^{(0)}_{2n}(0) = ce^{(1)}_{2n}(0) \), we find, as \( \xi \rightarrow 0 \) and \( \eta \rightarrow 0 \) or \( \pi, (p_+ \rightarrow 0) \),

\[ x^3 \frac{u^{(0)}_y}{3x} = -c(a/2p_+)\frac{1}{2} \begin{cases} \cos \left( \gamma_+/2 \right), \\ \sin \left( \gamma_-/2 \right), \end{cases} \quad (3.24) \]

where

\[ C = 2a \sum_{n=0}^{-\infty} B_n(q)M_{(2,0)}^{(-1)}(0,q)ce_{2n}(0,q), \]

which is a complex coefficient depending on \( q \). In the above derivation use has been made of the relation

\[ x \frac{3\xi}{3x} = \frac{\sinh 2\xi \cos^2 \eta}{2(\sinh^2 \xi + \sin^2 \eta)} \rightarrow \left( \frac{a}{2p_+} \right)^{\frac{1}{2}} \begin{cases} \cos \left( \gamma_+/2 \right), \\ \sin \left( \gamma_-/2 \right), \end{cases} \]

as \( p_+ \rightarrow 0 (\xi \rightarrow 0, \eta \rightarrow 0, \pi) \).

Equation (3.24) shows that \( u_1^{(1)} \) indeed contains a singular term at each edge as \( p_+ \rightarrow 0 \) which must be cancelled by a corresponding term in the complementary solution. Therefore, we choose

\[ u_1^{(1)} = C \left( a/2p_+ \right)^{\frac{1}{2}} \frac{t^3}{e} \cos \left( \gamma_+/2 \right) + (a/2p_-)^{\frac{1}{2}} \frac{t^3}{e} \sin \left( \gamma_-/2 \right), \quad (3.25) \]
Each term in brackets in the above represents a diverging wave from an edge of the strip, satisfying the Helmholtz equation and vanishing identically on the strip ($\gamma_+ = \pm \pi, \gamma_- = 0, 2\pi$). Substitution of (3.25) and $u^{(1)}_{\infty} = 0$ into (3.22) completes the first order solution.

The perturbation solution for the scattering by a rigid-fixed ribbon, accurate up to the first degree of $\varepsilon = (6 - 8\nu)^{-1}$, is

$$u = u^{(0)} + \varepsilon u^{(1)},$$

(3.26)

with $u^{(0)}$ given by (3.20) and $u^{(1)}$ by (3.22), in which $u^{(1)}_{\infty} = 0$ and $u^{(1)}_{\gamma\sigma}$ is determined in (3.25). The solution satisfies the boundary condition (3.19) and $u$ remains finite at the edges of the ribbon.

We note that in this section we discuss only scattering by a rigid and fixed obstacle. Under the impact of incident waves, the cylinder would translate and rotate as a rigid body unless it were constrained by additional forces. Thus in the absence of external constraining forces, the boundary conditions (3.8) and (3.19) are rather stringent, accounting for the abnormal scattering intensity as reported in Ref. 3.1.

A more realistic boundary condition would be to allow the rigid ribbon to move with the surrounding medium. Along the ribbon $\xi = 0$, $u_\xi = u_y$, and $u_z = u_x$. Thus instead of (3.19) the boundary conditions for a movable rigid ribbon are, at $y = 0, -a \leq x \leq a$,

$$u_x = U + x \cos \Theta,$$

(3.27)

$$u_y = V + x \sin \Theta,$$
where \( U \) and \( V \) are translations of the center of mass along the \( z \) and \( y \) axes respectively, and \( \theta \) the angle of rotation (Fig. 3.1). They are in turn determined by the kinetic equation for a rigid body:

\[
\begin{align*}
\ddot{m}U &= \int_{-a}^{a} [\sigma_{xy}] \, dx, \\
\ddot{m}V &= \int_{-a}^{a} [\sigma_{yy}] \, dx, \\
I \ddot{\theta} &= \int_{-a}^{a} [\sigma_{yy}] \, x \, dx,
\end{align*}
\]

(3.28)

where \( m \) is the mass, \( I = ma^2/3 \) the moment of inertia of the strip, and \( [\sigma_{xy}] \) and \( [\sigma_{yy}] \) indicate the difference of stresses on both sides of the strip, i.e.,

\[
[\sigma_{yy}] = \sigma_{yy}(x, +0, t) - \sigma_{yy}(x, -0, t).
\]

The stresses are caused by the incident and reflected waves.

The problem now becomes extremely complicated. Not only is the boundary condition (3.27) difficult to handle, but the calculation of the rigid body motion from (3.28) alone turns out to be a major task. By decomposing the displacement into even and odd parts about the plane \( y = 0 \), Kostov recast the problem of the motion of a ribbon into two mixed-boundary-value problems, and solved them by the Wiener-Hopf method. (3.5) He represented the incident wave by a step function in time, and calculated the acceleration and rotation of the ribbon as a function of time, the mass and angle of incidence being treated as parameters. This is so far the only available numerical data for the
motion of a rigid ribbon.

With the rigid body motion \( \ddot{u}, \ddot{v}, \) and \( \ddot{\theta} \) determinable, at least numerically, the scattering by a movable rigid ribbon presumably can be analyzed by the method described in Section 1.2 of Chapter II. There still remains a problem of what value should be assigned to the mass \( m \). If the ribbon is treated as the limiting case of an ellipse as \( \xi_0 \to 0 \), the total mass of the ribbon is zero because the total area of the ellipse diminishes to nothing. Only by letting the density of the ribbon approach infinity can we assign a finite but indeterminate value to the mass \( m \). This, however, does not seem to be very reasonable.

Equally baffling is the case of a cavity ribbon. For a stress-free elliptical cavity, the proper boundary conditions are

\[ \sigma_{\xi\xi} = \sigma_{\xi\eta} = 0, \text{ at } \xi = \xi_0, \]

where \( \sigma_{\xi\eta} \) is the total stress due to the incident and reflected waves. As \( \xi_0 \to 0 \), the cavity degenerates into a slit with finite width, called the cavity ribbon in Ref. 3.3. Because the slit has no dimension in thickness, the boundary condition above with \( \xi_0 = 0 \) remains valid so long as the stresses at the boundary tend to open the slit. Any oppositely-acting stress developed near the surface \( \xi_0 = 0 \), as in the case of a medium under oscillating loads, will close the opening. Once the slit is closed, the normal stress (compressive) is continuous across the two surfaces of the slit, which invalidates the originally assumed zero stress condition. Thus for a cavity ribbon, the steady wave scattering solution is meaningful when it is understood that a
pretension of sufficient magnitude has been applied so the cavity
never closes during the passage of a compressional wave.

4. INTEGRAL EQUATIONS FOR THE DIFFRACTION OF SH WAVES

BECAUSE OF THE COMPLEXITY of evaluating the Mathieu functions, so far
little is known of the numerical values of the stress concentration
factors around an elliptic inclusion. One may wonder whether the use
of integral equations would be a better approach than Mathieu functions
for determining numerical values. There is of course no difficulty in
setting up the appropriate integral equations for a particular prob-
lem; the difficulty lies in solving them. Experience indicates that
unless the integral equations in a diffraction problem can be solved
by the Hilbert-Schmidt method (see Subsection 2.3 of Chapter II), there
is little hope of finding its exact solution.

One exception is the case of diffraction of scalar waves by a
semi-infinite screen — the Sommerfeld problem, which if formulated
in terms of integral equations can be solved by applying the Weiner-
Hopf technique. However, the same problem has also been solved by
many other methods, of which the use of parabolic wave functions, as
discussed in the next chapter, is perhaps the simplest. Furthermore
we recall that the Hilbert-Schmidt method is applicable only when the
kernel of the integral equation can be represented by a series of
orthogonal functions suitable for the geometry of the scatterer. In
the case of an elliptic cylinder, the orthogonal functions are the
Mathieu functions. Thus the solution of the integral equation so
obtained is still in the form of a series of Mathieu functions, and nothing is gained.

On the other hand, if only an approximate answer is all that is expected, the integral equation formulation has great potential. In Chapter II, Section 2, we show how an integral equation can be reduced to a system of linear equations of the unknown quantities at discrete points at a bounding surface of arbitrary shape. If this reduction can be extended successfully to the case of coupled integral equations which have arisen invariably from the scattering of P or SV waves, we are then in a position to determine stress concentrations near an irregular bounding surface.

In this Section we discuss the integral equation method as applied to the scattering of SH waves by an elliptical cylinder. Its

![Diagram of an elliptical cylinder with labels and arrows showing the paths of P and Q waves, indicating the geometry of the problem.]

Fig. 4.1. Geometry of an Elliptical Cylinder
generalization to a cylinder of arbitrary shape is obvious.

The problem is no different from that discussed in Chapter II, Section 2 (the two-dimensional case). Let the scatterer be bounded by an elliptical cylindrical surface \( A \), with a projection \( \Gamma \) on the \( x-y \) plane (Fig. 4.1). On the closed curve \( \Gamma \), either \( u_z \) or \( \partial u_z / \partial n \) (= \( \partial u_z / \partial \xi \)) vanishes, corresponding to Dirichlet's and Neuman's boundary conditions, respectively. From (II-2.5b) and (II-2.21), the total wave outside the ellipse is given by:

1. **Rigid surface** \( (u_z = 0 \text{ on } \Gamma) \) —

\[
 u_z(r) = u_z^{(1)}(r) - \frac{i}{4} \int_\Gamma H_0^{(1)}(k|r - r_0|) \frac{\partial u_z(r_0)}{\partial n_0} \, ds_0. \tag{4.1a}
\]

At the surface \( \Gamma \), the normal derivative of \( u_z \), which is still unknown, satisfies the integral equation (II-2.24).

\[
 u_z^{(1)}(r) = \frac{i}{4} \int_\Gamma H_0^{(1)}(k|r - r_0|) \frac{\partial u_z(r_0)}{\partial n_0} \, ds_0, \quad r = r_0. \tag{4.2a}
\]

2. **Traction-free surface** \( (\partial u_z / \partial n = 0 \text{ on } \Gamma) \) —

\[
 u_z(r) = u_z^{(1)}(r) + \frac{i}{4} \int_\Gamma u_z(r_0) \frac{\partial}{\partial n_0} H_0^{(1)}(k|r - r_0|) \, ds_0. \tag{4.1b}
\]

The unknown displacement at the surface \( \Gamma \) is determined from the integral equation (II-2.26):

\[
 -\frac{\partial u_z^{(1)}(r)}{\partial n} = \left( \frac{i}{4} \right) \frac{\partial}{\partial n} \int_\Gamma u_z(r_0) \frac{\partial}{\partial n_0} H_0^{(1)}(k|r - r_0|) \, ds_0, \quad r = r_0. \tag{4.2b}
\]
For the elliptical boundary $\Gamma$, the integral equations (4.2a) or (4.2b) can be solved by the Hilbert-Schmidt method as illustrated in II-2.3. There the unknown functions and the kernel $\mathcal{H}_0^{(1)}(k | r - r_0 |)$ are to be expanded into a series of Mathieu functions. The solutions thus obtained should be identical to those shown in Section 2. Solving these integral equations by reducing them into a system of linear equations was also discussed in Section 2 of Chapter II. The rest of this Section concerns only the limiting case when $r_0 = 0$, at which the curve $\Gamma$ degenerates into the segment $(-a, a)$ in the $x$-axis.

4.1. Diffraction by a Finite Strip

At the beginning of this book, it was mentioned that by observing light passing through narrow slits in dark screens, Grimaldi found in the 17th century that the light beam was bent by the edge of the opening; he called this diffraction. The same problem occupies us today. The integral equation for the diffraction of light by a narrow slit, which was first formulated by Lord Rayleigh in 1897,\textsuperscript{(4.1)} is not much different from the diffraction of an SH wave by a strip of finite width but of infinitesimal thickness. Rayleigh also deduced an approximate solution for a normally incident light wave on a slit whose dimension was small in comparison with the wave length.

Because no exact solution for that integral equation was ever found except the representation of it by an infinite series of Mathieu functions, many other approximations followed. More than ten papers were cited in the Bouwkamp review article\textsuperscript{(2.4)} concerning the approximate solution of those integral equations. In the article "Theorie
der Beugung,(4.2) Hönle, Maue and Westpfahl presented the most thorough treatment of the derivation, alternate forms, and various approximate solutions of Rayleigh's integral equations. We might add that the search for the solution continued thereafter. New approximate solutions have been found, among others, by Boersma,(4.3) Ang and Knopoff,(4.4)(4.5) and by Loeber and Sih.(4.6) This last work discusses in particular the stress intensification factors at the tip of a slit during the passage of an SH wave.

Following Hönle, Maue, and Westpfahl, we will show first how various forms of Rayleigh's integral equation can be derived for the diffraction of an SH wave by a rigid strip or by an open crack. For low frequency and long wavelength scattering, Rayleigh's approximations and solutions by perturbation are then presented. The same perturbation method for low frequency scattering is also discussed in detail by Noble.(4.7)

A. Rayleigh's Integral Equation

In terms of Cartesian coordinates, the pertinent equations for the diffraction of an SH wave by a strip of width 2a are

$$(\gamma^2 + k^2)u_z(x,y) = 0, \quad u_z = u_z^{(t)} + u_z^{(v)}. \quad (4.3)$$

Boundary conditions:

Rigid Strip ---

$$u_z(x,0) = 0, \quad -a < x < a; \quad (4.4a)$$

Crack ---

$$\partial u_z(x,0)/\partial y = 0, \quad -a < x < a. \quad (4.4b)$$
Radiation conditions: as \( r = (x^2 + y^2)^{\frac{1}{2}} \to \infty \),

\[ \sqrt{r} u_z \to 0 \quad \text{and} \quad \sqrt{r} \left( \frac{\partial u_z}{\partial r} - iku_z \right) \to 0. \] (4.5)

Edge condition:

\[ u_z \text{ remains finite at } y = 0, \quad x = \pm a. \] (4.6)

The radiation condition follows Eq. (II-2.10), and the edge condition is imposed at the tips of the strip to ensure that the displacement field \( u_z \) is regular everywhere in the medium. In the equations above, the time factor \( e^{-i\omega t} \) has been omitted.

For the moment, no restriction is imposed on the form of the incident wave so long as it is harmonic in time. When we come later to the solving of the integral equations, we shall assume a normally incident plane wave:

\[ u_z^{(i)} = e^{iky}. \] (4.7)

As the elliptic boundary \( \Gamma \) shrinks to a line segment, \( |x| < a \), \( y = 0 \), the normal derivatives \( \partial / \partial y \) become \( \partial / \partial y \) on the plus side of the segment, and \( -\partial / \partial y \) on the minus side (Fig. 4.1). The Eqs. (4.1) and (4.2) take the following form:

1. Rigid strip \( u_z = 0 \) at \( y = 0, \ |x| < a \):

\[ u_z(x,y) = u_z^{(i)}(x,y) - \frac{i}{4} \int_{-a}^{a} H_0^{(1)} \left[ k\sqrt{(x - x_0)^2 + y^2} \right] \psi(x_0) \, dx_0, \] (4.8)

where
\[
\psi(x) = \frac{\partial u_z(x,0^+)}{\partial y} - \frac{\partial u_z(x,0^-)}{\partial y}
\]

is the jump of the shearing stresses across the rigid strip, and \( \psi(x) \) satisfies the integral equation

\[
u_z^{(2)}(x,0) = \frac{i}{4} \int_{-a}^{a} \frac{\partial u_z^{(1)}(y)}{\partial y} \psi(y) \, dy, \quad |x| < a.
\]

(2) Line crack (\( \frac{\partial u_z}{\partial y} = 0 \) at \( y = 0 \), \(|x| < a\)):

\[
u_z(x,y) = \nu_z^{(2)}(x,y) + \frac{i}{4} \int_{-a}^{a} \psi(y) \cdot \frac{\partial}{\partial y} \left[ \frac{1}{\sqrt{(x-x_o)^2 + (y-y_o)^2}} \right] \, dy \, dx_o.
\]

where

\[
\psi(x) = u_z(x,0^+) - u_z(x,0^-)
\]

is the jump of the displacement across the crack. The unknown function \( \psi(x) \) satisfies the integral equation

\[
-\frac{\partial u_z^{(i)}(x,y)}{\partial y} = \left( \frac{i}{4} \right) \frac{\partial}{\partial y} \int_{-a}^{a} \psi(y) \cdot \frac{\partial}{\partial y} \left[ \frac{1}{\sqrt{(x-x_o)^2 + (y-y_o)^2}} \right] \, dy \, dx_o, \quad |x| < a, y = 0.
\]
\[ \frac{3}{\delta y_o} H_0^{(1)}(k|r - r_o|) = - \frac{3}{\delta y} H_0^{(1)}(k|r - r_o|), \]

the equations above can be rewritten as

\[ u_z(x, y) = u_z^{(i)}(x, y) - \frac{i}{4} \frac{3}{\delta y} \int_{-a}^{a} \phi(x_o) H_0^{(1)}(k\sqrt{(x-x_o)^2 + y^2}) dx_o, \quad (4.12) \]

and

\[ \frac{\partial u_z^{(i)}(x, y)}{\partial y} = \left( \frac{i}{4} \frac{3}{\delta y} \right) \int_{-a}^{a} \phi(x_o) H_0^{(1)}(k\sqrt{(x-x_o)^2 + y^2}) dx_o, \quad y = 0. \]

Furthermore, because the Hankel function \( H_0^{(1)} \) satisfies the two-dimensional Helmholtz equation, the integral on the right-hand side of the above equation satisfies it also. The differential operation \( \partial^2 / \partial y^2 \) on the integral is equivalent to \( -\partial^2 / \partial x^2 - k^2 \). The unknown function of \( \phi(x) \) in (4.12) then satisfies the following integro-differential equation:

\[ -\frac{\partial u_z^{(i)}(x, 0)}{\partial y} = \frac{i}{4} \left( \frac{\partial^2}{\partial x^2} + k^2 \right) \int_{-a}^{a} \phi(x_o) H_0^{(1)}(k|x-x_o|) dx_o, \quad |x| < a. \quad (4.13) \]

This awesome looking equation can be simplified further if the function \( u_z^{(i)}(x, y) \) is given explicitly.

Equations (4.9) and (4.11) or (4.13) are called Rayleigh's integral equations.
B. Dual Integral Equations

The integrals and integral equations (4.8)-(4.9), and (4.12)-(4.13) can be transformed by noting that

$$
H_0^{(1)}(k\sqrt{x^2 + y^2}) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ik(x + \sqrt{1 - \alpha^2}y)} \frac{da}{\sqrt{1 - \alpha^2}}.
$$

(4.14)

(1) Rigid strip:

Substituting (4.14) in (4.8) yields

$$
u_2(x, y) = u_2^{(i)}(x, y) - \frac{i}{4\pi} \int_{-\infty}^{\infty} \overline{\psi}(\alpha) e^{ik(x + \sqrt{1 - \alpha^2}y)} \frac{da}{\sqrt{1 - \alpha^2}},
$$

(4.15)

with

$$
\overline{\psi}(\alpha) = \int_{-\alpha}^{\alpha} \psi(x_o) e^{-ikx_o} dx_o.
$$

(4.16)

and substituting (4.14) in (4.9) leads to

$$
u_2^{(i)}(x, 0) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \overline{\psi}(\alpha) e^{ikx} \frac{da}{\sqrt{1 - \alpha^2}}.
$$

(4.17a)

Rewriting (4.16) as

$$
\overline{\psi}(\alpha) = \int_{-\infty}^{\infty} \psi(x_o) e^{-ikx_o} dx_o.
$$

(4.17b)

with
\[ \psi(x_o) = \begin{cases} \psi(x_o), & |x_o| < a \\ 0, & |x_o| > a \end{cases} \]

from the Fourier integral transform, we obtain

\[ \psi(x_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\psi(\alpha)} e^{i k x_o \alpha} d\alpha. \]

Hence, the undetermined function \( \overline{\psi(\alpha)} \) in (4.15) and (4.16) must satisfy the following equation:

\[ \int_{-\infty}^{\infty} \overline{\psi(\alpha)} e^{i k x \alpha} d\alpha = 0, \quad |x| > a. \quad (4.17b) \]

Equations (4.17a) and (4.17b) constitute a dual-integral equation for the variable \( \overline{\psi(\alpha)} \). The total wave field \( u_2(x,y) \) is determined by substituting the solution for \( \overline{\psi(\alpha)} \) in the integral of (4.15).

Equations (4.15) and (4.17) could have been derived by applying the Fourier transform to the wave equation for the scattered wave and making use of the appropriate boundary conditions. Physically, it is clear that the second term in (4.15) represents the scattered wave by the rigid strip and the integral is a superposition of plane waves propagating in all directions, each with an amplitude \( \overline{\psi(\alpha)}/4\pi \).

(2) Line crack:

Equation (4.13) can be reduced to a dual-integral equation analogously. Substitution of (4.14) in (4.12) results in
\[ u_2(x, y) = u_2^{(i)}(x, y) - \frac{i}{4} \sqrt{1 - a^2 y} \int_{-\infty}^{\infty} \overline{\phi(a)} e^{ik(\alpha x + 1 - a^2 y)} \frac{da}{\sqrt{1 - a^2}}, \]

where

\[ \overline{\phi(a)} = \int_{-\infty}^{\infty} \phi(x_o) e^{-ik\alpha x_o} dx_o. \]

Carrying out the differentiation inside the integral, we obtain

\[ u_2(x, y) = u_2^{(i)}(x, y) + \frac{k}{4\pi} \int_{-\infty}^{\infty} \overline{\phi(a)} e^{ik(\alpha x + 1 - a^2 y)} da. \quad (4.18) \]

The unknown function \( \overline{\phi(a)} \) satisfies the following dual-integral equations, the first of which is obtained by substituting \( (4.14) \) into \( (4.13) \) and then carrying out the differentiation:

\[ k^2 \int_{-\infty}^{\infty} \sqrt{1 - a^2} \overline{\phi(a)} e^{ik\alpha x} da = 4\pi \frac{\partial u_2(x, 0)}{\partial y}, \quad |x| < a, \quad (4.19a) \]

\[ \int_{-\infty}^{\infty} \overline{\phi(a)} e^{ik\alpha x} da = 0, \quad |x| > a. \quad (4.19b) \]

The \( \overline{\phi(a)} \) has the same physical interpretation as that for \( \overline{\Psi(a)} \) in \( (4.15) \).

4.2. Approximate Solutions of Rayleigh's Integral Equation

Rayleigh's integral equation \( (4.9) \), which is the simplest one for all diffraction problems, has an exact solution in terms of a series of Mathieu functions. In elliptical coordinates, \( \psi(x_o) \) is proportional to the \( \omega(0, \eta)/\delta \) given by \( (2.9) \), with the incident wave part being
excluded. Exact solutions like this would not be very useful until extensive tables for Mathieu functions and modified functions were available, which began to appear only around 1940.

In his first paper on this subject, Rayleigh (4.1) set up the integral equation for the passage of waves through apertures or narrow slits in a plane screen, which had the same form as the scattering of waves by a rigid ribbon given by (4.9). For \( u_z^{(i)} = e^{iky} \), the equation simplifies to

\[
1 = \frac{i}{4} \int_{-a}^{a} \psi(x_o) \mathcal{H}_0^{(1)}(k|x - x_o|) \, dx_o, \quad |x| < a. \tag{4.20}
\]

From the known solution for the flow of incompressible fluids through a slit in an infinite plane, Rayleigh inferred that solutions of (4.20) might be of the form \( A(a^2 - x^2)^{-\frac{1}{2}} \), where \( A \) is some constant. He assumed further that \( k\alpha \) is very small, and that the secondary waves emitted by the sources among the slit are independent of \( x \). Thus \( \mathcal{H}_0^{(1)}(z) \) in (4.20) is approximated by its constant and logarithmic terms in its series expansion, i.e., \( \mathcal{H}_0^{(1)}(z) \approx (iz/\pi)[\gamma + \ln(z/2\alpha)] \), and \( x \) in (4.20) is set to be zero for the purpose of calculating coefficient \( A \). Equation (4.20) is then replaced by

\[
1 = -\frac{1}{2\pi} \int_{-a}^{a} \psi(x_o) \left[ \gamma + \ln \frac{k|x_o|}{2\alpha} \right] \, dx_o.
\]

By setting \( \psi(x_o) = A(a^2 - x_o^2)^{-\frac{1}{2}} \), this integration can be carried out with
\[
\int_{-\pi}^{\pi} \frac{1}{(a^2 - x_o^2)^{\frac{3}{2}}} \, dx_o = \pi,
\]
\[
\int_{-\pi}^{\pi} \frac{\ln |x_o|}{(a^2 - x_o^2)^{\frac{3}{2}}} \, dx_o = 2 \int_0^{\pi/2} \ln (a \sin \theta) \, d\theta = \pi \ln (a/2).
\]

Thus

\[
A = -2/[\gamma + \ln (ka/4i)].
\] (4.21)

The waves outside the ribbon are obtained from (4.8), with

\[
\psi(x_o) = -2[\gamma + \ln (ka/4i)]^{-1}(a^2 - x_o^2)^{-\frac{1}{2}}.
\]

The function can be removed from under the integral sign at distances far away from the ribbon, since the argument of the Hankel function

\[k\sqrt{(x-x_o)^2 + y^2} + k\sqrt{x^2 + y^2} = kr.\]

By making use of the asymptotic values of \(H_0^{(1)}(z)\) for large \(z\), i.e., \(H_0^{(1)}(z) \sim (2/\pi z)^{\frac{1}{2}}e^{iz}\), Eq. (4.8) is approximated by

\[
\psi(x,y) = e^{iky} - \frac{i}{4} \left[ \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{ikr} \int_{-\pi}^{\pi} \psi(x,o) \, dx_o \right].
\] (4.22)

The final answer is

\[
u_s(x,y) \approx e^{iky} - \frac{1}{\gamma + \ln (ka/4i)} \left(\frac{\alpha i}{2kr}\right)^{\frac{1}{2}} e^{ikr}.
\] (4.23)

The second term represents the scattered waves at a distance far from the ribbon.

Sixteen years later, not seeing any easy way to the solution by
use of elliptic coordinates, Lord Rayleigh fell back again upon approximate methods to improve the solution. In his second paper he assumed the width of the slit to be comparable to the wavelength — thus the Hankel function can no longer be approximated by the constant and logarithmic terms alone. Also, when \( k\alpha \) is greater, the resulting value of \( \psi(x_o) \) may not be the same over the slit, and one cannot set \( x = 0 \) in (4.8). Rayleigh then assumed \( \psi(x_o) = A(x^2 - x_o^2)^{-\frac{1}{2}} + B \); used more terms in the series expansion of the Hankel function, and numerically determined the constants \( A \) and \( B \). With that, he succeeded in explaining why the polarization of light is reversed upon the passage through a wider \( (k\alpha \approx 1.5) \) slit.

This line of approach can be generalized further by assuming

\[
\psi(x) = \frac{1}{a} \sum_{n=0}^{\infty} A_n \left(1 - \frac{x^2}{a^2}\right)^{n-\frac{1}{2}},
\]

as a solution for integral equations (4.9) and (4.13). A two-term approximation is discussed in Sommerfeld's Lecture on Theoretical Physics, Vol. 4 -- Addenda. Bouwkamp checked Sommerfeld's solution and, in addition, calculated the third term. On the assumption that \( u_2^{(i)} = e^{iky} \), the result is

\[
- \frac{1}{2} A_0 = \frac{1}{p} - \frac{1}{4} \left(1 - \frac{1}{2p}\right) (k\alpha)^2 + \frac{1}{128} \left(1 - \frac{9}{4p} + \frac{1}{p^2}\right) (k\alpha)^4 + \ldots,
\]

\[
- \frac{1}{2} A_1 = \frac{1}{2} \left(1 + \frac{1}{2p}\right) (k\alpha)^2 - \frac{1}{16} \left(1 - \frac{3}{4p}\right) (k\alpha)^4 + \ldots,
\]

\[
- \frac{1}{2} A_2 = \frac{1}{48} \left(1 + \frac{3}{4p}\right) (k\alpha)^4 + \ldots.
\]
while the parameter $p$ is defined by

$$p = \ln (\bar{\gamma} k a / 4) - i \pi / 2 = \ln (\bar{\gamma} k a / 4 i),$$

with $\ln \bar{\gamma} = 0.57721...$ (Euler's constant).

For a plane incident wave, the integro-differential equation (4.13) can be reduced to an integral equation. Assuming $u_3 (x, y) = e^{iky}$, we have

$$\left( \frac{d^2}{dx^2} + k^2 \right) \int_{-a}^{a} \phi (x_o) H_0^{(1)} (k |x - x_o|) \, dx_o = -4k. \quad (4.26)$$

Let $X(x)$ represent the integral of the unknown function with

$$X(x) = \int_{-a}^{a} \phi (x_o) H_0^{(1)} (k |x - x_o|) \, dx_o;$$

the differential equation $(d^2/dx^2 + k^2)X = -4k$ then has the general solution

$$X(x) = C_1 \cos kx + C_2 \sin kx - 4/k. \quad (4.27)$$

Because of the symmetry of the problem, the odd function in $x$ should be discarded. A new integral equation is thus derived:

$$\int_{-a}^{a} \phi (x_o) H_0^{(1)} (k |x - x_o|) \, dx_o = C_1 \cos kx - 4/k. \quad (4.28)$$

The unknown constant $C_1$ is determined by setting $x = 0$ in the above equation with the result.
\[ C_1 = \frac{4}{k} + \int_{-a}^{a} \phi(x_o) H_0^{(1)}(k|x_o|) \, dx_o. \]

Substitution of the above in (4.27) gives rise to another integral equation for \( \phi(x) \),

\[ \int_{-a}^{a} \phi(x_o) \left[ H_0^{(1)}(k|x - x_o|) - \cos kx H_0^{(1)}(k|x_o|) \right] \, dx_o = \frac{4}{k} (\cos kx - 1). \]  

(4.29)

We note that Eq. (4.28) has a symmetrical kernel in \( x \) and \( x_o \) but (4.29) does not.

Equation (4.29) can be solved by the same series solution

\[ \phi(x) = ik \alpha \sum_{n=1}^{\infty} B_n \left( \frac{1 - \frac{x^2}{a^2}}{a^2} \right)^{n-\frac{3}{2}}. \]  

(4.30)

The first three coefficients as calculated by Bouwkamp are

\[ \frac{1}{2} B_1 = 1 - \frac{1}{4} \left( p - \frac{1}{2} \right)(ka)^2 + \frac{1}{16} \left( p^2 - \frac{7}{8} p + \frac{7}{32} \right)(ka)^4 + \ldots, \]

\[ \frac{1}{2} B_2 = \frac{1}{12} (ka)^2 - \frac{1}{32} \left( p - \frac{1}{2} \right)(ka)^4 + \ldots, \]  

(4.31)

\[ \frac{1}{2} B_3 = \frac{1}{320} (ka)^4 + \ldots. \]

From the known values of \( \phi(x_o) \), the diffracted waves outside the crack are calculated from Eq. (4.12).

We shall not repeat the details in deriving solutions (4.25) and (4.31) here because they can be derived more systematically by a per-
turbation scheme which will be taken up in the next subsection.

4.3. Perturbation Solutions of Rayleigh's Integral Equation

For simplicity we shall assume an incident wave of the form

\[ u_z^{(i)} = e^{iky} \]

to illustrate the perturbation method for solving the Rayleigh's integral equation. As discussed by Hönl, Maue, and Westpfahl\(^{4.2}\) and by Noble,\(^{4.7}\) this method can also be extended to Eqs. (4.7a) and (4.7b), where the line integral is to be extended over a closed curve \( \Gamma \), and to the general three-dimensional cases.

In normally incident plane waves, Rayleigh's integral equations (4.9) and (4.13) are reduced to (4.20) and (4.28), which are repeated here:

\[ \int_{-a}^{a} \psi(x_o) H_0^{(1)}(k|x - x_o|) \, dx_o = -4i, \quad |x| < a; \quad (4.20) \]

\[ \int_{-a}^{a} k\phi(x_o) H_0^{(1)}(k|x - x_o|) \, dx_o = -4(1 + c \cos kx), \quad |x| < a. \quad (4.28) \]

Equation (4.28) is preferred to (4.29) because the former contains a symmetric kernel; the unknown constant \( c \) shall be fixed later by invoking the edge conditions.

Both (4.20) and (4.28) are of the form

\[ \frac{1}{2i} \int_{-a}^{a} f(x_o') H_0^{(1)}(k|x' - x_o'|) \, dx_o' = a\phi(x'), \quad |x'| < a, \quad (4.32) \]
which is an integral equation of the first kind for the unknown function \( f(x') \), and the kernel \( H_0^{(1)}(k|x' - x'|) \) has a logarithmic singularity at \( x' = x' \). The above integral equation can further be reduced to

\[
\frac{1}{2\pi} \int_{-1}^{1} f(x_0) H_0^{(1)}(\epsilon |x - x_0|) \, dx_0 = g(x), \quad |x| < 1, \tag{4.33}
\]

where \( x = x'/a \) and \( \epsilon = k\alpha \). Since the Hankel function has the following series expansion,

\[
\frac{1}{2\pi} H_0^{(1)}(z) = \frac{1}{\pi} \ln \frac{\gamma}{2z} - \frac{z^2}{4\pi} \left( \ln \frac{\gamma z}{2} - 1 \right) + \frac{z^4}{64\pi} \left( \ln \frac{\gamma z}{2} - \frac{3}{2} \right) + \ldots, \tag{4.34}
\]

the kernel in (4.33) can be expressed as a power series of \( \epsilon^2 \) by setting \( z = \epsilon |x - x'| \). This suggests that an expansion of the unknown function \( f(x) \) and the given function \( g(x) \) in terms of the perturbation parameter \( \epsilon \) as

\[
f(x) = f^{(0)}(x) + \epsilon^2 f^{(2)}(x) + \epsilon^4 f^{(4)}(x) + \ldots,
\]

\[
g(x) = g^{(0)}(x) + \epsilon^2 g^{(2)}(x) + \epsilon^4 g^{(4)}(x) + \ldots,
\]

will reduce (4.33) into a system of integral equations for \( f^{(n)}(x) \) by collecting like powers terms of \( \epsilon \). The perturbed integral equations are all of the form

\[
\int_{-1}^{1} f^{(n)}(x_0) \ln \left( \frac{\sqrt{2}}{2z} |x - x_0| \right) \, dx_0 = g^{(n)}(x), \tag{4.35}
\]
where \( G^{(n)}(x) \) is the combination of \( g^{(n)}(x) \) and integrals of \( f^{(n-2)}(x) \), \( f^{(n-4)}(x), \ldots \). It is known that Eq. (4.35) has the exact solution

\[
f^{(n)}(x) = \frac{1}{\pi \sqrt{1 - x^2}} \times \left\{ P \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{t - x} \frac{dG^{(n)}(t)}{dt} dt - \frac{1}{\ln (4t/\pi)} \int_{-1}^{1} \frac{G^{(n)}(t)}{\sqrt{1 - t^2}} dt \right\},
\]

(4.36)

where \( P \) denotes the principal value of the integral (see p. 428 of Ref. 4.10). With \( G^{(0)}(x) = g^{(0)}(x) \), the 0th order function \( f^{(0)}(x) \) can be found easily. The next higher order integral equation can be solved after the completion of evaluating \( G^{(2)}(x) \), which is a function of \( g^{(2)}(x) \) and an integration involving \( f^{(0)}(x) \):

\[
G^{(2)}(x) = g^{(2)}(x) - \int_{-1}^{1} f^{(0)}(t) \frac{1}{4\pi} (x - t)^2 [1 - \ln \frac{\pi}{2t} (x - t)] dt.
\]

The higher order equations can then be solved successively, at least in principle. It should now become obvious that in reality it is not easy to find the higher order solutions explicitly because of the long and involved integrations in \( G^{(n)}(x) \) and in (4.36).

To avoid this difficulty, the Hilbert-Schmidt method (Chapter II, subsection 2.3) can be applied to solve integral equations like (4.35). To that effect, we return to the original integral equation and substitute

\[
x = a \cos \xi, \quad x_0 = a \cos \xi_0,
\]
in (4.20) and obtain

$$\frac{1}{2\pi} \int_0^\pi \nu(\xi_0) H_0^{(1)}(\epsilon |\cos \xi - \cos \xi_0|) \, d\xi_0 = 1, \quad (4.37a)$$

with $\nu(\xi) = -\frac{i}{2\pi} \sin \xi \phi(a \cos \xi)$. Similarly, from (4.28), we derive

$$\frac{\epsilon}{2\pi} \int_0^\pi \omega(\xi_0) H_0^{(1)}(\epsilon |\cos \xi - \cos \xi_0|) \, d\xi_0 = 1 + C \cos (\epsilon \cos \xi), \quad (4.37b)$$

with $\omega(\xi) = -\frac{\epsilon}{2\pi} \sin \xi \phi(a \cos \xi)$.

The kernel in both equations is approximated by the series (4.34) with

$$\frac{1}{2\pi} H_0^{(1)}(\epsilon |\cos \xi - \cos \xi_0|) = K_0(\xi, \xi_0) + \epsilon^2 K_2(\xi, \xi_0) + \epsilon^4 K_4(\xi, \xi_0) + \ldots,$$

and

$$K_0(\xi, \xi_0) = \frac{1}{\pi} \ln \frac{\sqrt{\nu}}{2\pi} |\cos \xi - \cos \xi_0|,$$

$$K_2(\xi, \xi_0) = \frac{1}{4\pi} (\cos \xi - \cos \xi_0)^2 [1 - \pi K_0(\xi, \xi_0)],$$

$$K_4(\xi, \xi_0) = \frac{1}{64\pi} (\cos \xi - \cos \xi_0)^4 [\pi K_0(\xi, \xi_0) - \frac{3}{2}],$$

and so on.

Since $C$ is an unknown quantity, it is expanded also in a series of $\epsilon^2$ as

$$C = -1 + \epsilon^2 C_2 + \epsilon^4 C_4 + \ldots, \quad (4.39)$$
where $c_2$, $c_4$, ..., are undetermined constants. Thus the right-hand side of (4.37b) has the following series expansion:

$$1 + c \cos (\epsilon \cos \xi) = c^2 \left( \frac{1}{2} \cos^2 \xi + c_2 \right) + c^4 \left( -\frac{1}{24} \cos^4 \xi - \frac{1}{2} \cos^2 \xi c_2 + c_4 \right) + \ldots$$

Assuming in (4.37)

$$v(\xi) = v^{(0)}(\xi) + c^2 v^{(2)}(\xi) + \ldots, \quad (4.40a)$$

$$w(\xi) = w^{(1)}(\xi) + c^3 w^{(3)}(\xi) + \ldots, \quad (4.40b)$$

we then obtain the following integral equations for various orders of $v$ and $w$ by applying the regular perturbation procedure:

$$\int_0^\pi v^{(0)}(\xi_0) x_0(\xi, \xi_0) d\xi_0 = 1, \quad (4.41a)$$

$$\int_0^\pi v^{(2)}(\xi_0) x_0(\xi, \xi_0) d\xi_0 = -\int_0^\pi v^{(0)}(\xi_0) x_2(\xi, \xi_0) d\xi_0, \quad (4.41a)$$

$$\int_0^\pi v^{(4)}(\xi_0) x_0(\xi, \xi_0) d\xi_0 = -\int_0^\pi [v^{(2)}(\xi_0) x_2(\xi, \xi_0) + v^{(0)}(\xi_0) x_4(\xi, \xi_0)] d\xi_0, \quad (4.41a)$$

and so on.

$$\int_0^\pi w^{(1)}(\xi_0) x_0(\xi, \xi_0) d\xi_0 = c_2 + \frac{1}{2} \cos^2 \xi, \quad (4.41b)$$

$$\int_0^\pi w^{(3)}(\xi_0) x_0(\xi, \xi_0) d\xi_0 = c_4 - \frac{1}{2} c_2 \cos^2 \xi - \frac{1}{24} \cos^4 \xi$$

$$-\int_0^\pi w^{(1)}(\xi_0) x_2(\xi, \xi_0) d\xi_0.$$
\[ \int_{0}^{\pi} \psi^{(5)}(\xi, \xi) d\xi = C_6 - \frac{1}{2} C_4 \cos^2 \xi + \frac{1}{24} C_2 \cos^4 \xi + \frac{1}{720} \cos^6 \xi \]

\[ - \int_{0}^{\pi} \left[ \psi^{(3)}(\xi, \xi) d\xi + \psi^{(1)}(\xi, \xi) \right] d\xi, \]

and so on. All equations are of the form

\[ \frac{1}{\pi} \int_{0}^{\pi} F(\xi) \ln \left( \frac{\nu \epsilon}{2 \ell} |\cos \xi - \cos \xi_0| \right) d\xi = G(\xi), \quad (4.42) \]

which is the same as (4.35) when \( \epsilon \) in the latter is changed to \( \cos \xi \).

To solve the integral equation above by the Hilbert-Schmidt method, we note first that

\[ \ln \left( \frac{\nu \epsilon}{2 \ell} |\cos \xi - \cos \xi_0| \right) = P - 2 \sum_{m=1}^{\infty} \frac{\cos m\xi \cos m\xi_0}{m}, \quad (4.43) \]

with

\[ P = \ln \left( \frac{\nu \epsilon}{4 \ell} \right) = \ln \frac{\nu \epsilon}{4} - \frac{\pi \xi}{2}. \]

This is a special case of the formula

\[ \ln [2(\cos \xi - \cos \xi_0)] = -2 \sum_{m=1}^{\infty} \frac{\cos m\xi \cos m\xi_0}{m}, \quad (4.44) \]

which can be derived by setting \( z = e^{i\alpha} \) and \( z = e^{i\beta} \) successively in the identity

\[ \ln (1 - z) = -\sum_{m=1}^{\infty} (z^m/m). \]
The real parts of the two resulting equations are

$$\ln |2(1 - \cos \alpha)| = -2 \sum \cos m\alpha/m,$$

$$\ln |2(1 - \cos \beta)| = -2 \sum \cos m\beta/m.$$

Adding them, and setting $\alpha = \xi + \xi_0$ and $\beta = \xi - \xi_0$, yields Eq. (4.44).

Since $\cos m\xi_0$ ($m = 1, 2, \ldots$) are orthogonal in the interval $(0, \pi)$, Eq. (4.43) in essence supplies the eigenfunctions needed to separate the kernel in (4.42) into products of two parts, one depending only on $\xi$ and the other on $\xi_0$. Expanding $G(\xi)$ into a series of the orthogonal functions $\cos n\xi$, as

$$G(\xi) = \sum b_n \cos n\xi,$$  \hspace{1cm} (4.45)

and assuming that the unknown function $F(\xi)$ has the expansion

$$F(\xi) = \sum b_n \cos n\xi,$$  \hspace{1cm} (4.46)

we then can evaluate the unknown coefficients $b_n$ by completing the integrations in (4.42). With the aid of the orthogonality conditions

$$\frac{1}{\pi} \int_0^\pi \cos m\xi \cos n\xi \, d\xi = \begin{cases} 1, & (m = n = 0), \\ \delta_{mn}/2, & (m + n \neq 0), \end{cases}$$  \hspace{1cm} (4.47)

and

$$\frac{1}{\pi} \int_0^\pi \cos m\xi_0 \ln \left( \frac{\xi}{2\xi_0} \right) \left| \cos \xi - \cos \xi_0 \right| \, d\xi_0 = \begin{cases} \frac{\pi}{m}, & (m = 0), \\ -\frac{1}{m} \cos m\xi, & (m > 0), \end{cases}$$

the final results for $b_n$ in (4.46) are found as
\[ b_0 = B_0 p, \quad (4.48) \]

\[ b_n = -mB_n, \quad n > 0. \]

This completes the solving of the integral equation (4.42).

(1) Rigid Ribbon:

Returning to Eqs. (4.41a), since the inhomogeneous terms on the right-hand sides are expressible as series of \( \cos 2\nu \xi \), we assume solutions for all orders of \( \nu = 0 \) as

\[ \nu(\xi) = \sum_{\nu=0}^{\infty} b_{2\nu}(\xi) \cos 2\nu \xi. \]  \quad (4.49)

From (4.43) through (4.48), the coefficients \( b_{2\nu}^{(0)} \) for the 0th order solutions are found as

\[ b_{2\nu}^{(0)} = 0, \quad b_{2\nu}^{(0)} = 0, \quad \nu > 0. \]  \quad (4.50)

For the higher order equations, the inhomogeneous parts of (4.41a) are evaluated by expanding \( K_2(\xi, \xi_o) \), \( K_4(\xi, \xi_o) \), etc., also in Fourier cosine series. Substituting (4.43) in (4.38), and combining the products of cosines by the formula

\[ 2 \cos n \xi \cos m \xi = \cos (n + m) \xi + \cos (n - m) \xi, \]

we find

\[ K_2(\xi, \xi_o) = -\frac{p}{2\pi} - \frac{2p}{16\pi} (\cos 2\xi + \cos 2\xi_o) \]

\[ - \frac{1}{12\pi} (\cos 2\xi - \frac{1}{8} \cos 4\xi)(\cos 2\xi_o - \frac{1}{8} \cos 4\xi_o) + \ldots. \]
\[ X_4(\xi, \xi_0) = \frac{3}{256\pi} (3p - 1) + \frac{6p - 1}{192\pi} \cos 2\xi_n + \frac{1}{512\pi} (p + \frac{7}{12}) \cos 4\xi + \ldots, \]  

(4.51)

where only the terms which are needed in calculating \( \nu^{(2)} \) and \( \nu^{(4)} \) are written out. With \( \nu^{(0)}(\xi) = 1/p \), the second of Eqs. (4.41a) can be reduced to

\[ \frac{1}{\pi} \int_0^\pi \nu^{(2)}(\xi_0) \ln \left( \frac{\sqrt{\xi}}{2\xi} |\cos \xi - \cos \xi_0| \right) \, d\xi_0 = \frac{1}{4} + \left( \frac{1}{8} + \frac{1}{16p} \right) \cos 2\xi. \]

Again, by letting the solution be of the form (4.49), the coefficients are

\[ b_0^{(2)} = 1/4p, \quad b_2^{(2)} = -\frac{1}{4} \left( 1 + \frac{1}{2p} \right), \quad b_2^{(2)} = 0, \quad \nu > 1. \]

(4.52)

The next higher order integral equation is

\[ \int_0^\pi \nu^{(4)}(\xi_0) K_0(\xi, \xi_0) \, d\xi_0 = \frac{2 + 3p - 4p^2}{256p} + \frac{3 - 2p}{192p} \cos 2\xi - \frac{3 + 5p}{6144p^2} \cos 4\xi, \]

with the following coefficients for the series solution

\[ b_0^{(4)} = \frac{1}{64} \left( -1 + \frac{3}{4p} + \frac{1}{2p^2} \right), \]

\[ b_2^{(4)} = \frac{1}{48} \left( 1 - \frac{3}{2p} \right), \]

(4.53)

\[ b_4^{(4)} = \frac{1}{512} \left( \frac{4}{3} + \frac{1}{p} \right), \]

\[ b_{2n}^{(4)} = 0, \quad \nu > 2. \]
Substitutions of (4.50), (4.52), and (4.53) into (4.49) and then into (4.40a) complete the solution for \( \nu(\xi) \) in (4.37a), accurate up to the order of \((ka)^5\),

\[
\nu(\xi) = \frac{1}{p} + \frac{e^2}{4} \left[ \frac{1}{1p} - \left( 1 + \frac{1}{2p} \right) \cos 2\xi \right] \\
+ \frac{e^4}{16} \left[ \left( -\frac{1}{4} + \frac{3}{16p} + \frac{1}{8p^2} \right) + \left( \frac{1}{3} - \frac{1}{2p} \right) \cos 2\xi + \left( \frac{1}{24} + \frac{1}{32p^2} \right) \cos 4\xi \right] + \ldots
\]

(4.54)

In terms of the variable \( x \) and the function \( \psi(x) \) in (4.20), the solution is

\[
-\frac{1}{2} a \left( 1 - \frac{x^2}{a^2} \right)^2 \hat{\psi}(x) = \frac{1}{p} + \frac{(ka)^2}{4} \left[ \frac{1}{2p} - 1 + \left( 2 + \frac{1}{p} \right) \left( 1 - \frac{x^2}{a^2} \right) \right] \\
+ \frac{(ka)^4}{16} \left[ \frac{1}{8} - \frac{6}{32p} + \frac{1}{8p^2} + \left( -1 + \frac{3}{4p} \right) \left( 1 - \frac{x^2}{a^2} \right) \right] \\
+ \left( \frac{1}{3} + \frac{1}{4p} \right) \left( 1 - \frac{x^2}{a^2} \right)^2 \ldots
\]

(4.55)

where use has been made of the relations:

\[
\cos 2\xi = 1 - 2(1 - x^2/a^2), \quad \cos 4\xi = 2 \cos^2 2\xi - 1.
\]

The result above is the same as that given in (4.24) and (4.25).

(2) Line Crack:

In the previous analysis for \( \nu^n(\xi) \), the edge condition (4.6) was never mentioned. This is because \( \nu(\xi) = -\frac{1}{2} a \sin \xi \psi(a \cos \xi) \) and \( \psi(x) \) in (4.8) is the jump of shearing stresses across the rigid
ribbon. Even though the displacement must be regular at the tip of the ribbon, the stresses may be singular, thus no restriction is imposed on \( u(\xi) \) at \( \xi = 0, \tau \) \( (x = \pm a) \). However, when we come to the second group of equations of (4.41), the function \( w \) is related to \( \psi \) by \( w(\xi) = \frac{3}{2\xi} \sin \xi \, \phi(\alpha \cos \xi) \), and \( \phi(x) \) in (4.10) represents the jump of displacement at the crack. Hence at \( x = \pm a \), the edge condition (4.6) dictates that

\[
\phi(a) = \phi(-a) = 0. \tag{4.56}
\]

This condition creates additional complications for solving \( w^{(n)} \), since if a simple series solution like (4.49) is assumed, \( \phi(\alpha \cos \xi) = 2i \, w(\xi)/\sin \xi \) becomes singular at \( \xi = 0, \tau \).

In order to remove the singularity \( (\sin \xi)^{-1} \) from the solution, \( w^{(n)}(\xi) \) is assumed as

\[
w^{(n)}(\xi) = \frac{1}{2} \sum_{\nu=0}^{\infty} c_{2\nu+1}^{(n)} \left[ \cos 2\nu\xi - \cos (2\nu + 2)\xi \right]
= \sin \xi \sum_{\nu=0}^{\infty} c_{2\nu+1}^{(n)} \sin (2\nu + 1)\xi. \tag{4.57}
\]

If the inhomogeneous parts on the right-hand side of (4.41b) \( G^{(n)}(\xi) \) are expressed as Fourier cosine series,

\[
G^{(n)}(\xi) = \sum_{\nu=0}^{\infty} B_{2\nu}^{(n)} \cos 2\nu\xi, \tag{4.58}
\]

instead of (4.48), the coefficients \( c_{\nu}^{(n)} \) in (4.57) should have the following values:
\[ e_1^{(n)} = 2E_0^{(n)}/p, \]

\[ e_{2v+1}^{(n)} = e_{2v-1}^{(n)} - 4vB_2^{(n)}, \quad v > 0. \]

For the first equation of (4.41b),

\[ \int_0^\pi \omega^{(1)}(\xi) K_0(\xi,\xi') \, d\xi' = \phi^{(1)}(\xi) = (c_2 + \frac{1}{4}) + \frac{1}{2} \cos 2\xi, \]

we have

\[ c_1^{(1)} = \frac{2}{p} \left( c_2 + \frac{1}{4} \right), \]

\[ c_3^{(1)} = c_1^{(1)} - 1, \]

\[ c_{2v+1}^{(1)} = 0, \quad v > 1. \]

By setting \( c_3^{(1)} = 0 \), we finally obtain

\[ c_2 = \frac{p}{2} - \frac{1}{4}, \quad c_1^{(1)} = 1, \]

and

\[ \omega^{(1)}(\xi) = \frac{1}{2}(1 - \cos 2\xi). \]  

We list a few higher order solutions for \( \omega^{(n)} \) of (4.57) without further discussion:

\[ c_1^{(1)} = 1, \]

\[ c_1^{(3)} = -\frac{1}{4}(p - \frac{3}{4}), \quad c_3^{(3)} = -\frac{1}{48}, \]

\[ c_1^{(5)} = \frac{1}{16}(p^2 - \frac{5}{4}p + \frac{7}{16}), \quad c_3^{(5)} = \frac{1}{384}(3p - \frac{15}{8}), \quad c_5^{(5)} = \frac{1}{5120}. \]
The solution for \( \psi(\xi) \) in (4.40b), accurate up to the order of \( \varepsilon^6 \), is

\[
\psi(\xi) = \sin \xi \left\{ \varepsilon \sin \xi + \varepsilon^3 \left[ -\frac{1}{4} \left( p - \frac{3}{4} \right) \sin \xi - \frac{1}{48} \sin 3\xi \right] + \varepsilon^5 \left[ \frac{1}{16} \left( p^2 - \frac{8p}{4} + \frac{7}{16} \right) \sin \xi + \frac{1}{384} \left( 3p - \frac{15}{8} \right) \sin 3\xi + \frac{1}{5120} \sin 5\xi \right] + \ldots \right\}.
\]

(4.63)

Since

\[
\sin \xi = (1 - \frac{\xi^2}{a^2}),
\]

\[
\sin 3\xi = 3 \sin \xi - 4 \sin^3 \xi,
\]

\[
\sin 5\xi = 5 \sin \xi - 20 \sin^3 \xi + 16 \sin^5 \xi,
\]

the solution for \( \psi(x) \) in (4.28) is

\[
-\frac{1}{2} \varepsilon \psi(x) = \left(1 - \frac{\xi^2}{a^2}\right)^{\frac{3}{2}} (k\alpha) \left\{ 1 + (k\alpha)^2 \left[ \frac{1}{8} - \frac{p}{4} + \frac{1}{12} \left(1 - \frac{\xi^2}{a^2}\right) \right] + (k\alpha)^4 \left[ \frac{p^2}{16} - \frac{7p}{128} + \frac{7}{512} + \left( \frac{1}{64} - \frac{p}{32} \right) \left(1 - \frac{\xi^2}{a^2}\right) + \frac{1}{320} \left(1 - \frac{\xi^2}{a^2}\right)^2 \right] + \ldots \right\},
\]

(4.64)

which agrees with that given in (4.31).

Once the \( \psi(x) \) or \( \psi(x) \) are determined, solutions for waves diffracted at low frequencies by a rigid ribbon or by a line crack are completed by substituting (4.55) and (4.65) respectively into (4.8) and (4.10), and then by carrying out the integrations. Following this approach, Millar calculated in detail the transmission coefficients for the diffraction of acoustic waves by a slit. (4.11)
CHAPTER IV REFERENCES


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CHAPTER V

PARABOLIC CYLINDER PROBLEMS

NEXT TO THE PLANE surface, the parabolic cylindrical or the paraboloidal is perhaps the surface most frequently associated with the reflection of waves. We are all accustomed to the parabolic mirror reflecting light as a parallel beam when a source is placed at the focal point. The same phenomenon can be observed with acoustical waves. In this case, both the incident and reflected waves occur inside a parabola and constitute the "interior problem." If, on the other hand, a source is placed outside, part of the incident wave is reflected by the extended side of a parabola, and part is diffracted by the crown. This is then an "exterior problem" and is the main concern of this chapter.

As the focal length diminishes, a parabolic cylinder degenerates to a semi-infinite plane which, as a screen to wave propagation, divides the medium sharply into the illuminated and shadow regions. The shadow region is not completely "dark," only because the incident wave is diffracted at the edge of the screen. Although the diffraction of light was discussed by Fresnel in 1818, using the wave theory, a rigorous mathematical treatment was not known until 1896 when A. Sommerfeld found the exact solution to the diffrac-
tion of a plane wave by a perfectly conducting semi-infinite screen. This, together with the work of Rayleigh, Lamb, and Clebsch, is the pioneering mathematical study of scattering and diffraction of waves (Refs. 0.3, 0.4, 0.5 — see the historical introduction to Chapter I).

Sommerfeld's solution was expressed in terms of the multi-valued wave functions constructed from the solutions of the wave equation which were made single-valued on a preselected Riemann surface. In 1907, Lamb showed how the same solution can be obtained by integrating the wave equation in parabolic cylinder coordinates.\(^{(0.6)}\)

Lamb's solutions are expressed in terms of integrals like
\[
\int_0^{\infty} e^{it^2} dt,
\]
which can be converted to the product of parabolic cylinder wave functions discussed in this chapter.

A few years later, Epstein investigated the diffraction of electromagnetic wave by a partially conducting parabolic cylinder with an arbitrary focal length.\(^{(0.7)}\) The series-eigenfunction method was used and the results, which include Sommerfeld's as a limiting case, were expressed in terms of wave functions in parabolic cylinder coordinates. Part of Epstein's work was devoted to solutions of the wave equation in parabolic coordinates by separation of variables. The resulting ordinary differential equations, which Epstein analyzed a great deal, are known as Weber's equation, and hence the solutions are now referred to as Weber's function,\(^{(0.8,0.9)}\) or simply as the Parabolic Cylinder Function.\(^{(0.10,0.11,0.12)}\)

The series solution as obtained by Epstein converges rather slowly when the wavelength is short as compared with the characteristic dimension of a parabola — say, with the focal length. To analyze the
The asymptotic behavior of the Weber functions is discussed in detail in his paper, along with some new results.

Following Epstein, the diffraction of an elastic wave by a parabolic cylinder was treated by Thau, the results being reported in a dissertation and a series of papers. Because the P and SV waves are coupled at the parabolic boundary surface, again as in the case of an elliptic cylinder, the boundary conditions cannot be satisfied exactly when the series-eigenfunction method is applied. To overcome this difficulty, a perturbation method was developed which proved to be very useful in calculating stresses at low frequencies at the surface of the parabola. Much of this chapter is based on the dissertations by Epstein and by Thau. The new perturbation method has already been presented in Chapter II.

1. EQUATIONS IN PARABOLIC COORDINATES AND WEBER FUNCTIONS

We define the parabolic coordinates \((\xi, \eta, z)\) by the transformation

\[\xi + i\eta = [2(x + iy)]^{\frac{1}{2}}\text{ and } z = z.\]

To make this transformation single-valued, the positive \(x\)-axis \((y = 0, z > 0)\) is chosen as a branch cut. Thus

\[
x = \frac{1}{k}(\xi^2 - \eta^2), \quad -\infty < \xi < \infty,
\]

\[
y = \xi \eta, \quad 0 \leq \eta < \infty, \tag{1.1}
\]

\[
z = z, \quad -\infty < z < \infty.
\]
The inverse transformations in terms of Cartesian and polar coordinates are:

\[
\begin{align*}
\xi &= \pm \left[ (x^2 + y^2)^{\frac{1}{2}} + x \right]^{\frac{1}{2}} = \sqrt{2r} \cos (\theta/2), \\
\eta &= \pm \left[ (x^2 + y^2)^{\frac{1}{2}} - x \right]^{\frac{1}{2}} = \sqrt{2r} \sin (\theta/2), \\
r &= (x^2 + y^2)^{\frac{1}{2}} = \frac{1}{h_\xi}(\xi^2 + \eta^2), \quad 0 \leq r < \infty, \\
\theta &= \tan^{-1} (y/x) = 2 \tan^{-1} (\eta/\xi), \quad 0 \leq \theta < 2\pi.
\end{align*}
\]

(1.2)

Note that \(\xi\) and \(\eta\) have the dimension of the square root of length, and that \(\eta = 0\) specifies the branch line in the parabolic coordinate system. The scale factors in \(ds^2 = (h_\xi d\xi)^2 + (h_\eta d\eta)^2 + (dz)^2\) are given by

\[
\begin{align*}
h_\xi^2 &= h_\eta^2 = \xi^2 + \eta^2 = J^2, \\
h_\xi^2 &= 1.
\end{align*}
\]

(1.3)

The coordinate curves \(\eta = \text{constant}\) and \(\xi = \text{constant}\), are shown in Fig. 1.1. These are orthogonal families of confocal parabolas, the origin \((x = y = \xi = \eta = 0)\) being the common focal point. Note that \(\eta = \eta_0\) defines an entire parabola opening to the right, whereas \(\xi = \xi_0\) defines only the top half of a parabola opening to the left. The lower half is completed by \(\xi = -\xi_0\). The focal length which is the distance from the focal point 0 to the apex is \(\eta_0^2/2\), and the radius of curvature at the apex is \(\eta_0^2\). The degenerate parabola \(\eta = \eta_0 = 0\) (zero focal length) can be used to define a semi-infinite strip of discontinuity.
1.1. Weber's Equation and Solutions

The Helmholtz equation in parabolic cylinder coordinates $(\xi, \eta, z)$ can be written as

\[(\nabla^2 + \kappa^2)\varphi = \frac{1}{\xi^2} \left( \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} \right) + \frac{\partial^2 \varphi}{\partial z^2} + \kappa^2 \varphi = 0. \] \hspace{2cm} (1.4)

It can be solved by separating variables with

\[\varphi(\xi, \eta, z) = X(\xi) Y(\eta) e^{\pm\gamma z};\]

the separated equations are

\[d^2X/d\xi^2 + (a^2 \xi^2 - a)X = 0, \hspace{1cm} -\infty < \xi < \infty, \] \hspace{2cm} (1.5a)

\[d^2Y/d\eta^2 + (a^2 \eta^2 + a)Y = 0, \hspace{1cm} 0 < \eta < \infty, \] \hspace{2cm} (1.5b)

\[a^2 = \kappa^2 - \gamma^2, \]
where $\alpha$ is a separation constant. The solution of the above equations obviously depends on the wavelength $2\pi/\alpha$, but the parabolic coordinate system contains no characteristic length. It is thus convenient to remove the wave number $\alpha$ from the equations by introducing the new coordinates $u$ and $v$ as

$$u = \lambda \xi = \sqrt{-2i\alpha} \xi, \quad v = \bar{\lambda} \eta = \sqrt{2i\alpha} \eta,$$

with $\bar{\lambda} = \sqrt{2i\alpha}$ = complex conjugate of $\lambda$, and

$$\alpha = (2n + 1)\alpha.$$

The separated equations then take the identical form

$$\frac{d^2 \chi}{du^2} + (n + \frac{1}{2} - \frac{1}{4} u^2) \chi = 0, \quad (1.6a)$$

$$\frac{d^2 \gamma}{dv^2} + (n + \frac{1}{2} - \frac{1}{4} v^2) \gamma = 0. \quad (1.6b)$$

This is the standard form of Weber's equation. \((0.10)\)

(A) First Solution of Weber's Equation

We now consider the solution of the Weber equation

$$\frac{d^2 \omega}{dz^2} + (n + \frac{1}{2} - \frac{1}{4} \alpha^2) \omega = 0. \quad (1.7)$$

Let

$$\omega = e^{-z^2/4} \hat{\omega}(z),$$

then $\hat{\omega}(z)$ satisfies the following equation:
\[ \frac{d^2 w}{dz^2} - z \frac{dw}{dz} + nw = 0. \quad (1.8) \]

When \( n \) is a positive integer, the Hermite polynomial \( H_n(z) = 2^{-n/2} e^{z^2/2} (z + \sqrt{2}) \) is a solution \((0.11)\) where

\[
H_n(z) = (-1)^n e^{z^2/2} \frac{d^n (e^{-z^2})}{dz^n}
\]

\[
He_n(z) = (-1)^n e^{z^2/2} \frac{d^n (e^{-z^2})}{dz^n}
\]

\[= n! \sum_{m=0}^{N} \frac{(-1)^m}{m! \, 2^m (n-2m)!} z^{n-2m}, \quad (1.9)\]

where \( N \) is \( n/2 \) or \((n-1)/2\) according to \( n \) being even or odd, respectively. We shall denote, for positive integer \( n \), the first solution of \( w \) in \((1.7)\) as \( D_n(z) \) with

\[
D_n(z) = (-1)^n e^{z^2/4} \frac{d^n (e^{-z^2/2})}{dz^n}, \quad n > 0,
\]

\[= e^{-z^2/4} He_n(z), \quad n > 0. \quad (1.10)\]

It is known as a Weber function, or parabolic cylinder function.

The function defined by \((1.10)\) ceases to be meaningful when \( n \) is other than a positive integer. A general solution for arbitrary \( n \) can be found by first considering the following integral: \((0.10, 0.13)\)

\[
U_n(z) = \frac{1}{2\pi i} \int_{C_1} t^{-n-1} e^{-t^2/2 + zt} \, dt, \quad (1.11)
\]
Fig. 1.2. Contours for Integral Representation of Weber's Function

where $C_1$ is the keyhole-shaped contour in the complex $t$-plane as shown in Fig. 1.2. A branch cut along the negative real axis is introduced for non-integer $n$. The integral in (1.11) converges for $n < 0$.

For $n > 0$, convergence of the integral can be proved by successive integrations by parts. Substituting $W = U_n(z)$ in (1.8) we find

$$\left(\frac{d^2}{dz^2} - z \frac{d}{dz} + n\right) U_n(z) = \frac{1}{2\pi i} \int_{C_1} \left(t^{-n+1} - zt^{-n} + nt^{-n-1}e^{-t^2/2} + zt \right) dt$$

$$= -\frac{1}{2\pi i} \int_{C_1} \frac{d}{dt} \left(t^{-n}e^{-t^2/2} + zt \right) dt,$$

and the last integral vanishes for the chosen contour $C_1$. Thus $U_n(z)$ is a solution of (1.8) and so $e^{-z^2/4} U_n(z)$ is a solution of Weber's equation.

Defining
\[ D_n(z) = \Gamma(n+1)e^{-z^2/4}U_n(z) \]

\[ = \Gamma(n+1)e^{-z^2/4} \frac{1}{2\pi i} \int_{C_1} e^{-t^2/2} + zt \, dt \quad (1.12) \]

as Weber's function, where \( \Gamma(n+1) \) is the gamma function, we then have the first solution of (1.7) for arbitrary \( n \).

To see that the definition in (1.12) agrees with that in (1.9), we note that for positive integer \( n \), the integral \( U_n(z) \) can be evaluated as follows:

\[ U_n(z) = \frac{1}{2\pi i} \int_{C_1} e^{-t^2/2 + zt} \frac{dt}{t^{n+1}} \]

\[ = e^{z^2/2} \frac{1}{2\pi i} \int_{C} e^{-v^2/2} \frac{dv}{(z + v)^{n+1}}, \quad n \geq 0, \]

where the integral is over the complex \( v \)-plane along the path \( C \) around the pole \( v = -z \). The branch line is not needed for integer \( n \), and by the residue theorem,

\[ U_n(z) = e^{z^2/2} \frac{(-1)^n}{n!} \frac{d^n}{dz^n} e^{-z^2}, \quad n \geq 0. \quad (1.13) \]

Since \( \Gamma(n+1) = n! \) for integer \( n \), it is seen that the \( D_n(z) \) in (1.12) is indeed the same as that in (1.10) for positive integer \( n \).

Furthermore, for positive integer \( n \), \( U_n(z) \) in (1.12) can be reduced to an integral over a real variable if one notes the following identity:
\[ e^{-\frac{1}{2}(t-z)^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} + i(t-z)x \, dx, \quad x \text{ real.} \quad (1.14) \]

Applying the above formula to the integral in (1.12) and interchanging the order of integrations for \( t \) and \( x \), we obtain

\[ D_n(z) = \Gamma(n+1)e^{z^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} - i2x \left[ \frac{1}{2\pi i} \int_{C_1} t^{-n-1} e^{itx} \, dt \right] dx \]

\[ = \Gamma(n+1)e^{z^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} - i2x \left[ \frac{1}{n!} \left( ix \right)^n \right] dx, \quad n > 0. \]

Thus

\[ D_n(z) = \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{z^2/4} \int_{-\infty}^{\infty} x^n e^{-x^2/2} - i2x \, dx, \quad n \geq 0. \quad (1.15) \]

The integral is well suited for the numerical evaluation of \( D_n(z) \) because of the rapidly decaying exponential factor in the integrand.

When \( n \) is a negative integer, we note from Fig. 1.2 that

\[ U_n(z) = \frac{1}{2\pi i} \left( \int_{ab} + \int_{bc} + \int_{cd} \right)(t^{-n-1}e^{-t^2/2} + xt) \, dt \quad (n < 0) \]

\[ = \frac{1}{2\pi i} \left[ e^{i\pi n} \int_{0}^{\infty} x^{-n-1}e^{-x^2/2} - xz \, dx + 0 \right. \]

\[ + e^{-i\pi n} \int_{0}^{\infty} x^{-n-1}e^{-x^2/2} - xz \, dx \right] \]

\[ = -\frac{1}{\pi} \sin n\pi \int_{0}^{\infty} x^{-n-1}e^{-x^2/2} - xz \, dx. \]
Since $\Gamma(n+1)\Gamma(-n) = -\pi/\sin \pi n$ for any $n$, substitution of the above in (1.12) yields

$$D_n(z) = \frac{1}{\Gamma(-n)} e^{-z^2/4} \int_0^\infty x^{-n-1} e^{-x^2/2} - zx \, dx, \quad n < 0.$$  

Setting $n = -(m+1)$ with $m > 0$, we finally have

$$D_{-m-1}(z) = \frac{1}{m!} e^{-z^2/4} \int_0^\infty x^m e^{-x^2/2} - zx \, dx, \quad m > 0, \quad (1.16)$$

which is again a convergent integral for real variable $x$.

For purpose of analysis, it is sometimes convenient to change (1.16) to the following form: (0.12)

$$D_{-m-1}(z) = \sqrt{2} \frac{(-1)^m}{m!} e^{-z^2/4} \frac{d^m}{dz^m} \left[ e^{z^2/2} \text{erfc} \left( \frac{z}{\sqrt{2}} \right) \right], \quad (1.17)$$

where the complementary error function is given by

$$\text{erfc} (x) = \int_0^\infty e^{-t^2} \, dt.$$  

The derivation of the result above follows formula (1.16) with

$$D_{-m-1}(z) = \frac{1}{m!} e^{-z^2/4} \int_0^\infty x^m e^{-x^2/2} - zx \, dx$$

$$- \frac{(-1)^m}{m!} e^{-z^2/4} \frac{d^m}{dz^m} \int_0^\infty e^{-x^2/2} - zx \, dx$$

$$+ \frac{(-1)^m}{m!} e^{-z^2/4} \frac{d^m}{dz^m} \left[ e^{z^2/2} \int_0^\infty e^{-\tau^2/2} \, d\tau \right].$$
(B) Second Solution of Weber's Equation

Consider now the second solution of Weber's equation. Since the equation (1.7) is unaltered if we replace $n$ and $z$ simultaneously by $-n-1$ and $\pm iz$, respectively, it follows that $D_{-n-1}(\pm iz)$ are also solutions of Weber's equation; so is $D_n(-z)$. The integral representation of $D_{-n-1}(z)$ in (1.16) is still valid if $z$ is replaced by $\pm iz$; thus

$$D_{-n-1}(iz) = \frac{1}{n!} e^{z^2/4} \int_0^\infty x^{n-2} e^{-x^2/2} - ix x \, dx, \quad n > 0. \quad (1.18)$$

That the solution above is not proportional to the first one, $D_n(z)$ ($n > 0$), can be seen by comparing the integral in (1.18) with that in (1.15) and noting the difference in the lower limit of integration.

For arbitrary values of $n$, it is convenient to define the second solution of Weber's equation by the following integrals in the complex $t$-plane:

$$D_{-n-1}(-iz) = -(-i)^n \sqrt{2\pi n} e^{-z^2/4} V_n(z), \quad (1.19)$$

$$V_n(z) = \frac{1}{2\pi i} \int_{C_2} t^{-n-1} e^{-t^2/2} + zt \, dt,$$

or, using $D_{-n-1}(\pm iz)$ as the second solution, with

$$D_{-n-1}(iz) = -(-i)^n \sqrt{2\pi n} e^{-z^2/4} W_n(z), \quad (1.20)$$

$$W_n(z) = \frac{1}{2\pi i} \int_{C_3} t^{-n-1} e^{-t^2/2} + zt \, dt.$$
The contours \( C_2 \) and \( C_3 \) are also shown in Fig. 1.2. Using formula (1.14), we can reduce (1.20) to (1.18) for positive integer \( n \). Note that \( U_n(z) \) in (1.12) and \( V_n(z) \) and \( W_n(z) \) above are all defined by the same integrand but with different paths of integration. From Fig. 1.2 it is seen that

\[
U_n(z) + V_n(z) + W_n(z) = 0,
\]

and hence

\[
\frac{\sqrt{2\pi}}{\Gamma(n+1)} D_n(z) = (-i)^n D_{-n-1}(iz) + i^n D_{-n-1}(-iz).
\]

To summarize, we have shown that \( D_n(z) \) and \( D_{-n-1}(iz) \) are two independent solutions of Weber's equation

\[
\frac{d^2 \omega}{dz^2} + (n + \frac{1}{2} - \frac{1}{4} z^2) \omega = 0.
\]

For arbitrary values of \( n \), these two solutions are represented by integrals with respect to a complex variable \( t \):

\[
D_n(z) = \Gamma(n+1)e^{-z^2/4} \frac{1}{2\pi i} \int_{C_1} t^{-n-1}e^{-t^2/2 + zt} dt; \quad (1.12)
\]

\[
D_{-n-1}(iz) = -i^n \sqrt{2\pi} e^{-z^2/4} \frac{1}{2\pi i} \int_{C_3} t^{-n-1}e^{-t^2/2 + zt} dt, \quad (1.20)
\]

where the contours \( C_1 \) and \( C_3 \) are shown in Fig. 1.2. When \( n \) is a positive integer the complex integrals can be reduced to integrals with respect to a real variable \( z \).
\[ D_n(x) = \frac{i^n}{\sqrt{2\pi}} e^{2x/4} \int_{-\infty}^{\infty} x^n e^{-x^2/2} - iax \, dx, \quad n \geq 0, \quad (1.15) \]

\[ D_{-n-1}(iz) = \frac{1}{n!} e^{2x/4} \int_0^{\infty} x^n e^{-x^2/2} - iax \, dx, \quad n > 0. \quad (1.18) \]

Furthermore, the integrals above can be expressed as

\[ D_n(z) = (-1)^n e^{2z/4} \frac{d^n}{dz^n} (e^{-z^2/2}), \quad n \geq 0; \quad (1.10) \]

\[ D_{-n-1}(z) = \sqrt{2} \frac{(-1)^n}{n!} e^{-z^2/4} \frac{d^n}{dz^n} \left[ e^{z^2/2} \text{erfc} \left( \frac{z}{\sqrt{2}} \right) \right], \quad n > 0; \quad (1.17) \]

with

\[ \text{erfc}(v) = \int_v^{\infty} e^{-t^2} \, dt. \]

Numerical values for \( D_n(z) \) and \( D_n(iz) \) are tabulated in Refs. 0.11, 0.8, and 1.1 for real \( z \). (The symbols for Weber's function (Parabolic Cylinder Function) adopted in this volume, which are in agreement with most common references, differ from those in the Handbook of Mathematical Functions. As can be seen by comparing the integral representations, \( D_n(z) \) in this volume corresponds to \( U(-n-\frac{1}{2},z) \) in Chapter 19 of the Handbook.) For wave diffraction problems the arguments of the parabolic functions appear as \( z = \tau (1+i)z \), with \( z \) being real. This can be seen from the solutions \( D_n(u) \) and \( D_n(v) \) of Eqs. (1.5a,b), where \( u = \sqrt{-2i\alpha \xi} = (1-i)\sqrt{\alpha \xi} \) and \( v = \sqrt{2i\alpha n} = (1+i)\sqrt{\alpha n} \). Tables for \( D_n[(1+i)z] \) have been prepared by Kireyeva and Karpov, for \( \tau \) from 0 to 10 and for \( n \) from 0 to 2. (0.9) If we write
\[ D_p[(1+i)x] = U_p(x) + iV_p(x), \]

graphs for several \( U_p(x) \) and \( V_p(x) \) are shown in Fig. 1.3.

Fig. 1.3a. Weber Functions \( D_p[(1+i)x] = U_p(x) + iV_p(x) \)

Fig. 1.3b. Weber Functions $D_p((1+i)x) = U_p(x) + iV_p(x)$


1.2. Properties of Weber Functions

Many important properties of Weber functions can be deduced from their integral representations. We group those which are needed for the remaining discussion in this subsection for functions with integer order ($n$ or $m$).
(A) **Negative Argument**

\[
D_n(-z) = (-1)^n D_n(z), \quad \text{see Eq. (1.15)},
\]

\[
D_{-n-1}(-iz) = \sqrt{2\pi} \left( \frac{-i}{n!} \right)^n D_n(z) + (-1)^{n+1} D_{-n-1}(iz), \quad \text{see Eq. (1.21)},
\]

\[
D_n(\overline{z}) = \overline{D_n(z)}, \quad \text{see Eq. (1.12)}. \quad (1.22)
\]

(B) **Recursion Formulas**

Integrating Eq. (1.11) by parts, we have

\[
U_n(z) = \frac{-1}{2\pi i n} \left[ \frac{-n}{e^{t^2/2} + zt} \right]_{C_1} - \frac{1}{2\pi i} \int_{C_1} t^n(t-z)e^{-t^2/2 + zt} dt.
\]

In the above, the first term on the left-hand side vanishes at the extreme points of the contour \(C_1\), and the integral can be expressed in terms of \(U_{n-1}(z)\) and \(U_{n-2}(z)\). Hence

\[
nU_n(z) = -U_{n-2}(z) + zU_{n-1}(z).
\]

In view of the definition (1.12), we have derived the important recursion formula

\[
D_n(z) = -(n-1)D_{n-2}(z) + zD_{n-1}(z),
\]

or

\[
zD_n(z) = nD_{n-1}(z) + D_{n+1}(z). \quad (1.23a)
\]

Similarly, by differentiating (1.12) with respect to \(z\), we can show
\[ D'_n(z) = -\frac{n}{2} D_n(z) + nD_{n-1}(z) \]
\[ = \frac{z}{2} D_n(z) - D_{n+1}(z) \]
\[ = \frac{1}{2}[nD_{n-1}(z) - D_{n+1}(z)]. \]  
(1.23b)

(C) Series Expansions for Small Argument

By expanding \( e^{z^2/4} \) and \( e^{-iz} \) in (1.15) and (1.18) into Maclaurin series, and then carrying out the integration for the first few terms, we can derive the following results

\[ D_{2n}(z) = \frac{(-1)^n (2n)!}{2^n n!} \left[ 1 - \left( n + \frac{1}{4} \right) z^2 \right] + O(z^4), \]

\[ D_{2n+1}(z) = \frac{(-1)^n (2n+1)!}{2^n n!} z + O(z^3), \]

\[ D_{-n-1}(z) = \frac{(2n/2)!}{n!} \left[ \Gamma \left( \frac{n+1}{2} \right) - z \Gamma \left( \frac{n+3}{2} \right) + z^2 \left( \frac{n}{2} + \frac{1}{4} \right) \Gamma \left( \frac{n+1}{2} \right) + \ldots \right], \]

\[ D_{-2n-1}(z) = \frac{\sqrt{\pi/2}}{2^n n!} - \frac{n!}{(2n)!} z + \frac{(n + \frac{1}{4})\sqrt{\pi/2}}{2^n n!} z^2 + O(z^3), \]

\[ D_{-2n-2}(z) = \frac{n!}{(2n+1)!} \left[ \frac{\sqrt{\pi/2}}{2^n n!} z + \frac{(n + 3/4)\sqrt{\pi/2}}{(2n+1)!} z^2 + O(z^3) \right]. \]

In the derivations above use has been made of the relation

\[ \int_0^\infty x^m e^{-x^2/2} \, dx = 2^{-\frac{m+1}{2}} \sqrt{\pi} \Gamma \left( \frac{m+1}{2} \right), \]

and the duplication formula for the gamma function,
\[ \sqrt{\pi} \Gamma(2n) = 2^{2n-1} \Gamma(n) \Gamma(n + \frac{1}{2}). \]

From Eq. (1.24), it is seen that at \( z = 0, \)

\[ D_{2n}(0) = \frac{(-1)^n (2n)!}{2^n \cdot n!}, \quad D_{2n+1}(0) = 0 \]

\[ D_{-2n-1}(0) = \frac{\sqrt{\pi/2}}{2^n \cdot n!}, \quad D_{-2n-2}(0) = \frac{2^n - n!}{(2n+1)!}. \quad (1.25) \]

(D) Asymptotic Formulas

For large values of \(|z|\), the asymptotic expansions of \( D_n(z) \)

(|z| > |n|) for various ranges of \( \text{arg } z \) are:

\[ D_n(z) \sim e^{-z^2/4} z^n \left[ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot z^4} - \ldots \right], \]

\[ |\text{arg } z| < \frac{3\pi}{4}, \]

\[ D_n(z) \sim e^{-z^2/4} z^n \left[ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot z^4} - \ldots \right] \]

\[ - \frac{\sqrt{2 \pi}}{\Gamma(-n)} e^{\pi i} e^{z^2/4} z^{-n-1} \left[ 1 + \frac{n(n+1)(n+2)}{2z^2} + \ldots \right], \]

\[ \frac{5\pi}{4} > \text{arg } z > \frac{\pi}{4} \quad (1.26) \]

and in the range \(- \frac{\pi}{4} > \text{arg } z > - \frac{5}{4} \pi\), the second of the above expansions holds if \( e^{\pi i}\) is replaced by \( e^{-\pi i}\).(0.10,0.12)

For large order and smaller argument, Darwin's asymptotic expression is useful: (0.11)
\[ D_{-\eta-1}(z) \sim \left( \frac{4\pi^2}{n!} \right)^{\frac{1}{2}} \exp \left[ - \left( n + \frac{n}{2} \right) \left( z^2 + \ln(z + x) \right) \right], \quad (1.27) \]

with

\[ x = z/2\sqrt{n + \frac{1}{2}}, \quad X = \sqrt{1 + x^2}. \]

The general behavior of \( D_n(z) \) for large \( n \) has been analyzed in detail by Rice. (0.13)

(E) Orthogonality Property

From the differential equation (1.7), we see that for \( m \neq n \),

\[ D_m(z)D''_n(z) - D_n(z)D''_m(z) + (n-m)D_n(z)D'_m(z) = 0, \]

and so

\[ (m-n) \int_{-\infty}^{\infty} D_n(z)D_m(z) \, dz = \left[ D_m(z)D'_n(z) - D_n(z)D'_m(z) \right]_{-\infty}^{\infty}. \]

In view of (1.26), when \( z \) is a real variable the right-hand member vanishes at both limits. If \( z \) is complex, the path of the integration should originate and terminate at both ends of the real axis and \( |\arg(z)| \leq \pi/4 \), or \( 3\pi/4 \leq \arg(z) \leq 5\pi/4 \), (0.14) so that the right-hand member vanishes at the limits. In either case, we obtain for \( m \neq n \)

\[ \int_{-\infty}^{\infty} D_n(z)D_m(z) \, dz = 0. \]

When \( m = n \), we note first (Ref. 0.10) that

\[ (n+1) \int_{-\infty}^{\infty} [D_n(z)]^2 \, dz = \int_{-\infty}^{\infty} [D_{n+1}(z)]^2 \, dz, \]

and by mathematical induction,
Thus we have the orthogonality condition

\[ \int_{-\infty}^{\infty} D_m(z) D_n(z) \, dz = \begin{cases} 0, & m \neq n, \\ \sqrt{2\pi(n!)}, & m = n, \end{cases} \quad (1.28) \]

for \( m \) and \( n \) being integers.

It follows from the orthogonal properties that if the expansion of an arbitrary function \( f(z) \) of the form

\[ f(z) = \sum_{n=0}^{\infty} a_n D_n(z) \]

exists, the coefficients \( a_n \) are given by

\[ a_n = \frac{1}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} f(z) D_n(z) \, dz, \quad (1.29) \]

provided the integration of the series is permissible term by term.

A general expansion theorem for an absolutely integrable function of real variable \( f(x) \) is given by Cherry. (1.2)

Other properties of Weber functions can be deduced from the general properties of confluent hypergeometric functions, (1.3) of which the Weber function is a special case.
1.3. Wave Functions in Parabolic Coordinates

(A) Wave Functions

Since $D_n(z)$ and $D_{-n-1}(iz)$ are two independent solutions of Weber's equation, we can take the solutions for Eqs. (1.5a,b) as

$$
X(\xi) = D_n(\lambda \xi), \quad D_{-n-1}(\pm i\lambda \xi); \\
Y(\eta) = D_n(\overline{\lambda} \eta), \quad D_{-n-1}(\pm i\overline{\lambda} \eta).
$$

Thus solutions for the wave equation (1.4) can be written as

$$
\varphi(\xi, \eta, z) = \sum_{n=0}^{\infty} \left[ a_n D_n(\lambda \eta) D_n(\lambda \xi) + b_n D_{-n-1}(\lambda \eta) D_{-n-1}(\lambda \xi) \right. \\
+ \left. c_n D_n(\overline{\lambda} \eta) D_{-n-1}(\overline{\lambda} \xi) + d_n D_{-n-1}(\overline{\lambda} \eta) D_{-n-1}(\overline{\lambda} \xi) \right] e^{\pm i\gamma z},
$$

where

$$
\lambda = \sqrt{-2iz}, \quad \overline{\lambda} = \sqrt{-2iz} = i\lambda, \quad \alpha^2 = \kappa^2 - \gamma^2, \quad \kappa = \omega/c,
$$

and $a_n$, $b_n$, $c_n$, and $d_n$ are arbitrary constants. Although nowhere in the derivation of the solutions above is $n$ restricted to integers, we shall assume integer values for $n$ in anticipation of satisfying the boundary conditions.

Each of the four function products in (1.31) represents a type of wave which can be interpreted from the asymptotic formulas for large arguments. Making use of (1.2) and (1.26) we obtain as $\tau \to \infty$, 

\[ \varphi(\xi, \eta, z) \sim \sum_{n=0}^{\infty} \left[ a_n (2\alpha y)^n e^{i\alpha x} + b_n (\lambda \eta)^{-1} \left( \cot \frac{\alpha}{2} \right)^n e^{i\alpha r} ight. \\
+ c_n (\lambda \xi)^{-1} \left( \tan \frac{\eta}{2} \right)^2 e^{-i\alpha r} + d_n (2\alpha y)^{n-1} e^{-i\alpha r} \left] e^{i\gamma z} \right. \]

For the choice of time factor, \( e^{-i\omega t} \), it follows that the first product solution is a plane wave moving in the positive \( x \)-direction; the second is a radially outgoing wave; the third, a converging radial wave; and the last, a decaying plane wave moving in the negative \( x \)-direction. For the study of diffraction of a plane wave by the parabolic cylinder \( n = n_0 \) as shown in Fig. 1.1, only the first two solutions are needed.

Four more independent solutions can be constructed by interchanging \( \xi \) and \( \eta \) in (1.31). Their properties can be analyzed similarly at infinity, and the results are summarized, together with others, in Table 1.1. When the diffracting surface is a semi-infinite line \( (n_0 = 0) \), other wave functions which are not of the type of separated variables can also be used. Two of them are discussed in subsection 2.2.
Table 1.1

Wave Functions in Parabolic Cylinder Coordinates

Plane coordinates \((\xi, \eta)\) are shown in Fig. 1.1 and a time factor \(e^{-i\omega t}\) is assumed. The parameter \(\lambda\) and its complex conjugate depend on the wave number \(\alpha\) with \(\lambda = \sqrt{-2i\alpha}\) and \(\bar{\lambda} = \sqrt{2i\alpha}\).

<table>
<thead>
<tr>
<th>Product of Weber Functions</th>
<th>Asymptotic Behavior</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_n(\lambda \eta)D_n(\lambda \xi))</td>
<td>((2ay)^n e^{i\alpha x})</td>
<td>Wave in (+x) direction</td>
</tr>
<tr>
<td>(D_{n-1}(\lambda \eta)D_n(\lambda \xi))</td>
<td>((\lambda \eta)^{-1} \cot^n \frac{\theta}{2} e^{i\alpha r})</td>
<td>Diverging wave from a parabola (\eta = \eta_0)</td>
</tr>
<tr>
<td>(D_n(\lambda \xi)D_{n-1}(\lambda \xi))</td>
<td>((\lambda \xi)^{-1} \tan^n \frac{\theta}{2} e^{-i\alpha r})</td>
<td>Converging waves to a parabola (\xi = \xi_0^*)</td>
</tr>
<tr>
<td>(D_{n-1}(\lambda \eta)D_{n-1}(\lambda \xi))</td>
<td>((2ay)^{-n-1} e^{-i\alpha x})</td>
<td>Decaying wave in (-x) direction</td>
</tr>
<tr>
<td>(D_n(\lambda \xi)D_n(\lambda \eta))</td>
<td>((2ay)^n e^{-i\alpha x})</td>
<td>Wave in (-x) direction</td>
</tr>
<tr>
<td>(D_{n-1}(\lambda \xi)D_n(\lambda \eta))</td>
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<td>(D_n(\lambda \xi)D_{n-1}(\lambda \eta))</td>
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</tr>
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<td>((2ay)^{-n-1} e^{i\alpha x})</td>
<td>Decaying wave in (+x) direction</td>
</tr>
</tbody>
</table>

*The branch cut should be taken, in this case, along the negative \(x\)-axis.*
(B) Expansion of a Plane Wave in Series of Wave Functions

As discussed in Chapter III, Section 1, a plane harmonic wave with its wave normal at an angle $\phi_o$ with the $z$-axis, and the projection of the normal at an angle $\theta_o$ with the $x$-axis, can be represented in Cartesian coordinates as

$$\varphi = A \exp \{i k [\sin \phi_o (x \cos \theta_o + y \sin \theta_o) + \cos \phi_o z - \omega t] \}.$$ 

By setting $k \cos \phi_o = \gamma$ and $k \sin \phi_o = \alpha$, we have

$$\varphi = A \exp \{i \alpha (x \cos \theta_o + y \sin \theta_o) + i \gamma z - i \omega t \}, \quad (1.32)$$

where as in (1.5) $\alpha^2 + \gamma^2 = k^2$. It is readily seen that the above expression simplifies to

$$\varphi = A \exp \{i k (x \cos \theta_o + y \sin \theta_o) - i \omega t \}, \quad (1.33)$$

at $\varphi_o = \pi/2$, and no loss in generality results if only the expansion of $\exp \{i \alpha (x \cos \theta_o + y \sin \theta_o) \}$ in terms of Weber functions is considered.

Assume that the plane wave $\varphi$, when expressed in terms of $u$ and $v$ in (1.6), has the following expansion:

$$\exp \{i \alpha (x \cos \theta_o + y \sin \theta_o) \} = \exp \{i \alpha (-u^2 + v^2) \cos \theta_o + 2 i \alpha v \sin \theta_o \}$$

$$= \sum_{n=0}^{\infty} c_n D_n(u) D_n(v), \quad |\theta_o| < \frac{\pi}{2}.$$ 

Differentiating both sides with respect to $v$ yields
\[ \frac{1}{2}(-\nu \cos \theta_0 + i\mu \sin \theta_0) \exp \left( \frac{1}{\chi} \left[-(u^2 + v^2) \cos \theta_0 + 2iuv \sin \theta_0 \right] \right) \]

\[ = \sum_{n=0}^{\infty} a_{2n} D_n(u) D_n'(0). \]

As can be seen from (1.24) and (1.25):

\[ D_{2n+1}(0) = 0, \quad D_{2n}'(0) = 0. \]

Thus by setting \( \nu = 0 \) in the above two expansions, we obtain

\[ e^{\frac{1}{2}u^2 \cos \theta_0} = \sum_{n=0}^{\infty} a_{2n} D_{2n}(0) D_{2n}(u), \]

\[ \frac{i}{2}ue^{-\frac{1}{2}u^2 \cos \theta_0} = \sum_{n=0}^{\infty} a_{2n+1} D_{2n+1}'(0) D_{2n+1}(u). \]

From the orthogonality conditions of the \( D_n(u) \), the coefficients \( a_n \) can be evaluated as

\[ a_{2n} = \frac{1}{\sqrt{2\pi (2n)!}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2 \cos \theta_0} D_{2n}(u) \, du, \]

\[ a_{2n+1} = \frac{i}{2\sqrt{2\pi (2n+1)!}} \int_{-\infty}^{\infty} u e^{-\frac{1}{2}u^2 \cos \theta_0} D_{2n+1}'(u) \, du. \]

The first integral (\( \cos \theta_0 > 0 \)) is evaluated by replacing \( D_{2n}(u) \) by its integral representation (1.12), then interchanging the order of integrations and noting that

\[ \int_{-\infty}^{\infty} e^{-\alpha^2 u^2 + bu} \, du = \frac{\sqrt{\pi}}{\alpha} e^{b^2/4\alpha^2}, \]
and
\[ \frac{1}{2\pi i} \int_{C_1} \frac{e^{\beta t^2}}{t^{2n+1}} \, dt = \frac{\delta^n}{n!}. \]

Thus
\[ \int_{-\infty}^{\infty} e^{-\left(u^2 \cos \theta\right)/4} D_{2n}(u) \, du = \frac{\sqrt{2\pi} \sec \frac{\theta}{2} (2n)!}{2^n n!} \tan \frac{2n \theta}{2}, \quad |\theta| < \frac{\pi}{2}. \]

Similarly, using the formula
\[ \int_{-\infty}^{\infty} u e^{-\alpha u^2 + bu} \, du = \frac{\alpha}{\beta} \int_{-\infty}^{\infty} e^{-\alpha u^2} \, du = \frac{\sqrt{\pi} \beta}{2\alpha} e^{\beta^2/4\alpha}, \]

we calculate the second integral to be
\[ \int_{-\infty}^{\infty} u e^{-\left(u^2 \cos \theta\right)/4} D_{2n+1}(u) \, du = \frac{\sqrt{2\pi} \sec \frac{\theta}{2} (2n+1)!}{2^n n!} \tan \frac{2n+1 \theta}{2}, \quad |\theta| < \frac{\pi}{2}. \]

(1.35)

The initial values \( D_{2n}(0) \) and \( D_{2n+1}(0) \) are given in (1.24) and (1.25), thus
\[ a_{2n} = \sec \frac{\theta}{2} \frac{1}{(2n)!} \tan \frac{2n \theta}{2}, \]
\[ a_{2n+1} = \sec \frac{\theta}{2} \frac{1}{(2n+1)!} \tan \frac{2n+1 \theta}{2}. \]

with \( u = \lambda \xi \) and \( v = \lambda \eta \), the final form of the expansion of the plane wave is as follows
\[ \exp \left[iu(x \cos \theta + y \sin \theta)\right] = \sec \frac{\theta}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \tan \frac{\theta}{2}\right)^n D_n(\lambda \xi) D_n(\lambda \eta), \]
\[ |\theta| < \frac{\pi}{2}. \quad (1.36a) \]
Similarly, we can show that

\[ \exp \left[ i \alpha (x \cos \theta_o + y \sin \theta_o) \right] = \csc \frac{\theta_o}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \cot \frac{\theta_o}{2} \right)^n D_n(\lambda \xi) D_n(\lambda \eta), \]

\[ \frac{\pi}{2} < \theta_o < \frac{3\pi}{2}. \quad (1.36b) \]

Both series are uniformly and absolutely convergent within the indicated ranges of \( \theta_o \); neither converges at \( \theta_o = \pm \pi/2 \). \( (0.14) \) When \( \theta_o = 0, \pi \)

\[ e^{i\alpha x} = D_o(\lambda \xi)D_o(\lambda \eta), \]

\[ e^{-i\alpha x} = D_o(\lambda \xi)D_o(\lambda \eta). \quad (1.37) \]

Note that the series (1.36a) is the same as the first term of the wave function expansion (1.31) and has the proper asymptotic behavior for large arguments \( \xi \) and \( \eta \).

If we do not adhere to functions of the separated variables \( \xi \) and \( \eta \), the plane wave can be expressed in terms of Weber functions in a simple form. From (1.1), we note that

\[ \exp \left[ i \alpha (x \cos \theta_o + y \sin \theta_o) \right] \]

\[ = \exp \left\{ -\frac{1}{4} \left[ \sqrt{2i\alpha} \left( \xi \sin \frac{\theta_o}{2} - \eta \cos \frac{\theta_o}{2} \right) \right]^2 \right\} \]

\[ \times \exp \left\{ -\frac{1}{4} \left[ \sqrt{-2i\alpha} \left( \xi \cos \frac{\theta_o}{2} + \eta \sin \frac{\theta_o}{2} \right) \right]^2 \right\} \]

Since \( D_o(x) = e^{-x^2/4} \), it follows that:
\[
\exp \left[ i a (x \cos \theta_0 + y \sin \theta_0) \right] = D_\phi \left[ \lambda \left( \xi \sin \frac{\theta_0}{2} - \eta \cos \frac{\theta_0}{2} \right) \right] \\
\times D_\phi \left[ \lambda \left( \xi \cos \frac{\theta_0}{2} + \eta \sin \frac{\theta_0}{2} \right) \right].
\]

(1.38)

It is interesting to note that the formula above can also be derived by summing the series (1.36) according to the formula

\[
D_k \left[ \lambda \left( \eta \cos \frac{\theta_0}{2} + \xi \sin \frac{\theta_0}{2} \right) \right] D_\eta \left[ \lambda \left( \eta \sin \frac{\theta_0}{2} - \xi \cos \frac{\theta_0}{2} \right) \right] \\
= \left( \cos \frac{\theta_0}{2} \right)^k \sum_{n=0}^{\infty} \frac{k!}{n!(k-n)!} \tan \frac{\theta_0}{2} D_{k-n}(\lambda\eta) D_n(\lambda\xi). 
\]

(1.39)

This formula is valid for \( 0 \leq \theta < \pi/2 \) and was proved by Cherry. (1.2)

1.4. Elasticity Equations in Parabolic Cylinder Coordinates

We list without derivation the following equations in parabolic coordinates, \( \xi, \eta, z \) \((J^2 = \xi^2 + \eta^2)\):

(A) Displacement-Potential

\[
\begin{align*}
\psi_\xi &= \frac{1}{J} \left( \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta} + \zeta \frac{\partial^2 \psi}{\partial \xi \partial z} \right), \\
\psi_\eta &= \frac{1}{J} \left( \frac{\partial \psi}{\partial \eta} - \frac{\partial \psi}{\partial \xi} + \zeta \frac{\partial^2 \psi}{\partial \eta \partial z} \right), \\
\psi_z &= \frac{\partial \psi}{\partial z} + \zeta \frac{\partial^2 \psi}{\partial z^2} - \frac{\zeta}{c^2} \frac{\partial^2 \psi}{\partial t^2}.
\end{align*}
\]

(1.40)
(B) Strain-Displacement

\[ \varepsilon_{\xi\xi} = \frac{1}{\eta} \frac{\partial u_\xi}{\partial \xi} + \frac{\eta}{\eta^2} u_\eta, \]

\[ \varepsilon_{\eta\eta} = \frac{1}{\eta} \frac{\partial u_\eta}{\partial \eta} + \frac{\xi}{\xi^2} u_\xi, \]

\[ \varepsilon_{\xi\zeta} = \frac{\partial u_\zeta}{\partial \zeta}, \]

\[ \varepsilon_{\xi\eta} = \frac{1}{2\eta} \left( \frac{\partial u_\eta}{\partial \eta} + \frac{\partial u_\xi}{\partial \xi} - \frac{\eta}{\xi^2} u_\xi - \frac{\xi}{\eta^2} u_\eta \right), \]

\[ \varepsilon_{\eta\zeta} = \frac{1}{2\eta} \left( \frac{\partial u_\zeta}{\partial \zeta} + \eta \frac{\partial u_\eta}{\partial \eta} \right), \]

\[ \varepsilon_{\xi\zeta} = \frac{1}{2\eta} \left( \frac{\partial u_\zeta}{\partial \xi} + \eta \frac{\partial u_\zeta}{\partial \zeta} \right). \quad (1.41) \]

(C) Stress-Displacement

\[ \sigma_{\xi\xi} = \frac{(\lambda + 2\mu)}{J} \frac{\partial u_\xi}{\partial \xi} + \frac{\lambda}{J} \frac{\partial u_\eta}{\partial \eta} + \lambda \frac{\partial u_\zeta}{\partial \zeta} + \frac{\lambda}{J^2} \xi u_\xi + \frac{(\lambda + 2\mu)}{J^3} \eta u_\eta, \]

\[ \sigma_{\eta\eta} = \frac{\lambda}{J} \frac{\partial u_\xi}{\partial \xi} + \frac{(\lambda + 2\mu)}{J} \frac{\partial u_\eta}{\partial \eta} + \lambda \frac{\partial u_\zeta}{\partial \zeta} + \frac{(\lambda + 2\mu)}{J^3} \xi u_\xi + \frac{\lambda}{J^3} \eta u_\eta, \]

\[ \sigma_{\zeta\zeta} = \frac{\lambda}{J} \frac{\partial u_\xi}{\partial \xi} + \frac{\lambda}{J} \frac{\partial u_\eta}{\partial \eta} + (\lambda + 2\mu) \frac{\partial u_\zeta}{\partial \zeta} + \frac{\lambda}{J^3} \xi u_\xi + \frac{\lambda}{J^3} \eta u_\eta, \]

\[ \sigma_{\xi\eta} = \frac{\mu}{J} \left( \frac{\partial u_\eta}{\partial \eta} + \frac{\partial u_\xi}{\partial \xi} - \frac{\eta}{\xi^2} u_\xi - \frac{\xi}{\eta^2} u_\eta \right), \]

\[ \sigma_{\eta\zeta} = \frac{\mu}{J} \left( \frac{\partial u_\zeta}{\partial \zeta} + \eta \frac{\partial u_\eta}{\partial \eta} \right), \]

\[ \sigma_{\xi\zeta} = \frac{\mu}{J} \left( \frac{\partial u_\zeta}{\partial \xi} + \eta \frac{\partial u_\zeta}{\partial \zeta} \right). \quad (1.42) \]
(D) Stress-Displacement Potential Relation

\[
\sigma_{\xi\xi} = \mu \left( \frac{\lambda}{\mu} \frac{1}{\sigma_2^2} \frac{\partial^2}{\partial \xi^2} + \frac{2}{J^2} \frac{\partial^2}{\partial \xi^2} - \frac{2\xi}{J^2} \frac{\partial}{\partial \xi} + \frac{2n}{J^2} \frac{\partial}{\partial \eta} \right) \varphi
\]
\[+ \frac{2\nu}{j^2} \left( \frac{\partial^2}{\partial \xi \partial \eta} - \frac{\xi}{J^2} \frac{\partial}{\partial \eta} - \frac{n}{J^2} \frac{\partial}{\partial \xi} \right) \psi
\]
\[+ \frac{2\nu}{j^2} \left( 2 \frac{\partial^2}{\partial \xi \partial \eta} - \frac{2\xi}{J^2} \frac{\partial}{\partial \eta} + \frac{2n}{J^2} \frac{\partial}{\partial \xi} \right) x.
\]

\[
\sigma_{\eta\eta} = \mu \left( \frac{\lambda}{\mu} \frac{1}{\sigma_2^2} \frac{\partial^2}{\partial \xi^2} + \frac{2}{J^2} \frac{\partial^2}{\partial \eta^2} + \frac{2\xi}{J^2} \frac{\partial}{\partial \xi} - \frac{2n}{J^2} \frac{\partial}{\partial \eta} \right) \varphi
\]
\[+ \frac{2\nu}{j^2} \left( \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\xi}{J^2} \frac{\partial}{\partial \eta} + \frac{n}{J^2} \frac{\partial}{\partial \xi} \right) \psi
\]
\[+ \frac{2\nu}{j^2} \left( \frac{\partial^2}{\partial \xi \partial \eta} + \frac{2\xi}{J^2} \frac{\partial}{\partial \eta} - \frac{2n}{J^2} \frac{\partial}{\partial \xi} \right) x.
\]

\[
\sigma_{zz} = \mu \left( \frac{\lambda}{\mu} \frac{1}{\sigma_2^2} \frac{\partial^2}{\partial t^2} + 2 \frac{\partial^2}{\partial z^2} \right) \varphi - 2\nu \frac{a}{az} v_1^2 x.
\]

\[
\sigma_{\xi\eta} = \frac{\mu}{J^2} \left\{ 2 \left( \frac{\partial^2}{\partial \xi \partial \eta} - \frac{n}{J^2} \frac{\partial}{\partial \xi} - \frac{\xi}{J^2} \frac{\partial}{\partial \eta} \right) \varphi + \left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2} + \frac{2\xi}{J^2} \frac{\partial}{\partial \xi} - \frac{2n}{J^2} \frac{\partial}{\partial \eta} \right) \psi
\]
\[+ 2\nu \left( \frac{\partial^2}{\partial \xi \partial \eta} - \frac{n}{J^2} \frac{\partial}{\partial \xi} - \frac{\xi}{J^2} \frac{\partial}{\partial \eta} \right) x \right\}.
\]

\[
\sigma_{\xi z} = \frac{\mu}{J} \left\{ 2 \frac{\partial^2}{\partial \xi \partial z} + \frac{\partial^2}{\partial \eta \partial z} + 2 \frac{a}{az} \left( \frac{\partial^2}{\partial \eta^2} - \frac{1}{\sigma_2^2} \frac{\partial^2}{\partial t^2} \right) x \right\}.
\]

\[
\sigma_{nz} = \frac{\mu}{J} \left\{ 2 \frac{\partial^2}{\partial \eta \partial z} - \frac{\partial^2}{\partial \xi \partial z} + \frac{a}{az} \left( \frac{\partial^2}{\partial \eta^2} - \frac{1}{\sigma_2^2} \frac{\partial^2}{\partial t^2} \right) x \right\}
\]

(1.43)

and

\[
v_1^2 = \frac{1}{J^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right).
\]
2. DIFFRACTION OF SH WAVES

IN TERMS OF WEBER FUNCTIONS, solutions for the diffraction of a horizontally polarized shear wave by a parabolic cylinder can be derived readily; the cylinder can be either rigid or vacuous. However, the case of an elastic cylinder cannot be solved so easily because of the difficulty in matching the boundary conditions at the surface. This same difficulty is also encountered in the diffraction of electromagnetic waves by a conducting parabolic cylinder, which was the subject treated extensively by Epstein. (0.7) As the parabolic cylinder degenerates to a semi-infinite strip, the diffraction of SH waves is then analogous to the diffraction of light by a semi-infinite screen, first treated by Sommerfeld. (0.2)

Let the coordinate surface \( \eta = \eta_0 \) be the boundary of a parabolic cylinder (Fig. 1.1). A plane wave propagating in the surrounding medium is represented by

\[
 u_{(i)}(x, y, t) = u_0 e^{ik(x \cos \theta_0 + y \sin \theta_0 - c_s t)},
\]

\[ u_y^{(i)} = 0, \]

\[ u_x^{(i)} = 0, \quad kc_s = \omega. \]  

(2.1)

The propagation vector makes an angle \( \theta_0 \) with the \( x \)-axis, and is limited in the range \( 0 \leq \theta_0 < \pi/2 \). That the angle \( \theta_0 \) should be so limited is apparent from the geometry, for if \( \theta_0 > \pi/2 \), the plane wave would have been reflected by the parabolic surface at infinity and it could not retain the plane wave form while reaching the crown
portion. On account of the loadings which generate the above incident wave, and the geometry of the scatterer, the problem is one of anti-plane strain. The governing field equations are:

\[
(\nu^2 + k^2)\nu_z = 0,
\]

\[
\nu^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{J^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right), \quad \sigma^2 = \xi^2 + \eta^2,
\]

and

\[
\sigma_{\xi z} = \frac{1}{J} \frac{\partial \nu_z}{\partial \xi},
\]

\[
\sigma_{\eta z} = \frac{1}{J} \frac{\partial \nu_z}{\partial \eta}.
\]

The total displacement \( u_z(\xi, \eta, t) \) is the sum of the incident wave \( u_z^{(i)} \) and the scattered wave \( u_z^{(s)} \). In parabolic coordinates, the former — see Eq. (1.36) — can be written as

\[
u_z^{(i)} = \nu_0 \sec \frac{1}{2} \theta \sum_{n=0}^{\infty} \frac{\tan \frac{\theta}{2}}{n!} D_{\eta}(\lambda \eta) D_{\eta}(\lambda \xi),
\]

\[
\lambda = \sqrt{-2ik}, \quad \bar{\lambda} = \sqrt{2ik},
\]

and the latter — see Table (1.1) — as

\[
u_z^{(s)} = \nu_0 \sum_{n=0}^{\infty} \alpha_n D_{\eta, n-1}(\lambda \eta) D_{\eta}(\lambda \xi).
\]

The unknown coefficients \( \alpha_n \) are determined as usual by the boundary conditions (2.1).
2.1. Parabolic Cylinder

For a rigid parabolic insert, the proper boundary condition is

\[ \frac{\partial^2 u_z}{\partial t^2} = \int_{-\infty}^{\infty} \sigma_{nz} \xi \, d\xi \quad \text{at} \quad \eta = \eta_0. \]

In the above, the right-hand side is the total shear force per unit length (along the z direction) acting on the cylinder, and the left-hand side is the mass per unit length times the acceleration. In view of Eq. (2.3), and considering that the motion is harmonic in time, the equation above can be written as

\[ -m_0^2 u_z = \mu \int_{-\infty}^{\infty} \frac{\partial u_z}{\partial \eta} \, d\xi \quad \text{at} \quad \eta = \eta_0. \quad (2.6) \]

Since the area enclosed by the parabola \( \eta = \eta_0 \), and hence the mass \( m \), is infinite, and since the integral in (2.6) converges when the integrand is given by a uniformly convergent series of parabolic functions, the displacement \( u_z \) at the boundary must vanish. The boundary condition (2.6) is then reduced to

\[ u_z(\xi, \eta_0) = 0, \quad (2.7) \]

which is the same as if the cylinder is rigid and fixed.

Applying condition (2.7) to the sum of \( u_z^{(c)} \) and \( u_z^{(b)} \) in (2.4) and (2.5), we can determine the coefficients \( \alpha_n \). The total wave outside the rigid parabolic cylinder \( \eta = \eta_0 \) is:
\[
\nu_z = u_0 \sec \frac{\theta}{2} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \tan \frac{\theta}{2} \left[ \frac{D_n(\lambda \eta_0)}{D_{n-1}(\lambda \eta_0)} - \frac{D_n(\lambda \eta_0)}{D_{n-1}(\lambda \eta_0)} \right] D_n(\lambda \xi).
\]

(2.8)

The stress \(\sigma_{x \eta}\) and \(\sigma_{nz}\) can be calculated easily from the above, using (2.3).

The total stresses will be made dimensionless by comparing them with \(\sigma_0 = i k u_0\), which is the maximum shearing stress generated by the incident wave \(u_z^{(i)}\) alone. As the wave number \(k\) approaches zero, the shearing stress \(\sigma_{nz} = \sigma_{nz}/\sigma_0\) is of the order \(1/k\), (2.1) thus it becomes infinite in the static limit \((k = 0)\) for \(0 \leq \theta < \pi/2\). For other values of wave number, the absolute values of shearing stress on the boundary are shown in Fig. 2.1. The ordinate \(|\sigma_{nz}|\) may be

![Fig. 2.1. Normalized Shear Stress, |\(\sigma_{nz}\)|, vs. Square Root of Normalized Wave Number (or Normalized Focal Length), \(\kappa^{2}_n\), at the Surface of a Rigid Parabolic Insert for Various Angles, \(\theta_0\); Solid and Dashed Lines Denote Stress at the Base (\(\xi = 0\)), and Below the Focal Point (\(\xi = -\eta_0\)), Respectively]
considered as the dynamic stress concentration factor and the abscissa scale \( \sqrt{k\eta_0} \) may be interpreted as the square root of wave number for fixed \( \eta_0 \), or as the square root of focal length for fixed wave frequency.

If the surface, \( \eta = \eta_0 \), is free of stresses, the boundary condition is

\[
\sigma_{\eta z} = \mu \frac{\partial u_z}{\partial \eta} = 0, \quad \text{at } \eta = \eta_0.
\]

The total displacement then becomes

\[
u_z = u_0 \sec \frac{\theta}{2} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \tan \frac{\theta}{2} \left[ D_n(\lambda \eta) - \frac{i D_n'(\lambda \eta)}{D_{n-1}(\lambda \eta)} D_{n-1}(\lambda \eta) \right] D_n(\lambda \xi).
\]

(2.9)

**Fig. 2.2. Normalized Shear Stress, \(|\sigma_{zz}|\), vs. Square Root of Normalized Wave Number (or Normalized Focal Length), \(k^{1/2}\eta_0\), at the Tip (\(t=0\)) of a Parabolic Notch for Various Incident Angles, \(\theta_0\)**
The stress $\sigma_{xz}$ again is of the order $\sqrt{1/k}$ as $k \to 0$ when $\theta \neq 0$. Plots of the absolute value of normalized shear stress $|\sigma_{xz}| = |\sigma_{xz}/\sigma_o|$ are shown in Fig. 2.2.

### 2.2. A Semi-Infinite Crack or Rigid Ribbon

As the focal length, $\eta_0^2/2$, of a parabola $n = \eta_0$ diminishes, the parabolic surface degenerates to a semi-infinite line ($y = 0$ and $x > 0$). The solutions (2.8) and (2.9) can be simplified accordingly.

Consider the case of a semi-infinite rigid ribbon fixed at the positive $x$-axis. The total wave is given by (2.8) with $\eta_0 = 0$. Using the original expression (2.1) for the incident wave and (1.25) for the ratio $D_n(0)/D_{-n-1}(0)$, we obtain

$$u_x/u_o = e^{ik\left[\theta^2 - n^2\right]} + \frac{\eta_0}{2} \left[ \frac{\lambda}{n} \tan^{-1} \frac{\theta}{2} \right] D_{-2n-1}(\eta_0) D_{2n}(\lambda \xi).$$

Replacing the incident wave expression by the product of Weber functions as in (1.38) and summing the infinite series according to (1.39), we obtain

$$u_x/u_o = \left[ \frac{\lambda}{n} \left( \xi \sin \frac{\theta}{2} - n \cos \frac{\theta}{2} \right) \right] D_n \left[ \lambda \left( \xi \cos \frac{\theta}{2} + n \sin \frac{\theta}{2} \right) \right]$$

$$- \frac{1}{\sqrt{2} \pi} \left\{ D_0 \left[ \frac{\lambda}{n} \left( \xi \cos \frac{\theta}{2} + n \sin \frac{\theta}{2} \right) \right] D_{-1} \left[ \lambda \left( n \cos \frac{\theta}{2} - \xi \sin \frac{\theta}{2} \right) \right]$$

$$+ D_0 \left[ \frac{\lambda}{n} \left( \xi \cos \frac{\theta}{2} - n \sin \frac{\theta}{2} \right) \right] D_{-1} \left[ \lambda \left( \xi \sin \frac{\theta}{2} + n \cos \frac{\theta}{2} \right) \right] \right\}. $$
The first two terms can be combined further according to (1.21) with
\( n = 0 \); the final expression is

\[
\nu_2(\xi, \eta) = \nu_0 W_2(\xi, \eta; \lambda, \theta_0), \quad 0 \leq \theta_0 \leq \pi/2
\]  

(2.10)

where

\[
W_2(\xi, \eta; \lambda, \theta_0) = \frac{1}{\sqrt{2\pi}} \left\{ D_0 \left[ \lambda \left( \xi \cos \frac{\theta_0}{2} + \eta \sin \frac{\theta_0}{2} \right) \right] D_{-1} \left[ \lambda \left( \xi \sin \frac{\theta_0}{2} - \eta \cos \frac{\theta_0}{2} \right) \right] 
- D_0 \left[ \lambda \left( \xi \cos \frac{\theta_0}{2} - \eta \sin \frac{\theta_0}{2} \right) \right] D_{-1} \left[ \lambda \left( \xi \sin \frac{\theta_0}{2} + \eta \cos \frac{\theta_0}{2} \right) \right] \right\},
\]  

(2.11)

and

\[
\lambda = \sqrt{-2\bar{\kappa}}, \quad \bar{\lambda} = \sqrt{2\bar{\kappa}}.
\]

The solution above is an odd function of \( \eta \), demonstrating that the
boundary condition \( \nu_2(\xi, 0) = 0 \) is satisfied.

The stresses are

\[
\sigma_{\xi z} = \frac{\sigma_0}{\sqrt{2\pi}} \left[ \cos \frac{\theta}{2} \cos \theta_\theta W_2 - \sin \frac{\theta}{2} \sin \theta_\theta W_1 \right],
\]

\[
\sigma_{\eta s} = \frac{\sigma_0}{\sqrt{2\pi}} \left[ \cos \frac{\theta}{2} \sin \theta_\theta W_1 - \sin \frac{\theta}{2} \cos \theta_\theta W_2 - \frac{2 \cos (\theta_\theta/2)}{\sqrt{kr}} \epsilon^{ikr + \pi/4} \right],
\]

(2.12)

in which the polar coordinates \((r, \theta)\) defined in (1.2) have been used,
and the function \( W_1 \) is given by (2.14).

Similarly, the solution for a semi-infinite crack can be derived
by setting \( \eta_o = 0 \) in (2.9). The final result is

\[
u_2 / u_o = e^{i \lambda \left( \xi^2 - \eta^2 \right) \cos \theta_o + \xi \eta \sin \theta_o}
\]

\[
= \sqrt{\frac{2}{\pi}} \sec \frac{\theta_o}{2} \sum_{n=0}^{\infty} \left( \tan \frac{\theta_o}{2} \right)^{2n+1} D_{-2n-2}^{2n+1}(\lambda \xi) D_{2n+1}(\lambda \xi) \tag{2.13}
\]

\[
= \hat{W}_1(\xi, \eta; \lambda, \theta_o) \quad 0 \leq \theta_o < \pi/2,
\]

where

\[
\hat{W}_1(\xi, \eta; \lambda, \theta_o)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left\{ D_0 \left[ \lambda \left( \xi \cos \frac{\theta_o}{2} + \eta \sin \frac{\theta_o}{2} \right) \right] D_{-1} \left[ \lambda \left( \xi \sin \frac{\theta_o}{2} - \eta \cos \frac{\theta_o}{2} \right) \right] + D_0 \left[ \lambda \left( \xi \cos \frac{\theta_o}{2} - \eta \sin \frac{\theta_o}{2} \right) \right] D_{-1} \left[ \lambda \left( \xi \sin \frac{\theta_o}{2} + \eta \cos \frac{\theta_o}{2} \right) \right] \right\}. \tag{2.14}
\]

The stresses are

\[
\sigma_{\xi\xi} = \sigma_0 \sqrt{2\pi} \left[ \cos \frac{\theta_o}{2} \cos \theta_o \hat{W}_1 + \sin \frac{\theta_o}{2} \sin \theta_o \hat{W}_2 + \frac{2 \sin (\theta_o/2)}{\sqrt{kr}} e^{i(kr + \pi/4)} \right]. \tag{2.15}
\]

\[
\sigma_{\eta\eta} = \sigma_0 \sqrt{2\pi} \left[ \cos \frac{\theta_o}{2} \sin \theta_o \hat{W}_2 - \sin \frac{\theta_o}{2} \cos \theta_o \hat{W}_1 \right].
\]

The function \( \hat{W}_1 \) above is even in \( \eta \); thus it obviously satisfies the boundary condition for a semi-infinite crack \( \partial u_2 / \partial \eta = 0 \). It differs from \( \hat{W}_2 \) only by the sign between the two terms.

Although the two series that combined to yield \( \hat{W}_{1,2} \) are both divergent at \( \theta = \pi/2 \), the functions \( \hat{W}_1 \) and \( \hat{W}_2 \) as given by (2.11) and
(2.14) have well-defined values there. In view of (1.2), (1.10), and (1.17), the two functions \( \hat{w}_{1,2} \) can be expressed in polar coordinates as

\[
\hat{w}_{1,2}(r, \theta, \lambda, \theta_0) = \frac{1}{\sqrt{\pi}} \left[ e^{ikr \cos(\theta-\theta_0)} \int_{\alpha^-}^{\alpha^+} e^{-s^2} ds \right. \\
\left. \pm e^{ikr \cos(\theta+\theta_0)} \int_{\alpha^-}^{\alpha^+} e^{-s^2} ds \right] 
\]

(2.16)

with \( \alpha = \sqrt{-2ikr \sin \frac{1}{2}(\theta \pm \theta_0)} \).

This formula is very convenient in the numerical evaluation of \( \hat{w}_{1,2} \) because the integrals can be transformed to the standard form of Fresnel integral for which extensive tables are available. An integral solution like that in (2.16) but in parabolic coordinates was introduced by Lamb for the investigation of reflection of optical waves by a parabolic mirror. (0.6.2.2)

It is to be noted that the functions \( \hat{w}_1 \) and \( \hat{w}_2 \) are the famous Sommerfeld solutions for the diffraction of plane electromagnetic waves by a semi-infinite, perfectly conducting screen (see Section 11.5 of Ref. 2.3). \( \hat{w}_2 \) is the function for an electric field being polarized parallel to the \( z \)-axis, and \( \hat{w}_1 \) is for the one polarized perpendicular to the \( z \)-axis. Because the \( \xi \) and \( \eta \) are not separated, these two wave functions are not listed in Table 1.1. However, one could have obtained them directly from the wave equation without summing the series. By introducing
DIFFRACTION OF SH WAVES

\[ \xi' = \xi' = \xi \cos \frac{\theta_0}{2} + \eta \sin \frac{\theta_0}{2}, \]
\[ \eta' = \eta' = \xi \sin \frac{\theta_0}{2} + \eta \cos \frac{\theta_0}{2}, \]

the two-dimensional Helmholtz equation (2.2) is transformed to

\[ \frac{\partial^2 u_z}{\partial \xi'^2} + \frac{\partial^2 u_z}{\partial \eta'^2} + (\xi'^2 + \eta'^2)k^2u_z = 0, \]

which is the same as (2.2) in terms of the new variables \( \xi' \) and \( \eta' \).

One of its solutions, as can be seen from Table 1.1, is

\[ u_z = AD_0(\lambda \xi'')D_{-1}(\lambda \eta'') + BE_0(\lambda \xi'')D_{-1}(\lambda \eta''). \]  

(2.17)

If \( A = -B = 1/\sqrt{2\pi} \), \( u_z = \bar{u}'_2 \), whereas \( u_z = \bar{u}'_1 \) when \( A = B = 1/\sqrt{2\pi} \).

Using the various representations for Weber functions, the Sommerfeld solutions can thus be put in numerous forms. It is possible to construct many other wave functions of orders \( \eta \) and \( -\eta-1 \) in terms of the \( \eta' \) and \( \xi' \) variables. However, since the original coordinates \( \eta \) and \( \xi \) are not separated in the solution, these functions are not very useful for solving general diffraction problems with a boundary \( \eta = \eta_0 \neq 0 \).

Whether a plane SH wave is diffracted by a semi-infinite rigid ribbon (2.13) or by a semi-infinite crack (2.15), one of the normalized shearing stress components contains the factor \( (1/kr)^{\frac{\theta}{2}} \) and is singular at the tip. Calculation also shows that the stresses when normalized by \( \rho_0 \) decrease rapidly away from the tip except in a case where the incident wave is nearly parallel to a semi-infinite crack. (2.1) Since the solutions contain the factor \( \sqrt{k\xi} \), the change of stress as the wave
Fig. 2.3a. Constant Displacement Amplitude Trajectories for an Incident SH Wave at $\theta_0 = 90^\circ$ on a Semi-Infinite Crack

Fig. 2.3b. Constant Displacement Phase Trajectories for an Incident SH Wave at $\theta_0 = 90^\circ$ on a Semi-Infinite Crack
number $k$ increases is the same as that for increasing $\xi$. In Figs. (2.3a,b) we show the magnitudes and phase angles for the complex displacements for an incident SH wave at $\theta_0 = 90$ degrees on a semi-infinite crack. The contours of constant phases physically represent the total wave fronts.

3. DIFFRACTION OF P AND SV WAVES BY A SEMI-INFINITE PLANE

As can be seen from the previous section, the analysis of diffraction of waves by a semi-infinite plane which is a degenerate parabolic cylinder (zero focal length) is considerably simpler than the general case of a parabolic cylindrical surface. The study of diffraction of P or SV waves would naturally begin with the case of a semi-infinite plane. Diffraction problems with this type of geometry are also amenable to other analytical methods, like the Wiener-Hopf technique, and in the case of transient waves, to the application of homogeneous solutions in addition to those discussed in Chapter II.

The Wiener-Hopf technique has been applied very effectively in solving the diffraction of a scalar wave by a semi-infinite screen (see Noble, p. 96). (3.1) Extending it to vector wave problems, Shima (3.2) and Maue (3.3) investigated the diffraction of a harmonic wave and a rectangularly shaped pulse by a semi-infinite crack; Roseau (3.4) studied the diffraction of harmonic waves by a clamped half-plane. The technique was applied earlier by Maue (3.5) to investigate the wave motion generated by the sudden opening of a semi-infinite crack in a pre-stretched plate, and later by Baker (3.6) for calculating the stress
field in front of a constantly moving crack. Very recently, Freund and Achenbach (3.7) considered the diffraction of waves by a semi-infinite crack at the interface of two different media.

We shall not present the application of the Wiener-Hopf technique here. For steady waves, the series-eigenfunction approach is again adopted, which when combined with the perturbation technique discussed in Chapter II, Section 4, is useful in determining the stress or displacement field near the edge of the half-plane and at low frequencies. Two types of half-planes are considered— a rigid-smooth strip and a rigid-fixed (clamped) strip. Problems with a transient incident pulse are solved by using the homogeneous functions discussed in Section 5.

3.1. A Rigid-Smooth Strip

A rigid-smooth surface is one that is restrained from moving normally, but is free to slide in the direction tangent to the surface. For a parabolic cylinder, including the degenerate case $\eta_0 = 0$, the corresponding boundary conditions over the surface $n = \eta_0$ are:

\[ u_{\eta}(\xi, \eta_0, z, t) = 0, \]
\[ \sigma_{\eta\xi}(\xi, \eta_0, z, t) = 0, \]
\[ \sigma_{\eta z}(\xi, \eta_0, z, t) = 0. \]

For problems of plane strain, only the first two conditions remain to be satisfied and the displacement field is determined completely by taking
\[ \varphi = \varphi(\xi, \eta, t), \]
\[ \psi = \psi(\xi, \eta, t), \]
\[ \chi = 0, \]

in all equations of subsection 1.4.

Boundary conditions for a semi-infinite rigid-smooth strip can be defined in terms of parabolic coordinates as

\[ u_\eta(\xi, \eta, t) = 0 \quad \text{at} \quad \eta = \eta_0 = 0, \]
\[ \sigma_{\eta \xi}(\xi, \eta, t) = 0, \]

or in terms of Cartesian coordinates as

\[ u_y(x, y, t) = 0 \quad \text{at} \quad y = 0 \text{ and } x > 0, \]
\[ \sigma_{yx}(x, y, t) = 0. \]

In terms of displacement potentials — see Eq. (I-3.11) — the equations above are equivalent to

\[ u_y = \frac{3\varphi}{\partial y} - \frac{3\psi}{\partial x} = 0 \quad \text{at} \quad y = 0, x > 0, \]
\[ \sigma_{yx} = \mu \left( \frac{3\varphi}{\partial x} + \frac{3\psi}{\partial y} \right) = 0, \]

as \( \partial u_y / \partial x = 0 \) along the positive \( x \) axis. Differentiating the first with respect to \( x \) and then subtracting it from the second, we find at the semi-infinite plane,
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \]

Since \( \nabla^2 \psi = \sigma^{-2} \psi \), it follows that \( \ddot{\psi} = 0 \). Furthermore, if \( \psi \) is not a linear function of time \( t \), we find

\[ \psi(+x, 0, t) = 0. \quad (3.3a) \]

With \( \psi \) being a constant over the entire positive \( x \)-axis, \( \partial \psi / \partial x \) must be zero there. The condition \( u_y = 0 \) then yields

\[ \partial \varphi(+x, 0, t) / \partial y = 0. \quad (3.3b) \]

In terms of parabolic coordinates, we have the new set of boundary conditions:

\[ \psi(\xi, \eta, t) = 0 \quad \text{at} \quad \eta = \eta_0 = 0, \quad (3.4) \]

\[ \partial \varphi(\xi, \eta, t) / \partial \eta = 0, \]

which are equivalent to (3.2).

From the derivation of (3.2), we note the existence of a discontinuity at the origin \( x = 0 \) and \( y = 0 \) (\( \eta = 0 \) and \( \xi = 0 \)), which is also the edge of the semi-infinite strip. Hence additional conditions must be specified for the behavior of solutions at the origin. A first requirement is that the displacement must be a finite quantity everywhere in the entire region including the edge. Secondly, because the strain energy contained in any finite region surrounding the tip of the semi-infinite strip must remain bounded, the strain components should be at least of the order \( r^{-\frac{1}{2}} \) as \( r \to 0 \), where \( r = (\xi^2 + \eta^2)/2 \).
is the radial distance measured from the origin. It then follows
that the displacement component should be at least of the order $r^{-\frac{1}{2}}$
as $r \to 0$. We thus add to (3.3) the condition of regular displacement
at the edge — sometimes known as the edge condition. In Ref. 3.8 it
was established by examining the behavior of boundary conditions at
the crown of a smooth-rigid parabola as $\xi$ and $\eta \to 0$, that

$$u_{\xi, \eta} = 0(r^{\frac{1}{2}}) \quad \text{as} \ r \to 0. \quad (3.5)$$

Since the two potentials $\varphi$ and $\psi$ are uncoupled in the boundary
conditions (3.4), finding the solution for the diffracted waves be-
comes easy. An incident P wave (subscript $A$) at an angle $\theta_1$ to the
$x$-axis is represented by

$$\varphi_A(t) = \varphi_0 \exp \left[ i\alpha(x \cos \theta_1 + y \sin \theta_1 - c_P t) \right], \quad \alpha = \omega/c_P,$$

$$\psi_A(t) = 0$$

(3.6)

and similarly, an incident SV wave (subscript $B$) is given by

$$\varphi_B(t) = 0,$$

$$\psi_B(t) = \psi_0 \exp \left[ i\beta(x \cos \theta_2 + y \sin \theta_2 - c_S t) \right], \quad \beta = \omega/c_S.$$ 

(3.7)

In either case, scattered waves are constructed from solutions of the
wave equations for $\varphi$ and $\psi$, which satisfy boundary conditions (3.4)
and the edge condition (3.5).

As far as (3.4) is concerned, the solution procedure is the same
as that for determining the scattered SH waves. For the incident P
wave, the total wave is given by
\[ \varphi_A = \varphi_A^{(i)} + \varphi_A^{(s)}, \]
\[ \varphi_A^{(i)} = \psi_A^{(i)} + \psi_A^{(s)} = 0, \]

with \[
\frac{\partial \varphi_A^{(s)}}{\partial n} = - \frac{\partial \varphi_A^{(i)}}{\partial n} \quad \text{at } n = 0.
\]

Thus \( \varphi \) should have the same form as (2.13) which is the solution for the diffraction of an SH wave by a crack, with \( \mu_2 \) being replaced by \( \varphi \). For an incident SV wave,

\[ \varphi_B = 0, \]
\[ \psi_B = \psi_B^{(i)} + \psi_B^{(s)}, \]

with \[
\psi_B^{(s)} = - \psi_B^{(i)} \quad \text{at } n = 0.
\]

The solution for \( \psi_B \) is the same as \( \mu_2 \) in (2.10) for the scattering of an SH wave by a rigid strip.

Without further derivation, we write down the solutions which satisfy boundary conditions (3.4) and the wave equations. For an incident P wave at an angle \( \theta_1 \), they are

\[ \varphi_A = \varphi_{\omega_1}^{(i)}(\lambda_1, \theta_1), \quad \lambda_1 = (-2i\omega)^{\frac{1}{2}}, \]
\[ \psi_A = 0, \tag{3.8} \]
\[ u_\xi = - \frac{1}{j} \frac{\partial \phi_A}{\partial \xi} = - \frac{\lambda_1^2}{2} \psi_o \left[ \frac{\xi}{j} \cos \theta_1 W_1(\lambda_1, \theta_1) \right. \]
\[ + \frac{n}{j} \sin \theta_1 W_2(\lambda_1, \theta_1) + \frac{4}{(2\pi)^\frac{3}{2}} \lambda_1^2 D_0(\lambda_1 \eta) D_0(\lambda_1 \xi) \right], \quad (3.9) \]
\[ u_\eta = - \frac{1}{j} \frac{\partial \phi_A}{\partial \eta} = \frac{\lambda_2^2}{2} \psi_o \left[ \frac{n}{j} \cos \theta_2 W_1(\lambda_2, \theta_2) - \frac{\xi}{j} \sin \theta_2 W_2(\lambda_2, \theta_2) \right]. \]

For an incident SV wave at an angle \( \theta_2 \), we have
\[ \phi_B = 0, \quad (3.10) \]
\[ \psi_B = \psi \psi_2(\lambda_2, \theta_2), \quad \lambda_2 = (-2jB)^{-\frac{1}{2}}. \]
\[ u_\xi = - \frac{1}{j} \frac{\partial \psi_B}{\partial \xi} = \frac{\lambda_2^2}{2} \psi_o \left[ \frac{n}{j} \cos \theta_2 W_2(\lambda_2, \theta_2) \right. \]
\[ - \frac{\xi}{j} \sin \theta_2 W_1(\lambda_2, \theta_2) + \frac{4}{(2\pi)^\frac{3}{2}} \lambda_2^2 D_0(\lambda_2 \eta) D_0(\lambda_2 \xi) \right], \quad (3.11) \]
\[ u_\eta = - \frac{1}{j} \frac{\partial \psi_B}{\partial \eta} = \frac{\lambda_2^2}{2} \psi_o \left[ \frac{\xi}{j} \cos \theta_2 W_1(\lambda_2, \theta_2) + \frac{n}{j} \sin \theta_2 W_1(\lambda_2, \theta_2) \right]. \]

In the above, the \( W_{1,2} \) are the first and second Sommerfeld solutions for the diffraction of scalar waves by a semi-infinite screen, as defined previously by (2.14) and (2.11) respectively. They are repeated here for easy reference.

\[ W_{1,2}(\xi, \eta; \lambda_j, \theta_j) = (2\pi)^{-\frac{1}{2}} \left\{ D_0[\lambda_j(\xi \cos \theta_j + \eta \sin \theta_j)] \right. \]
\[ \times D_{-1}[\lambda_j(\xi \sin \theta_j - \eta \cos \theta_j)] \]
\[ \left. + D_0[\lambda_j(\xi \cos \theta_j - \eta \sin \theta_j)] D_{-1}[\lambda_j(\xi \sin \theta_j + \eta \cos \theta_j)] \right\}. \quad (3.12) \]
Careful examination shows that \( \xi(\xi, \eta) \) in both (3.9) and (3.11) become singular as \( \xi, \eta \to 0 \). The singularity which is caused by the factor \( 1/J = (\xi^2 + \eta^2)^{-\frac{3}{2}} \) in both (3.9) and (3.11) is of the order \( r^{-\frac{3}{2}} \) as \( r \to 0 \). These singular terms must be eliminated in order to satisfy the edge condition (3.5).

The wave functions listed in Table 1.1 and in Eq. (3.12) are all regular functions in the \( \xi-\eta \) plane. We must now construct outgoing wave functions which are singular at \( r = \frac{1}{2}(\xi^2 + \eta^2) = 0 \), and satisfy the boundary conditions at \( \eta = 0 \) and \( \xi \neq 0 \). The functions which meet all these requirements are of the form \( r^{-\frac{1}{2}} e^{i\alpha} \cos \frac{\alpha}{2} \theta \) or \( r^{-\frac{1}{2}} e^{i\beta} \sin \frac{\beta}{2} \theta \).

In parabolic coordinates, they become

\[
S_1(\lambda_1) = \frac{1}{\lambda_1 J} \left( \frac{\xi}{\lambda} \right) D_0(\lambda_1 \eta) D_0(\lambda_1 \xi) \]

\[
= \frac{\xi}{\lambda_1(\xi^2 + \eta^2)} e^{i\alpha(\xi^2 + \eta^2)/2},
\]

(3.13)

\[
S_2(\lambda_2) = \frac{1}{\lambda_2 J} \left( \frac{\eta}{\lambda} \right) D_0(\lambda_2 \eta) D_0(\lambda_2 \xi) \]

\[
= \frac{\eta}{\lambda_2(\xi^2 + \eta^2)} e^{i\beta(\xi^2 + \eta^2)/2}
\]

Both \( S_1 \) and \( S_2 \) are unbounded at \( \xi = \eta = 0 \). They will now be added to the previous solutions so as to eliminate the singularities in the expressions for displacements.

The subtraction of the singular terms so as to satisfy the edge condition is also discussed by Kostrov in his analysis of diffraction by a rigid-smooth wedge.
(A) Incident P Wave

In lieu of (3.8), the total wave field is taken as

\[ \psi_A = \varphi_0 \Psi_1(\lambda_1, \theta_1) + A_1 S_1(\lambda_1), \]

\[ \psi_A = \varphi_0 \Psi_2(\lambda_2). \]

The corresponding displacements, when written in terms of polar coordinates \((r, \theta)\), have the following values as \(k_1 r \to 0\):

\[ u_r = -\frac{\lambda_1}{r^{2/3}} \left[ \sqrt{\frac{32}{\pi}} \varphi_0 \sin \frac{1}{2} \theta_1 + A_1 + \sqrt{\lambda_2} + \frac{i \cos \theta}{k_1 r} (A_1 - A_2/\sqrt{\kappa}) \right] + \text{regular terms}, \]

\[ u_\theta = -\frac{\sin \theta}{\lambda_1 (2r)^{3/2}} (A_1 - A_2/\sqrt{\kappa}) + \text{regular terms}, \]

where \(\kappa = c_p/c_v\) as defined in Eq. (1-2.37). It becomes obvious that the singular terms are eliminated if \(A_1\) and \(A_2\) assume the following values

\[ A_1 = \frac{A_2}{\sqrt{\kappa}} = -\sqrt{\frac{32}{\pi}} \frac{\varphi_0 \sin \frac{1}{2} \theta_1}{1 + \kappa}. \]

Substitution of the above in (3.14) then completes the solution.

We have thus obtained an exact solution for the diffraction of an incident P wave by a semi-infinite smooth-rigid plane. The displace-ments are:
\[ u_\xi (\xi, \eta) = \frac{A_1 \lambda_1}{4w} \left[ (\kappa - i \cos \theta) \left( e^{i \alpha r} - e^{i \beta r} \right) - (e^{i \alpha r} - e^{i \beta r}) \cos \theta \right] \]
\[ - \frac{1}{2} \lambda_1^2 \varphi_0 \left[ \cos \frac{i}{2} \theta \cos \theta \tilde{\varphi}_1(\lambda_1, \theta_1) + \sin \frac{i}{2} \theta \sin \theta \tilde{\varphi}_2(\lambda_1, \theta_1) \right], \]
\[ u_\eta (\xi, \eta) = \frac{A_1 \lambda_1}{4w} \left[ (\kappa e^{i \alpha r} - e^{i \alpha r}) + \frac{i}{\alpha r} (e^{i \beta r} - e^{i \alpha r}) \right] \sin \theta \]
\[ + \frac{1}{2} \lambda_1^2 \varphi_0 \left[ \sin \frac{i}{2} \theta \cos \theta \tilde{\varphi}_1(\lambda_1, \theta_1) - \cos \frac{i}{2} \theta \sin \theta \tilde{\varphi}_2(\lambda_1, \theta_1) \right]. \]

(3.15)

They both are regular at \( \xi = \eta = 0 \) \((r = 0)\). Expressions for stresses are rather lengthy but the calculation is straightforward. Near the tip of the ribbon and at low frequencies \((\alpha r \ll 1)\), the two normal stresses are

\[ \frac{\sigma_{\xi\xi}}{\sigma_1} = (A_1/2\lambda_1 J) \left[ \frac{3}{2} (1 - \kappa^{-2}) \cos \frac{3}{2} \theta - (1 - \kappa^{-1}) \cos \frac{i}{2} \theta \right. \]
\[ + 4 - 2 \cos \theta - 4 \kappa^{-2} \sin^2 \frac{i}{2} \theta - 4(\kappa^{-1} + \kappa^{-2}) \cos^2 \frac{i}{2} \theta \right) \]
\[ \left. + i - (1 + \cos \theta \cos 2\theta_1)/\kappa^2 + O[(\alpha r)^2] \right], \]

(3.16)

\[ \frac{\sigma_{\eta\eta}}{\sigma_1} = 2(1 - \kappa^{-2})(1 + A_1/2\lambda_1 J) - \frac{\sigma_{\xi\xi}}{\sigma_1} + O[(\alpha r)^2], \]

where \( \sigma_1 = -i \kappa^2 \varphi_0 \) is the maximum value of the principal stress due to the incident wave alone. On account of the factor \( 1/\lambda_1 J \), both stresses are of the order \( O(1/\alpha r) \) as \( r \to 0 \).

Along top and bottom edges of a smooth rigid strip, the absolute values of normalized stress \( \frac{\sigma_{\eta\eta}}{\sigma_1} \) are shown on Fig. 3.1 as a function of \( \xi \sqrt{\theta}/2 \) which may be regarded as the normalized shear wave number or
the normalized distance along the strip. The incident P wave impinges perpendicularly on the strip (θ₁ = π/2). It is seen that the magnitude of the normal stress decays rapidly away from the tip and the stress concentration factor (σ_{ηη}/σ₁) at any fixed location decreases as the frequency of the incident wave increases.

(B) Incident SV Wave

For an incident SV wave the results (3.10) and (3.11) should be modified accordingly. The potentials are taken as

\[ \varphi_\theta = B_1 S_1(\lambda_1), \]

\[ \psi_\theta = \psi_{\theta 2}(\lambda_2, \theta_2) + B_2 S_2(\lambda_2). \]
The coefficients are found to be

\[ B_1 = \frac{B_2}{\sqrt{\kappa}} = \psi_0 \sqrt{\frac{27}{\pi}} \cos \frac{3\theta}{2} \]

The complete displacements are

\[ u_\xi (\xi, \eta) = \frac{\lambda_1 B_1}{4\omega} \left[ \left( \frac{i \cos \theta}{\ar} + 1 \right) \left( e^{i\theta r} - e^{i\alpha r} \right) + \left( \kappa e^{i\theta r} - e^{i\alpha r} \right) \cos \theta \right] \]

\[ + \frac{\lambda_2^2}{2} \sin \frac{\theta}{2} \cos \theta \omega_{12} \left( \omega \omega_{12} \right) - \cos \frac{\theta}{2} \sin \theta \omega_{12} \left( \omega \omega_{12} \right) \]

(3.18)

\[ u_\eta (\xi, \eta) = \frac{\lambda_1 B_1}{4\omega} \left[ \left( \frac{i \cos \theta}{\ar} + 1 \right) \left( e^{i\theta r} - e^{i\alpha r} \right) + \left( \kappa e^{i\theta r} - e^{i\alpha r} \right) \right] \sin \theta \]

\[ + \frac{\lambda_2^2}{2} \cos \frac{\theta}{2} \cos \theta \omega_{12} \left( \omega \omega_{12} \right) + \sin \frac{\theta}{2} \sin \theta \omega_{12} \left( \omega \omega_{12} \right) \]

As \( \theta r \to 0 \), the stresses are

\[ \sigma_{\xi\xi}/\sigma_2 = -B_1/2(\lambda_1 \omega) \left\{ \frac{3}{2} (1 - \kappa^{-2}) \cos \frac{3\theta}{2} + \kappa^{-1}(1 - \kappa^{-1}) \cos \frac{1\theta}{2} \right\} \]

\[ + \left[ 2 + 4(1 - \kappa^{-2}) \sin^2 \frac{1\theta}{2} + 2(1 + \kappa^{-1}) \cos \theta \right] \cos \frac{1\theta}{2} \]

\[ - \cos \theta \sin 2\theta + O(\sqrt{\theta r}) \]

(3.19)

\[ \sigma_{\eta\eta}/\sigma_2 = -2(1 - \kappa^{-2})(B_1/\lambda_1 \omega) \cos \frac{1\theta}{2} - \sigma_{\xi\xi}/\sigma_2 + O(\sqrt{\theta r}) \]

where \( \sigma_2 = \mu \omega^2 \psi_0 \) is the maximum value of the shear stress acting on the incident plane SV wave front. Again the stresses approach infinity in the order of \( \sqrt{\theta r} \) as \( r \to 0 \).
Fig. 3.2. Normalized Stresses Along Top and Bottom Edges of a Semi-Infinite Rigid Smooth Ribbon vs. Normalized Shear Wave Number (or Normalized Distance) for an Incident Shear Wave with $\nu = 0.25$. 

**a. Perpendicular Normal Stress**

**b. Parallel Normal Stress**
Along the length of the ribbon, the absolute values of the two normalized principal stresses \( \sigma_{\eta\eta}(\pm \xi, 0)/\sigma_2 \) and \( \sigma_{\xi\xi}(\pm \xi, 0)/\sigma_2 \) are shown in Figs. 3.2a,b respectively for two incident angles \( \theta_2 = 0 \) and \( \pi/2 \). The former, \( \sigma_{\eta\eta}(\xi, 0) = \sigma_{yy} \), is a normal stress perpendicular to the ribbon and the latter, \( \sigma_{\xi\xi}(\xi, 0) = \sigma_{xx} \), is parallel to the ribbon.

Contrary to the case of incident P waves, the stresses are continuous across the ribbon at \( \theta_2 = \pi/2 \). At zero-degree incidence, there is no "jump" in their values because of symmetry. In the case of 90-degree incidence, the incident normal stresses \( \sigma_{yy} \) and \( \sigma_{xx} \) are zero along \( y = 0 \), and the scattered stresses differ only in sign across the ribbon, thus the absolute values are equal. No distinction, then, is made for the (+) or (−) side in Figs. 3.2a,b.

3.2. A Rigid (Clamped) Strip

The boundary conditions for a semi-infinite rigid-fixed strip are:

\[
\begin{align*}
\tau_{\xi}(\xi, \eta, t) &= \frac{s}{i} (\partial \varphi/\partial \xi + \partial \psi/\partial \eta) = 0, \quad \text{at } \eta = 0. \quad (3.20) \\
\tau_{\eta}(\xi, \eta, t) &= \frac{s}{i} (\partial \varphi/\partial \eta - \partial \psi/\partial \xi) = 0.
\end{align*}
\]

Since the two wave potentials \( \varphi \) and \( \psi \) which are associated with two distinct wave velocities \( c_p \) and \( c_s \) occur simultaneously in the boundary conditions, they cannot be satisfied exactly if the series-eigenfunction method is applied. The method of Wiener-Hopf is not very effective either because, as can be seen in Ref. 3.4, in factoring the kernel into two functions which are analytic in two different but overlapping regions in a complex plane, the crucial step cannot con-
veniently be carried out. Obtaining numerical results by that method becomes complex and cumbersome.

An alternative would be to fall back on the series-eigenfunction method and to try to satisfy the boundary conditions approximately, as is done in the case of scattering of P or SV wave by a finite-length rigid ribbon (Section 3 of Chapter IV). Undoubtedly, some success is expected in calculating the stress concentration factor at points away from the edge of the semi-infinite ribbon, but not at the edge, because there the stresses approach infinity.

A combination of the newly developed perturbation technique with the series-eigenfunction approach proves to be very effective in estimating the stress intensification near the edge and at low frequencies. The method as presented in Section 4 of Chapter II was tested for the diffraction of a P wave by a rigid-smooth ribbon, and the results compare favorably with the exact figures shown in Fig. 3.1. \( (3.10) \)

For a semi-infinite rigid ribbon, the perturbation method may begin with Eqs. (4.16), (4.17), and (4.7) of Chapter II. Up to the first order of \( \varepsilon \), the displacement vector is given approximately by

\[ u = u^{(0)} + \varepsilon u^{(1)}, \quad \varepsilon = 1/(6 - 8\nu), \]

where the \( 0^{th} \) and first order displacements satisfy the following equations:

\[ (\nabla^2 + k^2)u^{(0)} = 0, \]

\[ (\nabla^2 + k^2)u^{(1)} = 2\nabla^2 u^{(0)} - 4\nabla \cdot u^{(0)}. \]
The boundary conditions are

\[ u^{(0)}(y) = 0, \]
\[ \text{at } y = 0, x > 0 (n = 0). \]  \hspace{1cm} (3.22)
\[ u^{(1)}(y) = 0, \]

Consider an incident P wave at an angle \( \theta_1 \), with the semi-infinite strip (Eq. 3.6):

\[ u^{(i)} = u_0 (e_1 \cos \theta_1 + e_2 \sin \theta_1) e^{i \alpha M}, \]
\[ M = x \cos \theta_1 + y \sin \theta_1. \]

With \( \kappa = k(1 - 2\varepsilon) \), the exponential function in the above can be expanded into a series

\[ e^{i \alpha M} = e^{i \kappa M} [1 - i \omega M + O(\varepsilon^2)]. \]

Thus the first two orders of incident displacements are

\[ u^{(0)}(i) = u_0 (e_1 \cos \theta_1 + e_2 \sin \theta_1) e^{i \kappa M}, \]  \hspace{1cm} (3.23)
\[ u^{(1)}(i) = -u_0 (e_1 \cos \theta_1 + e_2 \sin \theta_1) (i \omega M) e^{i \kappa M}. \]

Our problem is to find two scattered waves \( u^{(0)}(\varepsilon) \) and \( u^{(1)}(\varepsilon) \) satisfying Eqs. (3.21) and (3.22) for the total wave \( u^{(n)} = u^{(n)}(i) + u^{(n)}(\varepsilon) \). (3.11)

(A) 0th Order Solution

In Cartesian coordinate systems, the 0th order equations for the scattered wave are
\[(v^2 + k^2)u_x^{(0)}(\phi) = 0,\]
\[(v^2 + k^2)u_y^{(0)}(\phi) = 0.\]

In view of (3.22a), we have at the half plane,
\[u_x^{(0)}(\phi) = -u_x^{(0)}(\phi) = -u_0 \cos \theta_1 e^{ikx \cos \theta_1} \quad \text{at } y = 0, \quad x > 0.

Because the two components \(u_x^{(0)}\) and \(u_y^{(0)}\) are separated for each component in the field equation as well as in the boundary conditions, the solution is the same as for the diffraction of an SH wave by a semi-infinite rigid ribbon (Section 2).

In analogy to Eq. (2.10), we obtain, for the 0th order total wave,
\[u_x^{(0)} = u_0 \cos \theta_1 \tilde{w}_2(\xi, \eta; \lambda, \theta_1),\]
\[u_y^{(0)} = u_0 \sin \theta_1 \tilde{w}_2(\xi, \eta; \lambda, \theta_1),\]

where \(\lambda = \sqrt{-2ik} \). The function \(\tilde{w}_2\) which is comprised of both incident and scattered waves is given in (2.11) for parabolic coordinates and in (2.16) for polar coordinates.

(B) First Order Solution

The particular solution of the first order perturbation, Eq. (II-4.17), has the following Cartesian components in two-dimensional space:
\[ u_{px}^{(1)} = -x \frac{\partial u_x^{(0)}}{\partial x} + y \frac{\partial u_y^{(0)}}{\partial y} - 2y \frac{\partial u_u^{(0)}}{\partial x}, \]

\[ u_{py}^{(1)} = x \frac{\partial u_x^{(0)}}{\partial y} - y \frac{\partial u_y^{(0)}}{\partial x} - 2x \frac{\partial u_x^{(0)}}{\partial y}. \]

All terms except \(-2x\frac{\partial u_x^{(0)}}{\partial y}\) vanish at \(y = 0, \ x > 0\), because there \(u^{(0)}\) is identically zero.

Noting that

\[ \frac{\partial}{\partial \Theta} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \]

we rewrite the second of the above particular solutions as

\[ u_{py}^{(1)} = x \frac{\partial u_x^{(0)}}{\partial x} - y \frac{\partial u_y^{(0)}}{\partial y} - 2y \frac{\partial u_x^{(0)}}{\partial x} - 2x \frac{\partial u_x^{(0)}}{\partial y}, \]

and obtain a general solution for the first order equation (3.21):

\[ u_x^{(1)} = -x \frac{\partial u_x^{(0)}}{\partial x} - y \frac{\partial u_x^{(0)}}{\partial y} - 2y \frac{\partial u_y^{(0)}}{\partial x} + u_{\partial x}^{(1)}, \]

\[ u_y^{(1)} = x \frac{\partial u_y^{(0)}}{\partial x} - y \frac{\partial u_y^{(0)}}{\partial y} - 2x \frac{\partial u_x^{(0)}}{\partial x} - 2x \frac{\partial u_x^{(0)}}{\partial \Theta} + u_{\partial y}^{(1)}. \]

In the above derivations, use has been made of the relation

\[ u_x^{(0)} = u_x^{(0)(i)} + u_x^{(0)(s)} \]

and of the fact that \(\frac{\partial u_x^{(0)}}{\partial \Theta}\), which satisfies a Helmholtz equation, has been absorbed in the complementary solution \(u_x^{(1)}\). Although \(\frac{\partial u_x^{(0)}}{\partial \Theta}\) also satisfies a Helmholtz equation, it is not combined with \(u_{\partial y}^{(1)}\) because the complementary
solution must be a wave going outward toward infinity.

Since $u(0) = u(0)(i) + u(0)(s)$, Eq. (3.25) may be regarded as a superposition of the first order incident wave $u^{(1)}(i)$ and the first order scattered wave $u^{(1)}(s)$. The former is contained in the derivatives of $u^{(1)}(i)$ -- Eq. (3.23b). The latter consists of the derivatives of $u^{(0)}(s)$ and the complementary solution $u^{(1)}_c$. The boundary condition is still

$$u^{(1)} = u^{(1)}(i) + u^{(1)}(s) = 0 \quad \text{at} \quad y = 0, \; z > 0.$$ 

But in this case, it is more convenient not to separate explicitly the incident and scattered parts from Eq. (3.25).

The complementary solution $u^{(1)}_c$ can now be found by noting first that $\partial W_1(r; \lambda, \theta_1) / \partial y$ meets the boundary condition at $y = 0$ where $W_1$ is given by (2.14). Since $\partial / \partial \theta = x \partial / \partial y$ at $y = 0$, we choose

$$u^{(1)}_{cx} = 0,$$

$$u^{(1)}_{cy} = 2 \frac{\partial}{\partial \theta} \left[ - u_0 \cos \theta_1 W_1(r; \lambda, \theta_1) + u^{(0)}(i) \right].$$

It should be noted that since the quantity inside the brackets is the difference of the total wave field and the incident wave, the complementary solution $u^{(1)}_c$ indeed vanishes at $r \to \infty$. The final answer for the complete first order solution is

$$u^{(1)} = -x \frac{\partial u(0)}{\partial x} + y \frac{\partial u(0)}{\partial y} - 2y \frac{\partial u(0)}{\partial x},$$

$$u^{(1)} = x \frac{\partial u(0)}{\partial x} - y \frac{\partial u(0)}{\partial y} - 2y \frac{\partial u(0)}{\partial x} - 2u_0 \cos \theta_1 \frac{\partial W_1(r; \lambda, \theta_1)}{\partial \theta}.$$
It can be verified that each term in the solutions above vanishes at \( y = 0 \) \((x > 0)\). Because the 0\(^{th}\) order displacements are regular at the edge and

\[
\frac{\partial^2 w_{1,2}}{\partial y^2} = 0 \left( \frac{1}{\sqrt{y}} \right), \quad \frac{\partial^2 w_{1,2}}{\partial \theta^2} = 0(y), \quad \text{as } r \to 0,
\]

the first order displacement remains finite at the edge of the rigid plane.

With the displacement field being given by (3.24) and (3.26) and \( u = u^{(0)} + \varepsilon u^{(1)} \), the approximate values for stresses can be calculated from the perturbed Hooke's law. Near the edge \((r \to 0)\), the leading terms in stresses along the top and bottom sides of the rigid strip become

\[
\sigma_{xx}(x,0^{\pm}) \approx \frac{\nu}{1 - \nu} \sigma_{yy}(x,0^{\pm}),
\]

\[
\sigma_{yy}(x,0^{\pm}) \approx \sigma_o \left( \frac{2}{\eta k x} \right)^{\frac{1}{2}} \left[ (1 + \frac{1}{2} \varepsilon) \sin \theta_1 \cos \left( \frac{1}{2} \theta_1 \right) + \varepsilon \sin \left( \frac{3}{2} \theta_1 \right) \right] i(kx - \frac{3}{4} \pi),
\]

\[
\sigma_{xy}(x,0^{\pm}) \approx \sigma_o \left( \frac{2}{\eta k x} \right)^{\frac{1}{2}} \left( 1 - \frac{3}{2} \varepsilon \right) \cos \theta_1 \cos \left( \frac{1}{2} \theta_1 \right) i(kx - \frac{3}{4} \pi),
\]

where

\[
\sigma_o = \frac{1}{1 - \nu} \left( \frac{E}{\eta} \right) \frac{1 - \nu}{k},
\]

is the maximum principal stress of the incident P wave. The ratios \( \sigma_o^{1}/\sigma_o \) are then the dynamic stress intensification factors at the
SCATTERING OF P OR SV WAVES BY A PARABOLIC CYLINDER

edge of a rigid strip.

4. SCATTERING OF P OR SV WAVES BY A PARABOLIC CYLINDER

IN THIS SECTION an example is given of the scattering of P or SV waves by a parabolic cylinder. The main objective is to investigate the stress or displacement field near the crown portion of a parabola rather than to study how the incident wave is diffracted into the shadow zone or into the far field.

Up to now, stress concentration factors around a parabolic cylinder are known only for a rigid surface subjected to an incident P wave moving in the direction parallel to the symmetry axis of the parabola (parallel incidence). Although solutions for other types of boundary conditions or incident waves can be formulated in terms of the Weber functions, the finding of numerical results from these solutions is still a major task. The ensuing discussion is concerned with the solution of the scattering of a parallel-incident P wave by a rigid surface. The first solution is found by satisfying the boundary conditions approximately, and then it is supplemented by applying the perturbation technique. (4.1)

An incident P wave at an angle \( \theta_0 \) with respect to the \( x \)-axis can be represented by (1.36a).

\[
\psi'(r) = \frac{i a (x \cos \theta_0 + y \sin \theta_0)}{\varphi_o e^{r \frac{\theta_0}{2}}}
\]

\[
= \varphi_o \sec \frac{\theta_0}{2} \sum_{m=0}^{\infty} B_m D_m(\lambda_1 \eta) D_m(\lambda_1 \xi), \quad -\frac{\pi}{2} < \theta_0 < \frac{\pi}{2},
\]

\[
\psi'(r) = 0,
\]
with
\[ g_m = \frac{i^m}{m!} \tan \frac{\theta \omega}{2}, \quad \lambda_j = i \lambda_j, \quad \lambda_1 = \sqrt{-2i\alpha}, \quad \lambda_2 = \sqrt{-2i\beta}. \]

For a parallel-incident P wave, \( \theta = 0 \) and the series above reduce to one term:
\[ \varphi(i) = \varphi_0 D_0(\lambda_1 \eta)D_0(\lambda_1 \xi) = \varphi_0 e^{i \omega \eta}. \quad (4.1) \]

Scattered P and SV waves can be represented — see Table 1.1 — by
\[ \varphi(s) = \varphi_0 \sec \frac{\theta \omega}{2} \sum_{m=0}^{\infty} a_m D_m(\lambda_1 \eta) D_m(\lambda_1 \xi), \quad (4.2) \]
\[ \psi(s) = \varphi_0 \sec \frac{\theta \omega}{2} \sum_{m=0}^{\infty} b_m D_m(\lambda_2 \eta) D_m(\lambda_2 \xi), \]

where the unknown coefficients are to be determined from the boundary conditions for the total wave \( \varphi = \varphi(i) + \varphi(s) \) and \( \psi = \psi(s) \).

4.1. Series-Eigenfunction Solution

Boundary conditions for a rigid immobile parabolic cylinder described by \( \eta = \eta_0 \) are
\[ u_\xi = u_\xi(i) + u_\xi(s) = 0, \quad \text{at } \eta = \eta_0. \quad (4.3) \]
\[ u_\eta = u_\eta(i) + u_\eta(s) = 0, \]

We note that when the assumed wave potentials are substituted in the displacement-potential relation (1.40) \( \chi \equiv 0 \), and then in the boundary conditions above, the resulting equations involve the unknown
constants \( a_n \) and \( b_n \) as well as \( D_0(\lambda_1 \xi) \) and \( D_n(\lambda_2 \xi) \). Since \( \xi \) varies from \(-\infty\) to \(+\infty\) along \( n = n_c \), and the two Weber functions with distinct arguments are not orthogonal to each other, there is no simple way to solve these equations for \( a_n \) and \( b_n \).

To find the approximate values for \( a_n \) and \( b_n \) to a desired accuracy, one of the two Weber functions is to be expanded into a series of the other. Let

\[
D_m(\lambda_1 \xi) = \sum_{n=0}^{\infty} A_n^m D_n(\lambda_2 \xi). \tag{4.4}
\]

Applying the orthogonality condition (1.28), we find

\[
A_n^m = \frac{1}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} \lambda_2 D_m(\lambda_1 \xi) D_n(\lambda_2 \xi) \, d\xi.
\]

For a given pair of values for \( m \) and \( n \), the coefficients \( A_n^m \) can be evaluated numerically. Using (4.1) and (4.2), we obtain the total wave for the parallel incidence as

\[
\varphi = \varphi_0 D_0(\lambda_1 \eta) \sum_{n=0}^{\infty} A_n^0 D_n(\lambda_2 \xi) + \varphi_0 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m A_n^m D_{m-1}(\lambda_1 \eta) D_n(\lambda_2 \xi), \tag{4.5}
\]

\[
\psi = \psi_0 \sum_{m=0}^{\infty} b_m D_{m-1}(\lambda_2 \eta) D_m(\lambda_2 \xi).
\]

The displacement field is calculated from (1.41). With the derivatives of the \( D_n(\lambda \xi) \) being replaced by the third of equations (1.23b), and for the purpose of satisfying the boundary conditions,
we establish two equations for \( a_n \) and \( b_n \):

\[
\sum_{n=0}^{\infty} \left\{ 2b_{n-1}(\lambda_n) + [(n+1)a_{n+1}^0 - a_{n-1}^0] \right\} D_n(\lambda_1 \eta_o) + \sum_{m=0}^{\infty} a_m D_{n-m} - \frac{1}{2} \sqrt{\lambda} [(n+1)b_{n-1}^m D_{n-2}(\lambda_2 \eta_o) - b_{n-1}^m D_{n-2}(\lambda_2 \eta_o)]
\]

\[
+ \sum_{m=0}^{\infty} a_m D_{n-m} - \frac{1}{2} \sqrt{\lambda} [(n+1)b_{n+2}^m D_{n-1}(\lambda_2 \eta_o) - b_{n+2}^m D_{n-1}(\lambda_2 \eta_o)] \}
\]

From the orthogonality condition (1.28), it follows that each of the expressions in the braces in the two equations above must vanish identically. The first equation then yields the following relation:

\[
2D_{n-1}(\lambda_2 \eta_o) b_n = D_0(\lambda_1 \eta_o) [(n+1)a_{n+1}^0 - a_{n-1}^0]
\]

\[
+ \sum_{m=0}^{\infty} a_m D_{n-m} - \frac{1}{2} \sqrt{\lambda} [(n+1)b_{n+2}^m D_{n-1}(\lambda_2 \eta_o) - b_{n+2}^m D_{n-1}(\lambda_2 \eta_o)]
\]

Combining it with the second equation and thus eliminating \( b_n \), we obtain

\[
\sum_{m=0}^{\infty} a_m \left\{ a_m D_{n-m} - \frac{1}{2} \sqrt{\lambda} [(n+1)R_{n+2} \lambda_2 \eta_o (n+2)a_{n+2}^m - a_{n-2}^m] D_{n-m-1}(\lambda_1 \eta_o) \right\}
\]

\[
= \frac{1}{2} \sqrt{\lambda} [(n+1)R_{n+2} \lambda_2 \eta_o (n+2)a_{n+2}^0 - a_{n+2}^0] D_0(\lambda_1 \eta_o) - \frac{1}{2} D_0(\lambda_1 \eta_o) a_{n+2}^0
\]

\[
- R_n(\lambda_2 \eta_o)(a_{n-2}^0 - a_{n-2}^0) D_0(\lambda_1 \eta_o) - iD_0'(\lambda_1 \eta_o) a_{n+2}^0
\]

\[
= \frac{1}{2} \sqrt{\lambda} [(n+1)R_{n+2} \lambda_2 \eta_o (n+2)a_{n+2}^0 - a_{n+2}^0] D_0(\lambda_1 \eta_o) - \frac{1}{2} D_0(\lambda_1 \eta_o) a_{n+2}^0.
\]
where

\[ R_n(z) = D_{-n}(z)/D'_{-n}(z). \]

This lengthy equation ends the analysis provided by the series-eigen-function method. All quantities in the above system of equations except \( a_m \) are known and can be evaluated numerically with the right-hand members depending on the index \( n \), and the left-hand members depending on \( m \) and \( n \). Symbolically, we can represent that system of equations as

\[ \sum_{m=0}^{\infty} M_{nm}a_m = T_n, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (4.7)

Since \( M_{nm} \) is an infinite matrix, the unknown coefficients \( a_m \) can be calculated only approximately. One way is to truncate the matrix into a finite-size array and then invert the finite matrix to calculate the first few members of the column matrix \( a_m \).

At the surface \( \eta = \eta_0 \) of a rigid cylinder, \( \varepsilon_{zz} = 0 \) (\( \varepsilon_{x\delta} = \varepsilon_{z\eta} = \varepsilon_{zz} = 0 \) on account of plane strain). Since the stress \( \sigma_{ij} = \lambda \nabla^2 \phi_{ij} + 2\varepsilon_{ij} \) \((i, j = x, \eta)\), it can be shown that at a rigid surface \( \eta = \eta_0 \),

\[ \sigma_{\xi \xi} = \lambda \nabla^2 \phi, \]
\[ \sigma_{\eta \eta} = (\lambda + 2\mu) \nabla^2 \phi = \rho \ddot{\phi}, \]
\[ \sigma_{\xi \eta} = \mu \nabla^2 \psi. \]  \hspace{1cm} (4.8)

The normal stress to the surface when it is normalized by \( \sigma = -\mu \nabla^2 \phi_0 \), which is the maximum principal stress associated with incident wave
(4.1), is then given by
\[ \frac{\sigma_{n\eta}(\xi,\eta_0)}{\sigma_0} = \varphi \]
\[ = e^{i\alpha(\xi - \eta_0)/2} + \sum_{n=0}^{\infty} a_n D_{n-1}(\lambda_1 \eta_0) D_n(\lambda_1 \xi). \] (4.9)

Numerical calculations of the coefficients \(a_n\) and the stress concentration factor \(\sigma_{n\eta}/\sigma_0\) at \(\xi = 0\) are successful in the range of \(1.5 \leq \sqrt{\beta \eta_0} \leq 7.0\). For smaller values of \(\sqrt{\beta \eta_0}\), the convergence of the series is too slow and a large number of terms in the series must be summed in order to obtain the desired accuracy. However the number of terms obtainable in the series is limited by the size of the truncated matrix \(M_{nm}\) that can be inverted and the capability of calculating the elements of \(M_{nm}\) accurately. A \(33 \times 33\) matrix is used in the cal-

Fig. 4.1. Normalized Radial Stress \(|\sigma_{n\eta}|\) at the Apex of a Rigid Parabolic Cylinder vs. Normalized Shear Wave Number \(\sqrt{\beta \eta_0}\) (or Normalized Focal Length) for a Parallel Incident P Wave
calculation of $c_{\eta n}(0, \eta_0)/c_0$ accurate to three places. The result is shown graphically in Fig. 4.1. Since for $\sqrt{3}\eta_0 > 3$, the result agrees closely with that determined by the ray theory (reflection of a plane wave by a flat rigid surface at an oblique angle), the series-eigenfunction method provides in essence the answer to the stress concentration factor at the apex of the parabola for $1.5 < \sqrt{3}\eta_0 < 3$. This portion of the curve is marked exact, as the accuracy can be improved by increasing the size of the truncated matrix.

4.2. Perturbation Solution

The missing data near $\sqrt{3}\eta_0 = 0$ in the previous solution is retrieved by applying the perturbation technique discussed in Section 4, Chapter II. The incident wave is approximated by

$$\psi(i) \approx \psi(i)(0) + e\psi(i)(1), \quad \psi(i) = 0, \quad (4.10)$$

with

$$\psi(i)(0) = \varphi_0 e^{ikx} = \varphi_0 D_0(\lambda\eta)D_0(\lambda\xi),$$

and

$$\psi(i)(1) = ikx\varphi_0 e^{ikx},$$

and

$$k^2 = (a^2 + b^2)/2 = a^2/(1-2e), \quad \lambda = i\lambda, \quad \gamma = \sqrt{-2ik}.$$ 

The boundary conditions for various orders of displacements are

$$u^{(0)} = 0, \quad u^{(1)} = 0 \quad \text{at} \quad \eta = \eta_0.$$

It is most convenient to find the Cartesian components of the
displacement in the \(0^{th}\) order solution directly. We thus seek the scattered waves \(u_x^{(0)}(s)\) and \(u_y^{(0)}(s)\) satisfying the homogeneous Helmholtz equation:

\[
(\nabla^2 + k^2)u_x^{(0)}(s) = 0, \quad (\nabla^2 + k^2)u_y^{(0)}(s) = 0.
\]

The total wave is given by

\[
u_x^{(0)} = ik\psi \ e^{ikx} + u_x^{(0)}(s),
\]

\[
u_y^{(0)} = u_y^{(0)}(s),
\]

each of which should vanish at \(\eta = \eta_o\).

Since \(u_x^{(0)}\) and \(u_y^{(0)}\) are uncoupled in the field equation as well as in the boundary condition, the solution for each of them is the same as for a parallel incident \(SH\) wave striking a rigid parabolic inclusion (2.8). Thus we have

\[
u_x^{(0)} = ik\psi \frac{D_0(\lambda \eta_o) D_0(\lambda \xi)}{D_{-1}(\lambda \eta_o)} D_{-1}(\lambda \eta) D_0(\lambda \xi),
\]

\[
u_y^{(0)} = 0.
\]

Since

\[
\nabla \cdot \nu^{(0)} = -k^2 \psi^{(0)}, \quad (\nabla \times \nu^{(0)}) \cdot e_z = k^2 \psi^{(0)},
\]

where \(e_z\) is a unit vector in the \(z\)-direction, the corresponding \(0^{th}\) order potentials are
\[ \varphi^{(0)} = -i \frac{A_0 \eta}{\lambda (\eta^2 + \lambda^2)} D_0(\lambda \eta) D_0(\lambda \xi), \]  

\[ \psi^{(0)} = -\frac{A_0 \xi}{\lambda (\xi^2 + \lambda^2)} D_0(\lambda \eta) D_0(\lambda \xi), \]  

(4.12)

where \( A_0 = 2 \varphi D_0(\overline{\eta \eta \rho})/D_{-1}(\eta \rho). \) Because the terms with the coefficient \( A_0 \) are of the order \( r^{-\frac{1}{3}} \), there is a singularity in the potential as well as in the stresses — see Eq. (4.5) — near the tip \((r \to 0)\) if \( \eta_o = 0. \) The displacements are regular everywhere however, even for a degenerate parabola. The displacement components in parabolic coordinates can be computed by

\[ u_{\xi} = j^{-2}(\xi u_x + \eta u_y), \]

\[ u_{\eta} = j^{-2}(-\xi u_x + \eta u_y). \]

Unlike the case of a semi-infinite strip, the first order particular solution in the form of displacement components (II-4.17) does not satisfy the boundary conditions at \( \eta = \eta_o (\eta_o \neq 0) \). Hence the displacements, if expressed in terms of Cartesian coordinates, are coupled in the first order equations as well as in the boundary conditions. For such a case, it is easier to go back to the displacement-potential formulation in seeking the first order solution.

From (II-4.11) and (II-4.12) we find

\[ \varphi^{(1)} = \varphi^{(1)}_0 + \left[ A_0 \eta / 2 \lambda (\xi^2 + \eta^2) \right] D_0(\lambda \eta) D_0(\lambda \xi) - \frac{i}{2} (\xi^2 - \eta^2) u_x^{(0)}, \]

\[ \psi^{(1)} = \psi^{(1)}_0 + \left[ A_0 \xi / 2 \lambda (\xi^2 + \eta^2) \right] D_0(\lambda \eta) D_0(\lambda \xi) + \frac{\lambda \xi}{2} A_0 D_0(\lambda \eta) \overline{D_0(\lambda \xi)}, \]

(4.13)
where the operation

\[ 2r \cdot \nabla = \xi (\partial / \partial \xi) + \eta (\partial / \partial \eta) , \]

has been utilized. These solutions can be simplified since the second terms on the right-hand side of each equation satisfy a homogeneous Helmholtz equation and can be absorbed into the complementary solutions \( \varphi^{(1)}_{o} \) or \( \psi^{(1)}_{o} \). Thus

\[ \varphi^{(1)} = \varphi^{(1)}_{c} - \frac{1}{2} (\xi^2 - \eta^2) u^{(0)}_{x} , \]

\[ \psi^{(1)} = \psi^{(1)}_{c} + \frac{1}{2} \alpha_{0} \frac{D_{0} (\lambda \eta) D_{1} (\lambda \xi) - D_{1} (\lambda \eta) D_{0} (\lambda \xi)}{\lambda^2 - 1} . \]

Note that the first order incident wave \( (4.10) \) is contained in the term \(- \frac{1}{2} (\xi^2 - \eta^2) u^{(0)}_{x} = - \frac{1}{2} (\xi^2 - \eta^2) u^{(0)}_{x} \), and \( \varphi^{(1)}_{c} \) is to be an outgoing scattered wave. Also \( \psi^{(1)}_{c} \) is comprised only of outgoing waves because there is no incident shear wave in this problem. Using \( (4.14) \) as a guide, we assume

\[ \varphi^{(1)}_{c} = \{ A_{1} \xi / \lambda (\xi_{x} + \eta^2) \} \frac{D_{0} (\lambda \eta) D_{1} (\lambda \xi)}{\lambda^2 - 1} + \sum_{n=0}^{\infty} \alpha_{n} \frac{D_{n} (\lambda \eta) D_{n+1} (\lambda \xi)}{\lambda^2 - 1} , \]

\[ \psi^{(1)}_{c} = \{ -A_{1} \xi / \lambda (\xi_{x} + \eta^2) \} \frac{D_{0} (\lambda \eta) D_{1} (\lambda \xi)}{\lambda^2 - 1} + \sum_{n=0}^{\infty} \beta_{n} \frac{D_{n} (\lambda \eta) D_{n+1} (\lambda \xi)}{\lambda^2 - 1} , \]

where \( A_{1} \), \( \alpha_{n} \), and \( \beta_{n} \) are undetermined coefficients.

The infinite series portions are analogous to \( (4.2) \); except here, the wave numbers in \( \varphi^{(1)}_{c} \) and \( \psi^{(1)}_{c} \) are the same.

The first order displacements are
\[ u_{\xi}^{(1)} = \frac{1}{j} \left[ \frac{\partial \phi_{\xi}^{(1)}}{\partial \xi} + \frac{\partial \psi_{\xi}^{(1)}}{\partial \eta} - \frac{1}{8} \lambda^2 A_0 D_0(\lambda \eta) D_1(\lambda \xi) \right], \]

\[ u_{\eta}^{(1)} = \frac{1}{j} \left[ \frac{\partial \phi_{\eta}^{(1)}}{\partial \eta} - \frac{\partial \psi_{\eta}^{(1)}}{\partial \xi} + \frac{1}{4} \lambda A_0 D_0(\lambda \eta) D_2(\lambda \xi) - \frac{1}{8} \lambda^3 A_0 D_0(\lambda \eta) D_0(\lambda \xi) \right]. \]

In accordance with the boundary conditions \( u^{(1)} = 0 \), the expressions inside the brackets must vanish at \( \eta = \eta_0 \). Applying the orthogonality conditions for the Weber functions \( D_n(\lambda \xi) \), we obtain from the boundary conditions:

\[ a_0 = -2 \left[ (\lambda \eta_0 /4) + D_{-2}(\lambda \eta_0) / D_{-2}(\lambda \eta_0) \right] [D_{-1}(\lambda \eta_0)]^{-2}, \]

\[ \beta_1 = -[D_{-1}(\lambda \eta_0) D_{-2}(\lambda \eta_0)]^{-1}, \]

\[ A_1 = -A_0 \left\{ \left(\frac{\lambda \eta_0}{2}\right)^2 - \frac{1}{2} + \frac{D'_{-1}(\lambda \eta_0)}{D_{-1}(\lambda \eta_0)} \left[ \frac{\lambda \eta_0}{2} + 2 \frac{D'_{-2}(\lambda \eta_0)}{D_{-2}(\lambda \eta_0)} \right] \right\}, \quad (4.16) \]

\[ a_1 = \beta_0 = 0, \]

\[ a_n = \beta_n = 0, \quad n > 1. \]

This completes the first order perturbation.

The normalized stress \( \sigma_{\eta \eta} \) at \( \xi = 0, \eta = \eta_0 \), which is calculated according to

\[ \sigma_{\xi \xi} = \sigma_{\eta \eta} + \epsilon \sigma_{\xi \xi}^{(1)}, \]

is also shown in Fig. 4.1. The stress is normalized by the same factor \( \sigma_0 = -\mu \varepsilon_0^2 \sigma_0 \), and its value agrees closely with that obtained by
the series-eigenfunction over the limited range of overlap. The perturbation results for \( \sigma_{n} \) increase rapidly without bound beyond \( \sqrt{\beta \eta_{o}} > 1.5 \), so they are not valid for large \( \sqrt{\beta \eta_{o}} \). But the answer given here is in closed form, which can be used to calculate the stresses near \( \sqrt{\beta \eta_{o}} = 0 \).

5. HOMOGENEOUS SOLUTIONS

BY APPLYING THE FOURIER SYNTHESIS to steady state solutions derived in the previous sections, transient solutions for diffraction by a parabolic cylinder may be derived. There is, however, another method — the use of homogeneous functions — which can be applied directly to the diffraction by a half-plane (parabolic cylinder with zero focal length) without recourse to the Fourier synthesis and the steady state solutions.

A function \( f(x_{1},x_{2},...,x_{n}) \) is said to be homogeneous of degree \( m \) if it may be expressed in the form \( x_{1}^{m}f(x_{2}/x_{1},...,x_{n}/x_{1}) \). Homogeneous solutions of wave problems can be found if the data of the problem contain no characteristic length, or if the only characteristic length must be derived from a parameter to which the solution is proportional.

The method has been proved a powerful tool in supersonic aerofoil theory, where it is known as the method of conical flow, originally developed by Busemann in 1935 (see Chapter 7 of Ward, Ref. 5.1). For scalar wave problems, this method has been applied to the diffractions of a pulse by a half plane or a wedge by Davis, (5.2) Keller and Blank, (5.3) Miles, (5.4,5.5) and Filippov. (5.6) Filippov (5.7) and Miles (5.8) ex-
tended this method to the diffraction of P and SV waves.

When applying this method, the wave equation in two-dimensional space which contains a total of three independent variables is first reduced by a similarity transformation to either a Laplace equation in two dimensions or the wave equation in one dimension, both having only two independent variables. General solutions are then constructed with the aid of the theory of analytic functions or characteristics.

5.1. Diffraction of an SH Pulse by a Semi-Infinite Plane

For an incident SH wave described by

\[ u^{(i)}_z(x, y, t) = H[c_8^2 t - (x \cos \theta_0 + y \sin \theta_0)] \]

\[ = H[c_8^2 t - r \cos(\theta - \theta_0)] , \]  

(5.1)

there is a discontinuity (a jump of unity) of displacement at the wave surface defined by \( c_8^2 t - (x \cos \theta_0 + y \sin \theta_0) = \text{constant}. \) The incident wave is diffracted by a semi-infinite plane \((y = 0, \ x > 0)\), and the scattered wave \( u^{(s)}_z \) satisfies the wave equation

\[ \nabla^2 u_z = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u_z = \frac{1}{c_8^2} \frac{\partial^2 u_z}{\partial t^2} . \]

(5.2)

The total wave is given by

\[ u_z = u^{(i)}_z + u^{(s)}_z , \]

(5.3)

and the boundary conditions for a rigid-fixed strip or a crack are respectively:

...
\[ u_z = 0 \quad \text{or} \quad \sigma_{yy} = 0 \quad \text{at} \quad y = 0, \quad x > 0. \quad (5.4) \]

The initial conditions are

\[ u_z(r, \theta, 0) = \begin{cases} 
1, & r \cos (\theta - \theta_0) < 0, \\
0, & r \cos (\theta - \theta_0) > 0,
\end{cases} \quad (5.5) \]

with the understanding that the incident wave front strikes the edge of the half plane, at \( t = 0 \). The initial velocity \( u_z \) is zero in the undisturbed region.

Because neither the geometry of the half plane nor the boundary and initial conditions contain a characteristic length, the solution for \( u_z \) must be homogeneous functions in radial coordinates \( r \) and \( \sigma_{st} \).

Choosing \( \theta \) and

\[ \zeta = \frac{\sigma_{st} t}{r} \]

as the dimensionless, homogeneous coordinates, we can transform the wave equation (5.2) to

\[ (\zeta^2 - 1) \frac{\partial^2 u_z}{\partial \zeta^2} + \zeta \frac{\partial u_z}{\partial \zeta} + \frac{\partial^2 u_z}{\partial \theta^2} = 0, \quad (5.6) \]

(note that \( - r \partial / \partial r = \zeta \partial / \partial \zeta = t \partial / \partial t \)).

The equation above is either elliptic (\( \zeta > 1 \)) or hyperbolic (\( \zeta < 1 \)). If we regard \( \zeta^{-1} \) and \( \theta \) as polar coordinates (thereby referring all length to \( \sigma_{st} \)), the elliptic and hyperbolic domains correspond to the interior \( (r < \sigma_{st}) \) and exterior \( (r > \sigma_{st}) \) of the unit circle, \( r = \sigma_{st} \). The circle \( \zeta = 1 \) (or \( r = \sigma_{st} \)) represents a singular
wave front, across which the function \( u_z \) is continuous but its \( \zeta \)-derivative may be discontinuous. (5.3)

For \( \zeta > 1 \), we further set

\[
\sigma = \cosh^{-1} \zeta = \ln [\zeta + \sqrt{\zeta^2 - 1}]
\]

and reduce Eq. (5.6) to the Laplace equation

\[
\frac{\partial^2 u_z}{\partial \sigma^2} + \frac{\partial^2 u_z}{\partial \theta^2} = 0.
\]

This has a general solution

\[
u_z = \text{Re} \, G(\beta), \quad \zeta > 1, \quad \beta = \theta + i\sigma = \theta + i \cosh^{-1} \zeta.
\]

Similarly, for \( \zeta < 1 \), we reduce Eq. (5.6) to the one-dimensional wave equation

\[
\frac{\partial^2 u_z}{\partial \sigma^2} - \frac{\partial^2 u_z}{\partial \theta^2} = 0
\]

through the transformation

\[
\sigma = \cos^{-1} \zeta.
\]

This has a general solution

\[
u_z = G_+(\beta_+) + G_-(\beta_-), \quad (5.8)
\]
with

\[ \beta_\pm = \theta \pm \cos^{-1} \zeta \]

and \( G_\pm \) being arbitrary functions of the characteristic variables \( \beta_\pm \).

To determine the functions \( G(\beta) \) or \( G_\pm(\beta_\pm) \) we shall adopt the following procedure:

(a) Pose the solutions of the diffraction problem in the form of (5.7) and (5.8).

(b) Convert the initial conditions to conditions on the wave fronts.

(c) Express the coordinates that vary on the boundaries in terms of \( \beta \) and transform the boundary conditions accordingly.

(d) Determine the functions \( G \) or \( G_\pm \) that satisfy the boundary conditions and conditions at the wave fronts.

Fig. 5.1. Diffraction of an SH Pulse by a Half-Plane
HOMOGENEOUS SOLUTIONS

Figure 5.1 shows the half plane obstacle \((y = 0, x > 0)\) and the incident wave front at \(t > 0\). As soon as the incident wave front touches the tip of the half-plane at \(t = 0\), diffraction and reflection take place. We divide the entire \(x-y\) plane by the circle \(\zeta = 1\) \((c_\theta t = r)\) and the two characteristics \(\beta_- = 2\pi - \theta_o\) and \(\beta_+ = \theta_o\), the latter coinciding with the incident wave front. The functions \(G(\beta)\) or \(G_+(\beta_+)\) will be determined separately in each of the following zones:

I Incident wave zone.

II Specular reflected wave zone \((\zeta < 1)\).

III Shadow zone \((\zeta < 1)\).

IV Scattered wave zone \((\zeta > 1)\).

Outside Zone IV, the governing equation is hyperbolic, and inside it, elliptic. The two cases with different boundary conditions will now be treated separately. For convenience, the incident wave is represented in various zones as

\[
\psi(i) = \begin{cases} 
H(\theta - \theta_o - \cos^{-1}\zeta), & 0 < \theta < \theta_o, \\
H(\theta - \theta_o - \cos^{-1}\zeta), & \theta_o < \theta < \pi + \theta_o, \\
H(2\pi - \theta + \theta_o - \cos^{-1}\zeta), & \pi + \theta_o < \theta < 2\pi,
\end{cases} \quad (5.9)
\]

which is equivalent to \((5.1)\) when \(0 \leq \cos^{-1}\zeta \leq \pi/2\).

(A) Clamped Half-Plane

As dictated by the initial condition, there is no disturbance other than the incident wave in Zone I. Thus \(\psi_g^{(s)} = 0\) and \(\psi_s = \psi_s^{(i)} = 1\) in Zone I.
In Zones II and III, the equation is hyperbolic and the scattered wave has the form given by (5.8). Substituting it into the boundary conditions then yields

$$u_z = u_z^{(i)} + G_z^{(s)}(\theta) = 0 \quad \text{at} \quad \theta = 0 \text{ and } 2\pi. \quad (5.10)$$

Zone II is bounded by a negative characteristic and $\theta$ is limited between $2\pi - \theta_0$ and $2\pi$. Thus

$$u_z = H(\theta - \cos^{-1} \zeta) + G_z(2\pi - \cos^{-1} \zeta) = 0 \quad \text{at} \quad \theta = 2\pi,$$

and

$$G_z^{(s)}(\theta) = -H(\theta + \theta_0 - 2\pi - \cos^{-1} \zeta), \quad 2\pi - \theta_0 < \theta \leq 2\pi$$

$$= -H[\zeta - \cos(\theta - (2\pi - \theta_0))]. \quad (5.11)$$

We note that the expression in the right-hand side represents a step plane wave propagating at an angle $2\pi - \theta_0$ with the $x$-axis. Hence the total disturbance $u_z$ in Zone II is zero. This is expected, because in the case of the secular reflection by a rigid plane, an incident SH wave

$$H[\sigma_{xy}^t - (x \cos \theta_0 + y \sin \theta_0)]$$

is reflected as

$$-H[\sigma_{xy}^r - (x \cos \theta_0 - y \sin \theta_0)].$$

In the region where these two waves overlap, such as in Zone II, the resultant displacement vanishes.
In Zone III, the positive characteristics $\beta_+$ prevail and
$0 < \theta < \theta_0$. From Eq. (5.10) we deduce

$$u_2 = H(\theta_0 - \cos^{-1} \zeta) + G_+(\cos^{-1} \zeta) = 0 \quad \text{at} \quad \theta = 0.$$ 

The function $G_+$ is then determined as

$$G_+(\theta_+) = -H[\theta_0 - (\theta + \cos^{-1} \zeta)], \quad 0 < \theta < \theta_0,$$  

which cancels the incident wave, giving rise to zero total displacement in Zone III. Again this is expected because the incident wave is blocked by the half-plane and the wave diffracted by the edge of the half-plane has not yet spread into the region outside the circular front $\zeta = 1$ at this moment.

So far the solutions for Zones I, II, and III are so simple that it hardly justifies the elaborate transformation in deriving a homogeneous solution. Actually, the value of this method lies in the calculation of the scattered wave in Zone IV for which the governing equation is elliptic. Since the solutions for $u_2$ must be continuous across the wave front $\zeta = 1$, the problem is now reduced to finding an analytic function $u_2(\theta, s) = \text{Re} \ G(\theta + is)$ inside the circle.

The transformation used in (5.7) maps Zone IV together with the two sides of the half-plane onto the $z$ $(\theta + is)$ plane as a semi-infinite rectangular strip — Fig. 5.2a — the circular wave front corresponding to $s = 0$. The half-plane corresponds to the sides $\theta = 0$ and $2\pi$ and its edge ($r = 0$) becomes the point at infinity. Because the displacement should remain finite at the edge, function $G(\theta + is)$ must be bounded as $s \to \infty$. From the continuation of the
values for \( u_z \) in Zones I, II, and III to the circular wave front \( \zeta = 1 \), the following values at \( s = 0 \) are obtained:

\[
    u_z(\theta, 0) = \begin{cases} 
    0, & 0 < \theta < \theta_0, \\
    1, & \theta_0 < \theta < 2\pi - \theta_0, \\
    0, & 2\pi - \theta_0 < \theta < 2\pi.
    \end{cases} \quad (5.13a)
\]

In addition, the rigid boundary conditions at the half-plane prescribe the value for \( u_z \) at \( \theta = 0 \) and \( 2\pi \):

\[
    u_z(0, s) = u_z(2\pi, s) = 0. \quad (5.13b)
\]

With its value being given over its entire boundary, the analytic function inside the strip can be determined uniquely.

To find the analytic function, the interior of the strip in the \( \theta \)-plane is further mapped into the upper half of the \( z \)-plane by the transformation

\[
    z = -\cos \frac{i \theta}{2}, \quad z = x + iy.
\]
The boundary of the rectangular-shaped strip is mapped to coincide with the \( x \)-axis of the \( z \)-plane — Fig. 5.2b — and the points \( \beta = 0, 2\pi, i\infty \) become \( z = -1, +1, - \) respectively. Moreover, the boundary conditions (5.13) are transformed to

\[
\begin{align*}
    u_z(x + i0) &= \begin{cases} 
      0, & |z| > \cos \frac{\beta}{2} \\
      1, & |z| < \cos \frac{\beta}{2}
    \end{cases},
\end{align*}
\]

It is known that the analytic function \( u_z(x + iy) \) can be determined from its value at \( x + i0 \) by the following integral:

\[
u_z(z) = \Re \left[ \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{u_z(\xi + i0)}{\xi - z} d\xi \right]. \tag{5.14}\]

Carrying out the integration, and changing the variable back to \( \beta \), we obtain

\[
u_z(z) = \Re \left[ \frac{1}{i\pi} \ln \frac{z - \cos \frac{\beta}{2}}{z + \cos \frac{\beta}{2}} \right].
\]
\[ u_n(0) = \text{Re} \left[ \frac{1}{2\pi} \ln \frac{\cos \frac{\theta}{2} + \cos \frac{\theta}{2}}{\cos \frac{\theta}{4} - \cos \frac{\theta}{4}} \right], \]

\[ = \text{Re} \left\{ \frac{1}{2\pi} \left[ \ln \tan \left( \frac{\theta}{4} \right) + \ln \tan \left( \frac{\theta}{4} - \frac{\pi}{4} \right) \right] \right\}, \]

and

\[ u_2(0, \xi) = \frac{1}{2\pi} \left[ \tan^{-1} \frac{\sin \frac{\xi}{2} \sin \frac{\theta}{2}}{\sinh \left( \frac{\xi}{2} \right)} + \tan^{-1} \frac{\sin \left( \frac{\xi}{2} - \frac{\theta}{2} \right)}{\sinh \left( \frac{\xi}{2} \right)} \right]. \]

Finally, the diffracted wave in Zone IV is expressed by

\[ u_2(\varphi, \theta, 4) = \frac{1}{\pi} \left[ \tan^{-1} \frac{\sqrt{2} \sin \frac{\varphi}{2} \sin \frac{\theta}{2}}{(\varphi - \theta)} + \tan^{-1} \frac{\sqrt{2} \sin \frac{\varphi}{2} \sin \frac{\theta}{2}}{(\varphi - \theta - 2\pi)} \right], \]

provided that the principal value is assigned to the arc tangent, i.e., \(-\pi/2 < \tan^{-1} \xi < \pi/2\).

(B) Semi-Infinite Crack

Waves in the various zones as diffracted by a permanently opened crack along the \( x \) axis — Fig. 5.1 — can be determined analogously. The boundary condition is given by the second equation of (5.4), which in polar coordinates can be expressed as

\[ \sigma_{yx} = \mu \left[ \sin \varphi \frac{3m}{\rho \rho^2} + \cos \varphi \frac{3m}{\rho \rho^2} \right] = 0 \quad \text{at} \quad \varphi = 0 \quad \text{and} \quad 2\pi. \]

For \( m \neq 0 \), it is equivalent to
\[ \frac{\partial u}{\partial \theta} \bigg|_{\theta=0,2\pi} = 0. \tag{5.16} \]

As in the case of a rigid half plane, the total displacement in Zone I due to the incident step wave (5.9) is +1. In Zone III, the scattered wave \( u_{\gamma}^{(s)} \) is determined by the following equation:

\[ \frac{\partial u}{\partial \theta} \bigg|_{\theta=0} = -\delta(\theta - \cos^{-1} \zeta) + G_+'(\cos^{-1} \zeta) = 0. \]

Thus, for \( \theta > \theta > 0, \)

\[ G_+'(\theta) = \delta(\theta - \theta_+), \quad \theta_+ = \theta + \cos^{-1} \zeta, \tag{5.17} \]

and

\[ G_+(\theta) = -\delta(\theta - \theta_+). \]

It shows again that the scattered wave annuls the incident wave in the shadow zone. By a similar calculation, we can show that the specular reflected wave in Zone II is

\[ u_{\gamma}^{(s)} = G_- = \frac{-i}{2\pi} \left( \cos^{-1} \zeta - \cos^{-1} \zeta \right), \quad 2\pi - \theta < \theta < 2\pi, \tag{5.18} \]

and the total disturbance there is

\[ u_{\gamma}(\theta, \zeta) = 2. \]

For the evaluation of the scattered waves in the elliptic Zone IV, it is convenient to separate the scattered wave \( u^{(s)} \) from the
total wave. By continuing the previously calculated values for \( \nu_z(\theta, \xi) \) in Zones I, II, and III to the circumference of Zone IV, the following boundary conditions are established:

\[
\nu_z(\theta, \xi) = \begin{cases} 
-1, & 0 < \theta < \theta_0 \\
0, & \theta_0 < \theta < 2\pi - \theta_0 \\
1, & 2\pi - \theta_0 < \theta < 2\pi,
\end{cases}
\]  

(5.19)

and

\[
\frac{\partial \nu_z}{\partial \theta} = -\frac{\partial \nu_z}{\partial \xi} = 0 \quad \text{at} \quad \theta = 0 \text{ and } 2\pi \quad (\xi > 1).
\]

The harmonic function \( \nu_z(\theta) \) within the strip bounded by \( \theta = 0, 2\pi \) and \( s = 0 \) -- Fig. 5.2a -- can then be constructed by applying conformal mapping, as in the previous case.

A simpler approach is to construct the following series solution which satisfies the Laplace equation for Zone IV:

\[
\nu_z(\theta, 0) = \sum_{n=0}^{\infty} a_n \theta^{-(n+2)} \cos \frac{n\pi}{2}\xi.
\]

This obviously satisfies the boundary conditions at \( \theta = 0, 2\pi \) and remains bounded as \( \theta \to \infty \). From the orthogonality conditions of \( \cos \frac{n\pi}{2}\xi \) over the interval \( 0 \leq \xi < 2\pi \) and the boundary values \( \nu_z(\theta, \xi) \), the coefficients \( a_n \) are fixed as

\[
a_n = \frac{2}{\pi} \left( \cos \frac{n\pi}{2} - 1 \right) \sin \frac{n\pi}{2} \phi.
\]

Hence the scattered wave in Zone IV is given by
\[ u_2(\varepsilon, \phi) = -\frac{2}{\pi} \text{Im} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n} \left[ e^{\frac{i}{2}(\phi - \phi_0)} \right] - \frac{z_1}{2} \sin \frac{1}{2} \phi_0 \sin \frac{1}{2} \phi_0. \] (5.20)

The series above can be summed by first rewriting it in the following forms:

\[ u_2(\varepsilon, \phi) = -\frac{2}{\pi} \text{Im} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n} \left[ e^{\frac{i}{2}(\phi - \phi_0)} \right] - \frac{z_1}{2} \sin \frac{1}{2} \phi_0 \sin \frac{1}{2} \phi_0, \]

where \( z_1 = \exp \left(-\frac{1}{2}i(\phi - \phi_0)\right) \), \( z_2 = \exp \left(-\frac{1}{2}i(\phi - \phi_0)\right) \), and \( |z_1, z_2|^2 \leq 1 \). Noting that each series is exactly the expansion of \( \tanh^{-1} z \), one thus derives a closed form solution for the scattered waves in Zone IV:

\[ u_2(\varepsilon, \phi) = -\frac{2}{\pi} \text{Im} \left[ \tanh^{-1} z_1 - \tanh^{-1} z_2 \right] \]

\[ = -\frac{1}{\pi} \left[ \tan^{-1} \frac{\frac{1}{2}(\phi + \phi_0)}{\sinh (\phi_0/2)} - \tan^{-1} \frac{\frac{1}{2}(\phi - \phi_0)}{\sinh (\phi_0/2)} \right]. \] (5.21)

In terms of \( r, \theta, \phi \), the total wave in Zone IV is

\[ u_2(r, \theta, \phi) = 1 - \frac{1}{\pi} \left[ \tan^{-1} \frac{\frac{1}{2}(\theta + \phi_0)}{(\theta + \phi_0/2)} - \tan^{-1} \frac{\frac{1}{2}(\theta - \phi_0)}{(\theta + \phi_0/2)} \right], \] (5.22)

with principal values being assigned to the arctangent.

Figure 5.3 shows the contour lines of constant displacements as reported by Davis for a normally incident step wave \( (\phi_0 = \pi/2) \).
Fig. 5.3. Displacement Field \((u_{x})\) for a Normally Incident Step Wave as Diffracted by a Semi-Infinite Crack

We note that the displacement is discontinuous across the crack, but it remains finite at its tip. The corresponding stresses in Zone IV are

\[
\sigma_{\theta} = -\frac{\sqrt{\alpha_{c} t/r - 1}}{\sqrt{2\pi}} \left[ \frac{\cos \frac{\alpha}{2}(\theta + \theta_0)}{\alpha_{c} t - r \cos (\theta + \theta_0)} - \frac{\cos \frac{\alpha}{2}(\theta - \theta_0)}{\alpha_{c} t - r \cos (\theta - \theta_0)} \right],
\]

\[
\sigma_{\theta} = -\frac{\sqrt{\alpha_{c} t/r}}{\sqrt{2\pi\nu\alpha_{c}^2 t/r - 1}} \left[ \frac{\sin \frac{\alpha}{2}(\theta + \theta_0)}{\alpha_{c} t - r \cos (\theta + \theta_0)} - \frac{\sin \frac{\alpha}{2}(\theta - \theta_0)}{\alpha_{c} t - r \cos (\theta - \theta_0)} \right].
\]

Near the tip of the crack, they are singular in the order of \(r^{-\frac{\alpha}{2}}\) as \(r \to 0\).
5.2. Diffraction of a P Pulse by a Semi-Infinite Plane

In Section 3, the scattering of steady state waves by a rigid half-plane was analyzed by applying the perturbation method. Although the perturbation technique is effective in determining the behavior of the scattered waves near the tip of the plane, the overall diffraction phenomenon is obscured in this approach, where a common wave speed is assigned in each order of perturbation solution. The use of homogeneous solutions will supplement the missing information. It was mentioned that Miles applied this method to scattering of elastic waves and obtained solutions for both incident P and SV waves. In his solution the tractions are assumed to vanish at all times along the semi-infinite plane, thus it models a permanently-opened crack. Papadopoulos later re-examined the case of a crack which may be closed by compressive stresses arising from the scattered wave. (5.9, 5.10)

As in the SH case, we introduce two variables,

$$
\xi = c_p t/r, \quad \eta = c_s s/r,
$$

which together with the angle \( \theta \) are the dimensionless homogeneous coordinates. The two wave equations for the potentials \( \psi(r, \theta, t) \) and \( \phi(r, s, t) \) are then transformed to

\[
(\xi^2 - 1) \frac{\partial^2 \psi}{\partial \xi^2} + \xi \frac{\partial \psi}{\partial \xi} + \frac{\partial^2 \psi}{\partial \eta^2} = 0,
\]

(5.24)

\[
(\eta^2 - 1) \frac{\partial^2 \phi}{\partial \eta^2} + \eta \frac{\partial \phi}{\partial \eta} + \frac{\partial^2 \phi}{\partial \theta^2} = 0.
\]

The displacement-potential and stress-potential relations are trans-
formed to:

\[ u_n = \frac{1}{n} \left( -\xi \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta} \right), \]  
(5.25)

\[ u_0 = \frac{1}{n} \left( \frac{\partial \psi}{\partial \xi} + n \frac{\partial \phi}{\partial \eta} \right); \]

\[ c_{rr} = 2(\alpha + \nu) \frac{1}{\eta^2} \frac{\partial^2 \phi}{\partial \eta^2} - c_{00}, \]

\[ c_{00} = \frac{\nu}{\eta^2} \frac{3}{\alpha \eta} \left( 1 - 2\eta^2 \right) \frac{\partial \psi}{\partial \eta} + 2\eta \frac{\partial \phi}{\partial \eta} \right], \]  
(5.26)

\[ c_{r\eta} = \frac{\nu}{\eta^2} \frac{3}{\alpha \eta} \left[ -2\eta \frac{\partial \phi}{\partial \eta} + (1 - 2\eta^2) \frac{\partial \psi}{\partial \eta} \right]. \]

The equation for \( \psi(\xi, \eta) \) may be transformed further to a Laplace equation or to the wave equation according to \( \xi > 1 \) or \( \xi < 1 \) respectively. It then has a general solution in terms of an arbitrary analytic function \( F(\alpha) \):

\[ \psi = \text{Re} \, F(\alpha), \]  
(5.27)

with

\[ \alpha = \xi + i \cosh^{-1} \xi, \quad \text{if} \ \xi > 1, \]

\[ \alpha = \alpha_\xi = 0 + i \cos^{-1} \xi, \quad \text{if} \ \xi < 1. \]

Similarly, we use an arbitrary analytic function \( G(\beta) \) to represent the solution for \( \psi(\theta, \eta) \) as
\[ \psi(\theta, \varphi) = \text{Re} \, \mathcal{G}(\varphi), \]  

with

\[ \varphi = \theta + i \cos^{-1} \eta, \quad \eta > 1, \]

and

\[ \beta = \beta_\perp = \theta + i \cos^{-1} \eta, \quad \eta < 1. \]

Substituting the results above into (5.25), we obtain

\[ u_\| = r^{-1} \text{Re} \left[ -i \xi (\xi^2 - 1)^{-\frac{1}{2}} p'(\alpha) + \mathcal{G}'(\beta) \right], \]

\[ u_0 = r^{-1} \text{Re} \left[ p'(\alpha) + i \eta (\eta^2 - 1)^{-\frac{1}{2}} \mathcal{G}'(\beta) \right], \]  

for points in the elliptic domains. On the other hand, if \( \alpha \) is continued as \( \alpha_\perp \) in the hyperbolic domain, the factor \( i(\xi^2 - 1)^{-\frac{1}{2}} \) should be replaced by \( i(1 - \xi^2)^{-\frac{1}{2}} \), and similarly for \( \beta \).

The remaining discussion concerns the determination of the functions \( p'(\alpha) \) and \( \mathcal{G}(\beta) \) from the initial and boundary conditions for the case of an incident step P wave:

\[ \varphi^{(i)} = \mathcal{P} \left[ \alpha_{\|} \theta - (x \cos \theta_\perp + y \sin \theta_\perp) \right], \]

\[ \psi^{(i)} = 0. \]  

If the crack extends over the entire \( x \)-axis, the incident plane wave would be reflected as a P and an SV wave with angle \( \theta_\perp \) and \( \theta_\parallel \) respectively, where

\[ \cos \theta_\perp = k \cos \theta_\perp, \quad k = \frac{c_P}{c_\parallel}. \]
The semi-infinite crack will reflect the incident wave just the same, except that near the tip of the crack there will be additional scattered waves. Both P and SV waves are generated at the tip and the cylindrical wave fronts radiate from the tip \( (r = 0) \) with velocities \( c_p \) and \( c_s \) respectively. While the P wave front which is defined by \( r = c_p t (\xi = 1) \) moves along the surface of the crack, additional shear waves are excited in order to satisfy the boundary condition. According to Huygen's principle, at the instant \( t = t \), each of the shear waves may be regarded as a cylindrical wave having its center at \( r = c_p t \) and \( \theta = 0 \), and having a radius \( c_s (t - r) \) -- Fig. 5.4. The envelope of these waves forms the head wave or Mach envelope. A perpendicular to it makes an angle \( \theta = \cos^{-1} (c_s / c_p) \) with the line of the crack.

![Fig. 5.4. Mach Envelope of Shear Waves Excited by P Wave Front Moving Along \( \theta = 0 \)](image)

The incident P wave front \((\theta + \cos^{-1} \xi = \theta_1)\), the two specular reflected wave fronts \((\theta - \cos^{-1} \xi = 2\eta - \theta_1 \text{ and } \theta - \cos^{-1} \eta = 2\eta - \theta_2)\), the two cylindrical scattered wave fronts \((\xi = 1 \text{ and } \eta = 1)\), and the Mach envelopes \((\theta + \cos^{-1} \eta = \theta_1 \text{ and } \theta - \cos^{-1} \eta = 2\eta - \theta_1)\) divide the
entire region into the following six zones -- Fig. 5.5:

I Incident wave zone.

II$_1$ P wave zone of specular reflection.

II$_2$ SV wave zone of specular reflection.

III Shadow zone.

IV$_1$ P wave scattering zone ($\xi > 1$).

IV$_2$ SV wave scattering zone.

---

In Zone I, there is no other disturbance than the incident P wave $\varphi(\xi)$ (5.30). In Zone II$_1$, the total wave is a combination of $\varphi(\xi)$ and the specular P wave reflection given by

$$
\varphi^{(p)} = H_{11} B\left[(\xi - \cos^{-1} \xi) - (2\pi - \theta_1)\right].
$$

(5.31)

The specular SV wave reflection is
\[ \psi^{(n)} = R_{21} R_{11} [\theta - \cos^{-1} \nu] - (2\pi - \beta_2), \] (5.32)

where the reflection coefficients \( R_{11} \) and \( R_{21} \) are given by

\[ R_{11} = (-\nu^2 \cos^2 \theta_1 \sin 2\theta_1 \sin 2\theta_2) / U, \]
\[ R_{21} = 2 \sin 2\theta_1 \cos 2\theta_2 / U, \]
\[ U = \nu^2 \cos^2 \theta_2 + \sin 2\theta_1 \sin 2\theta_2. \]

These are the same as the reflection coefficients for plane harmonic waves impinging on a free surface. Thus in zone II, the total wave is made of two parts,

\[ \psi = \psi^{(i)} + \psi^{(n)}, \quad \psi = \psi^{(n)}. \] (5.33)

Without scattering by the edge of the crack, the region behind the crack and the wave normal passing through the origin would be in total tranquility. The edge diffracts part of the incident wave into that region, however, and thus only the area bounded by the circular arc \( \xi = 1 \), the surface of the crack \((x > c_n, \xi)\), and the incident wave front are in the shadow (Zone III). Mathematically, it is found that the scattered wave \( \psi^{(n)} \) constructed from the solutions of (5.27), (5.28), and the boundary conditions just cancels the incident wave \( \psi^{(i)} \), leaving no disturbance in Zone III.

Constructing the solutions for the scattering Zone IV is indeed rather difficult. The total wave is given by
\[ \varphi = \varphi^{(s)} + \varphi^{(c)}, \quad (5.34) \]

\[ \psi = \psi^{(c)}, \]

with \( \varphi^{(s)} = \text{Re} \, P(\alpha) \) and \( \varphi^{(c)} = \text{Re} \, G(\beta) \), as given in (5.27) and (5.28). For \( \varphi^{(c)} \) and \( \psi^{(c)} \), the zero disturbance initial conditions dictate that (a) \( \varphi^{(c)} = 0 \) \( \varphi = \varphi^{(c)} \) on that portion of the P wave scattering circle \( \xi = 1 \) intercepted between the incident wave front and the specularly reflected P wave front, and (b) \( \psi^{(c)} = 0 \) on the Mach envelope \( \theta + \cos^{-1} \eta = c_c \) and on the portion of the SV wave scattering circle \( (n = 1) \) intercepted between this Mach envelope and the specularly reflected SV wave front. Since \( \alpha = 0 \) on \( \xi = 1 \), and \( \beta = 0 \) on \( n = 1 \), these two initial conditions impose the following restrictions on \( P(\alpha) \) and \( G(\beta) \):

\[
\begin{align*}
\text{Re} \, P(\alpha) &= 0, & 0_1 < \alpha &< 2\pi - 0_1, \\
\text{Re} \, G(\beta) &= 0, & 0_c &< \beta < 2\pi - 0_2.
\end{align*}
\]

(5.35)

The boundary conditions require \( q_{\beta 0} \) and \( q_{\beta 0} \) to vanish on the upper and lower surfaces of the crack, which are defined by \( \theta = 0 \) and \( 2\pi \) respectively. Integrating (5.26) with respect to \( \eta \) (from \( t = 0 \) to an arbitrary line \( t \)) yields

\[
(1 - 2\eta^2) \frac{\partial \varphi}{\partial \eta} + 2\eta \frac{\partial \varphi}{\partial \theta} = 0, \quad \text{at } \theta = 0, 2\pi.
\]

(5.36)

\[
-2\eta \frac{\partial \psi}{\partial \theta} + (1 - 2\eta^2) \frac{\partial \psi}{\partial \eta} = 0,
\]
At the edge of the crack, the displacements should remain bounded and have the order of magnitude

$$\kappa_{\gamma,0} = O(r^3) \quad \text{as} \ r \to 0.$$  \hspace{1cm} (5.37)

Miller, Ref. 5.8, found the solutions satisfying Eqs. (5.35), (5.36), and (5.37) as

$$F'(a) = \frac{C_{11}\sin \left(\alpha/2\right)\left(\gamma^2 \cos^2 \alpha - 1\right) + C_{21}\sin 2\alpha\left(1 - \gamma \cos \alpha\right)/2}{\eta \gamma \left(\eta_R - \cos \alpha\right) \left(\cos \beta - \cos \theta_1\right)}.$$  \hspace{1cm} (5.38)

$$C_0(\cos \theta_2)
= L(- \cos \theta_2)/(1 - \gamma^2)(\eta_R + \cos \theta_2),$$

$$\gamma = \kappa^{-1} = a_e/a_p,$$

and \(L\) is defined by the integral

$$L(x) = \exp \left[ \frac{1}{c} \int_0^1 \frac{\lambda(t)}{\int_0^t \frac{d\xi}{\sqrt{1 - \eta}} \right],$$

$$\chi(x) = \tan^{-1} \left[ \frac{x^2 - \frac{\gamma^2}{2} - \frac{\gamma^2}{2}}{(x^2 - \frac{\gamma^2}{2})^2} \right], \quad 0 \leq x \leq \frac{\pi}{2}.$$
The constants $\xi_R$ and $\eta_R$ are given by

$$
\xi_R = \frac{c_p}{c_R}, \quad \eta_R = \frac{c_\theta}{c_R},
$$

where $c_R$ is the speed for Rayleigh surface wave over a half-space, which is a root of the transcendental equation

$$
D \equiv (2\gamma^2 \cos^2 \alpha - 1)^2 + 4\gamma^3 \sin \alpha \cos \alpha \sqrt{1 - \gamma^2 \cos^2 \alpha} = 0.
$$

(5.39)

Numerical results for the functions $L(x)$ and $\chi(x)$ are given in Miles' paper for $\kappa = \gamma^{-1} = \sqrt{3}$. Substitution of $F'(\alpha)$ and $S'(\beta)$ in (5.29) completes the solutions for the scattered waves in Zones $IV_1$ and $IV_2$.

Additional results for displacements at the scattered wave fronts $\xi = 1$ and $\eta = 1$ (and the Mach envelopes) and on the surfaces of the crack are given in Miles' paper, where the case of a rigid half-plane is also discussed. Diffraction of a P wave by a wedge has been studied recently by Knopoff. (5.11)
CHAPTER V REFERENCES


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CHAPTER VI

SPHERICAL INCLUSION PROBLEM

THE PROBLEMS we have been discussing in the preceding three chapters are two-dimensional ones. They can be considered either as plane stress problems or as plane strain problems, through the choice of an appropriate set of elastic constants. In this final chapter we will discuss some of the simpler elastic wave scattering and dynamic response problems in three dimensions.

Due to their geometry, spherical inclusion problems are perhaps the easiest three-dimensional problems to treat, hence, they are treated extensively in the literature. The earliest investigation of the scattering of sound waves by a rigid spherical obstacle was done by Rayleigh in 1872. The general treatment of scattering of elastic waves by spherical obstacles, however, was not accomplished until 1927 by Sezawa. After Sezawa's work there appears to have been very little done until the late 1950's and mid-1960's, when the scattering of elastic waves by a sphere again received attention from various disciplines of engineering and physical science. Among the works appearing in that period, some were intended for geophysical application (Refs. 0.2 through 0.5). Some were intended for the study of propagation of ultrasonic pulses in crystalline alloys con-
taining precipitates (Refs. 0.6, 0.7); still others investigated the
dynamic stresses around spherical cavities and rigid inclusions due to
standing waves (Ref. 0.8). The references quoted so far have dealt
with harmonic inputs either as travelling waves or as standing waves.
The solution for the transient response of a spherical inclusion did
not appear until quite recently (Refs. 0.9, 0.10).

We shall, in the next few sections of this chapter, follow the
manner of presentation that we have used throughout the book to pre-
sent some of the basic equations in spherical coordinates, steady
state scattering phenomenon, and transient behavior of a spherical
obstacle.

1. BASIC EQUATIONS IN SPHERICAL COORDINATES

1.1. Governing Equations

For spherical coordinates, Fig. 1.1, the transformation which
relates \(x, y, z\) to \(r, \theta, \phi\) and the scale factors are as follows.

Transformation:

\[
\begin{align*}
\xi_1 &= r, \\
\xi_2 &= \theta, \\
\xi_3 &= \phi,
\end{align*}
\]

\[
x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta.
\]

The scale factors are

\[
\begin{align*}
h_r &= 1, \\
h_\theta &= r, \\
h_\phi &= r \sin \theta.
\end{align*}
\]
Substitution of Eqs. (1.1) and (1.2) into Eq. (1.2.52) gives the following equations for spherical coordinates:

\[ \nabla \mathbf{f} = \mathbf{e}_r \frac{\partial \mathbf{f}}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \mathbf{f}}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial \mathbf{f}}{\partial \phi}; \]

\[ \mathbf{v} \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \mathbf{v} \cdot \mathbf{f}) + \frac{1}{r} \frac{\partial}{\partial \theta} (r \sin \theta \mathbf{v} \cdot \mathbf{f}) + \frac{1}{r \sin \theta} \frac{\partial \mathbf{f}}{\partial \phi}; \quad (1.3) \]

\[ \nabla \times \mathbf{f} = \mathbf{e}_r \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial \mathbf{f}}{\partial \phi}) - \frac{\partial \mathbf{f} \theta}{\partial \phi} \right] + \mathbf{e}_\theta \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial \mathbf{f} \theta}{\partial r} - \frac{1}{\sin \theta} \left( r \mathbf{v} \cdot \mathbf{f} \right) \right] \]

\[ \mathbf{e}_\phi \frac{1}{r} \left[ \frac{\partial}{\partial r} (r \mathbf{v} \cdot \mathbf{f}) - \frac{\partial \mathbf{f}}{\partial \phi} \right]; \]

\[ \nabla^2 \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \mathbf{f}}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \mathbf{f}}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \mathbf{f}}{\partial \phi^2}. \]
The scalar wave equation in the expanded form is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2},
\]

(1.4)

which is common to \( \phi, \psi, \) and \( \chi, \) the three displacement potentials, with \( c = c_\phi \) for \( \phi, \) and \( c = c_\psi \) for \( \psi \) and \( \chi. \)

The displacements due to \( \phi, \psi, \) and \( \chi \) in spherical coordinates are given in Eq. (1-2.61), with \( \nu = r, \) \( c_\phi = \frac{e_r}{r}, \) as

\[
1 = \nu \frac{\partial f}{\partial \nu},
\]

\[
\nu \frac{\partial f}{\partial \nu} \times \mathbf{e}_\nu,
\]

(1.5)

\[
\mathbf{N} = i\nu \left[ \frac{\partial f}{\partial \nu} \right] - \lambda \mathbf{e}_\nu r\nu^2 \chi.
\]

The displacements in \( r, \theta, \phi \) directions in terms of the three displacement potentials are

\[
\nu_r = \frac{\partial f}{\partial r} + i \left[ \frac{\partial^2 (r\chi)}{\partial \nu^2} - r \nu^2 \chi \right];
\]

\[
\nu_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \theta} + \lambda \frac{1}{r} \frac{\partial^2 (r\chi)}{\partial \phi^2};
\]

\[
\nu_\phi = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} - \frac{1}{r} \frac{\partial f}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial^2 (r\chi)}{\partial \phi^2}.
\]

(1.6)

The stresses are

\[
\sigma_{rr} = \frac{\lambda}{2} \nu^{-2} \phi + 2 \mu \frac{\partial^2 \nu}{\partial \nu^2} + 2 \mu \frac{\partial}{\partial r} \left[ \frac{\partial^2 (r\chi)}{\partial \nu^2} - r \nu^2 \chi \right];
\]

(1.7a)
\[
\sigma_{\theta\theta} = \lambda \nu^2 \varphi + 2\mu \left( \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \phi^2} \right) + 2\mu \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \theta} \right] \\
+ 2\mu \left[ \frac{1}{r^2} \frac{\partial^3 \varphi}{\partial \theta^3} + \frac{1}{r} \left( \frac{\partial^2 \varphi}{\partial \theta^2} - \nu \varphi \right) \right];
\]

\[
\psi_{\phi \phi} = \lambda \nu^2 \varphi + 2\mu \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \phi} + \frac{1}{r \sin \theta} \cot \theta \frac{\partial \varphi}{\partial \phi} \right] \\
+ 2\mu \left[ \frac{\cot \theta \frac{\partial^2 \varphi}{\partial \theta^2}}{r^2 \sin \theta} - \frac{1}{r^2 \sin \theta} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right] \\
+ 2\mu \left[ \frac{1}{r^2 \sin \theta} \frac{\partial^3 \varphi}{\partial \phi \partial \theta^2} + \frac{1}{r} \left( \frac{\partial^2 \varphi}{\partial \theta^2} - \nu \varphi \right) + \frac{\cot \theta}{r^2} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right];
\]

\[
\alpha_{r\theta} = \frac{2\mu}{r} \left[ \frac{\varphi^2}{\partial \theta} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] - \frac{1}{r} \left[ \frac{1}{r \sin \theta} \frac{\partial^2 \varphi}{\partial \theta^2} \right] - r \frac{3}{r} \left( \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \theta} \right) \\
+ \frac{2\mu}{r} \left[ \frac{3}{\partial \theta} \left( \frac{\partial^2 \varphi}{\partial \theta^2} - \nu \varphi \right) - \frac{1}{r} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right] + r \frac{3}{r} \left( \frac{1}{r \sin \theta} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right); \quad \text{(1.7c)}
\]

\[
\sigma_{r\phi} = 2\mu \left[ \frac{1}{r \sin \theta} \frac{\partial^2 \varphi}{\partial \theta \partial \phi} - \frac{1}{r^2 \sin \theta} \frac{\partial \varphi}{\partial \phi} \right] + \nu \left[ \frac{2}{r^2} \frac{\partial \varphi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right] \\
+ \nu \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{2}{\partial \phi \partial \theta^2} - \nu \varphi \right) - \frac{2}{r^2 \sin \theta} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \right]; \quad \text{(1.7d)}
\]

\[
\alpha_{\phi \phi} = 2\mu \left[ \frac{1}{r \sin \theta} \frac{\partial^2 \varphi}{\partial \theta \partial \phi} - \frac{\cot \theta}{r^2 \sin \theta} \frac{\partial \varphi}{\partial \phi} \right] \\
+ \nu \left[ \cot \theta \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \phi^2} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial^2 \varphi}{\partial \phi \partial \theta} \\
+ 2\mu \left[ \frac{3}{r^2 \sin \theta} \frac{\partial^2 \varphi}{\partial \theta \partial \phi^2} - \frac{\cot \theta}{r^2 \sin \theta} \frac{\partial \varphi}{\partial \phi} \right]. \quad \text{(1.7f)}
\]
1.2. Spherical Wave Function

The general solution for steady state will be presented in this section. No generality is lost by discussing only the steady state solution since the transient solution is obtainable through Fourier analysis, as we have amply demonstrated in the previous chapters.

Consider that \( \psi \) is of the form \( \psi = \psi(r, \theta, \phi) e^{-i\omega t} \), then \( \psi(r, \theta, \phi) \) must satisfy the Helmholtz equation in spherical coordinates:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\omega^2}{c^2} \psi = 0.
\]

(1.6)

The equation is separable -- thus by letting \( \psi(r, \theta, \phi) = \psi_1(r) \psi_2(\theta) \psi_3(\phi) \), one finds

\[
r^2 \frac{d^2 \psi_1}{dr^2} + 2r \frac{d \psi_1}{dr} + \left( k^2 r^2 - \frac{\omega^2}{c^2} \right) \psi_1 = 0,
\]

(1.9)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \psi_2}{d\theta} \right) + \left( r^2 - \frac{\omega^2}{c^2} \sin^2 \theta \right) \psi_2 = 0,
\]

(1.10)

\[
\frac{d^2 \psi_3}{d\phi^2} + \frac{\omega^2}{c^2} \psi_3 = 0,
\]

(1.11)

where \( p \) and \( q \) are separation constants, and \( k = \omega/c \).

At once, we have the solution for \( \psi_3(\phi) \) as \( e^{i\omega \phi} \). Since the solution must be single-value, it is necessary that \( \psi_3(\phi) \) be periodic in \( \phi \) with a period of \( 2\pi \), and \( q \) therefore must be an integer, say \( m = 0, \pm 1, \pm 2, \ldots \), etc.

Letting \( p^2 = \nu(\nu+1) \), and \( \nu = \cos \theta \), Eq. (1.10) becomes
\[(1 - \mu^2) \frac{d^2 f_2}{d\mu^2} - 2\nu \frac{df_2}{d\mu} + \left[ \nu(\nu + 1) - \frac{\nu^2}{(1-\mu^2)} \right] f_2 = 0, \quad (1.12)\]

which is known as the associated Legendre equation. The associated Legendre equation is characterized by regular singularities at \( \mu = \pm 1 \), \( \mu = 0 \). The two solutions to Eqs. (1.19), \( P_n^m(\mu) \) and \( Q_n^m(\mu) \), are the Legendre functions, which are a series in ascending powers of \( \mu \). The series however, does not, in general, converge at \( \mu = \pm 1 \). Since the range of \( \theta \) of physical interest is \( 0 \leq \theta \leq \pi \), including the points \( \mu = \pm 1 \), we must choose values of \( \nu \) such that the series converges at \( \mu = \pm 1 \). This is accomplished by choosing \( \nu = n \), where \( n \) is an integer. Then one of the series breaks off after a finite number of terms and has a finite number at \( \mu = \pm 1 \). It is known as the associated Legendre polynomial \( P_n^m(\mu) \). The other solution \( Q_n^m(\mu) \) is singular, however, at \( \mu = \pm 1 \); it is usually excluded from problems of physical interest.

More on this will be given later.

Equation (1.9) can be transformed into a Bessel equation of half order by assuming \( f_1 = \frac{1}{(kr)^{\frac{1}{2}}} R(r) \). Substituting this expression into Eq. (2.9), we obtain

\[r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left[ k^2 r^2 - (n+\frac{1}{2})^2 \right] R = 0. \quad (1.13)\]

The solution is the cylinder function \( \xi_{n+\frac{1}{2}} \), i.e., \( \xi_{n+\frac{1}{2}}(kr) \) is any one of \( j_{n+\frac{1}{2}}(kr) \), \( y_{n+\frac{1}{2}}(kr) \), \( h_{n+\frac{1}{2}}^{(1)}(kr) \), or \( h_{n+\frac{1}{2}}^{(2)}(kr) \). The appropriate choice of the function is determined by the physical problem under consideration. The solution of the radial equation, (1.9), is then
\begin{equation}
J_1(kr) = \frac{1}{\sqrt{kr}} \mathcal{E}_{n+\frac{1}{2}}(kr).
\end{equation}

The elementary wave function for a harmonically timed varying wave which is single-value and finite on a spherical surface is characterized by the spherical wave function

\begin{equation}
\varphi = \frac{1}{\sqrt{kr}} \mathcal{E}_{n+\frac{1}{2}}(kr) Y_n^m(\nu) e^{i\frac{2m\phi}{r}} e^{-2\alpha r}.
\end{equation}

1.3. Some Properties of the Spherical Bessel Functions

The radial functions \( \sqrt{\frac{1}{kr}} \mathcal{E}_{n+\frac{1}{2}}(kr) \) which occur in the spherical wave equation are usually replaced by the spherical Bessel functions. They are defined as:

\begin{equation}
\begin{align*}
J_n(kr) &= \sqrt{\frac{\pi}{2kr}} J_{n+\frac{1}{2}}(kr), \\
Y_n(kr) &= \sqrt{\frac{\pi}{2kr}} Y_{n+\frac{1}{2}}(kr), \\
\tilde{h}_n^{(1,2)}(kr) &= \sqrt{\frac{\pi}{2kr}} H_{n+\frac{1}{2}}^{(1,2)}(kr).
\end{align*}
\end{equation}

The factor \( r^{-\frac{1}{2}} \) used in the definition is purely for simplicity in asymptotic expansions. For the purpose of commonality, we shall use \( J_n, Y_n, \tilde{h}_n^{(1)}, \) and \( \tilde{h}_n^{(2)} \) in the wave function. These functions will be referred to as spherical Bessel functions of the first, second, third, and fourth kinds, and shall be denoted by \( \tilde{g}_i \) where \( i = 1,2,3,4. \)
The spherical Bessel functions as defined in Eq. (1.16) have a unique feature in that they are expressible in terms of a finite number of algebraic and trigonometrical functions. The series which defines \( h_n^{(1)}(z) \) is

\[
h_n^{(1)}(z) = c_n z^{-\frac{n}{2}} e^{-i \frac{n}{2} \frac{z}{2}} \sum_{m=0}^{\infty} \frac{(m+n)!}{m! (n-m)!} \left( \frac{2}{z} \right)^m,
\]

which is valid for all values of \( z > 0 \). The real part of \( h_n^{(1)}(z) \) is \( j_n(z) \), and the imaginary part is \( y_n(z) \). \( h_n^{(2)} \) is the complex conjugate of \( h_n^{(1)}(z) \). Some useful formulas, and other representations of the spherical Bessel functions are:

\[
\begin{align*}
    j_n(z) &= \left( -i \right)^n \frac{1}{2^n \Gamma(\frac{n}{2})} \left( \frac{z}{2} \right)^n \sin \frac{z}{2}, \\
    y_n(z) &= - (1)^n \frac{1}{2^n \Gamma(\frac{n}{2})} \left( \frac{z}{2} \right)^n \cos \frac{z}{2}, \\
    h_n^{(1)}(z) &= -i (1)^n \frac{1}{2^n \Gamma(\frac{n}{2})} \left( \frac{z}{2} \right)^n \frac{iz}{2}, \\
    h_n^{(2)}(z) &= i (1)^n \frac{1}{2^n \Gamma(\frac{n}{2})} \left( \frac{z}{2} \right)^n \frac{-iz}{2}.
\end{align*}
\]

It follows, for example, for \( n = 0, 1 \), that the various spherical Bessel functions are:

\[
\begin{align*}
    j_0(z) &= \frac{\sin \frac{z}{2}}{z} , & y_0(z) &= - \frac{\cos \frac{z}{2}}{z} , \\
    j_1(z) &= z^2 \left( \sin \frac{z}{2} - \frac{z}{2} \cos \frac{z}{2} \right), & y_1(z) &= -z^{-2} \left( \cos \frac{z}{2} + z \sin \frac{z}{2} \right), \\
    h_0^{(1)}(z) &= -1 \frac{e^iz}{z} , & h_0^{(2)}(z) &= i \frac{e^{-iz}}{z} , \\
    h_1^{(1)}(z) &= iz^{-2} \left( i z - 1 \right) e^{iz} , & h_1^{(2)}(z) &= -iz^{-2} \left( -iz - 1 \right) e^{-iz} .
\end{align*}
\]
The recurrence formulas for the spherical Bessel functions follow directly from Chapter III, Eq. (2.36). Given below are some of the recurrences for \( \delta_n^{(\ell)} \):

\[
\frac{(2n+1)}{z} \delta_n^{(\ell)} = \delta_{n-1}^{(\ell)} + \delta_{n+1}^{(\ell)},
\]

\[
\frac{d}{dz} \delta_n^{(\ell)} = \frac{-1}{2n+1} \left[ n \delta_{n-1}^{(\ell)}(z) - (n+1) \delta_{n+1}^{(\ell)}(z) \right],
\]

\[
\frac{d}{dz} \left( z^{-n-1} \delta_n^{(\ell)}(z) \right) = n \delta_{n-1}^{(\ell)}(z),
\]

\[
\frac{d}{dz} \left( z^{-n} \delta_n^{(\ell)}(z) \right) = -n \delta_{n+1}^{(\ell)}(z).
\] (1.20)

Except for the arbitrary factor of \( z/2 \), the Wronskian conditions given in Eqs. (III-2.39) and (III-2.40) also apply here. As in the cylindrical case the choice of \( j_n, y_n, h_n^{(1)}, \) or \( h_n^{(2)} \) is dependent upon the physics of the problem. The functions of the first and second kind are used to represent standing waves, while the third and fourth kinds are used for travelling waves.

1.4. Some Properties of the Legendre Polynomial

Returning now briefly to the associated Legendre polynomial, when \( \kappa = 0 \) we have the Legendre polynomial \( P_n(u) \), which satisfies the Legendre equation

\[
(1-u^2) \frac{d^2 P_n(u)}{du^2} - 2u \frac{d P_n(u)}{du} + n(n+1) P_n(u) = 0.
\] (1.21)

The Legendre polynomial is a polynomial of degree \( n \), as represented by
\[ P_n^m(\mu) = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{1 \cdot 2 \cdot 3 \cdots n} \left\{ \mu - \frac{n(n-1)}{2(2n-1)} \mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdots (2n-1)(2n-3)} \mu^{n-4} - \cdots \right\}. \] (1.22)

If we now differentiate Eq. (1.21) \( m \) times with respect to \( \mu \), we obtain

\[ (1-\mu^2) \frac{d^2\nu}{d\mu^2} - 2(n+1)\nu \frac{d\nu}{d\mu} + \{n(n+1) - m(m+1)\} \nu = 0, \] (1.23)

where \( \nu = \frac{d^m P_n}{d\mu^m} \). And finally, by letting \( \nu = (1-\mu^2)^{-m/2} P_n^{m}(\mu) \), we obtain the associated Legendre equation

\[ (1-\nu^2) \frac{d^2 P_n^m(\nu)}{d\nu^2} - 2\nu \frac{dP_n^m(\nu)}{d\nu} + \{n(n+1) - \frac{\nu^2}{1-\nu^2} \} \nu P_n^m(\nu) = 0. \] (1.24)

Thus we obtain the following relationship between the associated Legendre polynomial and the Legendre polynomial:

\[ P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m}. \] (1.25)

As we have noted before, the functions \( P_n^m(\mu) \) and \( P_n(\mu) \) are finite at \( \mu = \pm 1 \). The definition of the associated Legendre polynomial above holds only when \( n \) and \( m \) are positive integers, and \( P_n^m(\mu) \) is a polynomial of degree \( n - m \) which vanishes identically if \( m > n \).

Because of the extensive literature available on the Legendre functions we shall list only some of the relations which will be needed in the ensuing discussions.
Recurrence relationships:

\[(1-\mu^2) \frac{dP^m_n}{d\mu} = (n+m)P^m_{n-1} - n\mu P^m_n, \quad (1.26a)\]

\[(1-\mu^2) \frac{dP^m_n}{d\mu} = (n+1)\mu P^m_n - (n-m+1)P^m_{n+1}, \quad (1.26b)\]

\[(2n+1)\mu P^m_n(\mu) = (n-m+1)P^m_{n+1}(\mu) + (n+m)P^m_{n-1}(\mu). \quad (1.26c)\]

The associated Legendre polynomials are orthogonal functions. It can be shown that

\[\int_{-1}^{1} P^n_m(\mu) P^n_k(\mu) d\mu = 0, \quad \int_{-1}^{1} P^n_m(\mu) P^n_m(\mu) d\mu = 2, \quad \int_{-1}^{1} [P^n_m(\mu)]^2 d\mu = 2n+1 \frac{(n+m)!}{(n-m)!}. \quad (1.27a)\]

when \(n \neq k\) or \(m \neq k\), and that

A few additional remarks are in order before we discuss specific problems. For most applications we shall choose the real \(\mu\)-dependent function, i.e., \(\cos \pi \phi\) or \(\sin \pi \phi\). The products of \(\cos m\phi P^m_n(\mu)\) and \(\sin m\phi P^m_n(\mu)\) are periodic for a constant radius \(\pi\), and the indices \(m\) and \(n\) determine the number of nodal lines. For this reason the functions \(\cos m\phi P^m_n(\mu)\) and \(\sin m\phi P^m_n(\mu)\) are sometimes called tesseral harmonics of the \(n\)th degree and \(m\)th order. When \(m = 0\) we have \(P^m_n(\mu)\), sometimes called a zonal harmonic. Here again the choice of the tesseral harmonic will depend upon even or odd behavior of the prob-
lem under consideration. This is best illustrated by the examples which follow.

2. ELASTIC INCLUSION AND LIMITING CASES

We shall begin our discussion of spherical problems by considering an elastic inclusion embedded in infinite elastic medium. As will be seen shortly, the various special cases, i.e., the rigid, vacuous, or fluid inclusion, can all be obtained from the solution of an elastic inclusion by appropriate limiting analyses.

Consider a plane P wave propagating in the infinite medium. When the plane wave impinges on the surface of an elastic inclusion, two types of waves (compressional and shear) are reflected back into the medium, and two waves are refracted into the inclusion. For the convenience of the ensuing discussion, the infinite solid is designated as medium 1 and the spherical inclusion as medium 2. All material constants \( \lambda, \mu, \rho \) in each medium will be distinguished by subscript 1 or 2, accordingly. The potentials, displacements, and stresses associated with the incident waves will be designated by a superscript \( (i) \); those with reflected waves, by \( (r) \); and those with the refracted waves, by \( (s) \).

Let the incident plane P wave propagate in the positive \( z \) direction and be represented by

\[
\varphi(i) = \psi_0 e^{i(z \frac{\lambda}{\mu} - \omega t)}, \tag{2.1}
\]

where \( \alpha_1 \) is the compressional wave number in medium 1 and \( \omega \) is a cir-
cular frequency. For the plane $P$ wave, the potentials are symmetric
about the $z$ axis; thus they are independent of the spherical coordi-
nate $\phi$. The spherical wave functions, therefore, will be independent of
$\phi$. It follows that $m = 0$ in $r^n (\nu)$.

The two reflected waves, which are outward propagating, can be
represented by

$$
\varphi(r) = \sum_{\nu=0}^{\infty} A_{\nu} J_{\nu}^{(1)} (\alpha_{\nu} r) P_{\nu} (\mu) e^{-i \omega t},
$$

(2.2)

$$
\chi(r) = \sum_{\nu=0}^{\infty} B_{\nu} Y_{\nu}^{(1)} (\beta_{\nu} r) P_{\nu} (\mu) e^{-i \omega t},
$$

The refracted waves, being confined in the spherical scatterer,
are standing waves. They are represented by

$$
\varphi'(r) = \sum_{\nu=0}^{\infty} C_{\nu} J_{\nu} (\alpha_{\nu} r) P_{\nu} (\mu) e^{-i \omega t},
$$

(2.3)

$$
\chi'(r) = \sum_{\nu=0}^{\infty} D_{\nu} Y_{\nu} (\beta_{\nu} r) P_{\nu} (\mu) e^{-i \omega t}.
$$

Following our previous discussions $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are compres-
sional and shear wave numbers in the mediums 1 and 2. $A_{\nu}$, $B_{\nu}$, $C_{\nu}$,
and $D_{\nu}$ are expansion coefficients to be determined by the boundary
conditions.

The choice of the various spherical Bessel functions in Eqs.
(2.2) and (2.3) is made clear if one observes the definition of
$J_{\nu}^{(1)} (z)$ and $Y_{\nu} (z)$ in Eq. (1.18). It is apparent that because of the
positive exponential in $J_{\nu}^{(1)} (z)$, the product of $J_{\nu}^{(1)} (z) e^{-i \omega t}$ repre-
sents an outward propagating wave. Similarly, \( j_{\kappa}(\nu) e^{-i\omega t} \) represents a standing wave. Determination of the unknown coefficients will be facilitated by first expanding the incident wave into spherical coordinates. Since \( P_{\nu}^{m}(\theta) \) forms a complete system of orthogonal functions, we may express any continuous function \( \varphi(r, \theta, \phi) \) which is finite at \( r = 0 \) as

\[
\varphi(r, \theta, \phi) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} a_{\nu m} j_{\nu}(k r) P_{\nu}^{m}(\theta) e^{i m \phi},
\]

(2.4)

In the axisymmetric case \( m = 0 \) we have

\[
\varphi(r, \theta) = \sum_{\nu=0}^{\infty} a_{\nu} j_{\nu}(k r) P_{\nu}(\theta),
\]

(2.5)

In particular, when \( \varphi(r, \theta) = \varphi_{0} e^{i (a - \omega t - \omega t)} \), where \( a = r \cos \theta \) and the time factor \( e^{-i\omega t} \) is omitted, we have

\[
e^{i a r \cos \theta} = \sum_{\nu=0}^{\infty} a_{\nu} j_{\nu}(a r) P_{\nu}(\theta).
\]

(2.6)

To determine the coefficients \( a_{\nu} \), we multiply Eq. (2.6) by \( P_{\nu}(\theta) \sin \theta \) and integrate from 0 to \( r \). By applying the orthogonality condition of \( P_{\nu}(\theta) \), Eq. (1.27), we obtain

\[
\int_{0}^{\pi} e^{i a r \cos \theta} P_{\nu}(\cos \theta) \sin \theta \, d\theta = \frac{2}{2\nu + 1} a_{\nu} j_{\nu}(a r).
\]

(2.7)

Expressing the left-hand side of the equation above in terms of \( \nu = \cos \theta \), we have
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\[
\int_0^1 e^{i a_1 r \cos \theta} P_n(\cos \theta) \sin \theta \, d\theta = \int_{-1}^1 e^{i a_1 r \mu} P_n(\mu) \, d\mu,
\]

which yields

\[
\int_{-1}^1 e^{i a_1 r \mu} P_n(\mu) \, d\mu = 2i e^{i a_1 r \nu} \mathcal{J}_n(a_1 r).
\]

It follows that

\[ a_n = (2n+1)i \nu, \]

and finally

\[
\int_{-1}^1 e^{i a_1 r \cos \theta} \psi(\theta) \, d\theta = \psi(0) \sum_{n=0}^{\infty} (2n+1)i \nu \mathcal{J}_n(a_1 r) P_n(\mu).
\]

Having expanded the incident wave in spherical coordinates, the unknown coefficients \( A_n, B_n, C_n, \) and \( D_n \) can be evaluated by imposing the continuity conditions at the interface between elastic inclusion and medium. Thus at \( r = a, \)

\[
\begin{align*}
\nu_\nu^{(i)} + \nu_\nu^{(r)} &= \nu_\nu^{(r)}, \\
\nu_\theta^{(i)} + \nu_\theta^{(r)} &= \nu_\theta^{(r)}, \\
\sigma_{rr}^{(i)} + \sigma_{rr}^{(r)} &= \sigma_{rr}^{(r)}, \\
\sigma_{\nu\nu}^{(i)} + \sigma_{\nu\nu}^{(r)} &= \sigma_{\nu\nu}^{(r)}, \\
\sigma_{\nu\theta}^{(i)} + \sigma_{\nu\theta}^{(r)} &= \sigma_{\nu\theta}^{(r)},
\end{align*}
\]
in which \( \psi_n^{(1)} \), \( \psi_n^{(2)} \), and \( \psi_n^{(3)} \), etc., denote the radial and tangential displacements, and the radial and shearing stresses due to the incident, reflected, and refracted waves respectively.

Substituting Eqs. (2.2), (2.3), and (2.10) into the displacement-potential and stress-displacement relationships, we obtain displace-
ments and stresses in terms of the potentials in the medium and the inclusion as follows:

In medium 1, due to the incident and reflected waves:

\[
\begin{align*}
\psi_{r1} &= \frac{1}{r} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \Lambda_n \psi_n^{e(2)} + \Gamma_n \psi_n^{e(3)} \right) \frac{p_n}{r} (\cos \theta); \\
\psi_{\theta1} &= \frac{1}{r} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \Lambda_n \psi_n^{e(2)} + \Gamma_n \psi_n^{e(3)} \right) \frac{\partial p_n}{\partial \theta} (\cos \theta); \\
\sigma_{rr1} &= \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \Lambda_n \psi_n^{e(2)} + \Gamma_n \psi_n^{e(3)} \right) \frac{p_n}{r} (\cos \theta); \\
\sigma_{\theta\theta1} &= \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \Lambda_n \psi_n^{e(2)} + \Gamma_n \psi_n^{e(3)} \right) \frac{\partial p_n}{\partial \theta} (\cos \theta).
\end{align*}
\]

In medium 2, due to the refracted waves:

\[
\begin{align*}
\psi_{r2} &= \frac{1}{r} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \psi_n^{e(2)} \right) \frac{r_n}{r} (\cos \theta); \\
\psi_{\theta2} &= \frac{1}{r} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \psi_n^{e(2)} \right) \frac{\partial r_n}{\partial \theta} (\cos \theta); \\
\sigma_{rr2} &= \frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \psi_n^{e(2)} \right) \frac{r_n}{r} (\cos \theta); \\
\sigma_{\theta\theta2} &= \frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} \left( \psi_n^{e(1)} + \psi_n^{e(2)} \right) \frac{\partial r_n}{\partial \theta} (\cos \theta).
\end{align*}
\]
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where

\[ e^{(\varepsilon)}_{11}(a) = \left( \frac{n^2 - n - (\beta r)^2}{2} \right)^2 e^{(\varepsilon)}_n(a r) + 2a r \delta^{(\varepsilon)}_{n+1}(a r), \]

\[ e^{(\varepsilon)}_{12}(a) = n (n+1) \left[ (n-1) e^{(\varepsilon)}_n(a r) - \beta r \delta^{(\varepsilon)}_{n+1}(a r) \right], \]

\[ e^{(\varepsilon)}_{41}(a) = (n-1) e^{(\varepsilon)}_n(a r) - \beta r \delta^{(\varepsilon)}_{n+1}(a r), \]

\[ e^{(\varepsilon)}_{42}(a) = -\left( \frac{n^2 - 1 - (\alpha r)^2}{2} \right)^2 e^{(\varepsilon)}_n(2 a r) - \beta r \delta^{(\varepsilon)}_{n+1}(2 a r), \]

\[ e^{(\varepsilon)}_{71}(a) = \beta r e^{(\varepsilon)}_n(a r) - \alpha r \delta^{(\varepsilon)}_{n+1}(a r), \]

\[ e^{(\varepsilon)}_{72}(a) = n (n+1) \delta^{(\varepsilon)}_n(\alpha r), \]

\[ e^{(\varepsilon)}_{81}(a) = \delta^{(\varepsilon)}_n(a r), \]

\[ e^{(\varepsilon)}_{82}(a) = -(n+1) \delta^{(\varepsilon)}_n(\beta r) + \beta r \delta^{(\varepsilon)}_{n+1}(\beta r). \]

\[ e^{(\varepsilon)}_{jk} \] and \( e^{(\varepsilon)}_{jk} \) are used to denote that portion of the stresses and displacements which involve only the spherical Bessel functions corresponding to various waves. Superscript \( (\varepsilon) \) is used to denote the kind of spherical Bessel function according to the discussion given in the preceding section; subscript \( j \) is used to denote particular parts of the stresses and displacements (see Eq. 2.12); and subscript \( k = 1, 2 \), refers to potentials \( \psi \) and \( \chi \) respectively. The general expressions for the stresses and displacements in terms of the displacement potentials, including \( \varepsilon \) dependency, are given in the appendix of this
The Legendre polynomials \( P_n(\cos \theta) \), and the associate Legendre polynomials

\[
\frac{dP_n}{d\theta} = -P_{n+1}(\cos \theta),
\]

each form a complete orthogonal set. Therefore, when we let \( r = a \) in Eqs. (2.12) and (2.13) and substitute them into the continuity condition in Eq. (2.11), the coefficients of like Legendre polynomials on both sides of Eq. (2.11) must be equal for each value of \( n \). This results in four simultaneous algebraic equations, sufficient to determine the four unknown coefficients \( A_n, E_n, C_n, \) and \( D_n \).

In matrix form, these equations are:

\[
\begin{pmatrix}
E_{11}^{(3)}(\alpha_1) & E_{12}^{(3)}(\beta_1) & -E_{11}^{(1)}(\alpha_2) & -E_{12}^{(1)}(\beta_2) \\
E_{41}^{(3)}(\alpha_1) & E_{42}^{(3)}(\beta_1) & -E_{41}^{(1)}(\alpha_2) & -E_{42}^{(1)}(\beta_2) \\
E_{71}^{(3)}(\alpha_1) & E_{72}^{(3)}(\beta_1) & -E_{71}^{(1)}(\alpha_2) & -E_{72}^{(1)}(\beta_2) \\
E_{81}^{(3)}(\alpha_1) & E_{82}^{(3)}(\beta_1) & -E_{81}^{(1)}(\alpha_2) & -E_{82}^{(1)}(\beta_2)
\end{pmatrix}
\begin{pmatrix}
A_n \\
E_n \\
C_n \\
D_n
\end{pmatrix}
= \varphi_0^{(n)}(2n+1)
\begin{pmatrix}
E_{11}^{(1)}(\alpha_1) \\
E_{41}^{(1)}(\alpha_1) \\
E_{71}^{(1)}(\alpha_1) \\
E_{81}^{(1)}(\alpha_1)
\end{pmatrix}
\tag{2.14}
\]

where \( E_{11}^{(3)}(\alpha_1) \) represents \( E_{11}^{(3)}(\alpha_1 r) \) evaluated at \( r = a \) ... etc.,

\( \bar{\mu} = \mu_2 / \mu_1 \) is the ratio of the shear modulus of the inclusion to that of the medium. Except for the limiting cases when the arguments of the spherical Bessel functions become very small or very large, these
equations can best be solved numerically.

We shall now examine some of the special cases where the inclusion is either fluid, rigid, or vacuous. We may consider these types of inclusions as limiting cases of the general elastic inclusion. The solution for these types of inclusions, therefore, can be derived from Eq. (2.14) provided a proper limiting analysis is carried out.

Fluid Inclusion

If a cavity in an elastic solid is filled with an inviscid fluid, only compressional waves are refracted into the fluid. The boundary conditions at \( r = a \) are reduced from Eq. (2.11) to

\[
\begin{align*}
\sigma_{rr}^{(f)} + \sigma_{\phi\phi}^{(f)} &= \sigma_{\tau\phi}^{(f)} \\
\sigma_{\phi\phi}^{(f)} &= \sigma_{\tau\phi}^{(f)} \\
\sigma_{\tau\phi}^{(f)} &= 0,
\end{align*}
\]

(2.15)

The condition of continuity in circumferential displacements cannot be imposed because of the inviscid assumption. The displacements and stresses in the elastic medium are obviously still the same as before. In the fluid, the inviscid assumption implies \( u_2 \to 0 \) and \( B_2 = 0 \) (no refracted shear wave). The stresses in the fluid, then, are

\[
\sigma_{\tau\phi}^{(f)} = \sigma_{\phi\phi}^{(f)} = \lambda_2 \psi^{(f)}.
\]

(2.16)

It should be noted further that as \( u_2 \to 0, B_2 \to \infty \), but
\[ w_2^2 \rho_2 = \omega^2 \rho_2 = \nu \xi_1^2 \xi_2 / \phi_1, \]

(2.17)

remains finite. Using now the relationship in (2.17) and (2.15) we obtain the following matrix formula for the determination of \( k_n, H_n, \) and \( C_n: \)

\[
\begin{bmatrix}
E_{11}^{(3)}(a_1) & E_{12}^{(3)}(a_1) & -E_{11}^{(1)}(a_2) \\
E_{41}^{(3)}(a_1) & E_{42}^{(3)}(a_1) & 0 \\
E_{71}^{(3)}(a_1) & E_{72}^{(3)}(a_1) & -E_{71}^{(1)}(a_2)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n \\
C_n
\end{bmatrix}
= \begin{bmatrix}
E_{11}^{(1)}(a_1) \\
E_{41}^{(1)}(a_1) \\
E_{71}^{(1)}(a_1)
\end{bmatrix}
\]

(2.18)

where

\[
E_{11}^{(1)}(a_2) = \frac{\xi_1^2}{\xi_2^2 / \xi_1} \beta_2^2 \gamma_2 \eta_2(a_2). \]

(2.19)

The same equation, as we have indicated earlier, can be deduced from the case of an elastic inclusion. Recall that \( \beta_2 \rightarrow \infty \) as \( \lambda_2 \rightarrow 0; \) hence the spherical Bessel function with argument \( \beta_2 \xi_2 \) in matrix (2.14) can be approximated by the asymptotic formula

\[
j_n(\xi_2) \sim \xi_2^{-1} \cos \left[ \xi_2 - \frac{1}{2}(n+1)\pi \right], \quad \xi_2 \rightarrow \infty.
\]

Then it is clear that as \( \beta_2 \xi_2 \rightarrow \infty, \)

\[
j_n(\beta_2 \xi_2) = 0(\epsilon),
\]

\[
\beta_2 \xi_2^{-1} \eta_1(n+1)(\beta_2 \xi_2) = 0(\epsilon^0),
\]

\[
\eta_2 = 0(\epsilon^2),
\]
where \( \varepsilon \) is an infinitesimal quantity \( 1/\varepsilon_n \), and "\( O(\varepsilon) \)" means "the order of magnitude of \( \varepsilon \)." Applying these limits to all elements of the matrix that contain \( \nu_2 \) and \( \beta_2 \), we obtain

\[
\begin{align*}
\tilde{\varepsilon}^{(1)}_{11}(\nu_2) &= E^{(1)}_{11}(\nu_2), & E^{(1)}_{12}(\beta_2) &= 0(\varepsilon), \\
\tilde{\varepsilon}^{(1)}_{12}(\beta_2) &= 0(\varepsilon^2), & E^{(1)}_{22}(\beta_2) &= 0(\varepsilon^0), \\
\tilde{\varepsilon}^{(1)}_{41}(\nu_2) &= 0(\varepsilon^2), & E^{(1)}_{42}(\beta_2) &= 0(\varepsilon^0) \\
\tilde{\varepsilon}^{(1)}_{42}(\beta_2) &= 0(\varepsilon).
\end{align*}
\]

If we now neglect all elements in the matrix with an order magnitude higher than \( \varepsilon^0 \), Eq. (2.14) becomes

\[
\begin{bmatrix}
E^{(3)}_{11}(\nu_1) & E^{(3)}_{12}(\beta_1) & -E^{(1)}_{11}(\nu_2) & 0 \\
E^{(3)}_{41}(\nu_1) & E^{(3)}_{42}(\beta_1) & 0 & 0 \\
E^{(3)}_{71}(\nu_1) & E^{(3)}_{72}(\beta_1) & E^{(1)}_{71}(\nu_2) & 0 \\
E^{(3)}_{81}(\nu_1) & E^{(3)}_{82}(\beta_1) & E^{(1)}_{81}(\nu_2) & 0(\varepsilon^0)
\end{bmatrix}
\begin{bmatrix}
A \eta \\
B \eta \\
C \eta \\
D \eta 
\end{bmatrix}
= \begin{bmatrix}
E^{(1)}_{11}(\nu_1) \\
E^{(1)}_{41}(\nu_1) \\
E^{(1)}_{71}(\nu_1) \\
E^{(1)}_{81}(\nu_1)
\end{bmatrix}
\]

\[(2.20)\]

These equations can easily be reduced to Eq. (2.18) for the determination of \( A \eta, B \eta, \) and \( C \eta \).
Rigid Inclusion

When the inclusion is a perfect rigid sphere, by definition the distance between any two points in the sphere remains constant at all times. But because the surrounding medium is elastic, the sphere will translate as a rigid body under the impact of incident waves. We have already discussed, in Chapter III, how the rigid-body motion of a cylindrical inclusion can be determined by summing forces acting on the inclusion; in what follows we shall show how the solution for the rigid spherical inclusion may be obtained by a limiting analysis similar to the one we have done in the fluid inclusion case.

Although there can be no waves refracted into the rigid inclusion, we may not however suppress all the \( C_n \)'s and \( B_n \)'s for the reason that there is rigid body translation. This becomes clear if we note that in Eqs. (2.13a) and (2.13b), for a constant \( \gamma \), there are two terms in the series having the form

\[
\begin{align*}
\nu_x &= a_1 P_1 (\cos \theta) = a_1 \cos \theta, \\
\nu_y &= b_1 \left( dP_1 (\cos \theta) / d\theta \right) = -b_1 \sin \theta,
\end{align*}
\]

(2.21)

where \( a_1 \) and \( b_1 \) are constants. By transforming the displacements into \( z-q \) coordinates — see Fig. 1.1 —

\[
\begin{align*}
\nu_z &= \frac{1}{2} (z_1 + b_1) + \frac{1}{2} (a_1 - b_1) \cos 2\theta, \\
\nu_q &= \frac{1}{2} (a_1 - b_1) \sin 2\theta,
\end{align*}
\]

we see clearly that there is a rigid body translation \( \frac{1}{2} (a_1 + b_1) \) in the
z direction. We should, therefore, in our limiting analysis, expect to find all \( C_n \) and \( D_n \) suppressed, except for the \( n = 1 \) term.

Now, for a rigid inclusion we let \( \lambda_2 \to \infty, \mu_2 \to \infty \) for \( \rho_2 \) finite to obtain the following order of magnitudes of spherical Bessel functions:

\[
\alpha_2 \alpha = O(\varepsilon), \quad \beta_2 \beta = O(\varepsilon),
\]

\[
\widetilde{u} = O(\varepsilon^{-2}), \quad \beta_2 \alpha_{n+1} \beta_2 \alpha = O(\varepsilon^{n+2}),
\]

\[
\hat{j}_n (\alpha_2 \alpha) = O(\varepsilon^n).
\]

If these limiting values are substituted into the matrix elements of Eq. (2.14), one finds for \( n \neq 1 \)

\[
E^{(1)}_{71} (\alpha_2) = E^{(1)}_{72} (\beta_2) = E^{(1)}_{81} (\alpha_2) = E^{(1)}_{82} (\beta_2) = O(\varepsilon^n),
\]

\[
\tilde{\Psi}^{(1)}_{71} (\alpha_2) = \tilde{\Psi}^{(1)}_{72} (\beta_2) = \tilde{\Psi}^{(1)}_{81} (\alpha_2) = \tilde{\Psi}^{(1)}_{82} (\beta_2) = O(\varepsilon^{n-2});
\]

and for \( n = 1 \)

\[
E^{(1)}_{71} (\alpha_2) = E^{(1)}_{72} (\beta_2) = E^{(1)}_{81} (\alpha_2) = E^{(1)}_{82} (\beta_2) = O(\varepsilon),
\]

\[
\tilde{\Psi}^{(1)}_{71} (\alpha_2) = \tilde{\Psi}^{(1)}_{72} (\beta_2) = \tilde{\Psi}^{(1)}_{81} (\alpha_2) = \tilde{\Psi}^{(1)}_{82} (\beta_2) = O(\varepsilon).
\]

It follows then for \( n \neq 1 \) that the matrix in Eq. (2.14) can be reduced to
\[
\begin{bmatrix}
E^{(3)}_{71}(a_1) & E^{(3)}_{72}(b_1) \\
E^{(3)}_{81}(a_1) & E^{(3)}_{82}(b_1)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix} = -\rho \imath_n^{(2n+1)} \begin{bmatrix}
E^{(1)}_{71}(a_1) \\
E^{(1)}_{81}(a_1)
\end{bmatrix}
\]

when \(E^{(1)}_{71} \ldots E^{(1)}_{82}\) are neglected in comparison with \(\Xi E^{(1)}_{11}(a_2) \ldots \Xi E^{(1)}_{42}(b_2)\). For \(n = 1\), because of the order of magnitudes being the same, the complete \(4 \times 4\) matrix must be used with the spherical Bessel functions in the medium 2, i.e., with the third and fourth columns replaced by the corresponding small argument approximations.

Carrying out the necessary order-of-magnitude analyses, one finds, for \(n = 1\), that the elements of the matrix which resulted from \(E^{(3)}_{71}(a_1), E^{(3)}_{72}(b_1), E^{(3)}_{81}(a_1), E^{(3)}_{82}(b_1), E^{(1)}_{71}(a_1),\) and \(E^{(1)}_{81}(a_1)\) are as follows:

\[
e^{(3)}_{71}(a_1) = (1 - \eta)h_1(a_1 a) - a_1 a h_2(a_1 a),
\]

\[
e^{(3)}_{81}(a_1) = (1 - \eta)h_1(a_1 a),
\]

\[
e^{(3)}_{72}(b_1) = 2(1 - \eta)h_1(b_1 a),
\]

\[
e^{(3)}_{82}(b_1) = 2(1 - \eta)h_1(b_1 a) - 2 a h_2(b_1 a),
\]

\[
e^{(1)}_{71}(a_1) = (1 - \eta)j_1(a_1 a) - a_1 a j_2(aa),
\]

\[
e^{(1)}_{81}(a_1) = (1 - \eta)h_1(a_1 a).
\]

And \(A_1\) and \(B_1\) are determined by...
Although in this limiting process $C_n$ and $E_n$ are treated as finite numbers, the refracted waves still vanish because of the small values for $\alpha_2$ and $\beta_2$ which appear in the series representation of $\varphi(f)$ and $\psi(f)$.

Spherical Cavity

Unlike the rigid inclusion, the case of a spherical cavity presents no complications in analysis. The reduction of the general solution to the cavity case follows a limiting process similar to that in the two preceding cases, with $\lambda_2 \to 0$, $\mu_2 \to 0$, and $\rho_2 \to 0$. The matrix for the determination of $A_n$ and $E_n$ is

$$
\begin{bmatrix}
E_{11}^{(3)}(\alpha_1) & E_{12}^{(3)}(\beta_1) \\
E_{41}^{(3)}(\alpha_1) & E_{42}^{(3)}(\beta_1)
\end{bmatrix}
\begin{bmatrix}
A_n \\
E_n
\end{bmatrix}
= -2\Phi_2^{(2)}(2n+1)
\begin{bmatrix}
E_{11}^{(1)}(\alpha_1) \\
E_{41}^{(1)}(\alpha_1)
\end{bmatrix}.
$$

We have illustrated in the foregoing limiting processes how each of the various special cases of inclusion -- i.e., the rigid, vacuous, and fluid inclusions -- can be derived from the solution of an elastic inclusion. The important part of the limiting analysis is, of course, to keep track of the order of magnitude of the terms in Eq. (2.14) as the elastic constants go either to zero or to infinity.
In the following sections we will present some numerical results of spherical inclusions subjected either to plane waves or to spherical waves.

3. DYNAMIC STRESSES AROUND A SPHERICAL CAVITY

IN THIS SECTION, the dynamic stress amplification factors around a cavity subject to a plane incident wave and a spherical incident wave will be discussed.

3.1. Incident Plane P Wave

Consider first the case of a plane incident P wave. The unknown expansion coefficients of the two reflected waves are already determined by Eq. (2.25). To determine the stress distribution at the boundary of a cavity one need consider only the hoop stresses, since all others vanish identically. Two hoop stresses, $\sigma_{66}$ and $\sigma_{\phi\phi}$, exist at the boundary. They are determined by substituting the expansion coefficients $A_n$ and $B_n$, Eq. (2.25), into the following expressions, and by letting $n = \alpha$:

$$
\sigma_{66} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left( \phi_n \sigma_{21}^{(1)} + A_n \sigma_{21}^{(3)} + B_n \sigma_{22}^{(3)} \right) \cot \theta \eta_n^1 (\cos \theta)
$$

+ $(\phi_n \sigma_{21}^{(1)} + A_n \sigma_{21}^{(3)} + B_n \sigma_{22}^{(3)}) \cot \theta \eta_n^1 (\cos \theta)$, \hspace{1cm} (3.1)

$$
\sigma_{\phi\phi} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left( \phi_n \sigma_{31}^{(1)} + A_n \sigma_{31}^{(3)} + B_n \sigma_{32}^{(3)} \right) \cot \theta \eta_n^1 (\cos \theta)
$$

+ $(\phi_n \sigma_{31}^{(1)} + A_n \sigma_{31}^{(3)} + B_n \sigma_{32}^{(3)}) \cot \theta \eta_n^1 (\cos \theta)$, \hspace{1cm} (3.2)
where

\[ \varepsilon_{21}^{(i)} = (-n^2 - \beta^2 r^2/2 + \alpha^2 r^2) \delta_{n}^{(i)}(ar) - \alpha n \delta_{n+1}^{(i)}(ar), \]

\[ \delta_{21}^{(i)} = \delta_{n}^{(i)}(ar), \]

\[ \varepsilon_{22}^{(i)} = -n(n + 1) \left[ \alpha n \delta_{n}^{(i)}(br) - \beta n \delta_{n+1}^{(i)}(br) \right], \]

\[ \delta_{32}^{(i)} = (n + 1) \delta_{n}^{(i)}(br) - \beta n \delta_{n+1}^{(i)}(br), \]

\[ \varepsilon_{31}^{(i)} = \left[ n - \frac{\beta^2 r^2}{2} + (ar)^2 \right] \delta_{n}^{(i)}(ar) - \alpha n \delta_{n+1}^{(i)}(ar), \]

\[ \delta_{31}^{(i)} = \delta_{n}^{(i)}(ar), \]

\[ \varepsilon_{32}^{(i)} = n(n + 1) \delta_{n}^{(i)}(ar), \]

\[ \delta_{32}^{(i)} = -(n + 1) \delta_{n}^{(i)}(br) + 2\beta n \delta_{n+1}^{(i)}(br). \]

The equations for the general case, including \( \epsilon \)-dependency, are given in the appendix.

By normalizing Eqs. (3.1) and (3.2) by the incident wave's stress magnitude, \( \sigma_0 = \mu \beta^2 \phi_0 \), we now obtain the dynamic stress concentration factors (SCFD) \( \sigma_{\theta \theta}^{*} \) and \( \sigma_{\phi \phi}^{*} \) around the cavity. The analogous static solution for this problem was first presented by Southwell and Gough, (3.1) and was later examined by Goodier, (3.2).

As is evident from Eqs. (3.1) and (3.2), the dynamic stress concentration factors around a spherical cavity will depend upon the Poisson ratio of the medium and the wave number. The behavior of \( \sigma_{\theta \theta}^{*} \)
and $\sigma_{\phi\phi}$ is shown in Figs. 3.1 through 3.6 as a function of the wave number $\alpha \alpha$ and Poisson's ratio $\nu$. Figures 3.1 and 3.2 give the distributions of $|\sigma_{66}^*|$ and $|\sigma_{\phi\phi}^*|$ as a function of the angle $\theta$ at two wave numbers, $\alpha \alpha = 0.1, 3.0$. The wavelength corresponding to $\alpha \alpha = 0.1$ is approximately 30 times the diameter of the cavity, while for $\alpha \alpha = 3.0$ the wavelength is about the same as the diameter. Thus for $\alpha \alpha = 0.1$ the dynamic solution should approach the static solution, and for $\alpha \alpha = 3.0$ the scattering phenomenon should dominate. This is evident
Fig. 3.2. Real and Imaginary Parts of $c^*_{\theta\theta}$ vs. $\theta$ on the Cavity at Two Wave Numbers $\alpha \omega = 0.1, 3.0$

from the results. We note that for $\alpha \omega = 0.1$ both $|c^*_{\theta\theta}|$ and $|c^*_{\phi\phi}|$, as shown in Fig. 3.1, closely resemble the static solution, and for $\alpha \omega = 3.0$ the maximum of $|c^*_{\theta\theta}|$ has shifted toward the incident side of the cavity while the maximum of $|c^*_{\phi\phi}|$ is now at $\theta = 0$. An interesting and potentially important phenomenon for brittle materials is the large negative stress field around the cavity (negative stress means the stress has a sign opposite to the incident stress). This is shown in Fig. 3.2, where both the real part and the imaginary part of $c^*_{\theta\theta}$.
exhibit large negative stresses, having the same intensity as the stresses in the incident wave.

Fig. 3.3. Stress Concentration Factor $|\sigma_{00}^*|$ vs. Wave Number at $\theta = \pi/2$ for Various Poisson Ratios

Fig. 3.4. Stress Concentration Factor $|\sigma_{00}^*|$ vs. Wave Number at $\theta = \pi$ for Various Poisson Ratios
Figures 3.3 and 3.4 give the behavior of $|\sigma^*_{00}|$ at $\theta = \pi/2$ and $\pi$, respectively, as a function of $\omega \alpha$ and $\nu$. The dynamic stress concentration factors are 10% to 15% higher than the static value for Poisson's ratio less than 0.40. This is similar to the two-dimensional cavity problem. However, when Poisson's ratio is greater than 0.40, there is a marked increase in SCFD over the corresponding static values. This phenomenon was not observed in the two-dimensional problem.

![Graph showing stress concentration factor vs. wave number for various Poisson ratios.](image)

**Fig. 3.5. Stress Concentration Factor $|\sigma^*_{00}|$ vs. Wave Number at $\theta = \pi/2$ for Various Poisson Ratios**

The behavior of $|\sigma^*_{00}|$ at $\theta = \pi/2$ is shown in Fig. 3.5. We note the similarity of behavior between $|\sigma^*_{00}|$ and $|\sigma^*_{0\theta}|$ as a function of $\omega \alpha$. However, the Poisson ratio effect on the dynamic simplification is more pronounced in $|\sigma^*_{00}|$ than in the case of $|\sigma^*_{0\theta}|$.

From the foregoing discussion it is apparent that maximum dynamic stress concentration factors may shift around the cavity as the wave
number changes, as shown in Figs. 3.1 and 3.2, and there are several stationary values in the range of \( \alpha \) calculated, as evident from Figs. 3.3 through 3.5. The maximum of all maxima for any wave number and angle as a function of Poisson's ratio is shown in Fig. 3.6. Here we note that the static stress concentration decreases monotonically as \( \nu \) increases; however, the dynamic factors first decrease and then increase without bound as \( \nu \to 0.5 \). This phenomenon was first observed in 1958 by Nishimura, (3.3) in his analysis of the dynamic stress amplification around a spherical cavity due to a standing P wave.

![Graph](image)

*Fig. 3.6. Maximum Dynamic Stress Concentration Factor \(|\sigma^*_{80}|\) vs. Poisson Ratio*
3.2. Spherical Incident Wave

Let us now examine the effects of the incident wave's curvature on the dynamic stress concentration factors for a spherical cavity. The simplest example is that of a harmonic center of dilatation located at $O$, which is at an arbitrary distance $r_o$ from $O$ — see Fig. 3.7.

![Diagram of a point source](image)

**Fig. 3.7. Geometry of a Point Source**

Thus the effects of the incident wave's curvature on the dynamic stress concentration factors can be examined by observing the effects of $r_o$ on the stresses.

The harmonic spheroidal wave generated by the center of dilatation may be represented by
\[ \varphi^{(i)}(\vec{r}) = \varphi_o \frac{e^{i(\vec{a} \cdot \vec{r} - \omega t)}}{r}. \]  

(3.3)

In this representation the harmonically time-varying spherical wave is propagating outward from the source at \( \vec{0} \).

As the incident spherical wave impacts on the cavity of radius \( a \) centered at the origin \( 0 \), the incoming spherical compressional wave is reflected into both compressional and shear waves:

\[ \varphi^{(r)}(\vec{r}) = \sum_{n=0}^{\infty} A_n \tilde{h}_n (\vec{a} \cdot \vec{r}) P_n (\cos \theta), \]

(3.4)

\[ \psi^{(s)}(\vec{r}) = \sum_{n=0}^{\infty} B_n \tilde{h}_n (\vec{b} \cdot \vec{r}) P_n (\cos \theta), \]

where \( A_n \) and \( B_n \) are the expansion coefficients to be determined by the traction-free boundary conditions at \( r = a \). We note here that the reflected waves are expressed as outward propagating spherical waves centered at the origin \( 0 \).

To determine the unknown expansion coefficients we proceed first to expand the incident wave in the \((r, \theta, \phi)\) coordinates. We note first that Eq. (3.3) may be written as

\[ \varphi^{(i)}(\vec{r}) = \varphi_o \frac{e^{i(\vec{a} \cdot \vec{r} - \omega t)}}{r} = i \alpha \varphi_o h_0 (\vec{a} \cdot \vec{r}) e^{-i\omega t}, \]  

(3.5)

and

\[ \frac{r^2}{\tilde{r}^2} = \frac{r_o^2}{\tilde{r}_o^2} + r^2 + 2r \tilde{r} \cos \theta. \]  

(3.6)
Equations (3.5) and (3.6) together can be used to expand the incident wave in terms of the \((r, \theta, \phi)\) coordinates. First of all

\[
\lambda \hat{h}_\nu(\omega r) = \left\{
\begin{array}{ll}
\sum_{n=0}^{\infty} (i\omega)(2n+1)\hat{Y}_n(\omega r)h_n(\omega \sigma)P_n(\cos \theta), & r > r_0 \\
\sum_{n=0}^{\infty} (i\omega)(2n+1)\hat{Y}_n(\omega r)\hat{Y}_n(\omega \sigma)P_n(\cos \theta), & r < r_0
\end{array}
\right.
\] (3.7)

which is known as the Addition Theorem for \(h_\nu(\alpha)\). The appropriate expansion to use for the determination of \(I_\eta\) and \(E_\kappa\) is Eq. (3.7a), since at \(r = a\), \(r\) is always less than \(r_0\). Therefore in the region \(r < r_0\), \(\varphi(t)\) is

\[
\varphi(t)(\alpha r) = \varphi_0 \sum_{n=0}^{\infty} (i\omega)(2n+1)\hat{Y}_n(\omega \sigma)\hat{Y}_n(\omega \sigma)P_n(\cos \theta), \quad r < r_0,
\] (3.8)

where the time factor \(e^{-i\omega t}\) has been omitted.

Comparing this with the plane incident wave case we note that the only difference between the two cases is in the series representations. We recall from Eq. (2.10) that in the plane wave case

\[
\varphi^{(P)} = \varphi_0^{(P)} e^{i\omega \alpha \cos \theta} = \varphi_0^{(P)} \sum_{n=0}^{\infty} i^n(2n+1)\hat{Y}_n(\omega \sigma)P_n(\cos \theta),
\]

where the superscript \((P)\) is used to denote a plane wave. It is apparent then that the only difference between the two series is that for the spherical wave we have \(\hat{Y}_n(\omega \sigma)\) instead of \(i^n\) for the plane wave. Obviously the scattering analyses in Section 2 are applicable for the spherical incident wave.
Before discussing the scattering phenomena, we shall examine in detail the stress fields of a spherical wave and several limiting cases and compare them with those for a plane wave. Consider first that the source is located at a large distance \( r_o \) from the origin \( O \). If one observes the incident wave in the immediate neighborhood of the cavity, i.e., \( r_o \gg r \), the phase of Eq. (3.3) can be approximated by

\[
\varphi = \alpha_o \left[ 1 + \left( \frac{r}{r_o} \right)^2 + \frac{2r}{r_o} \cos \theta \right]^{1/2},
\]

\[
\approx \alpha_o \left[ 1 + \left( \frac{r}{r_o} \right) \cos \theta \right]. \tag{3.9}
\]

Thus as the ratio \( r/r_o \to 0 \), the spherical wave potential reduces to

\[
\varphi(t) = \left( \frac{\varphi_o \varepsilon}{r_o} \right) e^{i\alpha r \cos \theta - \omega t} = \varphi_o^{(P)} e^{i(\alpha r \cos \theta - \omega t)}, \tag{3.10}
\]

where

\[
\varphi_o^{(P)} = \frac{\varphi_o \varepsilon}{r_o}
\]

is assumed to be finite. If we call \( \varphi_o^{(P)} \) a coefficient for the plane wave, then we note that Eq. (3.10) is the representation of a plane wave propagating in the positive \( z \) direction. Thus it appears that if the region of observation is sufficiently far from the source, the observer cannot detect the spherical shape of the incident waves. This leads to the belief that an incident wave can be approximated
by a plane wave when the source is at a large distance from the observer. The validity of this point will now be examined.

The stress components generated by a plane wave propagating in the z direction, Eq. (2.1), are

\[
\sigma_{zz}^{(p)} = -\beta^2 \mu_0 \psi_0^{(p)},
\]

\[
\sigma_{xx}^{(p)} = \sigma_{yy}^{(p)} = \frac{\nu}{1 - \nu} \sigma_{zz}^{(p)}.
\]

At zero frequency \(\alpha, \beta \to 0\), the corresponding static stress field is

\[
\sigma_{zz}^{(p)} = -\sigma_F,
\]

\[
\sigma_{xx}^{(p)} = \sigma_{yy}^{(p)} = \frac{\nu}{1 - \nu} \sigma_{zz}^{(p)},
\]

where \(\sigma_F = \beta^2 \mu_0 \psi_0^{(p)}\), which is taken to be a nonzero constant.

The stresses generated by the spherical waves in Eq. (3.3) can be computed from Eq. (1.7). If the stresses are expressed in the coordinate system with \(\overline{0}\) as its origin, then

\[
\sigma_{zz}^{(y)} = \frac{\mu_0 \psi_0}{r^3} \left[4 - \beta^2 \frac{2-2}{r^3} - 4i\alpha r \right] e^{i\alpha r},
\]

\[
\sigma_{\phi\phi}^{(y)} = \frac{\mu_0 \psi_0}{r^3} \left[-2 + 2i\alpha r - (\alpha^2 - 2)\alpha^2 \frac{2-2}{r^3} \right] e^{i\alpha r},
\]

and so on, where \(\alpha^2 = (c_\phi/c_\phi)^2\). Now using the relationship between \(\overline{r}\) and \((r, \theta)\) in Eq. (3.6), the incident stress field is expressed in the coordinate system with \(0\) as its origin. The stresses become
\[ \sigma_{rr}^{(i)} = 2 \mu \varphi \rho_o^{-3} \left( \frac{r_o}{r} \right) \left[ - \frac{\kappa^2 a^2 r^2}{2} - 2i \omega \left( \frac{\rho_o}{r} \right) + (a^2 r^2 \sin^2 \theta + 2) \left( \frac{\rho_o}{r} \right)^2 \right] e^{iar} \]

\[ + 3i \omega \rho_o \sin^2 \theta \left( \frac{\rho_o}{r} \right)^3 - 3 \sin^2 \theta \left( \frac{\rho_o}{r} \right)^4 \] e^{iar} \quad (3.14a)

\[ \sigma_{\theta\theta}^{(i)} = 2 \mu \varphi \rho_o^{-3} \left( \frac{r_o}{r} \right) \left[ (1 - \kappa^2/2)a^2 r^2 + i \omega \left( \frac{\rho_o}{r} \right) - (a^2 r^2 \sin^2 \theta + 1) \right] \]

\[ \times \left( \frac{r_o}{r} \right)^2 - 3i \omega \rho_o \sin^2 \theta \left( \frac{\rho_o}{r} \right)^3 + 3 \sin^2 \theta \left( \frac{\rho_o}{r} \right)^4 \] e^{iar} \quad (3.14b)

and so on.

Comparison of these stresses with those of plane waves will now be made. First, we shall examine the incident stress field when the distance \( r_o \) is large as compared to \( r \); then we shall examine the long wavelength limit. When the oscillating source \( (\alpha \neq 0) \) is moved to infinity, we find that, by using Eq. (3.9) and letting \( r_o/r \to \infty \) in Eqs. (3.14a) and (3.14b), the stresses approach

\[ \sigma_{rr}^{(i)} \to -2a^2 \mu \varphi \rho_o^{-3} [(\kappa^2/2) - \sin^2 \theta] e^{iar \cos \theta} \quad (3.15a) \]

\[ \sigma_{\theta\theta}^{(i)} \to 2a^2 \mu \varphi \rho_o^{-3} [(1 - \kappa^2/2) - \sin^2 \theta] e^{iar \cos \theta} \quad (3.15b) \]

where

\[ \varphi \rho_o^{(P)} = \frac{\varphi \rho_o}{r_o}, \]

and so on. Note that Eq. (3.15) contains those terms in Eq. (3.14) involving \((ar_o)^2\), since they dominate for large \( ar_o \). The stresses given by (3.15), when converted to Cartesian coordinates, agree with
those given in (3.11) which are for the plane wave.

If we let \( a \to 0 \) \((\omega \to 0)\) we find that Eq. (3.14) becomes

\[
\sigma_{rr}^{(z)} = 2\mu r^{-3} \phi_0 \left[ 2 - 3 \sin^2 \theta \left( \frac{r_0}{r} \right)^2 \right],
\]

\[
\sigma_{\theta\theta}^{(z)} = -2\mu r^{-3} \phi_0 \left[ 1 - 3 \sin^2 \theta \left( \frac{r_0}{r} \right)^2 \right],
\]

(3.16)

and so on. These stresses are the same as the stresses created by a static center of dilatation defined (see Ref. 3.4 and 3.6) by

\[ u = v \phi, \quad \psi = \frac{\phi_0}{\bar{r}}. \]

If this static source is then moved to infinity, that is \( r_0/\bar{r} \to 1 \), we find the stresses in the Cartesian coordinates as

\[
\sigma_{zz} = -\sigma_0, \quad \sigma_{xx} = \sigma_{yy} = \frac{1}{3} \sigma_0,
\]

(3.17)

where \( \sigma_0 = -4\mu r^{-3} \phi_0 \) remains finite.

From the state of stress shown in Eq. (3.17) it is apparent that the static stress field derived from a center of dilatation is independent of Poisson's ratio, which is entirely different than the static state of plane waves -- see Eq. (3.12). This difference in static limiting values indicates that a spherical wave with small curvature (large \( r_0/r \)) may not be approximated by a plane wave at low frequencies. Further insight will be gained if we examine the limiting value of the series in Eq. (3.8) for large \( \omega r_0 \). Spherical Hankel functions with large argument have asymptotic values like
\[ h_n(z) \xrightarrow{z \to \infty} (i)^{-n-1} e^{i z / z}, \quad z > n. \]

Replacing \( h_n(\omega) \) by its asymptotic value and setting

\[ \psi(\nu) = \frac{\omega \nu}{\nu}, \]

we find that \( \psi(\nu) \) approaches \( \psi(\nu) \) as \( \omega \nu \) increases without bound.

There is still some ambiguity about the meaning of large \( \omega \nu \) as \( \alpha = (2\pi/\lambda) \) is the reciprocal of a fixed wavelength \( \lambda \), whereas \( \nu \) can be large or small depending on the location of the origin. We may clarify this point by recalling that the region of observation is in the neighborhood of the origin. Referring to Fig. 3.7 we see that for an observer at point \( P \), if \( \nu \ll \nu_0 \), then \( \omega \nu_0 \approx \omega \nu = 2\pi \nu / \lambda \). Large \( \omega \nu_0 \), then, means the wavelength is much shorter than the distance \( \nu \) between the source and the observer. Therefore, in the limit \( \omega \nu_0 \to \infty \), spherical waves are the same as plane waves so long as \( \alpha \neq 0 \). On the other hand, if \( \alpha \to 0 \) as \( \nu_0 \to \infty \), the product \( \omega \nu_0 \) is indeterminate, and it may be either zero, a constant, or infinity. When it is taken as infinity, the asymptotic value of the solution is the static state of plane waves. If we consider that the product approaches zero, the results of Eq. (3.16), and the equation's far-distance limit \( (\nu / \nu_0 \to 0) \), is the same as that for a static center of dilatation. Therefore, by taking different final values for \( \omega \nu \), we can arrive at two physically distinct problems.

The effects of wave curvature on the dynamic stress concentration
factors around the cavity will now be examined. First we shall normalize the stresses around the cavity by the stresses in the incident wave. The stress in the incident wave which we shall use for the normalization is $\sigma_{\theta\theta}^{(i)}$, since it is the maximum principal stress in the direction of propagation and since it is analogous to $\sigma_{zz}^{(i)}$ in the plane wave case. The difference lies, of course, in the fact that $\sigma_{\theta\theta}^{(i)}$ will decrease as a function of $r$, while $\sigma_{zz}^{(i)}$ is constant throughout the medium.

At the surface of the cavity, the nonzero principal stresses are $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$. The dynamic stress concentration factors are therefore defined as

$$
\sigma_{\theta\theta}^* = \left| \frac{\sigma_{\theta\theta}}{\sigma_{\theta\theta}^{(i)}} \right|_{r=a}, \quad \sigma_{\phi\phi}^* = \left| \frac{\sigma_{\phi\phi}}{\sigma_{\phi\phi}^{(i)}} \right|_{r=a}.
$$

(3.16)

The static stress concentration factors due to a center of dilatation, which Eq. (3.18) should reduce to when $a \to 0$, have been obtained by Bose (3.5) and Moon and Pan (3.6). They are

$$
\sigma_{\theta\theta} = \sigma_0 \left[ 1 - 3 \left( \frac{\rho}{a} \right)^2 \sin^2 \theta \right] \left( \frac{a}{r} \right)^3 + \sigma_{\theta\theta}(I); \quad (3.19a)
$$

$$
\sigma_{\theta\theta}(I) = \sigma_0 \sum_{n=0}^{\infty} \gamma_n \left\{ \eta \left( \frac{3\eta^2}{n} + 2n - \nu(2n + 1) \right) \frac{\mu}{n} \cos \theta \right\} \left( \frac{a}{r} \right)^3 + \sigma_{\theta\theta}(I) \cot \theta \frac{d\Gamma_n}{d\theta}; \quad (3.19b)
$$

$$
\sigma_{\phi\phi} = \sigma_0 \left( \frac{\alpha}{r} \right)^3 + \sigma_{\phi\phi}(I); \quad (3.20a)
$$
\begin{align*}
\sigma_{\phi\phi}(I) &= - \sigma_0 \sum_{n = 0}^{\infty} \gamma_n \left\{ n[3n + 1 - 2\nu(2n + 1)^2]P_n(\cos \theta) \right. \\
&\quad \left. + [3n + 2 - 2\nu(2n + 1)]\cot \theta \frac{dP_n(\cos \theta)}{d\theta} \right\}, \quad (3.20b)
\end{align*}

where

\[ \gamma_n = \left( \frac{a}{r_0} \right)^{n+1} \frac{(n - 1)/n^2 + n + 1 - \nu(2n + 1)}{.} \]

and

\[ \sigma_0 = \frac{2\mu \psi_o}{a^3}. \]

The first terms in \( \sigma_{\theta\theta} \) and \( \sigma_{\phi\phi} \) are those due to the center of dilatation alone in an infinite medium. The terms \( \sigma_{\theta\theta}(I) \) and \( \sigma_{\phi\phi}(I) \) are the image stresses added to meet the traction-free conditions on the cavity. As a point of interest, we will compare the static stress concentration factors around the cavity due to a center of dilatation at large \( r_o/r \), and the stress field corresponding to the static state of a plane wave.

Letting \( r/r_o \to 0 \) in Eqs. (3.19) and (3.20), we find that the maximum stress concentration due to a center of dilatation is at \( \theta = \pi/2 \), and it is

\[ \frac{\sigma_{\theta\theta}}{\sigma_o} = (SCF)_{\theta} = \frac{15(2 - \nu)}{2(7 - 5\nu)}. \]

The maximum stress concentration due to the static state of a plane wave is also at \( \theta = \pi/2 \), and it is

\[ (SCP)_{\theta} = \frac{3(9 - 16\nu + 5\nu^2)}{2(1 - \nu)(7 - 5\nu)}. \]
For \( \nu = 0.3 \) we find \((SCF)_S = 2.32\) and \((SCF)_P = 1.81\). \((SCF)_S\) is 28 per cent larger than \((SCF)_P\). This increase in maximum value of stress concentration is due entirely to the difference in the applied stress fields for a center of dilatation and for the static state of a plane wave. Figure 3.8 illustrates the static stress concentration factors for values of \( r_o/a = 4, 50 \). For the case of \( r_o/a = 4 \), since both \( \sigma_{PP}^{(i)} \)

\[ \sigma_{PP}^{(i)} = \frac{a_0}{a} \sigma_{op}^{(i)} \]

\[ = 10a^{3} \sigma_{op}^{(i)} / \mu \psi_0 \]

Fig. 3.8. Stress Concentration Factors vs. \( \theta \) on the Cavity, Static Case \( (\nu = 0.3)\)

and \( \sigma_{00} \) vary from point to point along the circumference of the cavity, both \( \sigma_{00} \) and \( (SCF)_S \) are shown. When \( r_o/a = 50 \), the variation of \( c_{PP}^{(i)} \) is very small and is ignored. As can be seen, the stress distribution due to a center of dilatation at an arbitrary distance differs substantially from the static limit due to a plane wave — see Figs. 3.1
and 3.8. The static stress factors in Fig. 3.8 should be the static limit \((\alpha = 0)\) for dynamic stress concentration factors.

![Diagram showing stress concentration factors](image)

**Fig. 3.9. Stress Concentration Factor vs. \(\theta\) on the Cavity, Spherical Wave Case \((\nu = 0.3)\)**

Figures 3.9 and 3.10 show the dynamic stress concentration factors around the cavity for \(r_0/a = 50\) and 4, respectively, for \(\nu = 0.30\). Because of the variation of \(\sigma_{\theta \theta}(\xi)\) we have chosen to show both \(\sigma_{\theta \theta}^*\) and \(\sigma_{\phi \phi}^*\) as well as \(\sigma_{\theta \theta}\) and \(\sigma_{\phi \phi}\). Absolute values are used to give the largest value at each point during one complete cycle.

In general, we note there is no symmetry about \(\theta = \pm \pi/2\) for spherical waves. Only in the case of large \(r_0/a\) and small \(\alpha\) is there a near symmetry, as Fig. 3.9 shows. Variations of \(\sigma_{\theta \theta}^*\) and \(\sigma_{\phi \phi}^*\) as a function of \(\alpha\) and \(r_0/a\) are shown in Fig. 3.11 \((\nu = 0.3)\) with a portion of small wave numbers enlarged in Fig. 3.12. The curve
Curvature of incident wave

Fig. 3.10. Stress Concentration Factor vs. $\theta$ on the Cavity, Spherical Wave Case ($\nu = 0.3$)

Fig. 3.11. Stress Concentration Factor vs. Wave Number for Plane and Spherical Waves ($\nu = 0.3$)
shown in Fig. 3.11 for \( r_o/a = 3 \) is at 115 degrees, which is near the point of maximum stress for this value of \( r_o/a \). Figure 3.12 shows different values of \( r_o/a \) ranging from 3 to 50 in the range of \( \alpha \) from 0 to 1.0. Because the positions of maximum stress on the cavity vary with \( r_o/a \), the curves are computed at various positions near the point where the maximum may occur for each \( r_o/a \).

For a plane wave, \( |\sigma_{00}^k| \) reaches a maximum value of 1.95 at \( \alpha \) = 0.55 for \( v = 0.25 \), as shown in Fig. 3.3. For an incident spherical wave, \( |\sigma_{00}^k| \) depends on \( r_o/a \) as well as on \( \alpha \). As is expected, all curves begin at \( \alpha = 0 \) with the static values shown in Table 3.1, the lowest being 2.32 for \( r_o/a = \infty \) which as noted before is 28 percent higher than the plane wave cases. As \( \alpha \) increases each curve reaches a peak and then gradually approaches that for a plane wave. The particular wave number at which a plane wave approximation may be
Table 3.1

<table>
<thead>
<tr>
<th>$r_0/\alpha$</th>
<th>$\theta$, deg</th>
<th>$-\sigma_{56}/\sigma_{RR}$</th>
</tr>
</thead>
<tbody>
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<td>$\infty$</td>
<td>90</td>
<td>2.318</td>
</tr>
<tr>
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</tr>
<tr>
<td>4</td>
<td>70</td>
<td>2.385</td>
</tr>
</tbody>
</table>

applied is dependent upon the values of $r_0/\alpha$. The larger $r_0/\alpha$, the lower of $\alpha \alpha$. We note that when $r_0/\alpha = 50$ the spherical wave solution becomes almost indistinguishable from the plane wave solution at $\alpha \alpha \approx 0.15$; however, at $r_0/\alpha = 6$, the value of $\alpha \alpha$ is approximately 1.0. All calculations indicate that a plane wave approximation is valid if the wavelength $\lambda$ is much shorter than the distance $r_0$ between the source and the center of the cavity.

4. TRANSIENT RESPONSE OF SPHERICAL INCLUSIONS

4.1. Transient Motion of a Rigid Inclusion

We shall conclude our discussion of the spherical problems by examining some transient phenomena. There are only a few transient spherical solutions existing in the literature, and most of them have appeared only very recently.

We will discuss here a transient problem which is of theoretical interest and which has some practical importance in the design of accelerometers and velocity gauges for measuring ground shock.
In Section 2 we obtained the steady state solution for the rigid inclusion problem. It was shown that the rigid inclusion has a rigid body translational motion, resulting from the impact of the incident wave. It was further demonstrated that the rigid body motion is contributed only by the $\kappa = 1$ term in the series in Eq. (2.13a,b). The unknown coefficients $A_1$ and $B_1$ as determined by Eq. (2.14), after considerable amount of algebraic simplification, are:

$$A_1 = -\frac{3\psi^2}{\lambda_1} \left[ a_\alpha^2 j_2(\alpha \sigma) h_2(\beta \alpha) - 2(1 - \overline{\rho}) a_\alpha j_2(\alpha \sigma) h_1(\beta \alpha) - (1 - \eta) 2a_\alpha j_1(\alpha \sigma) h_2(\beta \alpha) \right];$$  \hspace{1cm} (4.1a)

$$B_1 = \frac{3\psi(1-\overline{\rho})}{\alpha a h_\lambda};$$ \hspace{1cm} (4.1b)

$$\lambda_1 = 2(1 - \overline{\rho}) a_\alpha h_1(\beta \alpha) h_2(\alpha \sigma) - (1 - \overline{\rho}) 6a_\alpha h_1(\alpha \sigma) h_2(\beta \alpha) + a_\beta a^2 h_2(\alpha \sigma) h_2(\beta \alpha);$$ \hspace{1cm} (4.1c)

where $\overline{\rho}$ is the density ratio of the medium to the inclusion and

$$\kappa = \rho / \sigma.$$ \hspace{1cm} The resulting rigid body motion of the inclusion due to an incident harmonic dilatational wave is

$$u_\gamma = \frac{3\kappa_1 \psi h_2(\beta \alpha) e^{-i\omega t}}{a \left\{ + 2(1 - \overline{\rho}) a_\alpha h_1(\beta \alpha) h_2(\alpha \sigma) - (1 - \overline{\rho}) 6a_\alpha h_1(\alpha \sigma) h_2(\beta \alpha) + a_\beta a^2 h_2(\alpha \sigma) h_2(\beta \alpha) \right\}}.$$

(4.2)

Normalizing $u_\gamma$ with respect to the magnitude of the displacement
in the incident wave, which is \( i \varphi_0 a \), we obtain the normalized displacement

\[
\overline{u}_z = \frac{-3i \mu \omega \kappa_2 (8\alpha)}{a a^3} e^{-i \omega t}.
\]  

(4.3)

It is of some interest to note that both wave numbers are present in Eq. (4.3). However, if \( \overline{\varrho} = 1 \), that is, if the mass density of the medium and inclusion are the same, then Eq. (4.3) degenerates into

\[
\overline{u}_z = \frac{-3i \varrho e^{-i \omega t}}{a a^3} \kappa_2 (a \varrho).
\]  

(4.4)

It is clear, then, that for \( \overline{\varrho} = 1 \) the rigid body motion of the inclusion is independent of all material properties, and is only a function of the incident wave number \( (a \varrho) \). Furthermore, as shown in both Eqs. (4.1b) and (4.4), the scattered shear wave vanishes for \( \overline{\varrho} = 1 \).

Returning now to the case of arbitrary \( \overline{\varrho} \), Eq. (4.3) will be expressed in another form, for reasons that shall soon be apparent. We recall from Eq. (1.18) that the spherical function of order \( n \) can be expressed as

\[
\kappa_n^{(1)}(\zeta) = -i (-1)^n \zeta^n \left( \frac{d}{\zeta d \zeta} \right)^n e^{i \zeta}.
\]

Thus

\[
h_1(\zeta) = -i \zeta^{-2} (i \zeta - 1) e^{i \zeta},
\]  

(4.5)

\[
h_2(\zeta) = -i \zeta^{-3} (-\zeta^2 - 3i \zeta + 3) e^{i \zeta}.
\]
Substituting Eq. (4.5) into Eq. (4.3), we obtain

\[ \xi_{\alpha} = \chi e^{-i\omega t}, \]  

(4.6)

where

\[ \chi = 3\zeta (-\kappa^2 \alpha^2 \alpha^2 - 3i\kappa \alpha + 3)\alpha^{-1}b^2 \alpha^4 + i\kappa[(2\kappa + 1) + \bar{\beta}(\kappa + 2)] \]

\[ \alpha^3 - [(2\kappa^2 + 1) + \bar{\rho}(\kappa^2 + 9\kappa + 2)]\alpha^2 - 9i\zeta(\kappa + 1)a\alpha + 9\rho \]

(4.7)

\[ \text{Fig. 4.1. Re} \ \overline{U}(\alpha \alpha) \ vs. \ \alpha \alpha \ (\text{Spherical Inclusion}) \]

According to the Fourier synthesis method outlined in Chapter II, the function \( \chi \) is the admittance function that can be used to determine the transient response of the inclusion. The behavior of \( \chi \) as a function of the wave number \( \alpha \alpha \) is shown in Figs. 4.1 and 4.2 for \( \nu = .25 \). Before we proceed to determine the transient response a further note is appropriate. The function \( \chi \) is derived for the displace-
Fig. 4.2. Im \( \bar{U}(oo) \) vs. oo (Spherical Inclusion)

ment; quite frequently, however, the particle velocity is of more interest — thus if we differentiate Eq. (4.6) with respect to the time, we obtain the velocity of the inclusion as

\[
\frac{du}{dt} = -i\omega \bar{U}.
\]

Now if we normalize \( \frac{du}{dt} \) with respect to the magnitude of the incident particle wave, \( |\bar{U}| \), we find that the admittance function \( \chi \) is the same for displacement and velocity. Carrying out the reasoning one step further, it can be shown that the same \( \chi \) applies for the acceleration also.

Having determined the admittance, it is now possible to determine the transient response.

Consider a decaying shock propagating in the positive \( s \) direction. At the plane \( s = 0 \) (origin of the sphere), the particle velocity of a decaying wave may be expressed as
\[ v(t) = \begin{cases} 
0, & t < 0, \\
\nu e^{-\lambda t}, & t > 0.
\end{cases} \quad (4.8) \]

The Fourier transform of the decaying shock is

\[ \tilde{v}(\omega) = \frac{\nu(\omega)}{\nu_0} = \frac{1}{\sqrt{2\pi}} \frac{i}{\omega + i\lambda}. \quad (4.9) \]

It follows that the transient response of the rigid inclusion is

\[ \tilde{\nu}_e(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(\omega) \tilde{v}(\omega) e^{-i\omega t} \, dw. \quad (4.10) \]

Equation (4.10) can be made into a dimensionless form by letting

\[ \tau = \frac{c_p}{a} t, \]
\[ \lambda_c = \frac{a}{c_p} \lambda, \]
\[ dw = \frac{c_p}{a} d(\omega c). \]

It follows that the dimensionless velocity of the inclusion is

\[ \tilde{\nu}_e(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3i\omega(-\kappa^2 a^2 \omega^2 - 3i\kappa \omega + 3)e^{-i(1 + \tau)\omega\omega} d(\omega a)}{(\omega a + i\lambda_c) \left[ \kappa^2 a^4 \omega^4 + i\kappa[(2\kappa + 1) + \bar{\rho}(\kappa + 2)]a^2 \omega^2 + 9 \right]} \]
\[ - \left[ (2\kappa^2 + 1) + \bar{\rho}(\kappa^2 + 9\kappa + 2) \right] a^2 \omega^2 - 9 \kappa(\kappa + 1) a^2 + \bar{\rho} \right]. \quad (4.11) \]
Equation (4.11) can be integrated by the method of contour integration. Letting

$$\alpha = \Re \zeta,$$

in which

$$\zeta = \xi + i\eta,$$

the integral shown in Eq. (4.11) becomes

$$\overline{u}(\tau) = \frac{2\pi}{\tau} \int_{\mathcal{C}} (-\kappa^2 \zeta^2 - 3i\kappa\zeta + 3)e^{-i(1+i)\zeta} \frac{d\zeta}{(\zeta + i\lambda_0)\zeta} \left\{ \kappa^2 \zeta^4 + i\kappa \right\}$$

$$\times \left[ \frac{\sqrt{\rho(\kappa + 2)}}{\sqrt{\rho(\kappa + 2)}} \right]^{3} - \left[ \frac{\sqrt{\rho(\kappa^2 + 1)}}{\sqrt{\rho(\kappa^2 + 1)}} \right]^{2}$$

$$\times \zeta^2 - i9\sqrt{\rho(\kappa + 1)}\zeta + 9\sqrt{\rho},$$

(4.12)

where $\mathcal{C}$ is an appropriate contour to be determined by the singularities of the integrand. The integrand in this case is regular except for a finite number of poles. In general there are five simple poles. One pole due to the term $1/(\zeta + i\lambda_0)$ is contributed by the incident wave -- see Eq. (4.9) -- and the other four poles, due to the fourth-order polynomial, are contributed by the admittance function $\chi$.

Here we denote the location of the five poles as $\zeta = -i\lambda_0, \lambda_1, \lambda_2, \lambda_3,$ and $\lambda_4$, where $\lambda_1, \ldots, \lambda_4$ are the roots obtained by setting the fourth-order polynomial in the denominator of the integrand equal to zero; $-i\lambda_0$ is due to the incident pulse.

Denoting the integrand by $f(\zeta)$, we note that $f(\zeta)$ can first be reduced to the following form:
\[
f(\zeta) = \frac{(-\kappa^2 \zeta^2 - 3i \kappa \zeta + 3)e^{-\zeta(1 + \tau)}}{\kappa^2(\zeta + i\lambda_0)(\zeta^2 + p_1 \zeta + q_1)(\zeta^2 + p_2 \zeta + q_2)},
\]

where

\[
p_1 + p_2 = \frac{i}{\kappa} [(2\kappa + 1) + \bar{\rho}(\kappa + 2)],
\]

\[
q_1 + q_2 + p_1^2 = -\frac{1}{\kappa^2} [(2\kappa^2 + 1) + \bar{\rho}(\kappa^2 + 9\kappa + 2)],
\]

\[
p_1 q_2 + p_2 q_1 = -9i\rho \frac{(\kappa + 1)}{\kappa^2},
\]

\[
q_1 q_2 = \frac{9\rho}{\kappa^2}.
\]

Once \(p_1, p_2, q_1,\) and \(q_2\) are determined the poles of \(f(\zeta)\) are found. Equation (4.14) indicates that \(p_1, ..., q_2\) are functions of \(\rho\) and \(\kappa\) only. It follows then that the locations of the poles are only a function of \(\rho\) and \(\kappa\).

Examination of Eq. (4.12) shows that the roots \(\lambda_1, ..., \lambda_4\) can be either case 1 — two complex pairs, i.e. \((\lambda_1, -\lambda_1)\) and \((\lambda_2, -\lambda_2)\), where \(\lambda_1\) is the complex conjugate of \(\lambda_1\); or case 2 — one complex pair and two pure imaginary roots. Whether the roots are two complex pairs or one complex pair and two pure imaginary numbers will depend upon the values of \(\rho\) and \(\kappa\). Except for a special case, where a pole of order 2 may occur, all poles are simple and lie in the lower half of the \(\zeta\)-plane.

Having determined the types of the singularities in the integrand, the appropriate contour \(C\) for the contour integration is shown in Fig. 4.3. We remark in passing that the exponential \(e^{-i(1 + \tau)\zeta}\) in the
The integrand vanishes as $|\zeta| \to \infty$ on the upper semicircle when $(1 + \tau) < 0$, and on the lower semicircle when $(1 + \tau) > 0$. Since the integrand is regular except for a finite number of poles within $C$, Eq. (4.12) can be evaluated using Jordan's lemma and Cauchy's residue theorem to obtain

$$\int_C f(\zeta) \, d\zeta = 2\pi i \sum_{\eta=1}^{N} R_\eta, \quad \text{for } \tau > -1,$$

(4.15)

$$= 0, \quad \text{for } \tau < -1,$$

where $R_\eta$ is the residue. Finally, we have

$$\tilde{u}_z(\tau) = 3\rho \sum_{\eta=1}^{N} R_\eta, \quad \text{for } \tau > -1,$$

(4.16)
Also to be noted, for example, is that \( R(\lambda_1) = \overline{R(\lambda_1)} \). It follows then that \( R_2(\lambda_1) + R_3(\lambda_1) = 2 \Re R_2(\lambda_1) \).

Evaluations of residues are straightforward. By carrying out the necessary algebraic manipulations we find for case I that the roots are two complex pairs:

\[
\bar{u}_2(\tau) = \frac{3\pi}{\kappa^2} \left\{ \frac{(-\kappa \lambda_0^2 + 3\kappa \lambda_0 - 3)e^{-(1 + \tau)\lambda_0}}{(\lambda_0^2 - \bar{p}_1\lambda_0 - q_1)(\lambda_0^2 - \bar{p}_2\lambda - q_2)} \right. \\
+ \frac{2}{\xi} \sum_{n=1}^{\infty} \frac{\eta_n(1 + \tau)}{\xi_n(\lambda_0^2 - \xi_n\lambda_0 - q_n)^{1/2}} \cos \left( (1 + \tau)\xi_n \xi_n + \xi_n \right) \left( (1 + \tau)\xi_n \eta_n + \eta_n \right) \right\},
\]

where

\[
\bar{p}_n = -i\bar{p}_n, \\
\bar{a}_n = |a_n + i\eta_n|, \\
\bar{B}_n = |b_n + i\eta_n|, \\
\lambda_n = \xi_n + i\eta_n, \quad n = 1,2,
\]

in which

\[
a_n = \kappa^2 q_n + \frac{\lambda \bar{a}_n (\lambda \bar{a}_n - 3)}{2} + 3, \\
\lambda \bar{a}_n = \lambda \bar{a}_n (\lambda \bar{a}_n - 3), \\
b_1 = (q_2 - q_1) + \frac{\bar{p}_1(\bar{p}_2 - \bar{p}_1)}{2}, \\
b_2 = (q_1 - q_2) + \frac{\bar{p}_2(\bar{p}_1 - \bar{p}_2)}{2}, \\
\beta_1 = \xi_1(\bar{p}_2 - \bar{p}_1), \quad \beta_2 = \xi_2(\bar{p}_1 - \bar{p}_2),
\]

\[
\text{DIFFRACTION AND STRESS CONCENTRATIONS}
\]
and
\[
\varepsilon_{\eta} = \tan^{-1} \frac{\xi_{\eta}(\alpha_{\eta} \eta - \alpha_{n} \eta) + (\alpha_{n} \eta + \alpha_{\eta} \eta)(\lambda_{0} + \eta_{\eta})}{\xi_{\eta}(\alpha_{n} \eta + \alpha_{\eta} \eta) - (\alpha_{n} \eta - \alpha_{\eta} \eta)(\lambda_{0} + \eta_{\eta})}.
\]

For case II we find that the roots are one complex pair and two pure imaginary numbers:

\[
\frac{\overline{B}_{2}(\tau)}{\lambda} = \frac{3\nu}{\lambda} \left\{ \frac{(-\xi_{0}^{2} \lambda_{0}^{2} + 3\eta_{0} \lambda_{0} - 3)e}{(\lambda_{0}^{2} - \overline{p}_{1} \lambda_{0} - q_{1})(\lambda_{0}^{2} - \overline{p}_{2} \lambda_{0} - q_{2})} \right. \\
+ \frac{\eta_{1}(1 + \tau)}{\lambda_{1}} \frac{\cos[(1 + \tau)\xi_{1} + e_{1}]}{\xi_{1}(\lambda_{0}^{2} - \overline{p}_{1} \lambda_{0} - q_{1})^{2}} \\
+ \frac{(\xi_{1}^{2} \eta_{3}^{2} + 3\eta_{3} \eta_{3})e}{(\lambda_{0} + \eta_{3})(\eta_{3} - \eta_{4})(\eta_{3}^{2} + \overline{p}_{1} \eta_{3} - q_{1})} \\
+ \frac{(\xi_{1}^{2} \eta_{4}^{2} + 3\eta_{4} \eta_{4})e}{(\lambda_{0} + \eta_{4})(\eta_{4} - \eta_{3})(\eta_{4}^{2} + \overline{p}_{2} \eta_{4} - q_{2})} \right\},
\]

where \(A_{1}, B_{1}, \) and \(e_{1} \) and \(n_{1} \) are the same as defined before, and \(n_{3} \) and \(n_{4} \) are the pure imaginary roots, i.e., \(\lambda_{3} = i\eta_{3}, \lambda_{4} = i\eta_{4}.\)

In the foregoing discussion we have shown there are two distinct types of response for the rigid inclusion caused by the differences in the characteristics of the roots in the admittance function. The values which govern the response are \(\Gamma_{n}, q_{n}, \) and \(\lambda_{1}, \ldots, \lambda_{4}, \) and they are dependent on the values of \(\overline{\rho} \) and \(\kappa.\) Shown in Figs. 4.4 through 4.7 are the behaviors of \(\overline{\Gamma}_{n}, \overline{q}_{n}, \) and \(\lambda's \) as a function of some typical
Fig. 4.4. Coefficients of the Quadratic Factors $p_1$ and $q_1$ as Functions of the Density Ratio $\rho$

Fig. 4.5. Coefficients of the Quadratic Factors $p_2$ and $q_2$ as Functions of the Density Ratio $\rho$

Fig. 4.6. Root $\lambda_1$ of Fourth-Order Polynomial as a Function of the Density Ratio $\rho$
values of $\bar{\rho}$ and $\kappa$. In particular, we observe the behavior of the pair of the roots of the denominator of the admittance function $(\lambda_2, -\bar{\lambda}_2)$ as a function of $\bar{\rho}$ in Fig. 4.7. We note that the real part of this pair of roots vanishes and bifurcation occurs in the imaginary part as $\bar{\rho}$ increases beyond 1.4. The two branches are denoted by $\lambda_3$ and $\lambda_4$. The exact values of $\bar{\rho}$ where bifurcation occurs will depend on the value of $\kappa$. For the three values of $\kappa^2 (= 2.67, 3.0, 3.5)$ which correspond to Poisson's ratio of $\nu = 0.15, 0.25, 0.35$, the bi-
furcation point \( \rho_b \) lies between \( \bar{\rho} \) equal to 1.4 and 1.5, with higher values of \( \rho_b \) associated with higher values of \( \kappa \).

According to our earlier discussion, then, when \( \bar{\rho} < \rho_b \), Eq. (4.17) should be used for the transient response, and when \( \bar{\rho} > \rho_b \), Eq. (4.20) should be used. (The density ratio \( \rho_b \) is defined as the value of \( \rho \) at bifurcation.) Also, it should be noted that the argument \((1 + \tau)\) in the function \( \bar{u} \) indicates a time shift of a half-transit time, i.e., the motion starts at \( \tau > -1 \). This becomes clear if we recall that \( \tau = 0 \) is the time when the incident wave arrived at \( z = 0 \). However, the motion of the inclusion must begin at \( \tau = -1 \), when the wave has arrived at the incident side of the inclusion.

The behavior of \( \bar{u}(\tau) \), \( \bar{u}(z) \), and \( \bar{u}(\tau) \) for various decay rates and values of \( \bar{\rho} \) and \( \kappa \) are shown in Figs. 4.8 through 4.10. The effects of \( \bar{\rho} \) on the velocity response are apparent in Figs. 4.8a, b, c.
Even at $\bar{\rho} = 1$, the velocity of the inclusion differs substantially from that of the free field at early time. When $\bar{\rho} > 1$ there is apparently no overshoot, while for $\bar{\rho} < 1$ the velocity of inclusion can be greater than the free field values. The difference between
the displacements of the inclusion and the free field is small. The most pronounced effects are, however, in the acceleration response, as shown in Fig. 4.10. The free field acceleration for the pulse
given in Eq. (4.8) is

\[ \dot{y}(t) = \delta(t) - \lambda_0 \mathbf{e}^{-\lambda_0 t}, \]

where \( \delta(t) \) is the Dirac delta function. Thus the acceleration of the free field for an idealized pulse is characterized by a sudden jump and followed by a monotonically decreasing deceleration. The acceleration of the inclusion however does not exhibit the same discontinuous behavior as the free field. It always starts from zero at \( t = -1 \), reaches a positive peak and decreases, and then tends to oscillate about the free field acceleration curve.

4.2. Inverse Problem

Hitherto, we have treated the problem of a rigid inclusion in a known "free field" motion. It was shown that diffraction always occurs. As a result of diffraction, the motion of the inclusion is substantially different from that of the "free field." Furthermore, there is very little one can do to eliminate this diffraction phenomenon. Thus an interesting question arises — how can one extract the free field motion from the motion that includes the diffraction effects? We shall now address ourselves to this.

Let us assume that \( \bar{y}(\tau) \), the motion of the inclusion, is known, (for instance, there might be actual recorded data of the particle velocity or the acceleration). The Fourier transform of \( \bar{y}(\tau) \) is

\[ \bar{y}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{y}(\tau) e^{i\xi\tau} d\tau. \]  

(4.21)
But we recall that \( \overline{\mathcal{u}}(\tau) \) is the Faltung of the free field motion \( \overline{v}(\tau) \), and the admittance function \( \chi \).

It follows, then, that

\[
\overline{v}(\tau) = \frac{\overline{u}(\tau)}{\chi(\tau)},
\]

(4.22)

where \( \overline{v}(\tau) \) is the Fourier transform of the free field motion. If we take the Fourier inversion of Eq. (4.22), we obtain the "free field motion." Accordingly

\[
\overline{v}(\tau) = \int_{-\infty}^{\infty} \frac{\overline{u}(\zeta)}{\chi(\zeta)} e^{-i\zeta \tau} d\zeta.
\]

(4.23)

Substituting the expression of \( \chi(\zeta) \) in Eq. (4.23), we have

\[
\overline{v}(\tau) = \text{Re} \left\{ \frac{1}{3\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} \frac{\chi^2 \zeta^4 + \overline{\chi}(2\chi + 1) + (\chi + 2)\zeta^2}{(-\chi^2 \zeta^2 - 3i\chi + 3)} e^{-i\zeta (\tau - 1)} d\zeta \right\}
\]

\[
+ \left\{ \frac{(2\chi^2 + 1) + (\chi^2 + 9\chi + 2)\zeta^2 - y_0 \chi (\chi + 1)\zeta + y_0}{(-\chi^2 \zeta^2 - 3i\chi + 3)} \right\}
\]

\times \overline{u}(\zeta) e^{-i(\tau - 1)\zeta} d\zeta.
\]

(4.24)

Equation (4.24) is of the improper fraction type. However, we may divide the numerator by the denominator until the fraction becomes proper, i.e., the power of \( \zeta \) in the numerator becomes less than the power of \( \zeta \) in the denominator. Following this division, and using the relationship
\[ (-\zeta) \overline{\nu}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2 \overline{u}(\tau)}{d\tau^2} \, e^{-i\zeta \tau} \, d\tau, \]

Eq. (4.24) becomes

\[ \overline{v}(\tau) = \frac{1}{3\rho} \left[ d^2 \overline{u}(\tau - 1) + \frac{k(2 + \rho)}{k} + 2(\rho - 1) \frac{d\overline{u}(\tau - 1)}{d\tau} \right. \]

\[ + \frac{k^2(2 + \rho) + (1 - \rho)(4 - 6k)}{k^2} \overline{u}(\tau - 1) \]

\[ \left. + \int_{1}^{\tau} \hat{h}^*(\tau - 1 - (\hat{t} - 1)) \overline{u}(\hat{t} - 1) \, d\hat{t} \right], \quad (4.25) \]

where

\[ \hat{h}^*(\tau - 1) = 2(1 - \frac{\rho}{\kappa}) e^{\frac{2}{3}\kappa(\tau - 1)} \left[ \frac{\sqrt{3}}{2\kappa} \cos \frac{\sqrt{3}}{2\kappa} (\kappa - 1) \right. \]

\[ + (1 - \kappa^2) \sin \frac{\sqrt{3}}{2\kappa} (\kappa - 1) \right]. \quad (4.26) \]

If \( \rho = 1 \), then the expression for the free field motion becomes

\[ \overline{v}(\tau) = \frac{1}{3} \frac{d^2 \overline{u}(\tau - 1)}{d\tau^2} + \frac{d\overline{u}(\tau - 1)}{d\tau} + \overline{u}(\tau - 1). \quad (4.27) \]

It is of interest to note that the argument \((\tau - 1)\) in the function \( \overline{u} \) indicates a retardation of the half-transit time. We may recall that in the response problem the time \( \hat{t} = 0 \) was chosen when the incident wave had just arrived at \( z = 0 \). However, for the case of the inverse problem, we have chosen the time \( \tau = 0 \) when the wave had just arrived at the incident side of the inclusion. Therefore, as is to be expected, the free field history should lag one half-transit
time from the response of the inclusion.

Equation (4.25) also shows that the free field motion depends on the motions of the inclusion, on the first and second derivatives of the motion, and on the past history of the motion of the inclusion -- the last integral in Eq. (4.25). However, the effects of the past history are eliminated if we choose \( \rho = 1 \).

4.3. Stresses Around a Cavity -- Step Plane P Wave

Consider now another limiting case of a general spherical inclusion; the cavity. In Section 3, we discussed in detail the boundary stresses around a spherical cavity due to a plane harmonic-compressional wave. The stresses as given by Eqs. (3.1) and (3.2) are a function of the incident wave number, and they are called the admittance function of the system. It has been shown in Section 4 of Chapter I and in Section 5 of Chapter III that once the admittance function of the system is known, the transient solutions, in general, may be determined by means of Fourier synthesis, or other approximate methods (see Chapter III, Section 8). We shall now utilize the solution obtained in Section 3 to construct a transient solution of the stresses around a cavity due to a step plane P wave.

The incident step plane P wave may be represented as

\[
\sigma^{(i)}_{22} = H(\hat{t}),
\]

where \( \hat{t} = t - \left( \frac{z + a}{c} \right) \), and \( t \) is zero when the wave has just arrived at \( z = -a \). \( z = 0 \) is chosen as the origin of the spherical coordinate system. As was discussed in detail in Chapter I, Section 4, and in
Chapter III, Section 5, the transient response of a linear system $g(x, \hat{t})$ due to an input $f(\hat{t})$ is determined by:

$$g(x, \hat{t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x, \omega) F(\omega) e^{-i\omega \hat{t}} d\omega,$$

(4.28)

where $\chi$ is the admittance function and $F(\omega)$ is Fourier transform of the function $f(\hat{t})$. Thus the transient behavior of $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ due to a unit step pulse at the boundary of the cavity may be determined by substituting Eqs. (3.1) and (3.2), together with the Fourier transform of $F(\hat{t})$, into Eq. (4.28), and carrying out the necessary inversion. Here, we shall omit many of the details since they were presented in Chapter III, Section 4, and we merely state that the inversion is accomplished by a contour integration along an appropriate contour $C$. The choice of the contour $C$ is dictated by the singularities of the integrand. The singularities of the integrand in this case are a simple pole at the origin of the complex plane due to the incident wave, and simple poles at the roots of the following transcendental equation:

$$\begin{vmatrix}
E_{11}^{(3)}(\alpha a) & E_{12}^{(3)}(\alpha a) \\
E_{41}^{(3)}(\alpha a) & E_{42}^{(3)}(2\alpha a)
\end{vmatrix} = 0,$$

(4.29)

which is the determinant in Eq. (2.25) and the common denominator for Eqs. (3.1) and (3.2) contained within the coefficients $A_n$ and $B_n$.

Since the spherical Bessel function $s_n^{(i)}(\xi)$ does not have a branch
point, there are no branch points in the integrand. It is clear then that Eqs. (3.1) and (3.2) are meromorphic functions of $\zeta$ in the whole $\zeta$ plane (where $\zeta = \zeta + i\eta$, Re $\zeta = \alpha a$). (See notations given in Chapter III, Section 5.) It follows that the transient response of $\sigma_{60}(\tau)$ and $\sigma_{\phi}(\tau)$, where $\tau = \frac{a_p}{a} \tau$, is due to the residues of the simple poles located at the origin and at the roots of Eq. (4.29). Since the poles are all simple, computation of the residues is straightforward once the location of the poles is determined.

![Graph showing location of poles with $v = 0.25$](image)

**Fig. 4.11. Location of Poles, $v = 0.25$**

Figure 4.11 shows the location of the poles of Eq. (4.29) in a complex $\zeta$-plane for $n = 0...5$ and $v = 0.25$. We note that for $n = 0,1$ there is only one pair of poles, for $n = 2,3$ there are three pairs of poles and for $n = 4,5$ there are five pairs of poles. A similar finding was noted before in the cylindrical cavity case (Chapter III, Section 5). In addition to the similarity in the number of pairs of poles for each $n$, we notice also the similarities in the location of the poles in the
complex plane for a given \( \eta \). For example, the pair of poles closest to the origin \( (\zeta = 0) \) belongs to \( \eta = 2 \) mode, both for the cylindrical case and for the spherical case. In general, the poles for the spherical case are located farther away from the origin than those for the corresponding cylindrical case. If we recall that the real part of \((\zeta, \kappa)\) represents a natural frequency, where \((\zeta, \kappa)\) denotes the location of the \( k \)th pole of the \( n \)th mode, and the imaginary part represents the radiation damping, then the fact that the poles for the spherical cavity case are located farther away from the origin implies that the cavity has a higher frequency and damping for a given mode than its counterpart in the cylindrical case.

The residue from the pole at the origin yields the following dimensionless expression for

\[
\frac{\zeta^0_{\theta\theta}}{2(1-\nu)(7-5\nu)} \left[ 4 - 6\nu + 5\nu^2 - (5 - 10\nu) \cos 2\theta \right] H(\hat{\zeta}), \quad (4.30)
\]

\[
\frac{\zeta^0_{\phi\phi}}{2(1-\nu)(7-5\nu)} \left[ 1 - 9\nu - 5\nu^2 - (5\nu - 10\nu^2) \cos 2\theta \right] H(\hat{\zeta}), \quad (4.31)
\]

where \( H(\hat{\zeta}) \) is the unit-step function. Equations (4.30) and (4.31), without the item \( H(\hat{\zeta}) \), are the static solutions for \( \zeta_{\theta\theta} \) and \( \zeta_{\phi\phi} \) for the plane-strain incident wave case. This is to be expected, since the long-time solution for a unit-step loading should yield the static solution.

The behavior of dimensionless stresses, \((S_{CFD}, \zeta_{\theta\theta}(\tau))\), and \( \zeta_{\phi\phi}(\tau) \) is shown in Figs. 4.12 and 4.13 for three angular positions on the boundary of the cavity. The first arrival of the wave at the
Fig. 4.12. Transient behavior of $\sigma_{x0}$ at $\theta = 0, \pi/2, \pi$ for $\nu = 0.25$

Fig. 4.13. Transient behavior of $\sigma_{yy}$ at $\theta = 0, \pi/2, \pi$ for $\nu = 0.25$
cavity is at $\theta = \pi$, and the time of arrival is $\tau = 0$. The time of arrival at $\theta = \pi/2$ is $\tau = 1$. For points in the shadow zone of the cavity, the time of arrival is determined by Fermat's principle. Thus at $\theta = 0$, the first signal arrives at $\tau = 1 + \pi/2$. Numerical results shown in the two figures are obtained by summing all the residues from the $n = 0, \ldots, 5$ modes and from the residue of the pole at the origin. Because of the limited number of terms used in the calculation, these results are known to be inadequate for providing detailed information near the wave fronts. Methods similar to geometrical optics must be used to determine the early time behavior.

We note that most of the transient disturbances die out very rapidly; approximately four transit times. The excitation of the vibrating motions of the cavity due to impact of the incident wave are most evident at $\theta = 0, \pi$. In fact, the stresses at $\theta = \pi$ go negative after approximately one transit time, which is after the incident wave has engulfed the cavity completely. The maximum dynamic stress concentration factor, $\sigma_{oo}$, occurs at $\theta = \pi/2$, and has a value of 1.98. It occurs at $\tau \approx 4.5$. Comparing this with the static value of 1.83 it is noted that there is an overshoot of approximately 10%. Similar behavior of $\sigma_{\phi\phi}$ at $\theta = \pi/2$ is also noted. The behavior of $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ at $\theta = 0, \pi$ is identical due to the axisymmetry of the problem. What is of interest is that the dynamic stress values can be substantially higher than the static values as well as having negative values, a fact which may warrant consideration in the design of underground cavities in a brittle medium.

Some papers have appeared recently which treat the transient
phenomenon. Notable among them is the work by Norwood and Miklowitz on the early and late time behavior of the displacements on the boundary of a spherical cavity. In their study they employed the same techniques as in an earlier work by Miklowitz. Although the field abounds with various treatments of wave scattering by spherical inclusions, the treatment of dynamic stress concentration around a spherical inclusion has just begun.
APPENDIX

Formulas in Spherical Coordinates

Displacement -- Potential Relationships:

\[ u_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \left[ \frac{\partial^2 (\psi r)}{\partial \theta^2} - \frac{\partial^2 \psi}{\partial r^2} \right], \]

\[ u_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\psi \rho)}{\partial \phi} + \left( \frac{\rho}{r} \right) \frac{\partial^2 (\psi r)}{\partial \phi \partial r}, \]

\[ u_\phi = \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} - \frac{1}{r} \frac{\partial (\psi \rho)}{\partial \theta} + \left( \frac{\rho}{r \sin \theta} \right) \frac{\partial^2 (\psi r)}{\partial \phi \partial \theta}. \]

Strain -- Displacement Relationships:

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \]

\[ \varepsilon_{\theta \theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \]

\[ \varepsilon_{\phi \phi} = \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \cot \theta \frac{u_\theta}{r}, \]

\[ \varepsilon_{r \phi} = \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - u_\phi + \frac{\partial u_\phi}{\partial r} \right], \]

\[ \varepsilon_{r \theta} = \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right], \]

\[ \varepsilon_{\phi \theta} = \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \theta} - \cot \theta \frac{u_\phi}{r} + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right]. \]
Stress -- Displacement Potential Relationships:

(a) 
\[ \sigma_{rr} = \lambda \nabla^2 \phi + 2\nu \frac{\partial^2 \phi}{\partial r^2} + 2\mu \frac{\partial}{\partial r} \left[ \frac{\partial^2 (\psi r)}{\partial r^2} - \nu \nabla^2 \chi \right], \]

(b) 
\[ \sigma_{\theta \theta} = \lambda \nabla^2 \phi + 2\nu \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) + 2\mu \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{1}{r \sin \theta} \frac{\partial (\psi r)}{\partial \theta} \right] \]
\[ + 2\mu \frac{1}{r^2} \frac{\partial^3 (\psi r)}{\partial \theta^2 \partial r} + \frac{1}{r} \left( \frac{\partial^2 (\psi r)}{\partial r^2} - \nu \nabla^2 \chi \right), \]

(c) 
\[ \sigma_{\phi \phi} = \lambda \nabla^2 \phi + 2\nu \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \phi}{\partial \theta} \right] \]
\[ + 2\mu \frac{\cot \theta}{r^2 \sin \theta} \frac{\partial^2 (\psi r)}{\partial \theta^2} - \frac{1}{r^2 \sin \theta} \frac{\partial^2 (\psi r)}{\partial \theta \partial \phi} \]
\[ + 2\mu \frac{1}{r^2 \sin \theta} \frac{\partial^3 (\psi r)}{\partial \theta^2 \partial r} + \frac{1}{r} \left( \frac{\partial^2 (\psi r)}{\partial r^2} - \nu \nabla^2 \chi \right) + \cot \theta \frac{\partial}{r^2} \frac{\partial^2 (\psi r)}{\partial \theta \partial r}, \]

(d) 
\[ \sigma_{r \theta} = 2\mu \left[ \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right] - \frac{\mu}{r} \left[ \left( \frac{1}{r \sin \theta} \frac{\partial (\psi r)}{\partial \theta} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r \sin \theta} \frac{\partial (\psi r)}{\partial \theta} \right) \right] \]
\[ + \frac{\mu}{r} \left[ \frac{3}{3 \theta} \left( \frac{\partial^2 (\psi r)}{\partial r^2} - \nu \nabla^2 \chi \right) + \frac{1}{r} \frac{\partial^2 (\psi r)}{\partial \theta \partial r} + \frac{1}{r^2} \frac{\partial^2 (\psi r)}{\partial \theta \partial r} \right], \]

(e) 
\[ \sigma_{r \phi} = 2\nu \left[ \frac{1}{r \sin \theta} \frac{\partial^2 \phi}{\partial r \partial \phi} - \frac{1}{r^2 \sin \theta} \frac{\partial \phi}{\partial \phi} \right] + \nu \left[ \frac{2}{r^2 \sin \theta} \frac{\partial (\psi r)}{\partial \phi} - \frac{1}{r^2} \frac{\partial^2 (\psi r)}{\partial \theta \partial r} \right] \]
\[ + \mu \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial^2 (\psi r)}{\partial r^2} - \nu \nabla^2 \chi \right) + \frac{2}{r^2 \sin \theta} \frac{\partial^2 (\psi r)}{\partial \theta \partial r}, \]
\( \sigma_{\theta\phi} = 2\mu \left[ \frac{\cot \theta \, \frac{\partial^2 \phi}{\partial \theta^2}}{r^2 \sin \theta} - \frac{\cot \theta \, \frac{\partial \phi}{\partial \theta}}{r^2 \sin \theta} \frac{\partial \phi}{\partial \phi} \right] \\
+ \nu \left[ \frac{\cot \theta \, \frac{\partial^2 (r\psi)}{\partial \theta^2}}{r^2} - \frac{\cot \theta \, \frac{\partial^2 (r\psi)}{\partial \theta^2}}{r^2 \sin \theta} \frac{\partial \phi}{\partial \phi} \right] \\
+ 2\nu \left[ \frac{1}{r^2 \sin \theta} \frac{\partial^3 (r\chi)}{\partial \theta^3} - \frac{\cot \theta \, \frac{\partial^2 (r\chi)}{\partial \theta^2}}{r^2 \sin \theta} \frac{\partial \phi}{\partial \phi} \right]. \\

Components of Stresses Due to Spherical Wave Functions

Displacement Potentials:

\[ r = r_n^{(i)}(ar)P_n^m(\cos \theta)e^{i \omega t}, \]

\[ \psi = r_n^{(i)}(ar)Y_n^m(\cos \theta)e^{i \omega t}, \]

\[ x = r_n^{(i)}(ar)H_n^m(\cos \theta)e^{i \omega t}, \]

where \( r_n^{(i)} \) denotes spherical Bessel functions, \( (i) = 1,2,3,4 \), and is used to identify the different Bessel functions, with

\[ r_n^{(1)} = j_n, \quad r_n^{(2)} = y_n, \quad r_n^{(3)} = h_n^{(1)}, \quad \text{and} \quad r_n^{(4)} = h_n^{(2)} \]

and

\[ a = \omega / c_p \quad \text{compressional wave number,} \]

\[ b = \omega / c_s \quad \text{shear wave number.} \]
Stress -- Spherical Wave Function Relationship ($e^{-i\omega t}$ Omitted):

\[ \sigma_{rr} \text{ due to:} \]
\[ \varphi: \frac{2\mu}{r^2} \left\{ (n^2 - \frac{(\beta r)^2}{2}) \delta_n^{(i)}(ar) + 2ar \delta_{n+1}^{(i)}(ar) \right\} P_n^m(\nu) e^{im\phi}, \]
\[ \psi: \text{ none,} \]
\[ \chi: \frac{2\mu}{r^2} \left\{ n(n+1) \delta_n^{(i)}(\beta r) - \beta r \delta_{n+1}^{(i)}(\beta r) \right\} P_n^m(\nu) e^{im\phi}. \]

\[ \sigma_{\theta\theta} \text{ due to:} \]
\[ \varphi: \frac{2\mu}{r^2} \left\{ (\nu^2 - \frac{(\beta r)^2}{2} + (ar)^2) \delta_n^{(i)}(ar) - ar \delta_{n+1}^{(i)}(ar) \right\} P_n^m(\nu) \]
\[ + \frac{1}{\sin^2 \theta} \left[ \left( \nu^2 - \cos^2 \theta \right) P_n^m(\nu) \right] \delta_n^{(i)}(ar) e^{im\phi}, \]
\[ \psi: \frac{2\mu}{r^2} \left[ \frac{\delta_n^{(i)}(\beta r)}{\sin^2 \theta} \right] \left[ (n+1) \cos \theta P_n^m(\nu) - (n+m) P_{n-1}^m(\nu) \right] \delta_n^{(i)}(\beta r) e^{im\phi}, \]
\[ \chi: \frac{2\mu \delta_n^{(i)}(\beta r)}{r^2} \left\{ (n+1) \delta_n^{(i)}(\beta r) - \beta r \delta_{n+1}^{(i)}(\beta r) \right\} P_n^m(\nu) \]
\[ + \frac{1}{\sin^2 \theta} \left[ (n+1) \delta_n^{(i)}(\beta r) - \beta r \delta_{n+1}^{(i)}(\beta r) \right] \]
\[ \times \left[ \left( \nu^2 - \cos^2 \theta \right) P_n^m(\nu) + (n+m) \cos \theta P_{n-1}^m(\nu) \right] e^{im\phi}. \]

\[ \sigma_{\phi\phi} \text{ due to:} \]
\[ \varphi: \frac{2\mu}{r^2} \left\{ \left( n - \frac{(\beta r)^2}{2} + (ar)^2 \right) \delta_n^{(i)}(ar) - ar \delta_{n+1}^{(i)}(ar) \right\} P_n^m(\nu) \]
\[ + \frac{1}{\sin^2 \theta} \left[ (n \cos^2 \theta - m^2) P_n^m(\nu) (n+m) \cos \theta P_{n-1}^m(\nu) \right] \delta_n^{(i)}(ar) e^{im\phi}. \]
\[ \psi: \frac{2\mu}{r} \left[ \frac{i m}{\sin^2 \theta} \right] \left[ - (\n - 1) \cos \delta_{\n}^m (\mu) + (\n + m) \delta_{\n-1}^m (\mu) \right] \delta_{\n} (\varphi) e^{im\varphi}, \]

\[ \chi: \frac{2\mu}{r^2} \left[ \frac{1}{\sin^2 \theta} \right] \left\{ \n (\n + 1) \sin^2 \theta \delta_{\n} (\varphi) e^{im\varphi (\mu)} + \left[ (\n + 1) \delta_{\n} (\varphi) - \n \delta_{n+1} (\varphi) \right] \left[ (\n \cos^2 \theta - m^2) \delta_{\n}^m (\mu) \right] - (\n + m) \cos \delta_{\n}^m (\mu) \right\} e^{im\varphi}. \]

\( \sigma_{\varphi \theta} \) due to:

\[ \varphi: \frac{2\mu}{r^2} \left[ \frac{1}{\sin \theta} \right] \left[ (\n - 1) \delta_{\n} (\varphi) - \n \delta_{\n+1} (\varphi) \right] \left\{ \n (\n + m) \delta_{\n}^m (\mu) \right\} e^{im\varphi}. \]

\[ \chi: \frac{2\mu}{r^2} \frac{1}{\sin \theta} \left\{ \n^2 - 1 - \frac{\beta^2 r^2}{2} \right\} \delta_{\n} (\varphi) + \n \delta_{\n+1} (\varphi) \left\{ \n (\n + m) \delta_{\n}^m (\mu) \right\} e^{im\varphi}. \]

\( \sigma_{\varphi r} \) due to:

\[ \varphi: \frac{2\mu}{r^2} \left[ \frac{i m}{\sin \theta} \right] \left[ (\n - 1) \delta_{\n} (\varphi) - \n \delta_{\n} (\varphi) \right] \delta_{\n}^m (\mu) e^{im\varphi}. \]
\[ \psi: \frac{-\mu}{r \sin \theta}\left[(n-1)\varepsilon_{n}(\xi)(ar) - \delta_{n+1}(\xi)(ar)\right] \]
\[ \times \left[\eta \cos \theta \tilde{P}_{n}^{m}(\mu) - (n+m)\tilde{P}_{n-1}^{m}(\mu)\right] e^{im\phi}, \]
\[ \chi: \frac{2\mu}{r^{2}} \frac{i\eta}{\sin \theta} \left\{\left[n^{2} - 1 - \frac{(\xi r)^{2}}{2}\right] \delta_{n}(\xi)(\xi r) + \xi r \delta_{n+1}(\xi)(\xi r)\right\} \tilde{P}_{n}^{m}(\mu)e^{im\phi}. \]

\(\sigma_{\phi \phi}\) due to:

\[ \psi: \frac{2\mu}{r^{2}} \frac{i\eta}{\sin^{2} \theta}\left[(n-1)\cos \theta \tilde{P}_{n}^{m}(\nu) - (n+m)\tilde{P}_{n-1}^{m}(\nu)\right] \delta_{n}(\xi)(\xi r)e^{im\phi}, \]
\[ \chi: \frac{2\mu}{r} \frac{1}{\sin^{2} \theta} \left[(\frac{n(n-1)}{2} \sin^{2} \theta + \eta - m^{2})\tilde{P}_{n}^{m}(\mu) \right] \]
\[ - (n+m)\cos \theta \tilde{P}_{n-1}^{m}(\mu)\right]\delta_{n}(\xi)(\xi r)e^{im\phi}, \]

\[ \sigma_{\phi \phi}\) due to:

\[ \psi: \frac{2\mu}{r^{2}} \frac{i\eta}{\sin^{2} \theta}\left[(n+1)\delta_{n}(\xi)(\xi r) - \delta_{n+1}(\xi)(\xi r)\right] \]
\[ \times \left[(n-1)\cos \theta \tilde{P}_{n}^{m}(\nu) - (n+m)\tilde{P}_{n-1}^{m}(\nu)\right] e^{im\phi}. \]

Displacement -- Spherical Wave Function Relationships:

\(\nu_{n}\) due to:

\[ \psi: \frac{1}{r} \left[n \delta_{n}(\xi)(\xi r) - \eta \delta_{n+1}(\xi)(\xi r)\right] \tilde{P}_{n}^{m}(\mu)e^{im\phi}, \]
\[ \psi: \text{none}, \]
\[ \chi: \frac{2}{r} n(n+1) \delta_n^{\mu} (\partial_r) P_n^m(\mu) e^{im\phi}. \]

\( \nu_\theta \) due to:

\[ \varphi: \frac{1}{r \sin \theta} \left[ n \cos \theta P_n^m(\mu) - (n+m) P_n^m_{n-1}(\mu) \right] \delta_n^{(\mu)} (ar) e^{im\phi}, \]

\[ \psi: \frac{im}{\sin \theta} \delta_n^{(\mu)} (\partial_r) P_n^m(\mu) e^{im\phi}, \]

\[ \chi: \left( \frac{1}{r \sin \theta} \right) \left[ n \cos \theta P_n^m(\mu) - (n+m) P_n^m_{n-1}(\mu) \right] \delta_n^{(\mu)} (\partial_r) e^{im\phi}. \]

\( \nu_\phi \) due to:

\[ \varphi: \frac{2m}{r \sin \theta} \delta_n^{(\mu)} (\partial_r) P_n^m(\mu) e^{im\phi}, \]

\[ \psi: \frac{1}{\sin \theta} \left[ n \cos \theta P_n^m(\mu) - (n+m) P_n^m_{n-1}(\mu) \right] \delta_n^{(\mu)} (\partial_r) e^{im\phi}, \]

\[ \chi: \frac{r}{r \sin \theta} \left[ (n+1) \delta_n^{(\mu)} (\partial_r) - \partial_n \delta_{n+1}^{(\mu)} (\partial_r) \right] P_n^m(\mu) e^{im\phi}. \]
CHAPTER VI REFERENCES


0.2 Wolf, Alfred, "Motion of a Rigid Sphere in an Acoustic Wave Field," Geophysics, Vol. 10, 1945, p. 91.


