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MARCH 1971

ON INSENSITIVITIES IN URBAN REDISTRICTING AND FACILITY LOCATION

Richard C. Larson and Keith A. Stevenson

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PREFACE

This report grew out of the authors' work on urban service systems, specifically police patrol cars and ambulances, for which rapid responses to calls for service are required. It is part of a study of the deployment of emergency services, supported by the New York City-Rand Institute. Many of the mathematical details have been omitted in order to make the results generally accessible to urban researchers and planners concerned with policy implementation.

We have benefitted from the concurrent development of a model which is described in a companion report by G. Carter, J. Chaiken, and E. Ignall, "Response Areas for Two Emergency Units," New York City-Rand Institute, R-532, March 1971.

SUMMARY

In this report we consider one class of problems associated with urban service systems which dispatch vehicles from fixed facilities. Given the limited resources available, one important issue is the location of the facilities and the design of their response districts to minimize average response time in the face of spatially distributed demand patterns.

Simple initial calculations with spatially homogeneous demands suggest that the mean travel time resulting from totally random distribution of facilities in the region served is reduced by only 25% when the facilities are optimally distributed. This apparent insensitivity of mean travel time to facility location is pursued in detail by analyzing two classes of system involving a pair of facilities. In the first case we outline a procedure for determining the optimal location of a second facility, given a position for the first facility, when no inter-facility cooperation is allowed. The results are summarized in Figs. 4, 5 and 6. In the second case we examine the same region, but allow a simple form of inter-facility cooperation. The results are summarized in Figs. 8, 9 and 10.

These simple models suggest that due to insensitivities it may not be necessary to quantize geographical data finely and to try laboriously to find the "optimal" solutions to redistricting and facility location problems. Redistricting and facility location based on rather crude assumptions and an awareness of some of the heuristic properties illustrated by simple analytical models may yield mean travel times very near the minimum possible.

ACKNOWLEDGMENTS

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I. INTRODUCTION

There is a growing interest in research on the operating behavior of spatially distributed urban service systems. Police and fire departments, ambulance services, out-patient clinics, and taxi-cab fleets are examples of these systems. Typically the problem is to specify a set of criteria to characterize the operating behavior of the service system, to set standards of operation in terms of these criteria, and then to examine current operations to see if the standards are met. Preliminary research in this area has indicated several classes of problems which these systems commonly share.

One important class which will be discussed in this paper is that of area redistricting and facility location. In a fire department context, a region of a city may have N fire stations and the need for an additional station might be anticipated. The area redistricting part of the problem is of the form, "How can the region be partitioned into N fire districts to obtain best overall operation?" The facility location component of the problem is of the form, "Given that the present N fire stations cannot satisfy the specified standards of operation, even with an optimal redistricting, what is the best location for the $N + 1^{st}$ fire station?" Clearly, this class of problems is shared by many urban agencies.

Several algorithmic approaches to the area redistricting problem in public systems have been previously discussed, particularly the work of Hess, et al[1] on political redistricting in Delaware and of Gass[2] on determining police beats in Cleveland. Vitale[3] considered a similar problem applied to the transportation of goods. Facility location has often been examined in other contexts, especially optimal factory and warehouse location to minimize total transportation costs. Current algorithmic approaches to facility location are reviewed in a paper by Revelle, Marks, and Liebman,[4] which includes several "public sector location models." Recently Hogg[5] discussed an algorithm for locating fire stations; the algorithm determines the best combination of r station sites from a set of n alternative sites, where "best" denotes minimal travel time. Teitz[6] has discussed the need for a unified theory of urban facility locations.

In this paper we examine systems in which mobile units are stationed at facilities in an urban region and are dispatched to requests for service from points inside the region. We shall consider two classes of such systems in which average "travel time" or "response time" (i.e., the time required for a dispatched unit to reach the scene of the service request) is one meaningful measure of system performance,* and its minimization is an operational objective. Systems of this type include fire departments and ambulance services.

In the first class of systems (Section IV) the facilities operate totally independently of each other. A number of units located at each facility respond to requests for service from within the district associated with the facility. No units are dispatched across district boundaries, and calls arriving when all the units in that district are busy are serviced from outside the region.**

In the second class of systems (Section V) we consider a simple form of inter-facility cooperation. A single unit is located at each facility and responds primarily to requests for service from within its own district. A unit may, however, respond to a call in an adjacent district if the unit associated with that district is unavailable. We shall assume that calls arriving when all units are unavailable are answered from outside the region.

Rather than develop complex allocation algorithms, in this paper we utilize several analytical models to gain insight into such questions as:

(1) What is the approximate range of values of mean travel time that can be expected?

(2) How does the optimal position of one facility change as another facility is moved about a region?

(3) How sensitive is mean travel time to facility location?

(4) How does some form of inter-district cooperation affect optimal facility locations?

---

*In actual urban environments there are usually several, often conflicting, effectiveness measures that need to be considered.

**The mathematics of this model also apply to systems in which requests for service are generated throughout the district of the facility, but the individual associated with the request travels to the facility in his district. Examples include out-patient clinics, police precinct stations, and shopping centers. To simplify discussion, however, we shall always refer to the case in which response units travel from the facility to the service request.

In all our developments we assume that the travel distance between two points (x_1,y_1) and (x_2,y_2) is closely approximated by the "right-angle distance metic,"

$$d = | x_1 - x_2 | + | y_1 - y_2 |,$$

where the coordinate axes are defined parallel to directions of travel. If there are two different travel speeds, v_x and v_y, corresponding say to north-south and east-west movement, respectively, the travel time is

$$t = \frac{1}{v_x} | x_1 - x_2 | + \frac{1}{v_y} | y_1 - y_2 |.$$

Although the methods of this paper allow the speeds to differ, in our developments we assume for simplicity that $v_x = v_y = 1$, and thus we use the terms "travel time" and "travel distance" interchangeably.

In Section II, we find approximate upper and lower bounds for the mean travel distance performance of a system with N facilities distributed per unit area, spatially homogeneous demand, and no inter-district cooperation. Section III illustrates how to redistrict a region with N fixed facilities, arbitrary demand distribution, and no inter-district cooperation. Section IV uses previously-derived results and examines optimal facility locations and redistricting for a rectangular region with two facilities, homogeneous demand, and no inter-district cooperation. In an example, we continuously vary the position of facility 1 over a closed trajectory in one part of the region and examine the locus of optimal positions of facility 2 induced by the movement of facility 1. Section V treats the same example as Section IV, but includes a simplified form of inter-district cooperation suggested by Carter, Chaiken and Ignall.[7] A short discussion is given in Section VI.

The focus of this document is on insensitivities, both of the minimum expected travel time to the relative locations of the facilities, and of the optimal location of one facility to variations in the position of a neighboring facility. These suggest that in actual urban environments detailed simulations or complex allocation algorithms relying on finely quantized data may be gratuitous, and that intelligent intuition may be equally valuable for planning certain aspects of these systems.

II. APPROXIMATE BOUNDS ON PERFORMANCE

In this section we determine the approximate range of values of mean travel time that can be expected in a system with N facilities distributed per unit area, spatially homogeneous demand, and no inter-district cooperation. Each facility is located within a district and each call generated within the district is serviced by a unit assigned to that facility, if one is available. (Otherwise the call is serviced from outside the region.)

Assuming right angle travel distances, it is easy to show that minimal mean travel distance is obtained by positioning facilities and designing districts in the regular lattice pattern shown in Figure 1.* With this arrangement, the mean intra-district travel distance is equal to $\frac{2}{3}[2N]^{-1/2} \simeq 0.471\ N^{-1/2}$. This mean travel distance represents a lower bound to that which could be achieved by allocating N facilities per unit area.

Now, a reasonable upper bound can be obtained by positioning the facilities in a totally random manner, insuring that on the average there are N facilities per unit area.** Then, the facilities are distributed as a homogeneous Poisson process with parameter N (= average number of facilities per unit area).[9] Given our model assumptions, it is easy to show[10] that the mean*** travel distance from an incident to the nearest facility is $\frac{1}{4}[\frac{2\pi}{N}]^{1/2} \simeq 0.627\ N^{-1/2}$.

Comparing the two results, we see that an optimal positioning (and redistricting) reduces the mean travel distance over that obtained by totally random positioning by only about 25 percent.****

---

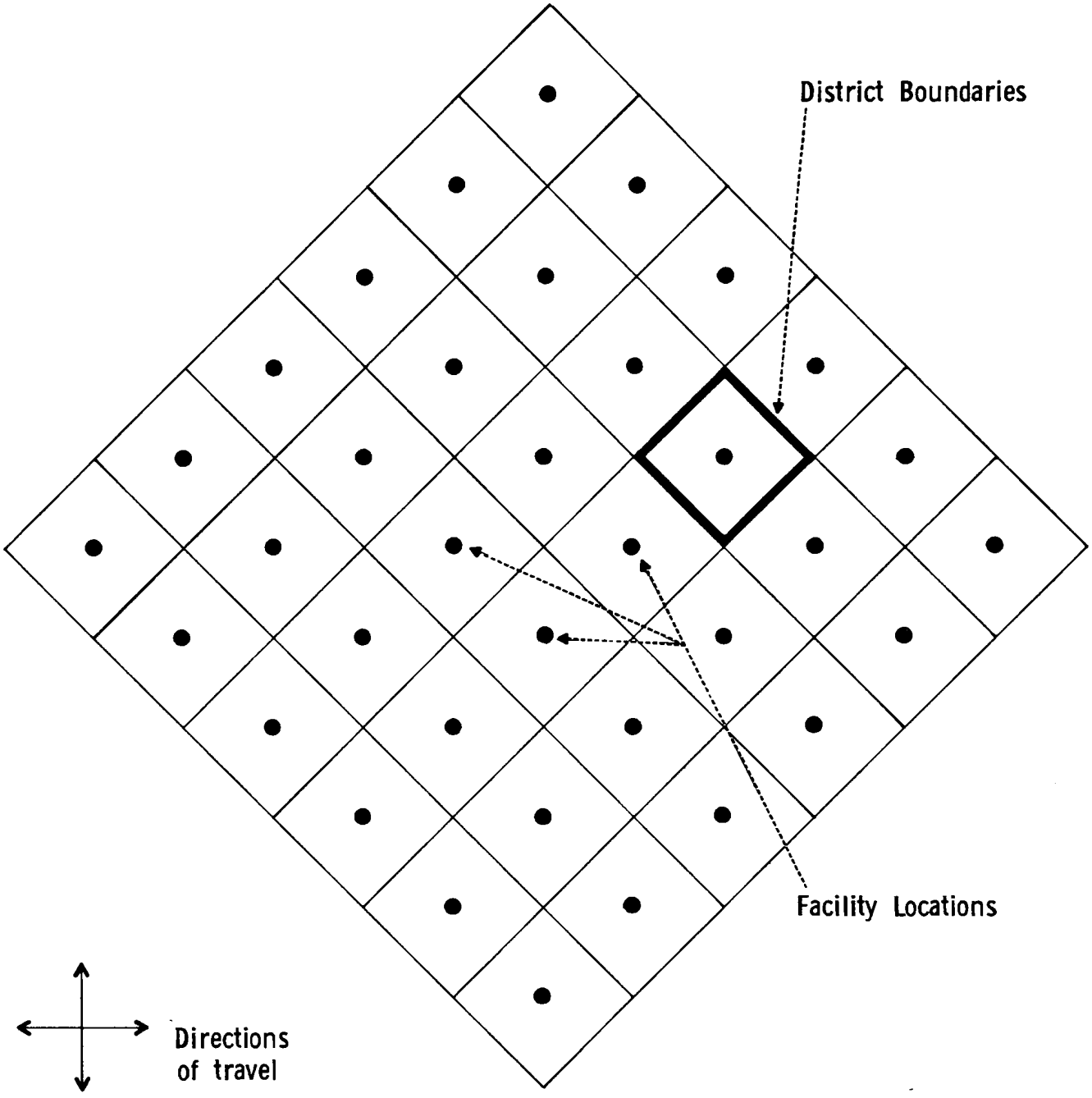
*Euclidean travel distances would give rise to the familiar hexagonal patterns often found in texts on "spatial economy" (c.f. Isard [8]).

**In some real systems facilities may cluster together because of advantages resulting from close proximity (e.g. hospitals sharing facilities and staff), causing the mean travel time to exceed the "upper bound" associated with randomly positioned facilities.

***The corresponding standard deviations are $\frac{1}{6}N^{-1/2} \simeq 0.167\ N^{-1/2}$ for optimal positioning and $\frac{1}{2}[\frac{4-\pi}{2N}]^{1/2} \simeq 0.328N^{-1/2}$ for totally random positioning.

****To further illustrate insensitivity to positioning, imagine that in one of the districts in Figure 1 the facility is moved a fraction ε of the (right-angle) distance from the center to the district boundary ($0 \leq \varepsilon \leq 1$). Then one finds that the mean travel distance in the district is increased to $[2/3 + \varepsilon^2 - 1/3\ \varepsilon^3]\ [2N]^{-1/2}$. An ε equal to 0.25, say, increases mean travel distance by only about 8.6 percent.

FIG. 1: REGULAR LATTICE OF DISTRICTS REPRESENTING OPTIMAL POSITIONING OF FACILITIES AND REDISTRICTING OF A HOMOGENEOUS REGION



There are N facilities per unit area.
The area of each district equals N^{-1}.
A facility is located in the center of each district.

III. DISTRICTING A REGION WITH N FIXED FACILITIES

Continuing the N facility system with no inter-district cooperation, we next confront the more realistic situation in which we do not have the freedom to position the facilities. We are interested in the problem of defining the districts associated with each facility, given the facility positions, so that response time is minimized. Regardless of the spatial distribution of demands, the "optimal district" for each facility in a region R consists of all points closer to that facility than to any other. (Points which are equidistant from 2 or more facilities can be arbitrarily assigned.)

One straightforward way to obtain an optimal district for facility i is to consider each other facility j ($j \neq i$) and construct an optimal boundary between i and j. If the coordinates of facility i are (x_i, y_i), then, using right-angle distance, the set of points equidistant from i and j is

$$S_{ij} = \{(x,y) \in R: |x - x_i| + |y - y_i| = |x - x_j| + |y - y_j|\}.$$

Usually the set S_{ij} consists of three connected straight line segments such as the "type 1" or "type 2" boundaries in Figures 2a and 2b, respectively; but the set can be more complicated (Figure 2c). Any line S'_{ij} which partitions the region R and which is included in S_{ij} is an optimal boundary between districts i and j. The entire district boundary around facility i (i = 1, ..., N) is constructed by combining "pieces" of the boundaries S'_{ij} between facilities i and j ($j \neq i$).* The districts associated with each facility for an example (N=5) are shown in Figure 3.

---

*More formally, we can construct optimal districts in a region R as follows. Define the set

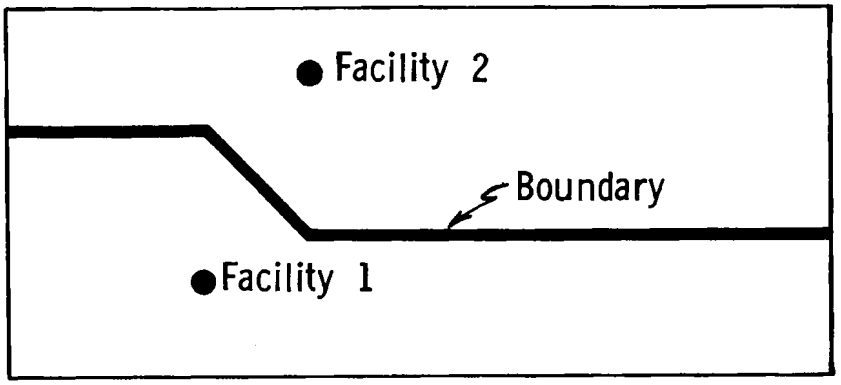
$$D(i,j) = \{ \underline{x} \in R: d(\underline{x}_i, \underline{x}) \leq d(\underline{x}_j, \underline{x}) \text{ if } i < j \text{ or } d(\underline{x}_i, \underline{x}) < d(\underline{x}_j, \underline{x}) \text{ if } i > j\}$$
$$i \neq j, \quad i,j = 1, \ldots, N$$

where $d(\underline{x},\underline{y})$ = "distance" between points $\underline{x}$ and $\underline{y}$
$\underline{x}_i$ = "location" of facility i

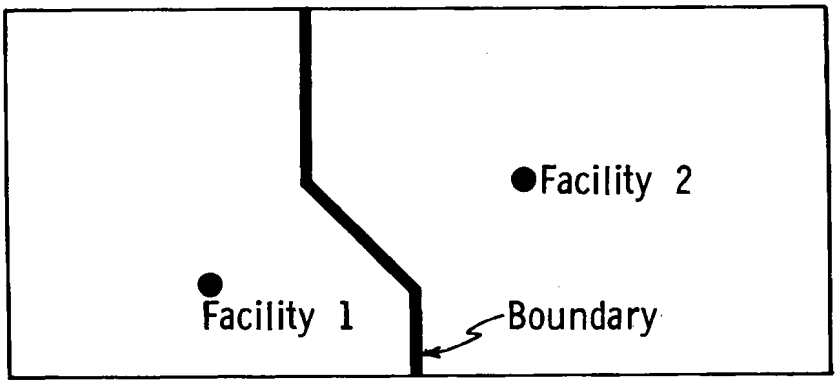
(The D(i,j) defined above arbitrarily assigns those points equidistant from the two facilities to facility i)
Then, one optimal district for facility i is

$$C_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{N} D(i,j) \text{ for } i=1,\ldots,N.$$

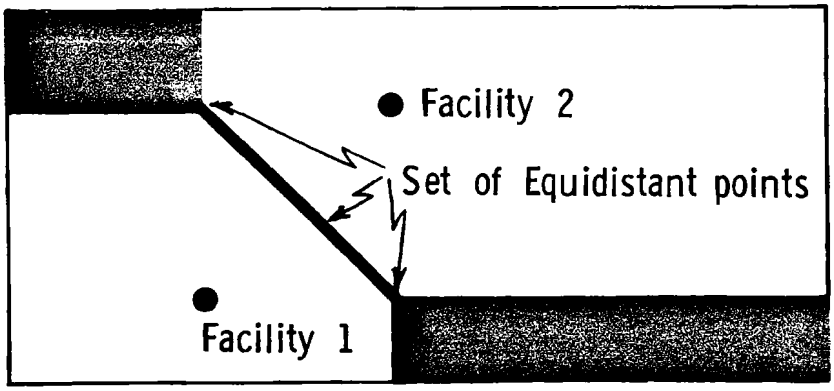
FIG. 2: SETS OF POINTS EQUIDISTANT FROM TWO FACILITIES



2 (a): Type 1 Boundary



2 (b): Type 2 Boundary



3 (c): Set of Points Equidistant from Both Facilities

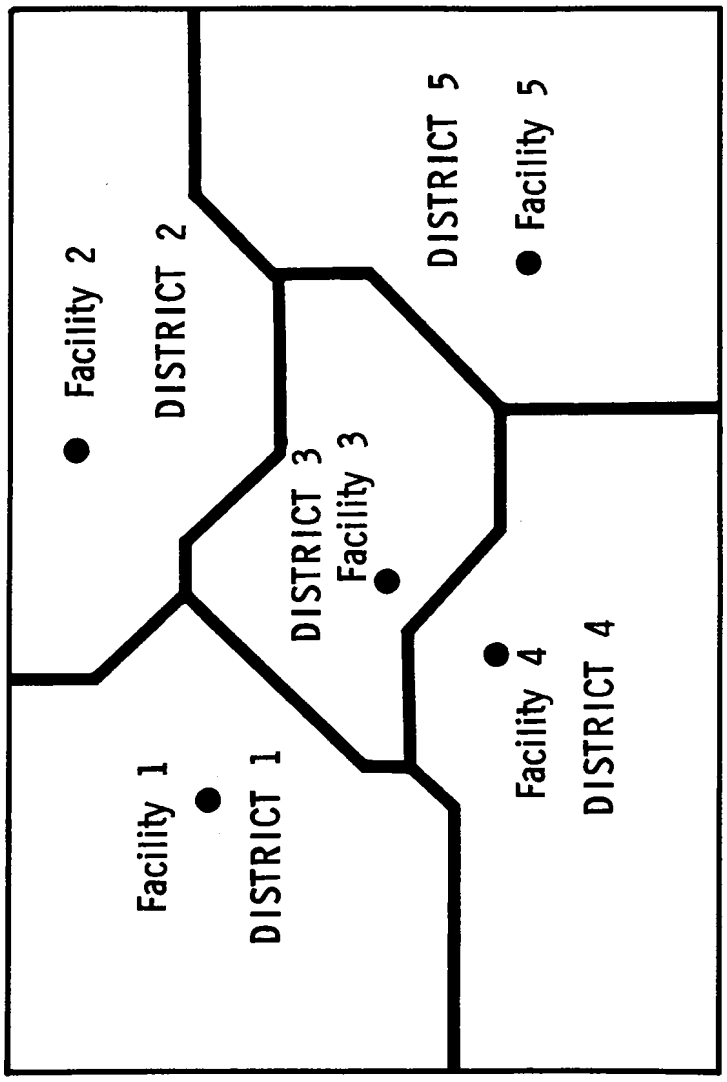


FIG. 3: (N = 5) AN OPTIMALLY PARTITIONED CITY

IV. FACILITY LOCATION: TWO INDEPENDENT DISTRICTS

To examine the sensitivity of the mean travel time (distance) to facility location when no inter-district cooperation is allowed, we consider here a simple two facility example. In particular, we consider an n x m rectangular region in which demands are uniformly distributed and in which facility 1 is located at (x_1,y_1). We are free to position facility 2 at any location (x_2,y_2) and to construct optimal districts. This situation might correspond to one in which an existing facility servicing the region was overburdened and a second facility was about to be introduced. The problem would be to locate the second facility so that the resulting average travel time was minimized.

It is elementary but tedious to obtain the mean travel time to the closest facility, given arbitrary positions (x_1,y_1) and (x_2,y_2), respectively, for the two facilities.* One can then find the position of facility 2, (x_2^o,y_2^o), that minimizes this mean travel time.**

---

*For the case which gives rise to a type 1 boundary ($x_2 \geq x_1$, $y_2 \geq y_1$, $y_2 - y_1 \geq x_2 - x_1$), the expected travel time is:

$$E[t] = \frac{1}{nm}\left[-\frac{1}{6}x_1^3 + \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_1x_2^2 + \frac{1}{6}x_2^3\right.$$

$$+\frac{1}{2}x_1^2y_1 + \frac{1}{2}x_1^2y_2 - \frac{1}{2}x_2^2y_2 - \frac{1}{2}x_2^2y_1$$

$$+\frac{3}{4}y_1^2n + \frac{3}{4}y_2^2n - \frac{1}{2}y_1x_1n - \frac{1}{2}y_1y_2n$$

$$-\frac{1}{2}y_2nx_1 + \frac{1}{2}ny_1x_2 - \frac{1}{4}x_1^2n + \frac{1}{2}nx_2y_2$$

$$+\frac{1}{2}x_1x_2n - \frac{1}{4}x_2^2n + \frac{1}{2}n^2m - nmx_2$$

$$\left.+\frac{1}{2}nm^2 - nmy_2 + x_2^2m\right].$$

For each of the other 7 possibilities for the placement of facilities, the expressions for E[t] can be obtained by a straightforward relabeling of coordinates.

**The minimization is done either by differential calculus or by a gradient search procedure. For each different boundary configuration (or equivalently, for each possibility for the placement of facilities), there may correspond a local minimum; care must be taken to consider each of these to arrive at the global minimum.

Performing the minimization we find that for certain locations of facility 1 the boundary resembles that shown in Figure 2a, whereas for other facility 1 locations it resembles that shown in Figure 2b. A continuous trajectory of facility 1 locations often induces a discontinuous trajectory of optimal facility 2 locations, the discontinuities occurring at points where the optimal boundary configuration changes say, from type 1 to type 2. The mean travel time is found to be continuous, however.

To indicate the qualitative properties of the results, we discuss as an example a region for which $n = 1$, $m = 2$. We continuously vary the position of facility 1 over a closed trajectory in the "southern" half of the region. We then examine the locus of optimal positions of facility 2 induced by varying facility 1 over the fixed trajectory.

In Figure 4, we see the effect of varying the position of facility 1 over a circular trajectory centered at (0.5, 0.5) and having radius 0.45. The most striking feature of Figure 4 is that for large changes in the position of facility 1, the change in the position of facility 2 to maintain optimality is relatively small.

In Figure 5, the trajectory of positions of facility 1 has been chosen to be a square centered at (0.5, 0.5) of dimension 0.9. This figure clearly indicates the discontinuous nature of the locus of optimal positions of facility 2. We see that for facility 1 "near" the top corners of the square the optimal location of facility 2 "jumps" to a region in which a type 2 (rather than a type 1) boundary is required.

Corresponding to the two trajectories for facility 1 in Figures 4 and 5, we have plotted in Figure 6 the expected travel time $\overline{T}$ when facility 2 is optimally located. The figure shows $\overline{T}$ as a function of θ, an angle defined from the point (0.5, 0.5) to the position of facility 1. Because of the symmetry of facility 1 locations about $x = 0.5$, we have graphed $\overline{T}$ only for values of θ less than 180°. For the circular trajectory, $\overline{T}$ varies between $\overline{T} \simeq 0.56$ ($\theta \simeq 140°$) and $\overline{T} \simeq 0.59$ ($\theta \simeq 65°$). For the square trajectory, the maximum value of $\overline{T}$ is about 0.65 ($\theta \simeq 45°$), the minimum approximately 0.57 ($\theta \simeq 0°$ or 180°). Considering the size of the facility 1 trajectories, $\overline{T}$ is not very sensitive to the facility 1 position.

FIG. 4: CORRESPONDING FACILITY LOCATIONS FOR MINIMUM EXPECTED TRAVEL TIME

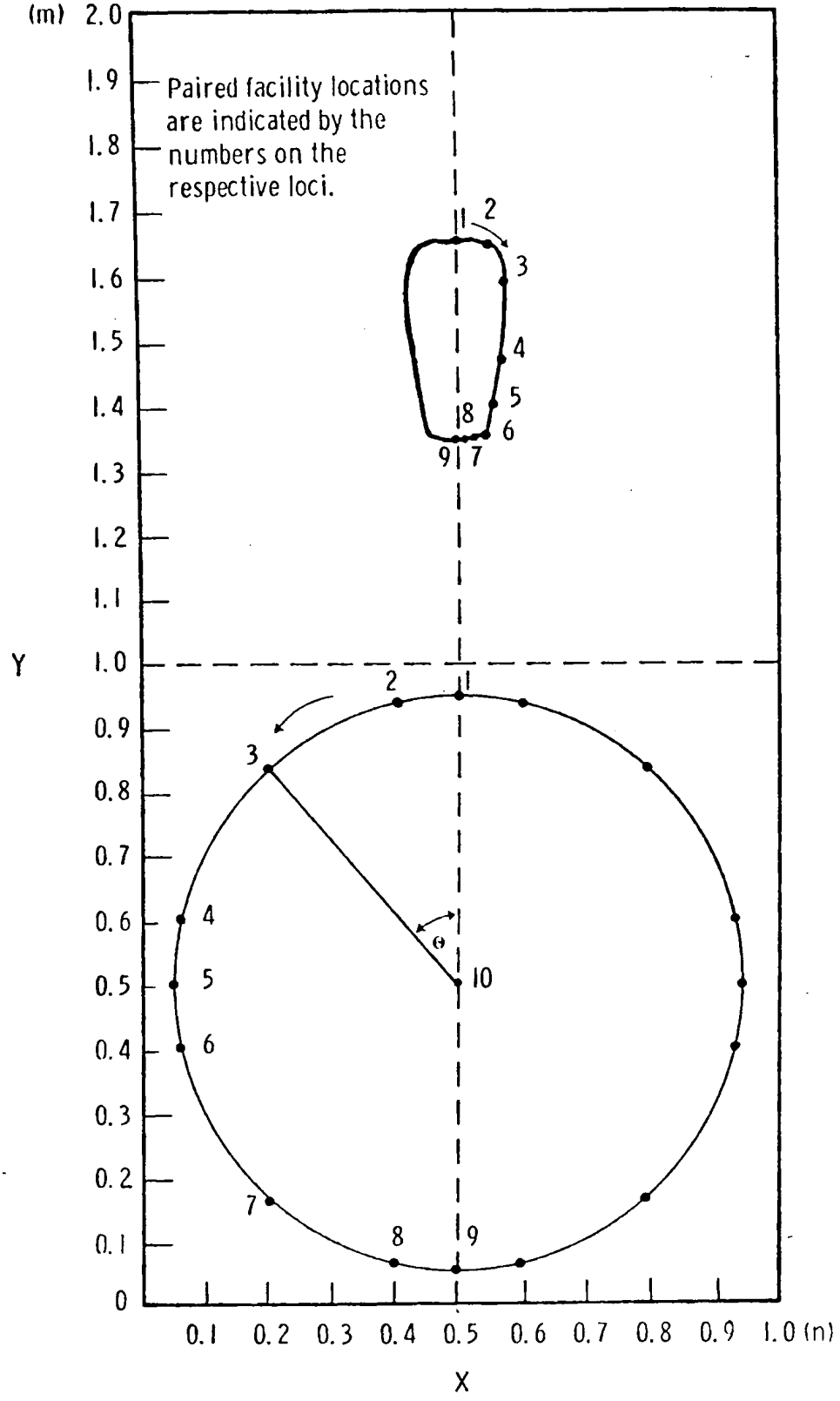


FIG. 5: CORRESPONDING FACILITY LOCATIONS FOR MINIMUM EXPECTED TRAVEL TIME

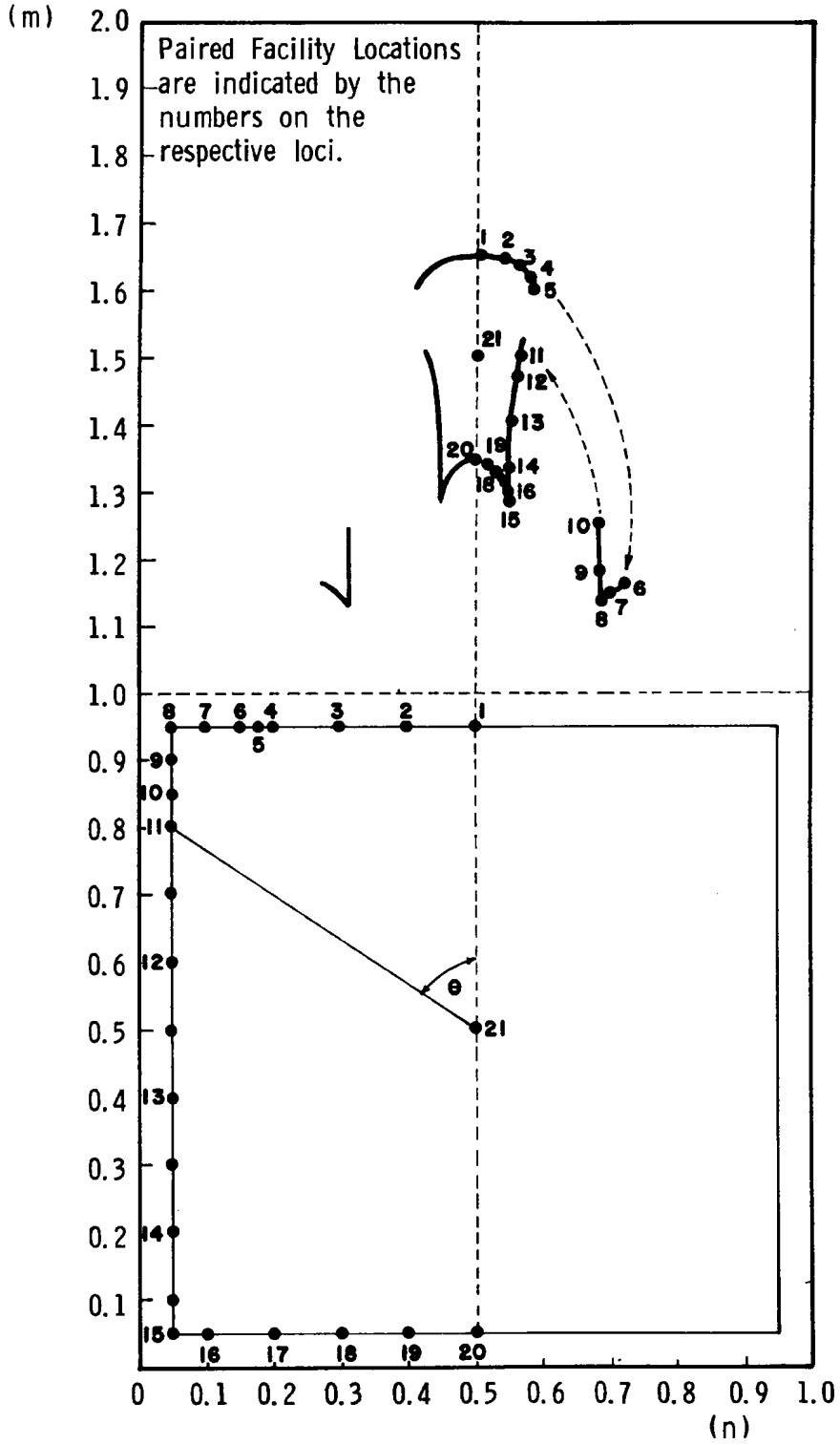
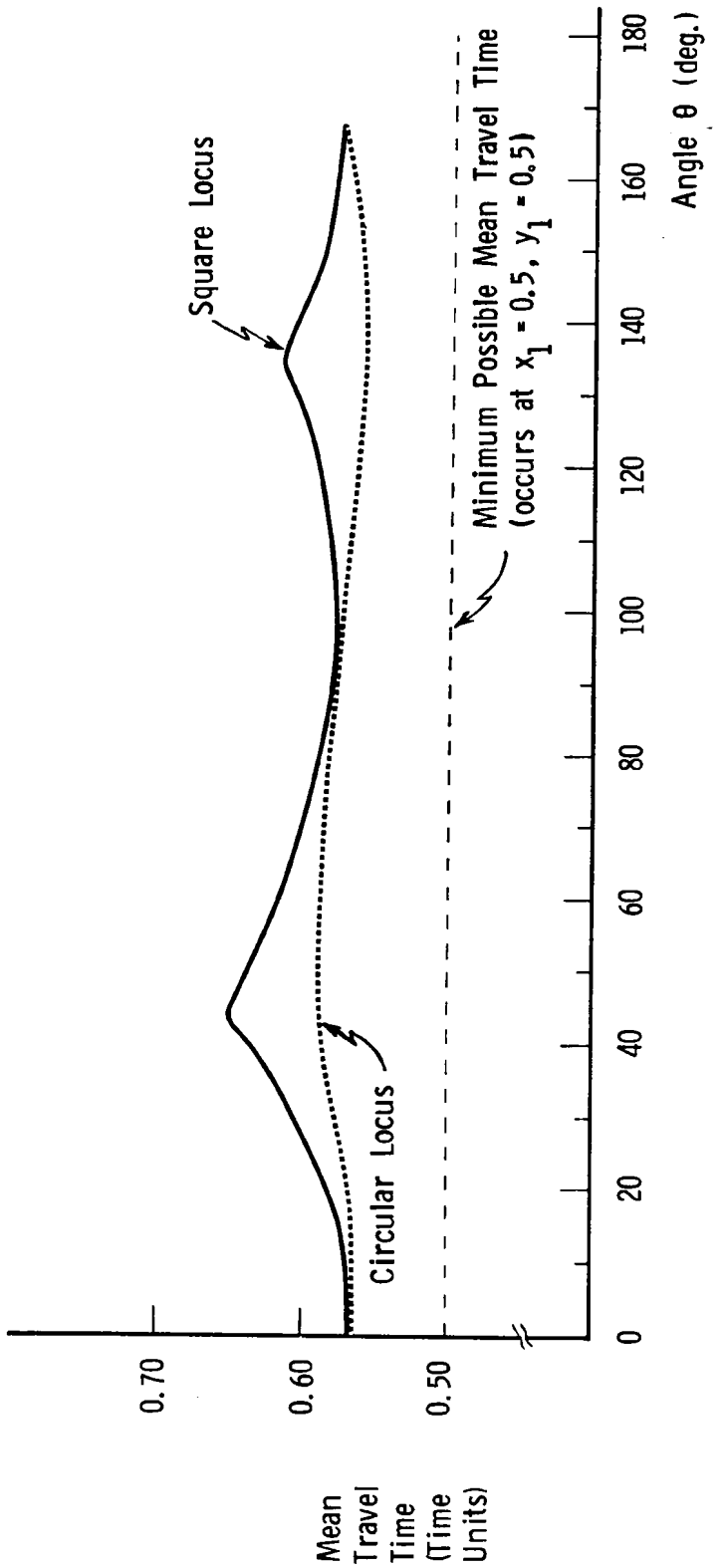


FIG. 6: MINIMAL MEAN TRAVEL TIME AS A FUNCTION OF FACILITY 1 POSITION



(1 Time Unit = Time to Traverse 1 Unit of Distance)

A similar insensitivity in $\bar{T}$ is found when we allow facility 2 to move away from its optimal position. As an example, for $x_1 = 0.5$, $y_1 = 0.5$, we know that the optimal facility 2 coordinates are $x_2^o = 0.5$, $y_2^o = 1.5$. Employing a sensitivity analysis, we find that changing x_2 to $x_2^o \pm 0.1$ increases $\bar{T}$ by only about 0.8 percent; changing y_2 to $y_2^o \pm 0.2$ increases $\bar{T}$ by only about 3 percent.

In this example, the only behavior we have observed which apparently suggests that the optimal coordinates (x_2^o, y_2^o) <u>are</u> sensitive to the particular values of the coordinates (x_1, y_1) is the discontinuous jump in the locus of (x_2^o, y_2^o) when (x_1, y_1) follows the square trajectory in Figure 5. However, if (x_2^o, y_2^o) is required to maintain a continuous path, so that an optimal type 1 boundary is always employed, the corresponding mean travel time is increased only slightly over the optimal (less than 4 percent) that would be achieved by allowing the jump. In fact, the observation that for certain (x_1, y_1) there are two almost equally desirable areas in which facility 2 may be located is an interesting and possibly useful consequence of the insensitivity of the mean travel distance.

V. FACILITY LOCATION: TWO COOPERATING DISTRICTS

In the previous section we described the situation in which a region R is divided into two districts, each containing one service facility. The unit(s) assigned to each facility serviced calls only in the facility's own district, and our primary concern was the mean travel time from the facility to points in its own district.

We now examine the same issues discussed in the previous section for the second class of systems described in Section I. Again the region is divided into two districts, R_1 and R_2, with which we associate facilities 1 and 2 and units 1 and 2, respectively. In this case, however, unit i has primary responsibility in R_i, responding to calls in R_i whenever it is available. Unit i, however, also responds to calls in R_j $(j \neq i)$ if unit j is unavailable. A model proposed by Carter, Chaiken and Ignall[7] effectively represents this situation. The model assumptions are as follows:*

1. The service units in two cooperating districts act as a two-server, zero line capacity, queueing system;

2. From any region $A \subseteq R$, calls arrive as a simple Poisson process at an average rate $\lambda(A)$; define $\lambda(R) = \lambda$;

3. The region R is partitioned into two districts, R_1 and R_2. A call arriving from district R_i $(i=1,2)$ is serviced by unit i, if available; otherwise it is serviced by the other unit, if available; if both units are busy, the call is lost;

4. The mean time required to service each call is μ^{-1}, regardless of the server, the position of the call, and the history of the system.**

*This model is an obvious over-simplification of most real, operating systems. A more realistic model could include (1) the possibility of queueing calls if both units were already busy; (2) the assignment of priorities to different types of calls; (3) the inclusion of more than 2 cooperating units in a region. Yet, even this simple model will illustrate some of the important system properties when system dynamics and inter-district cooperation are considered.

**The service time includes time traveling to and from the scene and the time spent at the scene. Thus, this assumption implies that travel time is a negligibly small fraction of the total service time. This simplification is remarkably good for most police operations, is adequate for fire operations, and appears to be inadequate for current ambulance service operations.

Computation of the mean travel time for the system operating in the steady-state requires the long-range probabilities of the system being in each of the following states:

AA: Both units available
AW: Unit 1 available, unit 2 working
WA: Unit 1 working, unit 2 available
WW: Both units working

Then, for instance, the fraction of calls lost is* P_{WW}, the fraction of time unit 1 is busy is $P_{WA} + P_{WW}$, and the fraction of dispatches which are "unit 1 to district 2" is $P_{AW}\ \lambda(R_2)/((1-P_{WW})\lambda)$. A measure of the system workload is $r = \frac{\lambda}{\mu}$.

From reference 7, an optimal boundary between the two districts is known to be a line on which all points are 2s units of distance closer to facility 1 than to facility 2 (for some s, possibly negative). For right-angle travel distances, an optimal boundary consists of points (x,y) satisfying an equation of the form

$$| x_1 - x | + | y_1 - y | = | x_2 - x | + | y_2 - y | - 2s, \qquad (1)$$

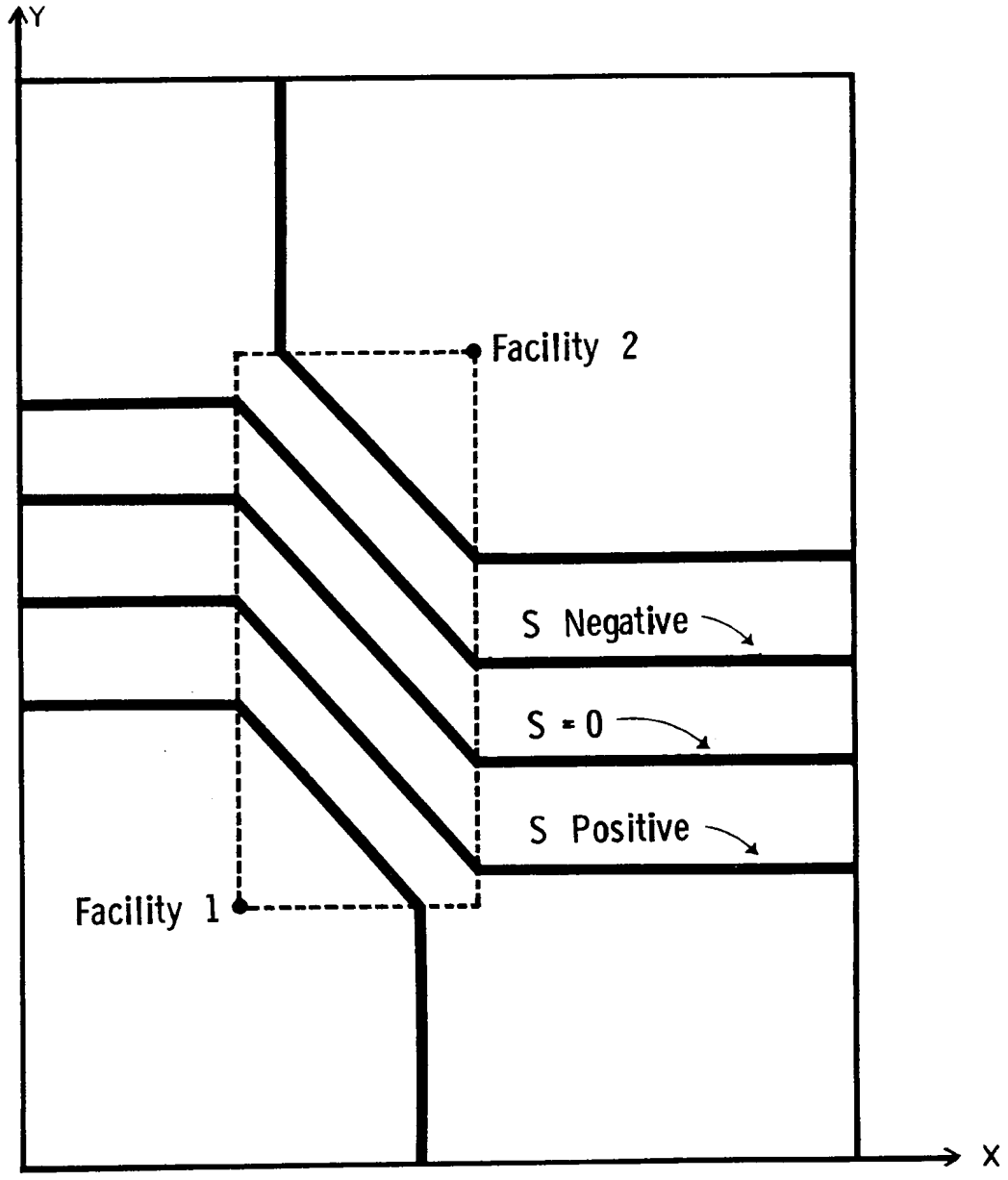
where (x_1,y_1) and (x_2,y_2) are the respective facility coordinates. Eq. (1) generates the family of boundaries illustrated in Figure 7.** Note that for sufficiently small $|s|$, these boundaries resemble the boundary found in Section IV, but vertically displaced a distance s.

With the positions of the two facilities fixed and the boundary displacement s determined, it is a straightforward though tedious task to calculate the mean travel time using the four probabilities listed above. In order to illustrate several qualitative features of the model we again examine a 1 x 2 rectangular region in which requests for service are uniformly distributed and facility 1 is located at (x_1,y_1). For a given value of r, we find the minimum mean travel time by positioning facility 2 arbitrarily at (x_2,y_2) and finding the optimal boundary

*The long-range probability of the system being in state Z is denoted P_Z.

**Given (x_1, y_1) and (x_2, y_2), there will be two values of s for which Eq. (1) does not yield unique boundaries but rather "regions" in which all points are 2s closer to facility 1 than to facility 2. This situation is analagous to that depicted in Figure 2c. Any line partitioning R and contained in such a "region" is an optimal boundary.

FIG. 7: FAMILY OF OPTIMAL BOUNDARIES
FOR TWO COOPERATING DISTRICTS



displacement s by a gradient search procedure.* Then using a second gradient search, facility 2 is repositioned in the direction of the maximum reduction in mean travel time, and a new optimal boundary displacement is found. The mean travel time is recalculated and the procedure is repeated until the optimal position (x_2^o, y_2^o) for facility 2 is reached. The mean travel time associated with this position of facility 2 is a minimum, given the position of facility 1 and the value of r.

For each of the values of r = 0.0, 0.4, 2.0, we have varied the position of facility 1 continuously over the square of dimension 0.9 centered at (0.5, 0.5) and computed the corresponding locus of optimal facility 2 positions. The situation in which $r \simeq 0.0$ corresponds to $P_{AA} \simeq 1$, in which case there is seldom any need for inter-district dispatching. Hence, the locus of optimal facility 2 positions is essentially the same as shown in Figure 5.

For r = 0.4, we notice in Figure 8 the same discontinuity observed in Figure 5 in the locus of (x_2^o, y_2^o) as (x_1, y_1) follows the square trajectory. In addition, the optimal location of facility 2 is even less sensitive to the location of facility 1 than we have observed for $r \simeq 0.0$; and the locus of optimal positions has shifted downward.

These last two observations are more pronounced when r = 2.0 (Figure 9). We see that as r increases the optimal position of facility 2 becomes less sensitive to the location of facility 1 and converges from both the type 1 and type 2 boundary regions to the point (0.5, 1.0). This phenomenon can be understood intuitively as follows. As r increases the probability P_{AA} that both service units are simultaneously available goes to zero. For large r, nearly all dispatch assignments are made to the one available unit, the other unit being busy on a previous assignment. Thus, the spatial distribution of incidents to which either service unit is assigned becomes uniform over the entire region. Clearly, the optimal position for a unit whose incidents are distributed uniformly over the region is at the center of the region (0.5, 1.0). Hence for large r, the

*Reference [7] and this document were developed concurrently and the authors found it more convenient to continue using an existing computer gradient search method than to reprogram for the calculation of the optimal s by Theorem 1 in that reference.

FIG. 8: CORRESPONDING FACILITY LOCATIONS FOR MINIMUM EXPECTED TRAVEL TIME WHEN r = 0.4

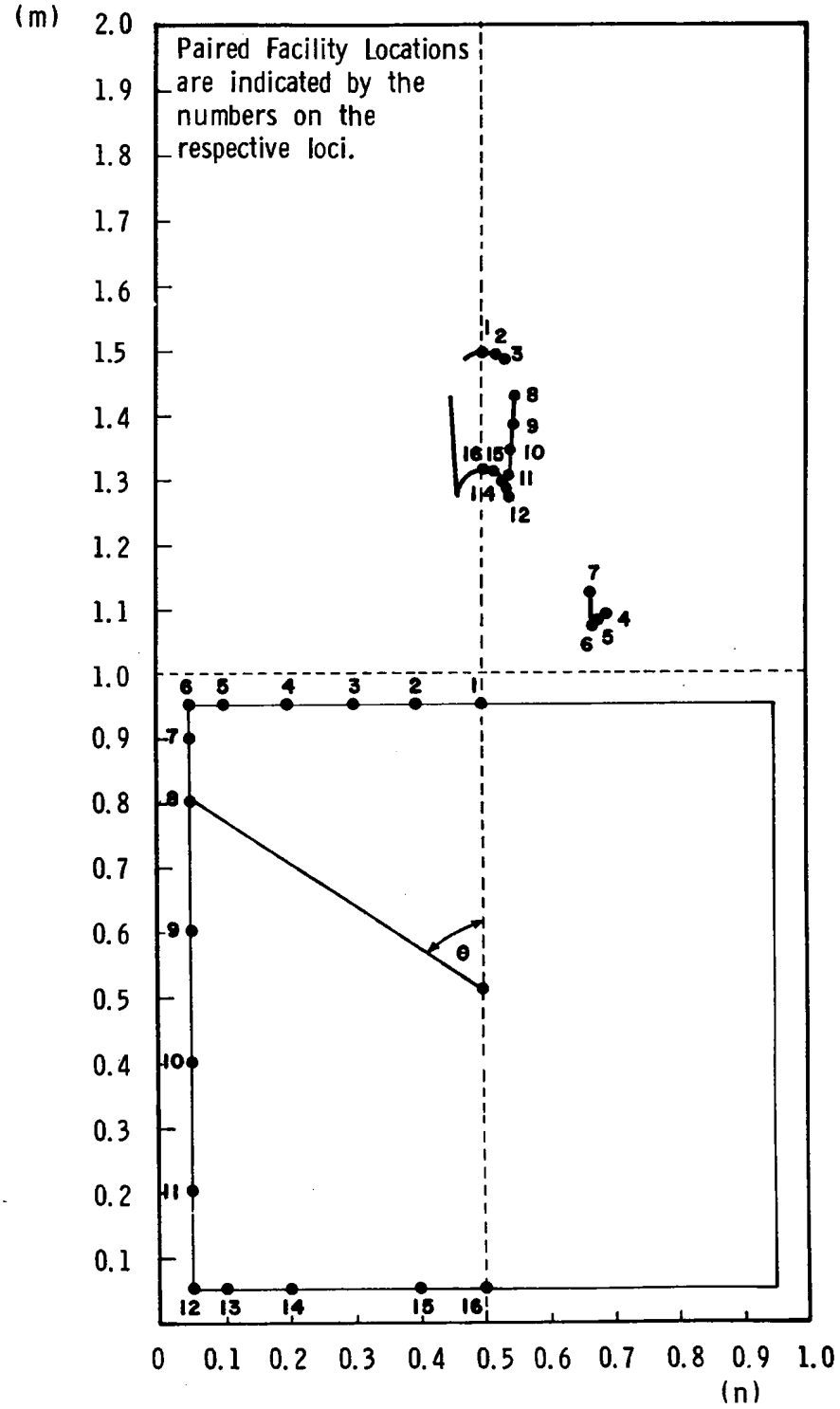
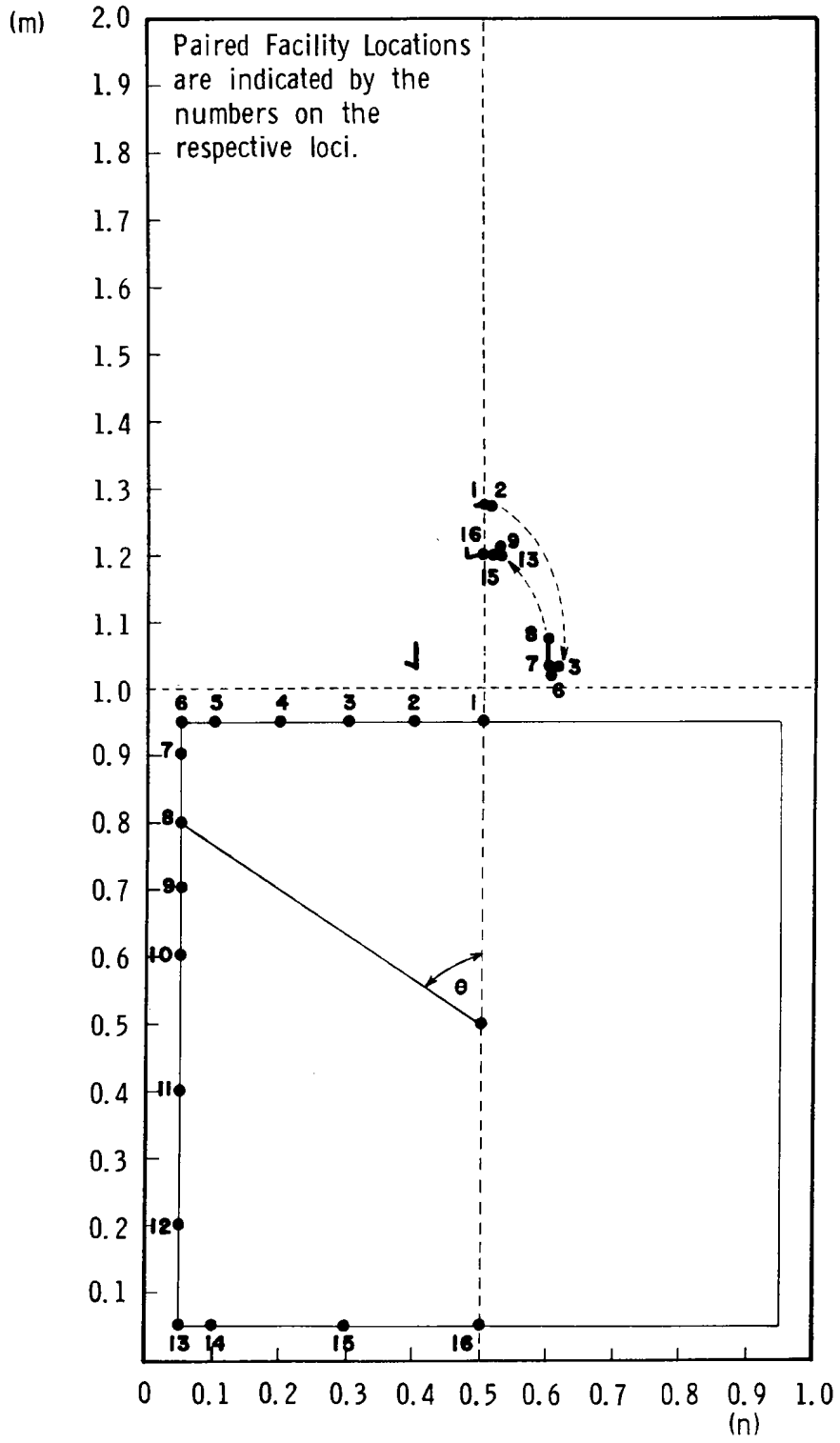


FIG. 9: CORRESPONDING FACILITY LOCATIONS FOR MINIMUM EXPECTED TRAVEL TIME WHEN r = 2.0

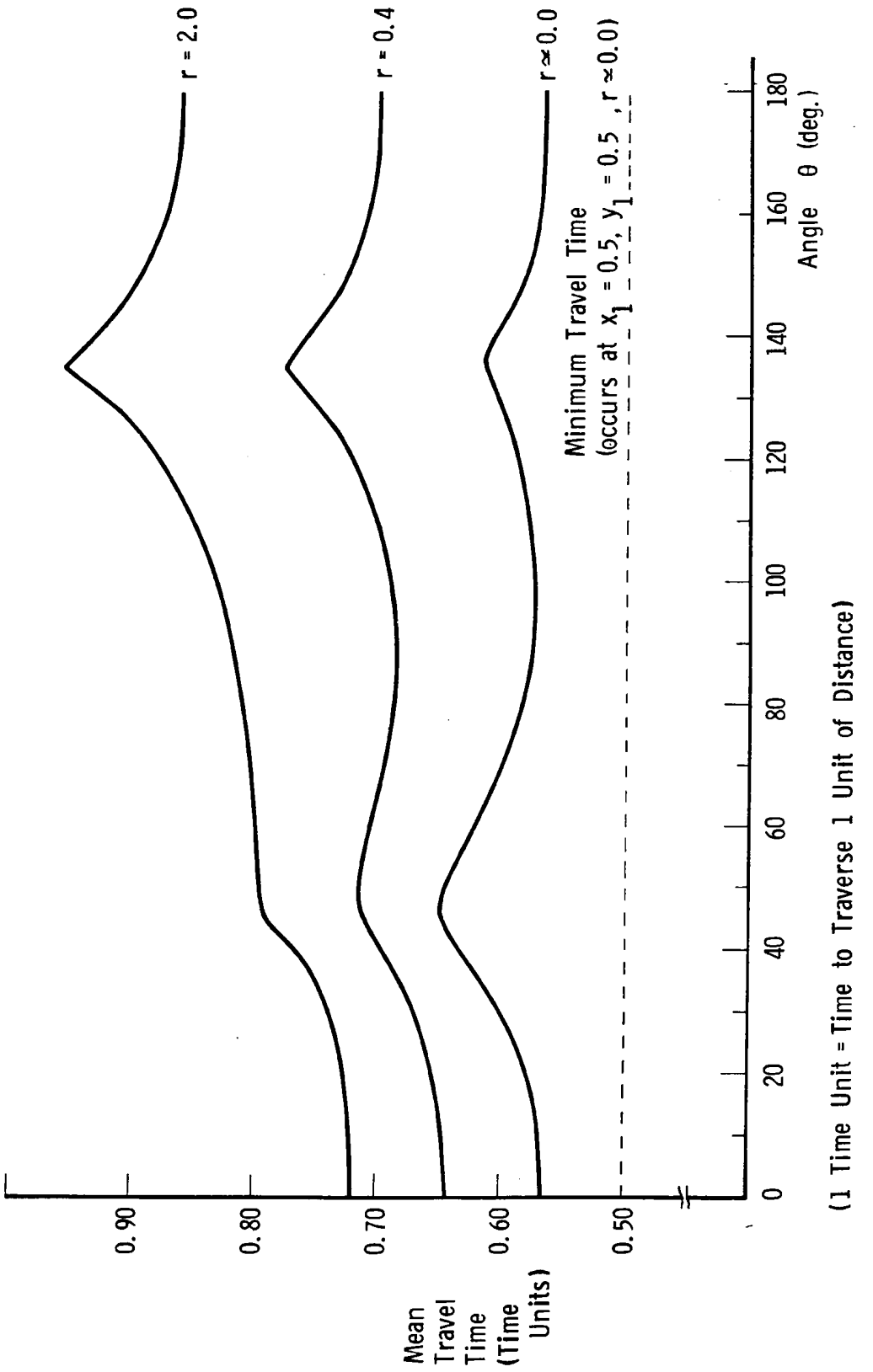


optimal location of facility 2 is near the center of the region, irrespective of the location of facility 1. Indeed, if we had the freedom to specify the position of facility 1, its optimal position would also converge to the midpoint (0.5, 1.0).

Following the method illustrated in Section IV, we plot the minimal expected travel time $\bar{T}$ vs. the angle θ of the position of facility 1 in Figure 10. Three different functions are plotted, corresponding to the three values of r (r = 0.0, 0.4, 2.0). When r ≃ 0.0, (i.e., no inter-district dispatching), the maximum $\bar{T}$ occurs for values of θ near 45° and 315°, corresponding to facility 1 near the top corners of the square. As r is increased, the effects of traveling long distances from (x_1,y_1) into district 2 become apparent, and the maximum $\bar{T}$ occurs for values of θ near 135° and 225°, corresponding to facility 1 near the bottom corners of the square. We notice that for r = 2.0, $\bar{T}$ increases both as we move away from the line x = 0.5, and as we move away from the line y = 0.95. This is consistent with the notion that for large r the optimal location of each facility is at the point (0.5, 1.0).

We again note that despite the large changes in the position of facility 1 implied by the square trajectory of side 0.9, the total variation in mean travel time (for small r) is reasonably small. Sensitivity analyses (similar to those reported in Section IV) also corroborate the notion that the mean travel time is quite insensitive to small variations about the optimal location of facility 2, given a location for facility 1.

FIG. 10: MINIMAL MEAN TRAVEL TIME AS A FUNCTION OF FACILITY 1 POSITION AND r



VI. CONCLUSION

This paper has used simple analytical models to shed some light on the questions raised on Section I. The results suggest that due to insensitivities it may not be necessary to undertake costly, time-consuming data collection and analyses to arrive at a reasonable solution to redistricting and facility location problems. Considering the diversity of political constraints and other nonquantifiable factors confronting an urban administrator, simple models which reveal through insensitivities a range of policy options may be more useful for planning purposes than complex algorithms that produce precise numerical "answers."

Recognizing that only a few special cases have been examined, we do not suggest that all redistricting and facility location problems can be "solved" with oversimplified models. Clearly, more work is required in further exploring the insensitivities of these models. However, in actual urban environments administrators are often required to make decisions without detailed analysis, and simple analytical models can provide important and useful insights to these decision-makers. Furthermore, when detailed analyses are contemplated, we would expect the insights afforded by these models to modify both the amount of analysis required and its direction by rapidly providing the analyst with an indication of the general nature of the solution.

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