COMPETITIVE OUTCOMES IN THE CORES OF MARKET GAMES

PREPARED FOR THE NATIONAL SCIENCE FOUNDATION

LLOYD S. SHAPLEY
MARTIN SHUBIK

R-1692-NSF
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PREFACE

This Report presents a contribution to the study of abstract market games, building on the model first introduced in RM-5671-PR: On Market Games (Shapley and Shubik, July 1968; see also Ref. [9]). It is part of a continuing investigation into the applications of game theory to economics, under a grant from the National Science Foundation; other recent publications include R-1476-NSF: An Example of a Slow-Converging Core, January 1974, and R-904/4-NSF: Game Theory in Economics--Chapter 4: Preferences and Utility, December 1974.

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SUMMARY

The competitive outcomes of an economic system are known, under quite general conditions, always to lie in the core of the associated cooperative game. It is shown here that every "market game" (i.e., one that arises from an exchange economy with money) can be represented by a "direct market" whose competitive outcomes completely fill the core. It is also shown that it can be represented by a market having any given core outcome as its unique competitive outcome, or, more generally, having any given compact convex subset of the core as its full set of competitive outcomes.
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1. INTRODUCTION

In a previous paper [9], the authors introduced a class of cooperative n-person games in characteristic-function form, called "market games," which come from trading economies in which the traders measure utility in money. This class of games was shown to coincide with the class of "totally balanced" games, i.e., games that have nonempty cores and all of whose subgames have nonempty cores as well.

In this report we shall compare the cores of market games with the competitive equilibria of the markets that they come from. We first consider the "direct market," previously introduced in [9], and discover that its competitive outcomes fill the entire core (Theorem 1). We then take an arbitrary point in the core and construct a market (actually a class of markets) which generates the given market game and which has the given core point as its only competitive outcome (Theorem 2). A modification of this construction yields any other closed convex subset of the core as the set of competitive outcomes.

Extensions of these results to games and markets without money (i.e., transferable utility) will be considered in a subsequent paper.
2. GAMES AND MARKETS

The reader is referred to [9] for a more extended discussion of the matters reviewed in this section.

A game, for our present purpose, will consist of a finite set N and a real-valued set function v, the latter defined on all the subsets of N and satisfying \( v(\emptyset) = 0 \). In the standard interpretation, N represents the set of players, and \( v(S) \) represents the "worth" of S, i.e., the total amount of utility (measured in some monetary unit) that the members of S can secure if they form a coalition and play the game without help from the other players.

Outcomes of the game are expressed as N-tuples of utility: \( \alpha = \{ \alpha_i : i \in N \} \), called payoff vectors. A payoff vector \( \alpha \) is said to be "feasible" if \( \sum_{i \in N} \alpha_i \leq v(N) \); "efficient" if \( \sum_{i \in N} \alpha_i = v(N) \); "individually rational" if \( \alpha_i \geq v(\{i\}) \) for each \( i \in N \); and "coalitionally rational" if \( \sum_{i \in S} \alpha_i \geq v(S) \) for each \( S \subseteq N \). The set of feasible, coalitionally rational payoff vectors is called the core of the game; thus, \( \alpha \) is in the core if and only if

\[
(1) \quad \alpha \cdot e^S \geq v(S), \text{ all } S \subseteq N, \text{ and } \alpha \cdot e^N = v(N),
\]

where \( e^S \) denotes the N-vector having \( e^S_i = 1 \) if \( i \in S \) and \( e^S_i = 0 \) if \( i \in N - S \). Geometrically, the core is a compact convex polyhedron, possibly empty. It is well known* that nonemptiness of the core is equivalent to the game being balanced, in the sense that

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*See [7].
(2) \[ \sum_{S \subseteq N} \gamma_S v(S) \leq v(N) \]

holds for every set of "weights" \( \gamma_S \geq 0, \ S \subseteq N \) such that

(3) \[ \sum_{S \subseteq N} \gamma_S e^S = e^N. \]

A game \( v \) is said to be **totally balanced** if it is balanced, and all
of its subgames, obtained by restricting \( v \) to the subsets of \( R, R \subseteq N \),
are also balanced.

A **market**, for our present purpose, will consist of a finite set
\( N \) of "traders"; a finite set \( M \) of "commodities"; an \( |M| \)-dimensional eu-
clidean orthant \( E^M_+ \) of "bundles"; and, for each \( i \in N \), an **initial bundle**
\( a^i \in E^M_+ \) and a continuous concave **utility function** \( u^i \) from \( E^m_+ \) to the
reals. In the interpretation, utility is understood to be measured
in units of money, and the traders may not only exchange the \( m \) com-
modities as initially supplied to them, but they may also transfer
money in any amount. The **final payoff**, which is what a trader seeks
to maximize, is therefore found by adding his net gain of money to
the utility of his final bundle of commodities.

A market generates a game in a natural way. First, for any
\( \emptyset \subseteq S \subseteq N \), let us define an **S-allocation** to be an indexed set
\( x^S = \{ x^i : i \in S \} \) of bundles in \( E^M_+ \). An S-allocation is said to be
**feasible** if

\[ *\text{See [8], pp. 807-808.} \]
\[ \sum_{i \in S} x^i = \sum_{i \in S} a^i. \]

To generate the game, we set \( v(\emptyset) = 0 \) and, for each \( \emptyset \subset S \subset N \), define

\[ v(S) = \max \sum_{i \in S} u^i(x^i), \]

letting the maximum run over all feasible \( S \)-allocations \( x^S \). Any \( v \) that can be represented in this way is called a market game.

By the "core" of a market we shall mean the core of the game it generates. Since market games are always balanced (indeed, the market games are precisely the totally balanced games), all markets have nonempty cores.

Obviously, many markets may generate the same game, while being dissimilar in other respects. In [9], a canonical representative for each class of "game-theoretically equivalent" markets was introduced, called the direct market because of its simple form and the fact that the players themselves are, in a sense, the commodities being brought and sold.

To define the direct market for any game \( v \), we first put the commodities \( M \) into one-to-one correspondence with the traders \( N \). The initial allocation is then given by \( a^i = e^i \), \( i \in N \). The traders all have identical utility functions, \( u^i = u \), given by

\[ u(x) = \max \sum_{S \subset N} \gamma_S v(S), \]
the maximum taken over all \( \{ \gamma_S \geq 0 : S \subseteq N \} \) satisfying*

\[
\sum_{S \subseteq N} \gamma_S e^S = x.
\]

An intuitive explanation of this market can be given in terms of group activities or implicit production processes; the reader is again referred to [9]. The technical justification for this construction, however, lies in the fact that if we start with a totally balanced set function \( v \), then taking the market game of the direct market of \( v \) gives us \( v \) back again. On the other hand, if we start with a \( v \) that is not totally balanced, then we get back the so-called "cover" of \( v \), which is the least totally balanced function that is greater than or equal to \( v \) for all \( S \).

From (4) and (5) and the definition of "balance" ((2), (3) relativized to \( R \)), one can verify** that the utility function of the direct market of a totally balanced game satisfies

\[
(6) \quad u(e^R) = v(R), \quad \text{all } R \subseteq N.
\]

This shows that \( u \) may be regarded as an extension of the set function \( v \)

*At least one such set of weights exists, since we can take \( \gamma[i] = x_i, \ i \in N, \) and all other \( \gamma_S = 0. \)

**See [9], Eq. (4-1).
to the domain of "fuzzy" sets, i.e., coalitions whose members participate at fractional levels of intensity.*

*This extension is continuous, concave, and positively homogeneous of degree 1, and is possible only when v is totally balanced. It may be contrasted with Owen's "multilinear" extension [6], which is always possible, for any v, and which is the appropriate extension for studying the "value" solution concept. See also Aubin [1] and Aumann and Shapley [2], Ch. IV, especially p. 166.
3. THE COMPETITIVE EQUILIBRIUM

The so-called "competitive" solution is not a game theory concept, but is based on the notion of an imposed schedule of prices which, if accepted by all the members of the economy, will make it possible to balance supply and demand in each commodity, clearing the market to everyone's satisfaction. In our present setting, we must remember that "money" or "transferable utility" is implicitly one of the commodities in exchange, so that the \( i \)th trader's complete utility functions takes the form

\[
U^i(x^i, s^i) = u^i(x^i) + s^i,
\]

where \( s^i \) denotes his final money balance. If we wish to keep this new commodity explicitly in view, a typical price schedule could be written as

\[
(\pi, l) = (\pi_1, \ldots, \pi_{|M|}, l).
\]

These prices serve to evaluate everything, including money, in terms of a new accounting unit; setting the price of money at 1 is just a convenient normalization.

Acting "competitively" in the face of (8), trader \( i \) will seek to maximize (7) subject to the budget constraint

\[
\pi \cdot x^i + s^i = \pi \cdot a^i + s^i_0,
\]

\( s^i_0 \) being his initial money balance. On the assumption of freely
transferable utility, $\xi^i$ is an unrestricted variable, so we may solve (9) and eliminate $\xi^i$ from (7). Trader i's goal can now be restated: to maximize

$$(10) \quad u^i(x^i) + \xi_0^i \pi^i (a^i - x^i)$$

where $x^i$ is chosen unrestrictedly from $E^M_+$. For the price schedule $(\pi, 1)$ to be in competitive equilibrium, there must exist a set of maximizing choices by the different traders that fit together to form a feasible $N$-allocation, since only then will the market be cleared to everyone's satisfaction.

By this roundabout path we have arrived at the definition we want.* A competitive solution for one of our markets is a pair $(\pi, z^N)$, where $\pi$ is an arbitrary $N$-vector,** and $z^N$ is a feasible $N$-allocation such that

$$(11) \quad u^i(z^i) - \pi \cdot z^i = \max_{x^i \in E^M_+} [u^i(x^i) - \pi \cdot x^i], \quad \text{all } i \in N.$$  

In words, each trader maximizes his "trading profit." Note that

---

*We could, of course, have stated and justified this definition directly, keeping the role of transferable utility hidden beneath the notational surface. We wished, however, to establish a firm connection with the standard definition of competitive equilibrium, and to clear up the (to some readers) mysterious absence of any budget constraint in (11).

**We have not been assuming that utilities are nondecreasing, and so we do not assume here that prices are nonnegative. This approach also entails using "$= "$ rather than "$\leq "$ in the budget condition (9). Note also that our prices are not just ratios, but have meaningful magnitudes; this is because of the hidden normalization at (8) that sets the price of money at 1.
we have omitted the terms $s^i$ and $\pi \cdot a^i$ appearing in (10), as it is
irrelevant to the maximization problem.

Moving to the payoff space, we shall call a vector $\alpha$ competitive
if it arises from a competitive solution $(\pi, z^N)$, thus:

$$\alpha^i = u^i(z^i) - \pi \cdot (z^i - a^i), \quad \text{all } i \in N.$$  

(12)

Using the fact that the competitive allocations are those that max-
imize total utility $\Sigma_{i \in N} u^i(x^i)$, it is not hard to establish that
the set of all competitive payoff vectors of a market is compact and
convex, and that it is contained in the core. The containment may
well be strict; indeed, there is often a unique competitive solution,*
whereas the core is typically a set of $|N| - 1$ dimensions. The fol-
lowing theorem shows, however, that in a direct market the set of
competitive vectors and the core coincide.

**THEOREM 1.** Every payoff vector in the core of a
game is competitive in the direct market of that game.

**Proof.** The idea will be to show that any core point can be used
as a competitive price vector for the direct market of the game.**

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*A sufficient condition for uniqueness of the competitive payoffs
is that the $u^i$ all be differentiable and that at least one competitive
allocation be strictly positive.

**If the game is not balanced its core is empty, so the theorem
is vacuously true. If the game is balanced but not totally balanced,
its core coincides with that of its cover (see [9], Lemma 3), and
hence with the core of its direct market.
Let $\alpha$ be in the core of $v$;

\[(13=1) \quad \alpha \cdot e^S \geq v(S), \text{ all } S \subseteq N, \quad \text{and } \alpha \cdot e^N = v(N).\]

Nonemptiness of the core means that $v$ is balanced;

\[(14=2) \quad \sum_{S \subseteq N} \gamma_S v(S) \leq v(N)\]

for all \(\{\gamma_S \geq 0 : S \subseteq N\}\) such that

\[(15=3) \quad \sum_{S \subseteq N} \gamma_S e^S = e^N.\]

By (6) and (13) we have

\[(16) \quad u(e^N) = v(N) = \alpha \cdot e^N.\]

In other words, the value of the total supply of goods available in the direct market is the same in utility terms as it is when computed using $\alpha$ as a price vector.

Next, take an arbitrary bundle $x \in E^M_+$, and let $\{\gamma_S : S \subseteq N\}$ be any set of nonnegative coefficients satisfying (5). Then, by (13) and (5),

\[\sum_{S \subseteq N} \gamma_S v(S) \leq \sum_{S \subseteq N} \gamma_S (\alpha \cdot e^S) = \alpha \cdot \sum_{S \subseteq N} \gamma_S e^S = \alpha \cdot x.\]
Hence

\[(17)\quad u(x) \leq \alpha \cdot x.\]

We can now show that \(\alpha\) is a competitive payoff vector. Define prices by \(\pi_i = \alpha^i, \ i \in N\). At these prices, a trader trying to choose \(x^i\) to maximize his "trading profit" \(u(x^i) - \pi \cdot x^i\) will find that he can't make it positive,* because of (17), but that he can make it zero by choosing \(x^i\) to be the bundle \(e^N\), because of (16). By the homogeneity of \(u\), any fraction \(f^i\) of that bundle is just as good. So we can construct a competitive solution \((\pi, z^N)\) by taking \(z^i = f^i e^N\), where the \(f^i\) are nonnegative numbers that sum to 1. Moreover, \((\pi, z^N)\) yields the desired payoff vector \(\alpha\), since we have (see (8))

\[u(z^i) - \pi \cdot z^i + \pi \cdot a^i = 0 + \pi_i = \alpha^i\]

for each \(i \in N\). This completes the proof of Theorem 1.

Theorem 1 tells us that every point in the core of a market game is competitive for at least one of its generating markets—namely, its direct market. The next theorem refines this result, by showing

*Indeed, if he could make it positive he could make it arbitrarily large, and no maximum would exist. But this is not the real issue; in general, with nonhomogeneous, concave utilities, a positive trading profit is quite possible at competitive equilibrium.
that for each core point there are markets that generate the given
game and for which only that core point is competitive.

**Theorem 2.** Among the markets that generate a given
totally balanced game, there is one that has any given
core point as its unique competitive payoff vector.

**Proof.** Let \( v \) be totally balanced, let \( d \) be a real number, and define

\[
    v_d(S) = v(S), \quad \text{all } S \subseteq N, \text{ and } v_d(N) = v(N) + d.
\]

The game \( v_d \) is obviously totally balanced if \( d \geq 0 \). Let \( u_d \) denote
the utility function for the direct market of \( v_d \) (see (4)). By (6) we have

\[
    u_d(s^S) = v_d(S), \quad \text{all } S \subseteq N.
\]

Let \( \alpha \) be in the core of \( v \), and define a modified utility function
\( u_{d,\alpha} \) by

\[
    u_{d,\alpha}(x) = \min(u_d(x), \alpha \cdot x).
\]

This is continuous and concave, and so it can be used in defining a
market with the same commodity space and initial bundles as a direct
market on \( N \). We shall show that for any positive \( d \) this market has
the properties claimed in the statement of the theorem, namely, (a)
that its market game is \( v \) and (b) that its unique competitive payoff vector is \( \alpha \).

(a) Since we have equal tastes and homogeneity of degree 1, the generated market game—call it \( w \)—is given by:

\[
(21) \quad w(S) = u_{d,\alpha}(\sum_{i \in S} a_i) = u_{d,\alpha}(e^S)
\]

(compare (6)). By (20) and (19), this means that

\[
w(S) = \min(v_d(S), \alpha \cdot e^S).
\]

By (18) and (1), we see that this is equal to \( v(S) \) both when \( S \neq N \) and when \( S = N \), and so \( w = v \) as claimed.

(b) As noted in the previous proof, the competitive solution maximizes total utility. Since \( u_{d,\alpha} \) is homogeneous of degree 1 and concave, any competitive price vector will therefore be the gradient of a linear support to \( u_{d,\alpha}(x) \) at \( x = e^N \) (or at any scalar multiple of \( e^N \)). But, by the definition (20) of \( u_{d,\alpha} \), when \( d \) is positive \( u_{d,\alpha}(x) \) is equal to the linear function \( \alpha \cdot x \) in at least a small neighborhood of \( x = e^N \), because we have

\[
u_d(e^N) = v_d(N) > v(N) = \alpha \cdot e^N
\]

and \( u_d \) is a continuous function. Hence the unique competitive prices are given by \( \pi_i = \alpha_i \), \( i \in N \), and as we saw in the previous proof these yield \( \alpha \) as the competitive payoff vector. This completes the proof of Theorem 2.

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*See [9], Eq. (3-3).
By a simple extension of this proof, a market can be constructed having any given closed convex subset of the core as its set of competitive payoff vectors. Indeed, it is only necessary to define
\[ u_{d,A}(x) = \min_{\alpha \in A} u_{d,\alpha}(x), \]
where \( A \) is the desired convex set. When \( A \) equals the core, this reduces to the direct market, independently of \( d \), and so unifies Theorems 1 and 2.
4. CONCLUDING REMARKS

The relation between the core and the competitive equilibrium can be explained in terms of the information that is lost in passing from a market to the game that it generates. In the first place, all details concerning the commodities and their distribution among the traders are suppressed, since the analysis of the market game takes place in the utility or payoff space, not the allocation space. In the second place, the game actually takes cognizance of only a finite number of the possible outcomes of the market process, i.e., the best result for each coalition. Most of the detailed preference information contained in the utility functions is ignored, as may be seen clearly in Eqs. (6) or (21) above, where the utilities are evaluated only at the vertices of a cube. This loss of information shows that the core is a blunt solution concept, and accounts for the many-to-one correspondence between markets and their market games. It is not surprising that we were able to find plenty of markets (without even looking beyond the special type of markets where the commodities "are" the traders) having just the competitive outcomes that we needed for our proofs.

In a subsequent note we plan to consider the competitive equilibria of markets without money, using the framework set forth in the work of Billera and Bixby.* Here too, although the space of games is far richer (the function v being set-valued), there is a great loss of information and a similar many-one relationship between markets and games. It turns out, however, that the locus of competitive payoffs is not the entire core of the game, but only a certain

"inner core." This inner core can also be characterized directly, without reference to the economic model. It should be emphasized, however, that our methods of analysis depend heavily on the assumption of concave utility functions, and hence, on the existence of cardinal utilities.* A radically new approach may be required before the characterization of "ordinal" market games and their competitive solutions can be clarified.

*See [3], pp. 129-130.
REFERENCES


