Theory of Normal Modes and Ultrasonic Spectral Analysis of the Scattering of Waves in Solids

Yih-Hsing Pao and C. C. Mow
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This research was supported in part by The Rand Corporation in an effort to better the basic understanding of some aspects of science and technology related to national security and welfare.

Y. H. Pao is Professor of Theoretical and Applied Mechanics, Cornell University, and a consultant to The Rand Corporation. His work is also supported by the Material Science Center of Cornell University. This report is documented in part as Material Science Center Report No. 2547.
SUMMARY

Spectral analysis of ultrasonic pulses in elastic solids has attracted wide attention in recent years as a tool of the quantitative nondestructive test method. So far, a complete analysis of the power spectra of the scattered pulses of any geometry is still lacking. This has hindered the general understanding and the application of ultrasonic spectroscopy to the detection of inclusions or flaws in solids.

This report presents a theory of the spectral analysis of the scattering of elastic waves and illustrates it with numerical results for the scattering by a circular cylindrical fluid inclusion in a solid. When the spectral frequencies are nearly equal to the real parts of the principal frequencies of the fluid inclusion in free vibration, the power spectrum of the scattered pulses undergoes a rapid rise and fall in magnitude because of the selective transmission of an incident wave. The conspicuous peaks and valleys of the backward and forward scattering spectra can be identified with the overtone frequencies of the two lowest normal modes of the cylinder, from which the characteristics of the fluid inclusion, the ratio of the wave speed to radius, can be determined. An application of spectral analysis to quantitative nondestructive testing of materials is discussed.
ACKNOWLEDGMENTS

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I. INTRODUCTION

As a tool of quantitative nondestructive testing of materials, the spectral analysis of ultrasonic pulses in elastic solids has attracted wide attention in recent years. Its possible applications to measurement of the thickness of a multilayered medium and detection of inclusions and flaws in solids were reviewed by Gericke. (1) For the case of multilayered media, the peaks of the transmission spectra of a normally incident wave have been attributed to the cavity resonance of plane waves in the elastic layers. (2) In a study of scattering by a cylindrical fluid inclusion, the multiple peaks of the power spectra of the scattered pulses were tentatively related to the resonance of the fluid inclusion. (3)

So far, a complete analysis of the power spectra of the scattered pulses by objects of any geometry is lacking. This has hindered the application of ultrasonic spectroscopy to the detection of inclusions or flaws in solids. Experimentally, it has been found that the detailed structure of the spectrum of scattered waves is very sensitive to the transducers used, the bonding conditions between a transducer and a specimen, the material properties of the matrix medium, the position of the transducer relative to the inclusion, and many other factors that cannot be easily controlled. The difficulty of producing repeatable data in laboratories motivated us to search for a theoretical basis for the spectrum.

Using a circular cylindrical fluid inclusion in an elastic solid as an example, we discuss a theory of the spectral analysis of scattered pulses and the interpretation of the spectrum from the theory of normal modes. The cylindrical geometry is chosen because the exact solutions for the scattering of elastic waves by a circular cylinder are known. For an incident plane harmonic wave, the solutions were given by White. (4) His work, along with many others, is summarized in a monograph by Pao and Mow. (5)

In Section II we collect all pertinent equations and results as contained in Ref. 5 (Chapter 3). The free oscillations and normal modes
of the fluid inclusion in an elastic solid are then analyzed in Section III, and the rather complicated frequency equations are solved to determine the complex-valued principal frequencies of the fluid-solid system. Numerical results are presented for a water inclusion in a body of aluminum.

In Section IV, we present a theory of scattered wave spectra. Based on Fourier analysis, we first show that the power spectrum of scattered pulses equals the power spectrum of the incident pulse multiplied by the steady state response function of the inclusion that is a representation of the scattered waves under the excitation of a monochromatic incident wave. For an incident pulse with the delta function in time dependence, the steady state response function is the same as the power spectrum. Since the power spectrum for a given incident pulse is known, in Section IV we determine the theoretical steady state response function of a fluid inclusion. The results exhibit the typical peak and valley structure of the experimentally determined power spectra of scattered pulses.\(^{(6)}\)

In Section V, we analyze the spectra in detail. When the incident wave frequency is nearly equal to the real part of the frequency of a normal mode, the magnitude of the steady state response function rises and falls sharply. Thus from the peak and valley structure, it is possible to identify the normal modes. In the case of backward scattering, the spectrum dips sharply at the overtone frequencies of the zeroth and first modes of the fluid inclusion. This is attributed to the total transmission of wave energy into the inclusion at approximately the resonating frequencies. In forward scattering, the spectra exhibit sharp peaks at the same overtones because of the strong radiation from the scatterer vibrating at resonance. When the receiver is at right angles to the transmitter, again the spectra peak at the overtone frequencies of the zeroth mode. These results are similar to the selective transmission of waves in layered media,\(^{(7)}\) briefly reviewed in the appendix.

Since the frequency difference of two overtones is approximately proportional to the ratio of the wave speed in the fluid inclusion to the radius of the inclusion,\(^{(3)}\) an unequivocal identification of overtones permits determination of either the wave speed or the radius of
the fluid inclusion from the power spectra. The application of this
technique to nondestructive testings is discussed in Section VI.

The resonance effect on the power spectra is analogous to the opti-
cal resonance in the light scattering by small particles. Since
optical probes usually measure the intensity of optical waves, many re-
sults in light scattering are expressed in terms of the total or differ-
etial scattering cross-sections. However, in ultrasonic experiments,
the piezoelectric transducers do not directly measure the sound wave
intensity. Instead, the voltage output of these transducers depends
on the stresses acting on the contact surface between the transducer
and the specimen. Hence we have calculated the power spectra of stress
pulses scattered by the cylinder.

The results shown in Section IV are compared with experimentally
measured power spectra. Although the detailed features of the theoreti-
cal and experimental spectra are quite different, the experimental
spectra do rise and fall at the theoretically predicted resonance fre-
quencies. Details are given in Ref. 9.
II. INCIDENT AND SCATTERED WAVES--STEADY STATE RESPONSE

In this section we briefly summarize the equations and formulae for the scattering of monochromatic waves by a circular cylindrical inclusion in a solid that are needed for the subsequent discussions. These equations and formulae are contained in Section 4, Chapter 3, of Ref. 5, and were based on original work by Mow and Workman. (10)

Consider a plane harmonic compressional wave (P-wave) propagating in the direction of the x-axis, with wave length $2\pi/a$ and circular frequency $\omega$. The three displacement components are

$$u_x^{(i)} = u_0 e^{i(\alpha x - \omega t)}$$

$$u_y^{(i)} = u_z^{(i)} = 0$$

(2.1)

where

$$\alpha = \omega/c_p, \quad \beta = \omega/c_s$$

The $c_p = [(\lambda + 2\mu)/\rho]^{\frac{1}{2}}$ is the speed of P-wave and $c_s = (\mu/\rho)^{\frac{1}{2}}$ the speed of S-wave (shear wave). The $\lambda$ and $\mu$ are Lamé's constants of the material; $\rho$ is the density. The nonvanishing components of the corresponding stress field are

$$\sigma_{xx}^{(i)} = -\sigma_o e^{i(\alpha x - \omega t)}$$

(2.2)

$$\sigma_{yy}^{(i)} = -[\lambda/(\lambda + 2\mu)] \sigma_o e^{i(\alpha x - \omega t)}$$

In the expressions, $u_0$ and $\sigma_o$ are the maximum displacement and principal stress of the incident wave. They are related to each other and to a third constant $\phi_o$ by
\[ \sigma_o = (\lambda + 2\mu) \alpha^2 \phi_o \] \[ u_o = i\alpha \phi_o, \quad (2.3) \]

For problems of circular cylindrical geometry, the wave field is expressed in polar coordinates \( r, \theta \) (Fig. 1),

\[ \sigma^{(1)}_{xx} = -c_o \sum \epsilon_n^1 J_n(\alpha r) \cos n\theta \ e^{-i\omega t}, \quad (2.4) \]

where \( J_n(\alpha r) \) is the Bessel function of the first kind, \( \epsilon_n^1 = 1 \) when \( n = 0 \) and \( \epsilon_n^1 = 2 \) when \( n \geq 1 \). Unless otherwise noted, the summation of all series is from \( n = 0 \) to \( \infty \).

The waves scattered by a fluid cylinder centered at the origin of the coordinates may be expressed as

\[ u^{(s)}(x; t) = u^{(s)}(r, \theta; \omega) e^{-i\omega t}, \quad (2.5) \]

\[ q^{(s)}(x; t) = q^{(s)}(r, \theta; \omega) e^{-i\omega t}, \]

where the coefficients \( u^{(s)}(r, \theta; \omega) \) and \( q^{(s)}(r, \theta; \omega) \) are known as the steady state response functions. Since the time factor \( e^{-i\omega t} \) appears in all solutions, we shall omit it and consider in the remainder of this section only the response functions.

The displacement and stress components of interest are:

\[ u^{(s)}_r(r, \theta; \omega) = \phi_o \rho^{-1} \sum A_n \epsilon^{(3)}_{71}(\alpha r) + B_n \epsilon^{(3)}_{72}(\beta r) \cos n\theta \]

\[ \sigma^{(s)}_{rr}(r, \theta; \omega) = 2\mu \phi_o \rho^{-2} \sum A_n \epsilon^{(3)}_{11}(\alpha r) + B_n \epsilon^{(3)}_{12}(\beta r) \cos n\theta \]

\[ \sigma^{(s)}_{r\theta}(r, \theta; \omega) = 2\mu \phi_o \rho^{-2} \sum A_n \epsilon^{(3)}_{41}(\alpha r) + B_n \epsilon^{(3)}_{42}(\beta r) \sin n\theta, \quad (2.6) \]
Fig. 1 — Geometry of the circular cylinder and receivers
where

\[ \varepsilon_{11}^{(3)}(ar) = \left( n^2 + n - \frac{1}{2} \beta^2 r^2 \right) H_n(ar) - ar H_{n-1}(ar) \]

\[ \varepsilon_{12}^{(3)}(\beta r) = -n(n + 1) H_n(\beta r) + n\beta r H_{n-1}(\beta r) \]

\[ \varepsilon_{41}^{(3)}(ar) = n(n + 1) H_n(ar) - nar H_{n-1}(ar) \]

\[ \varepsilon_{42}^{(3)}(\beta r) = -\left( n^2 + n - \frac{1}{2} \beta^2 r^2 \right) H_n(\beta r) + \beta r H_{n-1}(\beta r) \]

\[ \varepsilon_{71}^{(3)}(ar) = ar H_{n-1}(ar) - n H_n(ar) = ar H_n^\prime(ar) \]

\[ \varepsilon_{72}^{(3)}(\beta r) = n H_n(\beta r) \ldots \] (2.7)

The function \( H_n(z) = J_n(z) + i Y_n(z) \) is the Bessel function of the third kind, generally known as the Hankel function of the first kind.

The polar components of the incident wave (2.1) may be written as

\[ u_r^{(1)}(r, \theta; \omega) = \phi_0 r^{-1} \sum \varepsilon_n i_n \varepsilon_{71}^{(1)}(ar) \cos n\theta \]

\[ \sigma_{rr}^{(1)}(r, \theta; \omega) = \mu_0 r^{-2} \sum \varepsilon_n i_n \varepsilon_{11}^{(1)}(ar) \cos n\theta \]

\[ \sigma_{r\theta}^{(1)}(r, \theta; \omega) = \mu_0 r^{-2} \sum \varepsilon_n i_n \varepsilon_{41}^{(1)}(ar) \sin n\theta \ldots \] (2.8)

where \( \varepsilon_{jk}^{(1)}(ar) \) are obtained from (2.7) by substituting the Bessel functions of the first kind, \( J_n \), for that of the third kind, \( H_n \), in \( \varepsilon_{jk}^{(3)}(ar) \). Thus
\[ \varepsilon_{11}^{(1)}(ar) = \left( n^2 + n - \frac{1}{2} \beta^2 r^2 \right) J_n(ar) - ar J_{n-1}(ar), \text{ etc.} \] (2.9)

We denote the density and wave speed of the fluid inclusion by \( \rho_f \) and \( c_f \). The fluid is assumed to be inviscid and compressible, \( \dot{u}_r^{(f)} \) and \( \dot{u}_\theta^{(f)} \) being the velocity components and \( \sigma_{rr}^{(f)} = \sigma_{\theta\theta}^{(f)} \) the corresponding stresses (pressure). The ambient pressure of the fluid is set at zero, and the fluid carries no shearing stresses. Inside the fluid inclusion the refracted wave is a standing wave, which can be expressed as

\[ u_r^{(f)}(r, \theta; \omega) = \phi r^{-1} \sum C_n \varepsilon_{11}^{(1)}(a_f r) \cos n\theta \]

\[ \sigma_{rr}^{(f)}(r, \theta; \omega) = 2u_\phi r^{-2} \sum C_n \varepsilon_{11}^{(1)}(a_f r) \cos n\theta \] (2.10)

\[ \sigma_{r\theta}^{(f)}(r, \theta; \omega) = 0, \]

where

\[ \varepsilon_{11}^{(1)}(a_f r) = -n J_n(a_f r) + a_f r J_{n-1}(a_f r) \]

\[ \varepsilon_{11}^{(1)}(a_f r) = (\rho_f/2\rho) (\beta r)^2 J_n(a_f r), \] (2.11)

and \( a_f = \omega/c_f \).

The boundary conditions at the interface \( r = a \) are:

\[ u_r^{(1)} + u_r^{(s)} = u_r^{(f)} \]

\[ \sigma_{rr}^{(1)} + \sigma_{rr}^{(s)} = \sigma_{rr}^{(f)} \text{ at } r = a \] (2.12)

\[ \sigma_{r\theta}^{(1)} + \sigma_{r\theta}^{(s)} = 0. \]
By substituting (2.6), (2.8), and (2.10) into (2.12), we obtain a set of algebraic equations for the coefficients \(A_n\), \(B_n\), and \(C_n\). In matrix form, they are

\[
\begin{bmatrix}
\varepsilon_{11}^{(3)}(aa) & \varepsilon_{12}^{(3)}(\beta a) & -\varepsilon_{11}^{(1)}(\alpha_f a) \\
\varepsilon_{41}^{(3)}(aa) & \varepsilon_{42}^{(3)}(\beta a) & 0 \\
\varepsilon_{71}^{(3)}(aa) & \varepsilon_{72}^{(3)}(\beta a) & -\varepsilon_{71}^{(1)}(\alpha_f a)
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n \\
C_n
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{11}^{(1)}(aa) \\
\varepsilon_{41}^{(1)}(aa) \\
\varepsilon_{71}^{(1)}(aa)
\end{bmatrix}
\]

Substitution of \(A_n\) and \(B_n\) into (2.6) completes the solution for the scattered waves in the solid medium.
III. NORMAL MODES AND PRINCIPAL FREQUENCIES OF
THE FLUID CYLINDRICAL INCLUSION

There are two limiting cases for the vibration of the fluid inclusion in an elastic solid. First, the elastic solid is infinitely "rigid" compared with the fluid (rigid wall case); and second, the fluid is infinitely "soft" compared with the solid (cavity case). We discuss these two limiting cases first as they provide some insight into the complex interactions. In the rigid wall case, the free oscillation and normal modes of a fluid mass contained inside a rigid, circular cylindrical wall may be determined from the solutions (2.10) and the boundary conditions

\[ u^{(f)}_r = 0 \quad \text{at } r = a. \quad (3.1) \]

It leads to the frequency equation

\[ \Delta_f \equiv \frac{1}{3} \gamma_1^{(1)} (\alpha_f a) \equiv \alpha_f a J'_n (\alpha_f a) = 0. \quad (3.2) \]

The roots of the transcendental equation, \( J'_n(z) = 0 \), which are all real valued, are shown in Ref. 11 (Table 9.5). In Refs. 3 and 10, these solutions were used to estimate principal frequencies of a fluid cylinder in an elastic solid.

An infinitely extended solid with a cylindrical cavity may also "oscillate." Because the medium is unbounded, the "free oscillation" can be expressed only in the form of traveling waves (2.6). The boundary conditions are that the tractions generated by such traveling waves, if they ever exist, must vanish at the surface of the cavity; i.e.,

\[ \sigma^{(s)}_{rr} = 0 \quad \text{and} \quad \sigma^{(s)}_{r\theta} = 0 \quad \text{at } r = a. \quad (3.3) \]

When (2.6) is substituted into the above conditions, a frequency equation is derived from the system of homogeneous equations for \( A_n \) and \( B_n \).
\[ \Delta_s = \varepsilon^{(3)}_{11}(\alpha a) \varepsilon^{(3)}_{42}(\beta a) - \varepsilon^{(3)}_{12}(\beta a) \varepsilon^{(3)}_{41}(\alpha a) = 0 \quad (3.4) \]

The equation \( \Delta_s = 0 \) has real and imaginary parts, with the wave numbers \( \alpha a \) (\( \beta a = \alpha a \frac{c_p}{c_s} \)) as the unknown variable. It has no real roots and \( \alpha a \) must be complex valued. As shown in Ref. 12 (or p. 313 of Ref. 5), these roots are the poles of the Fourier integral solutions (or Laplace inverse transform integral) for the scattering of a pulse by a cavity.

When the cavity is filled with fluid, these two types of oscillations, the one for the fluid cylinder and the other for the surrounding solid, are coupled through the continuity conditions for displacements and tractions at the interface:

\[ u_r^{(e)} = u_r^{(f)}; \quad \sigma_{rr}^{(e)} = \sigma_{rr}^{(f)}; \quad \sigma_r^{(e)} = 0 \quad \text{at} \ r = a \quad (3.5) \]

Substitution of (2.6) and (2.10) into the above conditions gives rise to a system of homogeneous equations for \( A_n, B_n \), and \( C_n \), the homogeneous part of (2.13). The frequency equation of the coupled system is thus obtained by setting the determinant of the square matrix in (2.13) equal to zero:

\[ \Delta = - \varepsilon^{(3)}_{11}(\alpha a) \left[ \frac{\varepsilon^{(3)}_{41}(\alpha a)}{\varepsilon^{(3)}_{12}(\beta a)} - \frac{\varepsilon^{(3)}_{12}(\beta a)}{\varepsilon^{(3)}_{41}(\alpha a)} \right] 
\]

\[ - \varepsilon^{(1)}_{11}(\alpha a) \left[ \frac{\varepsilon^{(3)}_{11}(\alpha a)}{\varepsilon^{(3)}_{42}(\beta a)} - \frac{\varepsilon^{(3)}_{42}(\beta a)}{\varepsilon^{(3)}_{11}(\alpha a)} \right] = 0 \quad (3.6) \]

It is clear that when \( \varepsilon^{(1)}_{11}(\alpha a) = 0 \), (3.6) degenerates to two frequency equations, (3.2) and (3.4).

The roots \( \alpha a \) or \( \alpha a \) of (3.6) are always complex valued. We call the group of roots that are close to those of the fluid cylinder with a rigid wall, Eq. (3.2), the eigenvalues of the first class vibration of the fluid-solid system, and those that are close to the roots of the infinite medium with a cavity, Eq. (3.4), the eigenvalues of the second class vibration.
Since \( \omega = \alpha_f c_f = \alpha_c p \), complex eigenvalues also imply that the principal frequencies of the coupled system are complex valued. Numerically, we found that the eigenvalues of the first class have smaller imaginary parts than those of the second class. A small imaginary part in the principal frequencies implies that the fluid cylinder oscillates like a system with low damping, which is caused by the dissipation of energy (in the fluid) through radiation (in the solid).

As an example, the complex roots of (3.6) are calculated for a water-filled cylindrical cavity in an aluminum body. The pertinent material constants are:

\[
\begin{align*}
\text{Aluminum:} & \quad \rho = 2.70 \text{ gm/cm}^3, \\
& \quad c_p = 6380 \text{ m/s}, \quad c_s = 3140 \text{ m/s}; \\
\text{Water:} & \quad \rho_f = 1.00 \text{ gm/cm}^3, \quad c_f = 1494 \text{ m/s}.
\end{align*}
\tag{3.7}
\]

For each value of \( n \) in (3.6), there are an infinite number of roots. We shall designate the complex roots of \( \alpha a \) by \( (p_n, s + iq_n, s) \), which are called the eigenvalues of \( \alpha a \). The lower eigenvalues are listed in Table 1 for the first class vibration and in Table 2 for the second class.

The real parts of \( \alpha a(p_n, s) \) in Table 1 nearly equal the corresponding real roots of (3.2) when \( c_f a \) is converted to \( (c_p/c_f) a \). The difference is less than two-thousandth when \( s \geq 3 \) for all \( n \). For large \( s \), the values of the imaginary part of \( \alpha a(q_n, s) \) in Table 1 approach 0.0203, which is the imaginary part of the eigenvalues of a liquid layer between two solids, as discussed in the appendix. The complex roots in Table 2 may be compared to those for a cylindrical cavity as given in Table 5.1 of Ref. 5. There is, however, a difference in the value for Poisson's ratio \( \nu \) in these two calculations (\( \nu = 0.34 \) in this example). In addition to those shown in Table 2, there is another family of complex roots with negative real parts that are not shown.

We shall designate the vibrations with \( n = 0, 1, 2, \ldots \), the zeroth, first, second, and \( n \)th mode; for each \( n \), the case of \( s = 1 \) is the \( n \)th fundamental mode, and the case of \( s = 2, 3, 4, \ldots \) is the overtones of the \( n \)th mode. The number \( n \) equals the number of nodal diameters (\( u_r \))
<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.417</td>
<td>-1</td>
<td>-0.0160</td>
</tr>
<tr>
<td>2</td>
<td>0.890</td>
<td>-1</td>
<td>-0.0252</td>
<td>1.242</td>
</tr>
<tr>
<td>3</td>
<td>1.641</td>
<td>-1</td>
<td>-0.0227</td>
<td>2.382</td>
</tr>
<tr>
<td>4</td>
<td>3.120</td>
<td>-1</td>
<td>-0.0211</td>
<td>3.481</td>
</tr>
<tr>
<td>5</td>
<td>4.357</td>
<td>-1</td>
<td>-0.0209</td>
<td>4.219</td>
</tr>
<tr>
<td>6</td>
<td>4.956</td>
<td>-1</td>
<td>-0.0207</td>
<td>5.593</td>
</tr>
<tr>
<td>7</td>
<td>5.530</td>
<td>-1</td>
<td>-0.0206</td>
<td>6.048</td>
</tr>
</tbody>
</table>

Table 1: EIGENVALUES OF Qa (First class vibration)
Table 2

EIGENVALUES OF $\alpha_\alpha$

(Second class vibration)

<table>
<thead>
<tr>
<th>$\alpha_n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4317 -i 0.3578</td>
<td>1.0121 -i 0.6115</td>
<td>1.7728 -i 0.7125</td>
<td>2.5723 -i 0.7890</td>
<td>3.3994 -i 0.8577</td>
</tr>
<tr>
<td>2</td>
<td>0.4079 -i 1.7211</td>
<td>1.2388 -i 2.1515</td>
<td>2.0826 -i 2.4856</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>0.4174 -i 3.0404</td>
<td></td>
</tr>
</tbody>
</table>
or $u_\theta$ component), and $s$ equals the number of nodal circles with the boundary of the cylinder being counted as one nodal circle. Figure 2 depicts the magnitudes of either $u_r$ or $u_\theta$ of the first four fundamental modes.
Fig. 2 — The lowest four normal modes of the fluid inclusion 
(dashed lines indicate magnitude for either $u_r$ or $u_\theta$ component)
IV. SCATTERED PULSES AND THEIR POWER SPECTRA

An incident compressional pulse moving along the x-axis with speed $c_p$ may be represented by

$$u^{(i)}_x(x,t) = u_o f(t), \quad u^{(i)}_y = u^{(i)}_z = 0$$  (4.1)

where $f(t)$ is a sufficiently smooth function, and

$$\tau = t - (x + x_o)/c_p$$  (4.2)

is the local time, which is taken to be zero at the incident wave front. The specification of the quantity $x_o$ depends on the location $x$ at which the transient response is evaluated, and on the choice of the zero instant of the time scale $t$ (p. 305 of Ref. 5).

We denote the Fourier transform of $f(t)$ by $\tilde{f}(\omega)$,

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$  (4.3)

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} \, d\omega$$  (4.4)

When $\omega$ is the angular frequency, and $\tau$ the local time, $\tilde{f}(\tau)$ is known as the frequency spectrum of $f(\tau)$. Substituting (4.4) in (4.1), we find

$$u^{(i)}_x(x,t) = u_o \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i[\omega t - \alpha(x + x_o)]} \, d\omega$$  (4.5)

where $\alpha = \omega/c_p$.

While the transient scattered waves can formally be derived by applying the Fourier transform with respect to time to the original equations of motion and boundary conditions, they can also be constructed by applying the principle of superposition for a linear system. Comparing (4.5) with (2.1), we note that the transient incident
wave (4.1) is a superposition of steady-state waves (2.1), with the amplitude $u_o \tilde{f}(\omega) \exp(-i \omega t)$ distributed over a frequency range $d\omega$. Hence the scattered pulses are determinable from the steady-state response functions in (2.5) and (2.6) by the same superposition. For instance, the transient radial stresses of the scattered wave are

$$
\sigma_{rr}^{(s)}(x;t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) \sigma_{rr}^{(s)}(r,\theta;\omega) e^{-i(\omega t + \omega t)} d\omega,
$$

(4.6)

where the response function $\sigma_{rr}^{(s)}(r,\theta;\omega)$ is given by the second part of Eq. (2.6). Note that in (4.6) and the sequel, the transient response is distinguished by the parenthetical expression $(t)$ and the steady state response by $(\omega)$.

Comparing (4.6) with (4.4), we find the frequency spectrum of $\sigma_{rr}^{(s)}(x;t)$ directly:

$$
\tilde{\sigma}_{rr}^{(s)}(x;\omega) = \tilde{f}(\omega) \sigma_{rr}^{(s)}(r,\theta;\omega) e^{-i\omega t}.
$$

(4.7)

The absolute value of the frequency spectrum is the power spectrum. Hence the power spectrum of the scattered radial stress pulse is

$$
|\tilde{\sigma}_{rr}^{(s)}(x;\omega)| = |	ilde{f}(\omega)| \cdot |\sigma_{rr}^{(s)}(r,\theta;\omega)|.
$$

(4.8)

Equation (4.8) states that the power spectrum of the scattered pulse equals the product of the power spectrum of the incident pulse and the absolute value of the steady state response function. Since Eq. (4.8) is devoid of the factor $\exp(-i\omega t)$, the power spectrum of the scattered pulse is independent of the choice of the zero instant for the time scale.

To illustrate the salient features of power spectra of the scattered pulses, we calculated the response functions $|\sigma_{rr}^{(s)}(r,\theta;\omega)|$ as given in the second part of (2.6) for various values of $r$ and $\theta$. Again, we consider a water filled cylindrical hole in an aluminum block. Data for the
materials are given in (3.7). For convenience, the stress response function is normalized by $\sigma_o$ as given in (2.3), which is the maximum normal stress due to the incident harmonic wave as shown in (2.1).

The normalized response function is

$$\frac{\sigma^{(s)}_{rr}(r,\theta;\omega)}{\sigma_o} = 2 \sum_{n=0}^{\infty} \left[ A_n \frac{\varepsilon_{11}^{(3)}(ar)}{(\beta r)^2} + B_n \frac{\varepsilon_{12}^{(3)}(\beta r)}{(\beta r)^2} \right] \cos n\theta. \quad (4.9)$$

At a large distance from the center of the cylinder---i.e., $r/a \gg 1$---for a given value of $\beta a$ or $\alpha a$, we may replace the Hankel functions in $\varepsilon_{11}^{(3)}$ and $\varepsilon_{12}^{(3)}$ by their asymptotic expressions:

$$H_n(z) \approx \left(\frac{2}{\pi i z}\right)^{1/2} e^{-i(z - \pi/4 - n\pi/2)} \quad (4.10)$$

as $z \to \infty$ and when $z > n$.

From (2.7), we obtain, for $\beta r \to \infty$,

$$\frac{\sigma^{(s)}_{rr}(r,\theta;\omega)}{\sigma_o} \to 2 \sum_{n=0}^{\infty} \left[ A_n \frac{2}{n\pi ar} \left(\frac{c_s}{2} - \frac{1}{2}\right) e^{iar} + B_n \frac{2}{n\pi \beta r} \frac{1}{\beta r} e^{i\beta r} \right] \cos n\theta. \quad (4.11)$$

It is seen from the previous formula that the scattered S-waves, which are represented by $B_n$ terms, attenuate in the order of $(\beta r)^{-3/2}$, whereas the scattered P-waves attenuate only in the order of $(ar)^{-1/2}$.

Shown in Fig. 3 are $|\sigma^{(s)}_{rr}(r,\theta;\omega)/\sigma_o|$ in (4.9) for $\theta = \pi$, and at $r = 22.2a$, 10a, 5a, as a function of $\alpha a (0 < \alpha a \leq 5)$. If $|\hat{f}(\omega)|$ in (4.8) is taken to be unity, which corresponds to an impulse with the delta function in time, the graphs in Fig. 3 are also the normalized power spectra of the radial stresses in backward scattering. All graphs exhibit typical peak and valley structure with decreasing magnitudes as $r$ increases.
Fig. 3a — Normalized radial stresses versus frequencies due to backward scattering ($\theta = \pi$)

$r = 22.2a$
Fig. 3b — Normalized radial stresses versus frequencies due to backward scattering ($\theta = \pi$)

$r = 10a$, $\theta = \pi$

$\sigma_{rr}$

0.00 0.10 0.20 0.30 0.40

$\omega = \omega_0/c_p$

0.50 1.00 2.00 3.00 4.00 5.00

$n = 0, 2, 3$
Fig. 3c — Normalized radial stresses versus frequencies due to backward scattering ($\theta = \pi$)

$r = 5a$
We also used the asymptotic formula (4.11) to calculate \[ \left| \frac{\sigma_{rr}(r, \theta; \omega)}{\sigma_0} \right| \] for \( r \geq 10a \), and the numerical results were compared with those shown in Fig. 3. Even at \( r = 22.2a \), there is still an error of 5 percent in the magnitude of the spectrum.

For the purpose of comparison with experimental results reported separately, (9) we also calculated the spectra for the forward scattering (\( \theta = 0 \)) and the right angle scattering (\( \theta = \pi/2 \)) at \( r = R = 22.2a \). Experimentally, a 1/2 inch (1.270 cm) diameter wide-bandwidth piezoelectric transducer was used to generate an incident pulse. The same transducer was also used to receive the backward scattering. Another nearly identical transducer was used to record the scattered pulses at \( \theta = 0 \) and \( \pi/2 \) directions. Since the cross-sectional area of the transducer is considerably larger than the projected area of a 1/16-inch (0.1587 cm) diameter hole in an aluminum block, we need to know the average pressure action over the entire surface of the transducer.

With reference to Fig. 1, the average pressure may be calculated by the following formula:

\[
\left[ \frac{\sigma_{rr}(R, \theta; \omega)}{\sigma_0} \right]_{\text{ave}} = \frac{1}{2b} \int_{\theta-\delta_1}^{\theta+\delta_2} \left[ \frac{\sigma_{rr}(R, \theta; \omega)}{\sigma_0} \right] \text{R} \text{d}\theta
\]

\[
= \frac{R}{b} \sum_{n=0}^{\infty} A_n \left[ \frac{\varepsilon_{11}^{(3)}(\alpha R)}{(\beta R)^2} + B_n \frac{\varepsilon_{12}^{(3)}(\beta R)}{(\beta R)^2} \right] \frac{1}{n} \left[ \sin n(\theta + \delta_2) - \sin n(\theta - \delta_1) \right].
\]

(4.12)

The formula is only an approximation because the actual flat surface of the transducer has been represented by a portion of a cylindrical surface, and the contribution from \( \sigma_{r\theta} \) has been neglected. However, in view of the uncertainty of the bonding condition at the surface of an actual transducer, further refinement of the calculation is deemed unnecessary as long as \( R >> b \) and \( R >> a \).
Numerical results of $|\sigma_{rr}/\sigma_0|$ (ave) for $R = 22.2a$ and $b = 8.00a$, at $\theta = \pi$, $0$, $\pi/2$ are shown in Figs. 4, 5, and 6. The graphs are shown for the case of $\delta_1 = \delta_2 = \arctan (b/R) = 19.84$ degree. We also calculated the case of $\delta_1 = 0$ and $\delta_2 = \arctan (2b/R)$ to simulate an eccentrically placed receiver for backward scattering ($\theta = \pi$). The results differ slightly from those shown in Fig. 4, which is for a centrally located transducer. These graphs will be considered the theoretical power spectra of the averaged pressure pulse scattered by a water cylinder in an aluminum block, for an incident pulse with the delta time function.

The power spectra at three angular receiving locations are drastically different from each other. However, the spectrum of the averaged stress in Fig. 4 is almost the same as that in Fig. 3A, which is the spectrum of the stress at one point ($r = R$, $\theta = \pi$).

In addition to the absolute values, we also calculated the phase angle that is the arctan $[(\text{Im } \sigma_{rr})/(\text{Re } \sigma_{rr})]$ as a function of $\alpha a$. The results are omitted from reporting here because no significant conclusion can be deduced from them.
Fig. 4 - Spectrum of the normalized radial stresses averaged over the transducer at $r = 22.2a$ in backward scattering ($\theta = \pi$)
Fig. 5 – Spectrum of the normalized radial stresses averaged over the transducer at $r = 22$, in forward scattering ($\theta = 0$)
Fig. 6 - Spectrum of the normalized radial stresses averaged over the transducer at $r = 22.2a$ in right-angle scattering ($\theta = \pi/2$)
V. ANALYSIS OF POWER SPECTRA OF SCATTERED PULSES

From the maze-like graphs shown in Figs. 3-6, we seek some invariant features of the power spectra. The amplitudes of the spectra are very sensitive to the angular position of a receiver relative to a transmitter, and to the distance between the receiver and the scatterer. However, one invariant feature we have observed is the location (along the \( \alpha \)-axis) at which the spectral amplitude becomes either maximum or minimum.

LOCATIONS OF MAXIMA AND MINIMA OF SPECTRA

Pao and Sachse(3) indicated that the peaks of experimentally measured power spectra of scattered pulses seemed to occur at the principal frequencies of a fluid cylindrical inclusion, which were estimated from the eigenvalues of a fluid cylinder contained inside a rigid wall. In Section III, we showed that the principal frequencies can be precisely determined from (3.6), and we tabulated the characteristic values for \( \alpha \alpha \) of a water cylinder in an aluminum solid. The roots of (3.6) are all complex valued, and the real parts of the characteristic values of the first class vibration (Table 1) with \( n = 0, 1, 2, 3 \) are marked in Figs. 3-6 by small arrows or ticks.

At first glance, the distribution of the principal frequencies is so dense that one can relate almost any peak or valley of the spectrum to a principal mode of vibration (higher modes not yet shown in the figures). However, close examination of the overtones of the lower modes reveals some distinctive features of the spectra.

Consider first the spectrum for backward scattering, Figs. 3A and 4. Contrary to what was conjectured previously(3) the conspicuous minima of the spectra occur at locations that coincide with the real part of the eigenvalues for \( n = 0 \) and 1 modes. The one-to-one correspondence is striking, and there is no exception for all overtones calculated. No conclusive result can be deduced about the \( n = 2 \) mode; but some of the sharp peaks occur at the eigenvalues of the higher overtones of the \( n = 3 \) mode.
Next, in the spectrum of forward scattering (Fig. 5), the locations of maxima and minima seem to be just opposite to those in backward scattering (Fig. 4). Many peaks occur at \( n = 0 \) and \( 1 \) modes. Again, the coincidence of the peak locations with the eigenvalues of the overtones of the zeroth and first modes is one to one, and there is no exception.

Finally, consider right-angle scattering (Fig. 6). The sharp peaks occur regularly at the overtone eigenvalues of the \( n = 0 \) mode, but not at those of the first mode. Also, the spectrum exhibits no distinctive feature at the overtone eigenvalues of the third mode.

### Scattering Coefficients \( A_n \) and \( B_n \)

In order to understand the peak-and-valley structure of the power spectra and their relationships to the principal frequencies, we analyze the coefficients \( A_n \) and \( B_n \) in (4.9) and (4.12). The case of \( n = 0 \) is particularly simple since \( \xi_{12}^{(3)} \) and \( \xi_{73}^{(3)} \) vanish at \( n = 0 \). From (2.13), we find

\[
A_0 = \frac{-D_R(aa)}{D_R(aa) + iD_I(aa)}, \quad B_0 = 0 \quad (5.1)
\]

where

\[
D_R(aa) = \left\{ \begin{array}{l}
\xi_{11}^{(1)}(aa) \xi_{71}^{(1)}(a_f a) - \xi_{71}^{(1)}(aa) \bar{\xi}_{11}^{(1)}(a_f a) \\
\xi_{11}^{(1)}(aa) \xi_{71}^{(1)}(a_f a) - \xi_{71}^{(1)}(aa) \bar{\xi}_{11}^{(1)}(a_f a)
\end{array} \right\}_{n=0}
\]

\[
D_I(aa) = \left\{ \begin{array}{l}
\text{Im} \xi_{11}^{(3)}(aa) \xi_{71}^{(1)}(a_f a) - \text{Im} \xi_{71}^{(3)}(aa) \bar{\xi}_{11}^{(1)}(a_f a) \\
\text{Im} \xi_{11}^{(3)}(aa) \xi_{71}^{(1)}(a_f a) - \text{Im} \xi_{71}^{(3)}(aa) \bar{\xi}_{11}^{(1)}(a_f a)
\end{array} \right\}_{n=0}.
\]

Note that \( \text{Re} \xi_{ij}^{(3)} = \xi_{ij}^{(1)} \) where \( \text{Re} \) and \( \text{Im} \) denote "real part of" and "imaginary part of." The denominator \( D_R(aa) + i D_I(aa) \) has complex-valued zeros.
\[ \alpha a = p_{o,s} + iq_{o,s} \quad s = 1, 2, \ldots \] (5.3)

They are shown in Table 1 for the water-aluminum combination.

Introducing a new function \( \gamma(aa) \) with

\[ \tan \gamma = D_R/D_I \] (5.4)

we obtain from (5.1)

\[ A_o = -\tan \gamma/(\tan \gamma + 1) = \frac{1}{2} e^{i2\gamma} (1) . \] (5.5)

When \( \alpha a \) takes on real values, \( \gamma \) is real valued but \( A_o \) is complex.

Hence, as \( \gamma(aa) \) changes for increasing \( \alpha a \), \( \text{Im} A_o \) varies between 1/2 and \(-1/2\), and \( \text{Re} A_o \) varies between 0 and \(-1\). This behavior for \( A_o \) is analogous to that of the coefficients in the solution for scattering of light by cylindrical particles (Sections 10.2 and 15.3 of Ref. 8).

Figure 7 shows \( \text{Re} A_o \) (solid lines) and \( \text{Im} A_o \) (dashed lines) for real \( \alpha a \). We also indicate the real part of the complex zeros of (5.3) by small arrows, with \( p_{o,1} = 0.890 \), \( p_{o,2} = 1.641 \), \( p_{o,3} = 2.382 \) ... (see Table 1). Note that near \( \alpha a = p_{o,1} \), \( \text{Im} A_o \) changes from 0.0 to \(-0.5\) and \( \text{Re} A_o \) from 0.0 to \(-1.0\), all within a very narrow range of \( \alpha a \). Similar changes also occur at the zeros of the overtones, \( p_{o,2} \) ....

This means that near \( p_{o,s} \), \( |A_o| \) may be nearly zero and at its minimum or nearly one and at its maximum. If only the zeroth mode were excited, the power spectrum would exhibit a deep dip and then a sharp rise near the real part of the eigenvalues.

Analytically, the resonance of the zeroth mode occurs only when \( \alpha a = p_{o,s} + iq_{o,s} \), and \( D_R + i D_I \) in (5.1) vanishes at these complex zeros. However, at \( \alpha a = p_{o,s} \) neither \( D_R(p_{o,s}) \) nor \( D_I(p_{o,s}) \) vanishes, although they may be very small. When \( \alpha a = p_{o,1}^- \), a value slightly less than \( p_{o,1} \), \( D_R(p_{o,1}^-) \) vanishes but \( D_I(p_{o,1}^-) \) does not although it is very...
Fig. 7 - The spectrum of the scattering coefficient $A_0^*$ (real part in solid lines, imaginary part in dashed lines).
small. Thus $|A_0|$ in (5.1) attains the minimum magnitude of zero.

When $\alpha = p_{o,1}^+$, a value slightly larger than $P_{o,1}$, $D_1(p_{o,1}^+)$ vanishes but $D_R(p_{o,1}^+)$ does not. Thus $|A_o|$ attains its maximum value of one. A similar change of magnitudes for $|A_o|$ also occurs at $\alpha = p_{o,2}; p_{o,3}; \ldots$, etc.

Because of the complicated expressions, analogous analysis has not been carried out for the coefficients $A_n$ and $B_n$ for $n > o$. Instead, we have calculated them numerically for $n \leq 4$. Results for $A_1$ and $B_1$ are shown in Figs. 8A and 8B. All calculations substantiate the conclusion that near $\alpha = p_{n,s}^+$ where $p_{n,s}^+$ are the real parts of the eigenvalues of the imbedded fluid cylinder, $|A_n|$ or $|B_n|$ changes rapidly from its minimal value of nearly zero to its maximal value, or vice versa.

Calculations also show that $\max |B_n|$ is always less than $\max |A_n|$ for the same order $n$; and as $n$ increases, the envelopes for the coefficients $|A_n|$ and $|B_n|$ gradually decrease. The first conclusion signifies that the conversion of the P-wave to the S-wave mode in the scattering of an incident P-wave is of secondary importance. The second conclusion points out that only lower order modes need to be considered in the analysis of the power spectra.

Selective Transmission of the Fluid Cylinder

The mathematical analyses above show that the scattering coefficients $|A_n|$ and $|B_n|$ may attain both their maximum and minimum magnitudes in the neighborhood of $\alpha = p_{n,s}^+$, where $p_{n,s}^+$ is the real part of the eigenvalue of the nth mode and the sth overtone of the first class vibration. Since the magnitudes of $\sigma_{rr}(r,\theta;\omega)$ depend strongly on the values for $A_n$ and $B_n$, we can draw conclusions on the power spectra of the scattered waves from the behavior of the scattering coefficients.

Physically the peak and valley structure of the spectra may be interpreted as analogous to the selective transmission of waves through a multilayered medium. It is shown that when an incident P-wave impinges normally at a fluid layer that is sandwiched between two
Fig. 8A – The spectrum of the scattering coefficient $A_1$ (real part in solid lines, imaginary part in dashed lines).
Fig. 88 - The spectrum of the scattering coefficient $B_1$ (real part in solid lines, imaginary part in dashed lines)
semi-infinite solids, the P-wave may be totally reflected from the interface—hence no transmission—or completely refracted into the fluid and then the second solid—hence total transmission. This selective transmission depends on the incident wave frequency and on the principal frequencies of the fluid layer. For sake of completeness, we summarize the analysis of the solid-liquid-solid layered model in the appendix. Again the principal frequencies of the liquid layer are complex valued. The total transmission occurs when the incident wave frequency coincides with the real part of the resonance frequencies of the liquid layer.

This is precisely what happened in the case of back scattering (Fig. 4). When the incident wave frequency is nearly equal to $p_{o,s}$ or $p_{l,s}$, which is the real part of the principal frequency of an overtone of either the zeroth or first mode, total transmission through the fluid cylinder occurs. Since there is no energy scattered backward as the result of the total transmission at these two modes, the power spectra of the backward scattered waves exhibit sharp dips at these near-resonant frequencies.

Total transmission at near-resonance means that the magnitudes of the power spectra of forward scattering should become maximum at these frequencies. Indeed this is what we found in Fig. 5 where the peaks occur regularly at $\omega a \approx p_{o,s}$ and $p_{l,s}$. In the case of right-angle scattering (Fig. 6), the peaks also occur at $\omega a = p_{o,s}$, but not at $p_{l,s}$.

The absence of peak magnitudes at the overtones of the first mode in Fig. 6 is caused by the particular orientation of the receiver relative to the transmitter. As can be seen from (4.9) or (4.12) with $\delta_1 = \delta_2$, the scattered radial stress $\sigma_{rr}^{(s)}$ vanishes at $\theta = \pm \pi/2$ when $n$ is an odd integer. Thus even the cylinder is at near-resonance when excited by an incident wave, and the radial stress scattered waves associated with the odd-numbered modes are very weak in the directions of nodal diameters of the radial stress. This also explains the absence of peaks at $p_{3,s}$ in Fig. 6.

So far we are unable to relate the principal frequencies of the second class vibration (Table 2) to any significant feature of the
power spectra. Note that the $\alpha$ in Table 2 have much larger imaginary parts than those in Table 1. Since a large imaginary wave number means faster attenuation in distance for the scattered waves, the influence of these modes is perhaps not discernible, or is overshadowed by the modes of the first class, at large distances from the scatterer.

CONCLUSIONS

When the frequencies nearly equal the real parts of the principal frequencies of the fluid inclusion, the power spectrum of the scattered pulses undergoes a rapid rise and fall in magnitude, resulting in the typical peak-and-valley structure for the spectrum. At the overtone frequencies of the zeroth and first modes ($n = 0, 1$), the backward scattering spectrum (Fig. 4) exhibits sharp dips, whereas the forward scattering spectrum (Fig. 5) exhibits sharp peaks. This is caused by the selective transmission, for which the incident, reflected, and diffracted waves interfere constructively or destructively with each other at a frequency near the natural frequencies of the fluid inclusion. The net result is the total reflection or total transmission of a particular mode through the fluid. The absence of sharp peaks in the right-angle scattering spectrum (Fig. 6) at the overtone frequencies of odd-numbered modes is caused by the particular orientation of the nodal diameter for the radial stress of these modes.
VI. APPLICATIONS OF SPECTRAL ANALYSIS TO NONDESTRUCTIVE TESTINGS

The conclusions of the previous section have been substantiated with experimental observations. The multiple peaks or valleys in the scattering spectra can be identified with the principal frequencies of the overtones of the $n = 0$, or $n = 1$ modes. This will be useful in the quantitative nondestructive testing of a fluid-filled cylindrical cavity in solids.

Note first that the real parts of the principal frequencies of the first class vibration of the fluid filled cavity (Eq. (3.6) and Table 1) are nearly equal to the natural frequencies of a fluid cylinder contained in a rigid wall (Eq. (3.2)). As mentioned in Ref. 3, the roots of (3.2) for the nth mode with $s >> n$ are given approximately by Eq. (9.5.13) of Ref. 11:

$$(\alpha_f a)_{n,s} = \left(s + \frac{1}{2} n - \frac{3}{4}\right) \pi - \frac{4n^2 + 3}{8(s + n/2 - 3/4)\pi} + \ldots \quad (6.1)$$

For higher overtones of the same mode, the difference of two roots is

$$(\alpha_f a)_{n,s + 1} - (\alpha_f a)_{n,s} \approx \pi \quad (6.2)$$

With the relation $\alpha a = (c_f/c_p)\alpha_f a$, we can write (6.2) as

$$(\alpha a)_{n,s + 1} - (\alpha a)_{n,s} \approx \pi \frac{c_f}{c_p} \quad (6.3)$$

For a water-aluminum combination, $\pi \frac{c_f}{c_p} = 0.7356$. This nearly equals the differences of the tabulated $\alpha a$ (Table 1) for higher overtones.
Since $\alpha_f a = \omega a / c_f$, the circular frequency difference of two overtones is

$$\Delta \omega = \omega_{n,s+1} - \omega_{n,s} \equiv \pi \frac{c_f}{a}.$$  \hspace{1cm} (6.4)

Thus by measuring the averaged value of the frequency differences from the peaks or valleys of an experimentally measured power spectrum, one can determine the ratio of $c_f / a$ of the fluid inclusion from (6.4).

Then if one knows either the fluid properties or the radius of the inclusion, one can determine the other. The significance of this observation to nondestructive testing is that the important parameter to measure in the power spectrum is not the magnitude of the peaks or valleys of the power spectrum but the frequency difference that contains the significant information.

Pao and Sachse\(^{(3)}\) questioned the usefulness of (6.4) because the identification of the overtones of a particular mode from a power spectrum was termed "not easy" at that time. Here we have shown that the overtones of $n = 0$ and $n = 1$ modes can be clearly identified by comparing the spectra of the backward and forward scattering. In the event that only backward scattering data are obtainable, overtones of one of these two modes can be identified with less certainty by noting first that the spectral magnitude dips sharply only at the overtones of $n = 0$ and $1$ modes, and overtones of the same mode are almost uniformly spaced in the frequency ($\omega a$) axis.
Appendix

SELECTIVE TRANSMISSION OF A LIQUID LAYER

The solution for the transmission and reflection of elastic waves through a liquid layer sandwiched between two semi-infinite solids is given here. Only normal incidence of a plane P-wave is considered. Thus the final form of the solution is analogous to the three-layered liquid model (Section 3.3 of Ref. 7).

Let the amplitudes of the incident and reflected P-waves in the first solid be $U_0$ and $U_r$. Through the fluid layer of thickness $2a$, density $\rho_f$ and wave speed $c_f$, a P-wave of amplitude $U_t$ is transmitted into the second solid. The circular frequency of the incident and all other transmitted and reflected waves is $\omega$; the wave numbers in the solid and liquid are $\alpha = \omega/c_p$ and $\alpha_f = \omega/c_f$. In acoustics, it is customary to represent the properties of a material by its impedance $z$, with $z = \rho c_p$ for the solid (P-wave) and $z_f = \rho_f c_f$ for the fluid. The ratio of the amplitude of the reflected or transmitted wave to that of the incident wave is:

\[
\frac{U_r}{U_0} = \frac{-z^2 - 1}{z} \sin 2\alpha_f a + 2i \bar{z} \cos 2\alpha_f a} e^{-i2\alpha a} \quad (A.1)
\]

\[
\frac{U_t}{U_0} = \frac{-2i \bar{z}}{(z^2 + 1) \sin 2\alpha_f a + 2i \bar{z} \cos 2\alpha_f a} e^{-i2\alpha a} \quad (A.2)
\]

where $\bar{z} = z_p/z_f = (\rho_c p)/(\rho_f c_f)$ is the impedance ratio.

The principal frequencies of the normal modes of the fluid layer are given by the zeros of the denominator of Eq. (A.1) or (A.2). The complex-valued zeros are

\[
2\alpha_f a = p_k - iq_k \quad (A.3)
\]
where

\[ p_k = k\pi \quad k = 0, 1, 2, \ldots \quad (A.4) \]

\[ q_k = \text{arc tanh} \left[ \frac{z^2}{(z^2 + 1)} \right]. \]

For a water layer between two aluminum blocks, \( q_k = 0.1738 \). It is easily seen that when the incident wave has a frequency \( \omega = \alpha_f c_f = k\pi c_f / (2a) \), \( U_r = 0 \) and \( U_t \) reaches its maximum value of unity. Thus the spectrum of \( |U_r/U_o| \) shows sharp drops, whereas that of \( |U_t/U_o| \) shows sharp peaks. This is called selective transmission.

Since \( z \gg 1 \) for the case considered, \( |U_r/U_o| \) is analogous to \( |A_0| \) in Eq. (5.1) in the mathematical composition of the numerator and denominator. Hence the dips in Fig. 4 and peaks in Fig. 5 at the overtones of \( n = 0 \) mode are due to the same mechanism of selective transmission.

As can be seen from Eq. (A.1) and Eq. (A.2), the reflection and transmission coefficients reach their maximum values of \( (z^2 - 1) / (z^2 + 1) \) and \( 2z / (z^2 + 1) \) when \( 2\alpha_f a = (k + 1/2)\pi \). These frequencies are quite different from the real parts of the resonance frequencies of the liquid layer--\( p_k \) in Eq. (A.4). However, in the case of scattering by a fluid cylinder, the maxima of \( |A_0| \) in Eq. (5.1) occur in the region very near the resonance frequencies of the fluid cylinder.
REFERENCES


