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Generalizations of Palm's Theorem and Dyna-METRIC's Demand and Pipeline Variability

Manuel J. Carrillo

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PREFACE

This report states and proves the theoretical results that allow the Dyna-METRIC model to represent dynamic environments and parts failure processes with a range of variabilities. These results are extensions to the field of queueing theory and are therefore given in that context. References are provided for the reader interested in applications, such as in logistics systems of the military services. The work presented here should be of interest to logistics modelers as well as researchers in the field of queueing theory.

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SUMMARY

Palm's Theorem is first reviewed for nonhomogeneous Poisson arrivals and arbitrary service distributions and is then extended to compound Poisson arrivals and arbitrary initial conditions. For these cases it can represent processes where either the number of demands or the repair pipeline evidences extra-Poisson variability. For many service distributions of interest, the resulting probability distributions of the number in the queueing system have algebraic closed form solutions. Therefore, the results are attractive for applications in modeling dynamic environments or parts failure processes having a wide range of variabilities. One such application is in the field of logistics, where these results play a major role in the Dyna-METRIC model.

CONTENTS

I. INTRODUCTION

Palm's Theorem (Feller, 1968; Palm, 1943) for the $M/G/\infty$ queue[1] has served as a useful tool in modeling inventory problems in logistics models such as METRIC (Sherbrooke, 1968) and Mod-METRIC (Muckstadt, 1973). However, to fit its limited domain of applicability, time-dependent customer arrival rates have been approximated by the required constant rates. And even when the time-dependent behavior is modeled, arrivals with large variability (as measured by variance-to-mean ratios) have been modeled by Poisson processes (thus having a variance-to-mean ratio of unity). Many times this has caused loss of accuracy in representing the problem at hand. Such is the case in certain military logistics applications where models need to represent dynamic environments and extra-Poisson variation in both the parts failure and service processes.

This report reviews the extension of Palm's Theorem for time-dependent arrival rates and service distributions under Poisson input (Hillestad and Carrillo, 1980), and it provides further extensions to compound Poisson input. Processes can now be modeled where the number of either cumulative arrivals (demands) or customers in service (the repair pipeline) has variance-to-mean ratios greater than or equal to unity. In addition, this report treats a range of initial conditions of the queueing system, so that it can start empty or at a fixed or random level.

RAND's Dyna-METRIC model[2] can represent dynamic environments for logistics support systems of parts failure processes having a range of variabilities and initial conditions. Dyna-METRIC has found widespread use in policy analysis and evaluation both at RAND and in the Air Force.[3]

Section II introduces the nonhomogeneous Poisson queue with infinite servers, leading to a generalization of Palm's Theorem to nonhomogeneous Poisson input. Section III shows that comparable results hold in the case of compound Poisson input. Of special interest is the case where the "compounding distribution" is in the logarithmic family,

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[1]In queueing notation, the M stands for the customer interarrival times having the exponential or memoryless distribution, the G stands for arbitrary service distribution, and the ∞ stands for the infinite number of servers.

[2]Dyna-METRIC is a model developed at RAND and used by the Air Force to support policy analysis and evaluation of multi-echelon logistics systems.

[3]Other applications of the results given here are suggested in Crawford (1981); Hillestad (1982); Hillestad and Carrillo (1980).

which allows the treatment of arrival processes with variance-to-mean ratios greater than unity. Section IV relates the results of the previous sections to two-echelon repair systems. Finally, Secs. V and VI cover various initial conditions of the queueing system.

II. THE NONHOMOGENEOUS POISSON QUEUE

DEFINITION 1. Let $\{N(t), t \geq 0\}$ be the counting process of arrivals (e.g., failures of parts of a given type); the queueing system starts empty—$N(0) = 0$. This process has $\lambda(t)$ as intensity function, $\lambda(t) \geq 0$, and $m(t)$ as mean value function,

$$m(t) = E[N(t)] = \int_0^t \lambda(s)\, ds,$$

where, in general, $E[Y]$ denotes the expected value of a random variable Y.

DEFINITION 2. Corresponding to an arrival that occurs at time s, let the positive service (repair) time random variable Y have an arbitrary nonhomogeneous distribution depending on s, so that the probability that $Y \leq y$ is $P[Y \leq y] = G(s, s + y)$. The service times of any sequence of arrivals are assumed to be independent, and the distribution G is assumed to have a finite mean. When $G(s, s + y)$ is homogeneous (stationary), it will be denoted by $F(y)$.

Each arrival results in one "customer" in the nonhomogeneous Poisson queue, or one or more customers in the nonhomogeneous compound Poisson case (Sec. III). This report characterizes the transient behavior of the number of customers, $Z(t)$, in the queueing system, which in the present case of infinite servers is also the number in service.

As a point of departure, a generalization of Palm's Theorem is given in a form that describes the transient behavior of the $M/G/\infty$ queue (see Feller, 1968, p. 460; or Palm, 1943).

THEOREM 1 (Takacs, 1962, p. 160):
Suppose arrivals experience independent, identically distributed service times from a stationary distribution F as described in Definitions 1 and 2, and that the arrival times and service times are independent. For the special case of constant arrival intensity $\lambda(t) = c$, $c > 0$ for $t \geq 0$, and $\lambda(t) = 0$ otherwise (homogeneous Poisson input starting at time 0), the number of arrivals in service, $Z(t)$, has a Poisson distribution with mean

$$\Lambda(t) = \int_0^t c[1 - F(s)]\, ds.$$

COROLLARY 1 (Takacs, 1962, p. 160—Palm's Theorem):
Under the assumptions of Theorem 1, as $t \to \infty$, the limiting (steady state) distribution of the number of arrivals in service is Poisson with mean

$$\lim_{t \to \infty} \Lambda(t) = cE[Y].$$

Corollary 1 is the classical form of Palm's Theorem. Theorem 1 generalizes it to a case where the repair distribution is homogeneous and the arrival density takes a jump from 0 to c at time $t_0 = 0$. For processes that can be assumed to be Poisson, Corollary 1 has provided the basis for the computation of spare parts levels to cover the pipeline of parts in service.

Because a sum of Poisson random variables is Poisson with mean equal to the sum of the means, Theorem 1 suggests that homogeneity is rather unimportant—a lot of nonhomogeneous arrival processes can be approximated by sums of several arrival processes each having one jump. In fact (as will be shown in Theorem 2), homogeneity of neither the arrival process nor the repair process is that important.

The following lemma prepares the way for generalizing Palm's Theorem to nonhomogeneous Poisson arrivals.

LEMMA 1 (Problem 13 in Ross, 1970, p. 29):
For the arrivals referred to in Definition 1, if $N(t) = n$, then their n arrival times have the same distribution as the order statistics corresponding to n independent random variables each having the distribution

$$H(x) = \begin{cases} m(x)/m(t) & \text{for} \quad 0 \le x \le t, \\ 1 & \text{for} \quad x > t. \end{cases}$$

The proof is along the lines of a similar theorem for the stationary case (as given in Ross, 1970, p. 17, for example). It is now possible to prove the generalized Palm's Theorem for nonhomogeneous Poisson input.

THEOREM 2 (Hillestad and Carrillo, 1980):
Assume that the arrivals described in Definition 1 experience independent service times from a nonstationary distribution G as described in Definition 2. Furthermore, assume that the arrival and service times are independent. Then, the number of arrivals undergoing service, $Z(t)$, has a Poisson distribution with mean

$$\Lambda(t) = \int_0^t [1 - G(s,t)]\lambda(s)\, ds.$$

Hillestad and Carrillo (1980) provided the proof of this theorem, but it is also given here for completeness. An alternative proof appears in Crawford (1981). For a historical account on the initial conjecture see Crawford (1981); Hillestad (1982); Hillestad and Carrillo (1980).

Proof:

If there have been n arrivals by time t, then the probability that k customers are still in service at time t is given by

$$P[Z(t) = k \mid N(t) = n] = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for} \quad k = 0, 1, \ldots, n,$$

zero otherwise, and where

$$p = [m(t)]^{-1} \int_0^t [1 - G(s,t)]\lambda(s)\, ds$$

is the probability that an arbitrary arrival is still in service (by Lemma 1). Now, conditioning on the number of cumulative arrivals by time t yields

$$P[Z(t) = k] = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-m(t)} m^n(t)/n!$$

$$= e^{-m(t)}[pm(t)]^k/k! \sum_{n=k}^{\infty} [(1-p)m(t)]^{n-k}/(n-k)!$$

$$= e^{-m(t)}\Lambda^k(t)/k! e^{(1-p)m(t)}$$

$$= e^{-\Lambda(t)}\Lambda^k(t)/k!,$$

using the fact that $\Lambda(t) = pm(t)$.

Because Poisson-distributed random variables have the same mean and variance, the variance-to-mean ratios of both the number of arrivals, $X(t)$, and the number undergoing service, $Z(t)$, are equal to one.

As an example, consider a constant intensity function $\lambda(t) = c$, $c > 0$, and exponential service times with mean $E[Y] = 1/\mu$; $G(s, s + y) = F(y) = 1 - e^{-\mu y}$. Then $\Lambda(t) = (1 - e^{-\mu t})c/\mu$. As a variation of this example, consider the case where the servers are late and become operational at a fixed time $\tau > 0$. This leads to the non-stationary service distribution

$$G(s, t) = \begin{cases} 0 & \text{for} \quad 0 \le s \le t \le \tau, \\ 1 - e^{-\mu(t-\tau)} & \text{for} \quad 0 \le s \le \tau \le t, \\ 1 - e^{-\mu(t-s)} & \text{for} \quad \tau < s \le t. \end{cases}$$

For this case,

$$\Lambda(t) = \begin{cases} ct & \text{for} \quad 0 \le t \le \tau, \\ c\tau e^{-\mu(t-\tau)} + (1 - e^{-\mu(t-\tau)})c/\mu & \text{for} \quad t > \tau. \end{cases}$$

Closed-form algebraic representations of $\Lambda(t)$ for various situations of interest are given in Hillestad (1982); Hillestad and Carrillo (1980). Their ease of computation make them attractive for applications.

III. THE NONHOMOGENEOUS POISSON QUEUE WITH COMPOUND POISSON INPUT

The results presented above can be further generalized for nonhomogeneous compound Poisson input. The last section used the term *arrival* to correspond to the occurrence of a part's failure. The more general case is where the nth arrival results in the failure of a random number W_n of parts. In queueing parlance, W_n is the number of "customers" in an arriving batch, and $Z(t)$ is the total number of customers in service. It will be seen later that the $Z(t)$ process is useful to represent failure processes that have a variance-to-mean ratio greater than unity, a characteristic rather common in Air Force aircraft component removal data. Some definitions are required.

DEFINITION 3. Let W_n be the batch size of customers in the nth arrival. It is assumed that W_n, $n = 0, 1,...$ are independent, identically distributed random variables having the common compounding distribution $\{f_j = P[W = j], j = 0, 1,...\}$ having mean $E[W]$. Further it is assumed that the family $\{W_n\}$ is independent of the arrival process.

DEFINITION 4. Let $\{X(t), t \geq 0\}$ be the resulting counting process of customers, where

$$X(t) = \sum_{n=1}^{N(t)} W_n ,$$

and $N(t)$ is as given in Definition 1.

As is apparent from a conditioning argument on $N(t)$, under Definitions 1, 3, and 4, $X(t)$ has the compound Poisson distribution

$$P[X(t) = k] = \sum_{n=0}^{\infty} f_k^{(n)} e^{-m(t)} m^n(t)/n! \quad \text{for} \quad k = 0, 1,...,$$

where $\{f_k{}^{(n)}\}$ is the n-fold convolution of the compounding distribution $\{f_k\}$ with itself. The mean and variance of $X(t)$ are given by

$$E[X(t)] = E[E[X(t) \mid N(t)]] = m(t)E[W]$$

$$Var[X(t)] = E[Var[X(t) \mid N(t)]] + Var[E[X(t) \mid N(t)]]$$

$$= Var[W]m(t) + E^2[W]m(t)$$

$$= m(t)E[W^2].$$

Therefore, the variance-to-mean ratio of $X(t)$ is $E[W^2]/E[W]$, which is only dependent on characteristics of the compounding distribution and not on t or $m(t)$.

It is now possible to state and prove the generalization of Palm's Theorem for nonhomogeneous compound Poisson input.

THEOREM 3:
Let the arrivals, the compounding distribution, and the resulting nonhomogeneous compound Poisson process $\{X(t), t \geq 0\}$ be given by Definitions 1, 3, and 4, respectively. Also assume that all the customers in an arriving batch have equal service times from distribution G in Definition 2. (Note that this is a special case of the $M/G/\infty$ queue with batch arrivals and service.) Then the distribution of the total number of customers in service is compound Poisson with

$$P[Z(t) = k] = \sum_{j=0}^{\infty} f_k{}^{(j)} e^{-\Lambda(t)} \Lambda^j(t)/j! \quad \text{for} \quad k = 0, 1, \ldots,$$

and where the parameter $\Lambda(t)$ is defined as in Theorem 2 (and thus, $\Lambda(t)$ is here the expected number of batches in service).

Proof:
By Theorem 2, the number of batches in service at time t has a Poisson distribution with mean $\Lambda(t)$. Therefore, the number of customers in service, $Z(t)$, is nonhomogeneous compound Poisson with compounding distribution $\{f_j, j = 0, 1, \ldots\}$.

COROLLARY 2:
Under the assumptions of Theorem 3, the resulting variance-to-mean ratio of the number of customers in service, $Z(t)$, is also given by $E[W^2]/E[W]$.

COROLLARY 3:
Consider the special case of Theorem 3 having constant arrival intensity $\lambda(t) = c$, $t \geq 0$, $c > 0$, and stationary service distribution F given in Definition 2. As $t \rightarrow \infty$, the limiting (steady state) distribution of the number of customers in service is compound Poisson

$$\lim_{t \to \infty} P[Z(t) = k] = \sum_{j=0}^{\infty} f_k^{(j)} e^{-cE[Y]} (cE[Y])^j / j!.$$

The result given in Corollary 3 is found in Feeney and Sherbrooke (1966).

A special case of considerable importance in applications is the one in which the compounding distribution of the input is the logarithmic distribution.

DEFINITION 5. Let W have the logarithmic probability function[1]

$$P[W = j] = -(1-p)^j / (j \log p), \quad 0 < p < 1;\ j = 1, 2, \ldots$$

In this case, $E[W] = -(1/p - 1)/\log p$ and $E[W^2]/E[W] = 1/p$, which implies that the variance-to-mean ratio of either $X(t)$ or $Z(t)$ exceeds one and can take on arbitrarily large values for small values of p.

It is well-known (Feller, 1968, p. 291) that, in the case of the logarithmic distribution, $X(t)$ has a negative binomial distribution with parameters $r(t) = -m(t)/\log p$ and p. Specializing the previous results to this case yields the following theorem.

THEOREM 4:
Under the assumptions of Theorem 3, if the compounding distribution is logarithmic as in Definition 5, then

(a) the cumulative number of customer arrivals $X(t)$ has the negative binomial distribution with parameters $r(t) = -m(t)/\log p$ and p, and
(b) the number of customers in service, $Z(t)$, has the negative binomial distribution with parameters $R(t) = -\Lambda(t)/\log p$ and p; that is,

$$P[Z(t) = k] = \binom{R(t) + k - 1}{k} p^{R(t)} (1-p)^k \quad \text{for} \quad k = 0, 1, \ldots.$$

Proof:

It suffices to prove case (a), since the proof for case (b) is similar.

Define $A_X(s)$ to be the generating function of the probability distribution of some nonnegative discrete random variable X:

[1]The parameter p in this logarithmic distribution is unrelated to the p used in the proof of Theorem 2; also, $\log p$ is the natural logarithm of p.

$$A_X(s) = \sum_{j=0}^{\infty} s^j P[X = j]$$

For the nonhomogeneous compound Poisson process (see Ross, 1970, p. 23, for example):

$$A_{X(t)}(s) = A_{N(t)}(A_W(s)) = \exp[m(t)(A_W(s) - 1)]$$

Therefore,

$$A_{X(t)}(s) = [e^{-1} \exp \{\log(1 - (1 - p)s) \,/\log p\}]^{m(t)},$$

which after some manipulation leads to

$$A_{X(t)}(s) = [p/(1 - (1 - p)s)]^{-m(t)/\log p},$$

the generating function of the required negative binomial distribution.

Section I referred to the loss of accuracy resulting from using a model with Poisson arrival variability (variance-to-mean ratio equal to unity) when in fact the arrivals being modeled exhibited greater variabilities. With Theorem 4, it is now possible to see the effect in the number of customers in service for problems with compound Poisson input and a logarithmic compounding distribution. It suffices to compare the following two processes. The first, with variance-to-mean ratio equal to unity, is a nonhomogeneous Poisson process $\{N(t), t \geq 0\}$ with mean value function $m(t)$, as in Definition 1. The second is a nonhomogeneous negative binomial process with variance-to-mean ratio $1/p > 1$, and for which arrivals have mean value function $-m(t)(\log p)/(1/p - 1)$ so that both processes have $m(t)$ as the expected number of cumulative customers arriving by time t. For the first process, the number in service is Poisson with mean $\Lambda(t)$, whereas for the second, the number in service is negative binomial with parameters $\Lambda(t)/(1/p - 1)$ and p. Statistics for these two distributions can then be easily obtained and compared for an assessment of the magnitude of the error incurred in choosing the wrong model.

IV. TWO-ECHELON REPAIR SYSTEM APPLICATIONS

As noted at the beginning of this report, the results presented here have found application in the Dyna-METRIC model. Dyna-METRIC represents the failure and repair of components resulting from the operation of, say, a fleet of aircraft. In that context, repair of components occurs at either at the base level, where the aircraft operate, or at a remote service depot; this is known as a two-echelon repair system.

LEMMA 2:
Consider the nonhomogeneous Poisson process given in Definition 1. Let the failed part corresponding to an arrival be reparable at base level with fixed probability u and at depot level with probability $(1 - u)$, independent of where other previous failed parts are repaired. Then the one nonhomogeneous Poisson arrival stream with mean value function $m(t)$ becomes two independent nonhomogeneous Poisson streams with mean demand functions $um(t)$ for the base and $(1 - u)m(t)$ for the depot.

The proof of Lemma 2 is along the lines of a similar result for the homogeneous Poisson case (see, for example, Ross, 1972, p. 123).

The independence of the base and depot demand streams implies that the number of arrivals undergoing service at base and at the depot are independent and that the distribution of their sum can be found by a convolution of their distributions. For the case of nonhomogeneous Poisson input, the base or depot number of arrivals undergoing service each has a Poisson distribution. Therefore, the total in service (base plus depot) has a Poisson distribution with parameter equal to the sum of the parameters of the base and depot distributions.

For the case of nonhomogeneous compound Poisson input, the base or depot number of customers in service each has a compound Poisson distribution. If one allows the base to have a compounding distribution different from that of the depot, the total number of customers in service is not necessarily a compound Poisson having a stationary compounding distribution.

However, when base and depot have identical logarithmic compounding distributions, the distribution of the total number of customers in service is compound Poisson with the same compounding distribution—it has a negative binomial distribution with parameters $-[\Lambda_1(t) + \Lambda_2(t)]/\log p$ and p, where the subscripts correspond to the base and depot arrivals, respectively.

Sherbrooke (1966) dealt with two homogeneous independent compound Poisson processes (one for each base and depot) in a two-echelon environment. He noted that if the two compounding distributions are both geometric or both logarithmic, then the distribution of the total number of customers in repair (base plus depot) has a compounding distribution of the same form if and only if the parameters of both compounding distributions are identical. Clearly, that result has implications for the case of nonhomogeneous compound Poisson input. For example, if the compounding distribution is logarithmic and either the number of cumulative customers arrived or the number of customers undergoing service at the base had a variance-to-mean ratio different from those at the depot, then the distribution of the number of customers in repair at base and depot combined would no longer be negative binomial. Then one could no longer take advantage of the computational simplicity of the negative binomial distribution.

V. THE NONHOMOGENEOUS POISSON QUEUE WITH INITIAL CONDITIONS

This section relaxes the assumption that $N(0) = 0$—that is, that the queueing system starts empty. For that purpose, consider the total number of arrivals in service from two Poisson processes, one whose arrivals stop at $t = 0$ and another that starts at $t = 0$. The results below apply to the nonhomogeneous Poisson queue, but similar results for the nonhomogeneous compound Poisson queue are obtained in Sec. VI. To arrive at steady state results requires restriction to the case of a stationary distribution F of service time Y.

DEFINITION 6. Let $\{N_1(t), -T \le t < \infty\}$ be a homogeneous Poisson process with constant intensity function $\lambda_1(t) = c$ for $-T \le t < 0$, and zero otherwise. Furthermore, this system starts empty—$N_1(-T) = 0$.

DEFINITION 7. Also let $\{N_2(t), t \ge 0\}$ be a nonhomogeneous Poisson process with intensity function $\lambda_2(t)$, comparable to the process in Definition 1 and independent of the process in Definition 6.

The arrivals of these two processes have service times from a stationary distribution F as in Definition 2, consistent with the idea of having a common set of an infinite number of servers. Then $Z_1(t)$ and $Z_2(t)$, the number of arrivals in service from the first and second processes, are independent. It follows that the total number of arrivals in service, $Z(t) = Z_1(t) + Z_2(t)$, has a distribution given by the convolution of the corresponding distributions of the two processes.

This problem arises in military logistics applications. For example, an aircraft fleet may be flying at an approximately constant rate in peacetime, but then a military condition arises, at $t = 0$, where the intensity of flying increases and changes over time. The operation of the fleet of aircraft causes aircraft parts to fail and require repair; in such circumstances, these queueing models could represent the number of each part type undergoing repair at some positive time t (Hillestad and Carrillo, 1980).

Corollaries 4 and 5 below pertain to two types of initial conditions for one nonhomogeneous Poisson process that is preceded by another that is homogeneous Poisson. One set of initial conditions assumes knowledge of the number of arrivals in service at $t = 0$, whereas the other assumes that at $t = 0$ the number in service has a limiting (steady state) distribution. These results are applicable to all (measurable) stationary service distributions with a finite mean.

LEMMA 3:

Consider the arrival process in Definition 6 with stationary service distribution F in Definition 2. It is further assumed that the arrival and service processes are independent. If $Z_1(0) = n$, then, as $T \longrightarrow \infty$, the limiting distribution of the number of arrivals still in service at time t, $Z_1(t)$, $t \geq 0$, is binomial with parameters n and $[1 - F^*(t)]$, where

$$F^*(t) = [E[Y]]^{-1} \int_0^t [1 - F(s)]\, ds$$

is the equilibrium distribution of F (see Ross, 1970, p. 47).

Proof:

The proof of Lemma 3 is based on a result found in Takacs (1962, Theorem 2, p. 161) basically stating that, given $Z_1(0) = n$ and steady state, the probability that a typical customer in service at time 0 will depart by time t is given by the equilibrium distribution of F.

Note that the equilibrium distribution for an exponential is the same exponential. As a second example, consider the degenerate (deterministic) distribution

$$F(y) = 0 \text{ for } y < E[Y] \text{ and } F(y) = 1 \text{ for } y \geq E[Y].$$

The equilibrium distribution in this case is uniform with

$$F^*(y) = \begin{cases} 0 & \text{for} \quad y < 0, \\ y & \text{for} \quad 0 \leq y < E[Y], \\ 1 & \text{for} \quad y \geq E[Y]. \end{cases}$$

Furthermore, note that if F is exponential, Lemma 3 holds for any $T \geq 0$: the steady state requirement is not needed because of the memoryless property of the exponential distribution.

COROLLARY 4:

Consider the combined processes $\{N_1(t), -T \leq t < \infty\}$ and $\{N_2(t), t \geq 0\}$ given in Definitions 6 and 7, along with stationary service distribution F as in Definition 2. Again, the arrival process and the service times are assumed independent. If $Z(t) = Z_1(t) + Z_2(t)$, then, as $T \longrightarrow \infty$, the limiting distribution of $P[Z(t) = k \mid Z_1(0) = n]$ is the convolution of the binomial distribution in Lemma 3 (for $Z_1(t)$) with the Poisson distribution from Theorem 2 (for $Z_2(t)$ using $\lambda_2(t)$).

As an example, consider the special case of constant input rate $\lambda_2(t) = c$ and exponential service distribution F with mean $E[Y] = 1/\mu$ so that $\Lambda_2(t) = (1-e^{-\mu t})c/\mu$. Corollary 4 then leads to the transition probabilities $P_{ij}(t)$ for the $M/M/\infty$ queue as given in Saaty (1961, p. 100); Feller (1968, p. 481):

$$P_{in}(t) = \lim_{T\to\infty} P[Z(t) = n \mid Z(0) = i] =$$

$$\sum_{k=0}^{n} \binom{i}{k} e^{-k\mu t}(1 - e^{-\mu t})^{i-k}\, e^{-\Lambda_2(t)}[\Lambda_2(t)]^{n-k}/(n - k)!$$

$$\text{for} \quad i = 0, 1,\ldots;\ n = 0, 1,\ldots, i;\ t \geq 0.$$

For the case $n > i$, the above expression is valid if the upper limit of the summation index is changed to i.

Now we turn to different initial conditions, from one of known number of customers in the system at $t = 0$ to an assumption of random traffic.

LEMMA 4:
Consider $\{N_1(t), -T \leq t < \infty\}$ in Definition 6 and the stationary service distribution F in Definition 2. Assume that the arrival process and the service times are independent. Then, for $t \geq 0$,

$$\Lambda_1(t) = \int_t^{t+T} c[1 - F(s)]\, ds,$$

and, as $T \to \infty$, the limiting distribution of the number of arrivals in service $Z_1(t)$ is Poisson with mean

$$\lim_{T\to\infty} \Lambda_1(t) = cE[Y]\{1 - F^*(t)\}$$

Proof:
By Theorem 1,

$$\Lambda_1(t) = \int_{-T}^{0} c[1 - F(t - s)]\, ds = \int_t^{t+T} c[1 - F(s)]\, ds.$$

Therefore,

$$\lim_{T \to \infty} \Lambda_1(t) = cE[Y] - \int_0^t c[1 - F(s)]\,ds\ ,$$

thus completing the proof.

It is of interest that $F^*(t)$, the equilibrium distribution of F, plays a role in the limiting distributions of both types of initial conditions treated in this section. Also, for $t = 0$, the above result shows that the steady-state distribution of the number of arrivals in service is Poisson with parameter $cE[Y]$, in agreement with Corollary 1.

As an example, if F is exponential with mean $E[Y] = 1/\mu$, then

$$\Lambda_1(t) = e^{-\mu t}(1 - e^{-T/\mu})c/\mu \quad \text{and}$$

$$\lim_{T \to \infty} \Lambda_1(t) = e^{-\mu t} c/\mu.$$

If F is the degenerate (deterministic time) distribution,

$$\Lambda_1(t) = \begin{cases} cT & \text{for} \quad 0 \le t \le E[Y] - T, \\ c(E[Y] - t) & \text{for} \quad E[Y] - T < t \le E[Y], \\ 0 & \text{for} \quad t > E[Y]; \end{cases}$$

$$\lim_{T \to \infty} \Lambda_1(t) = \begin{cases} c(E[Y] - t) & \text{for} \quad 0 \le t \le E[Y], \\ 0 & \text{for} \quad t > E[Y]. \end{cases}$$

COROLLARY 5:

Consider the assumptions of Corollary 4. For the combined processes $\{N_1(t), -T \le t < \infty\}$ and $\{N_2(t), t \ge 0\}$, the distribution of the number of arrivals in service, $Z(t)$, is Poisson. For $t < 0, Z(t) = Z_1(t)$; for $t \ge 0, Z(t) = Z_1(t) + Z_2(t)$. In either case, the corresponding $\Lambda(t)$ are found by appropriate applications of Theorem 2, and the limiting form of $\Lambda_1(t)$ as $T \to \infty$ is given by Lemma 4.

Clearly, Corollary 5 is quite attractive for modeling because of its computational simplicity.

VI. THE NONHOMOGENEOUS COMPOUND POISSON QUEUE WITH INITIAL CONDITIONS

The results of Sec. V are valid for the nonhomogeneous Poisson processes that govern the arrival of batches in nonhomogeneous compound Poisson processes. In this case, the total number of customers in service (from the arriving batches), $Z(t) = Z_1(t) + Z_2(t)$, has a distribution given by the convolution of the corresponding distributions of the two processes.

The following corresponds to Lemma 3.

LEMMA 5:

Let the Poisson process and compounding distribution in Definitions 6 and 3, respectively, specify a compound Poisson process. Also, let resulting customers be provided service from a stationary service distribution F as in Definition 2; the arrival process is assumed independent of the service times. Then, for $t \geq 0$ and as $T \to \infty$, a limiting distribution for the number of customers still in service at time t is given by

$$\lim_{T\to\infty} P[Z_1(t) = k \mid Z_1(0) = n] =$$

$$\{\lim_{T\to\infty} P[Z_1(0) = n]\}^{-1}$$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} f_{n-k}^{(i-j)} \binom{i}{j} h^j (1-h)^{i-j} f_n^{(i)} e^{-cE[Y]} (cE[Y])^i / i!$$

$$\text{for} \quad h = 1 - F^*(t);\ k, n = 0, 1, \ldots$$

If the compounding distribution is logarithmic, then, as $T \to \infty$, the limiting distribution of the number of customers in service at $t = 0$ is negative binomial with parameters $-cE[Y]/\log p$ and p.

Proof:

Let $B(t)$ be the number of arrival batches in service at time $t > 0$. Then conditioning on $B(0) = i$ yields

$$P[Z_1(t) = k \mid Z_1(0) = n] =$$

$$\sum_{i=0}^{\infty} P[Z_1(t) = k \mid Z_1(0) = n, B(0) = i] P[B(0) = i \mid Z_1(0) = n].$$

But because of Bayes' formula,

$$\lim_{T \to \infty} P[B(0) = i \mid Z_1(0) = n] =$$

$$\{\lim_{T \to \infty} P[Z_1(0) = n]\}^{-1} f_n^{(i)} e^{-cE[Y]} (cE[Y])^i / i!$$

However,

$$P[Z_1(t) = k \mid Z_1(0) = n, B(0) = i] =$$

$$\sum_{j=0}^{i} P[Z_1(t) = k, B(t) = j \mid Z_1(0) = n, B(0) = i],$$

which leads to the desired result.

The following corresponds to Corollary 5.

COROLLARY 6:
Consider the two processes $\{N_1(t), -T \leq t < \infty\}$ and $\{N_2(t), t \geq 0\}$ in Definitions 6 and 7 having the identical logarithmic compounding distribution as specified in Definitions 3 and 5. The stationary service distribution F is given in Definition 2, and the service process is independent of the arrivals. Then, for the combined process, the distribution of the number $Z(t)$ of customers in service (from arriving batches) is negative binomial. For $t < 0, Z(t) = Z_1(t)$; for $t \geq 0$, $Z(t) = Z_1(t) + Z_2(t)$. In either case, the corresponding $\Lambda(t)$ are found by appropriate applications of Theorem 4 or Lemma 4. Thus $Z(t)$, $t \geq 0$, has a negative binomial distribution with parameters $-[\Lambda_1(t) + \Lambda_2(t)]/\log p$ and p.

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