Pulse Processes on Signed Digraphs: A Tool for Analyzing Energy Demand

T. A. Brown, F. S. Roberts and J. Spencer

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One of the concepts which can be used to analyze interacting factors in ecological systems (and other complex systems) is the notion of a signed digraph [4, 6]. The application of pulse processes on signed digraphs to analyzing the complex interactions underlying the growing demand for energy was introduced in R-756-NSF [9]. This Report presents a variety of mathematical techniques which can be applied to the analysis of pulse processes. Particular emphasis is placed on developing criteria for the concept of stability, which says that no variable (e.g., energy demand or environmental quality) becomes unboundedly large in absolute value. Closely related reports are R-927/1-NSF [7] and R-927/2-NSF [8], in which a specific energy demand signed digraph is constructed using the subjective judgments of groups of experts. The techniques developed here are applied in [8] to analyze pulse processes on that signed digraph.
SUMMARY

Many problems of society, including those related to energy use, air pollution, solid waste disposal, etc., seem amenable to formulation using the techniques of the rapidly growing field of graph theory, particularly the notion of a signed digraph. This Report considers the mathematical theory of predicting and controlling pulse processes in such digraphs. Section 2 introduces the basic definitions and in particular the notion of stability under a pulse process. Sections 3 through 5 show how classical techniques of linear analysis, combined with structural information from the digraph itself, enable one to characterize the stability and general long-term behavior of a signed digraph under a pulse process. Section 6 illustrates how the theory of linear recursion sequences can be used to settle questions about pulse processes by developing the theory of stability for a special, but very important, class of digraphs known as rosettes. In Section 7 it is shown that most of the results on signed digraphs apply to the more general concept of real-weighted digraphs.*

*The mathematical objects called "real-weighted digraphs" in this Report were called "weighted signed digraphs" in R-756-NSF [9].
ACKNOWLEDGMENT

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1. INTRODUCTION

The idea of studying energy demand and related environmental problems by means of signed digraphs was introduced in Roberts [9]. In applying the signed digraph approach, there are two quite distinct problems. The first is to build a signed digraph from real data, and the second is to analyze the signed digraph using various mathematical tools once it has been built. The first problem is considered in Roberts [7, 8], where an attempt is made to develop a methodology for building a signed digraph using the subjective judgments of experts as the data. The second problem is the subject of this Report.

The notion of a pulse process on a signed digraph was introduced in Roberts [9]. This Report develops in some detail the mathematics of pulse processes. Specifically, it introduces a notion of stability in a pulse process and gives criteria for stability.

Section 2 gives the definitions needed for the rest of the work. Sections 3 through 5 show how classical matrix theory can reduce the study of pulse processes to that of eigenvalues and eigenspaces. The approach developed in these sections is applied to a specific energy demand signed digraph in Roberts [8]. Section 6 applies the theory of linear recursion sequences to develop in detail the theory of stability in the class of signed digraphs called rosettes and advanced rosettes. Although it may appear
to the reader that this class is rather special, it has
turned out that many signed digraphs arising in the Rand
Energy Study are reducible to members of this class. For
example, the signed digraph constructed in Roberts [8] has
as its basic component an advanced rosette. Thus, the
techniques developed here for studying stability in advanced
rosettes enabled Roberts to determine exactly which "strat-
egies" would stabilize that signed digraph. Finally,
Section 7 summarizes the extent to which the results of
the preceding sections apply to digraphs which have real
numbers, rather than merely $\pm 1$, attached to the arcs. In
this Report we limit ourselves to the discussion of
mathematical tools to describe pulse processes. We refer
the reader to [8] for application of the tools developed
here.
2. BASIC DEFINITIONS

Pulse processes on signed digraphs were introduced in Roberts [9]. We begin by recalling the definition of a pulse process and follow a suggestion of [9] by formalizing and investigating a notion of stability in a pulse process.

A digraph $D$ consists of a set $N$ called the nodes or points and a binary relation $R$ on $N$ called the relation of adjacency. If $xRy$, we shall say there is a directed edge from $x$ to $y$. A signed digraph consists of a digraph together with an assignment of a sign $+$ or $-$ to each directed edge.

The reader is referred to Harary, Norman, and Cartwright [4] for a discussion of digraphs and signed digraphs. The following terminology of [4] is used in the sequel.

(a) A sequence in a signed digraph $D$ is a sequence of nodes $x_1, x_2, \ldots, x_n$ so that for all $i$, $x_i R x_{i+1}$.

(b) The sequence is a path if the nodes are distinct; the sequence is a cycle if, in addition, $x_n R x_1$.

(c) The sign of a sequence is the product of the signs of its edges.

(d) The length of the sequence is $n - 1$.

To introduce the notion of pulse process, suppose the signed digraph has nodes $x_1, x_2, \ldots, x_n$. Suppose each node $x_i$ attains a value $v_i(t)$ at times $t = 0, 1, 2, \ldots$. The value $v_i(t+1)$ is determined from $v_i(t)$, from an outside
pulse \( p_i^o(t+1) \) introduced at node \( x_i \) at time \( t+1 \), and from information about whether other nodes \( x_j \) adjacent to \( x_i \) went up or down at the last time period. In particular, we define

\[
v_i(t+1) = v_i(t) + p_i^o(t+1) + \sum_j sgn(x_j,x_i) p_j(t), \quad (*)
\]

where

\[
sgn(x_j,x_i) = \begin{cases} 
+1 & \text{if } x_j, x_i \text{ is } + \\
-1 & \text{if } x_j, x_i \text{ is } - \\
0 & \text{if there is no edge } x_j, x_i,
\end{cases}
\]

and

\[
p_j(t) = \begin{cases} 
v_j(t) - v_j(t-1) & \text{if } t > 0 \\
p_j^o(0) & \text{if } t = 0.
\end{cases}
\]

The quantity \( p_j(t) \) will be called the pulse at node \( x_j \) at time \( t \). A pulse process on a signed digraph \( D \) is defined by the rule \( (*) \), an initial vector of values

\[
V(0) = (v_1(0), v_2(0), \ldots, v_n(0)),
\]

and by vectors giving the outside pulse introduced at each
node at each time period. We shall denote these vectors by

\[ P^0(t) = (p_1^0(t), p_2^0(t), \ldots, p_n^0(t)). \]

We shall also use the pulse vector \( P(t) = (p_1(t), \ldots, p_n(t)) \).*

If \( P^0(t) = 0 \) for \( t > 0 \), the pulse process is called autonomous. An autonomous pulse process for which \( V(0) = 0 \) and \( P^0(0) \) has one entry 1 and all other entries 0 is called simple.

If the initial pulse is at node \( x_i \), let us denote by \( p^t(x_i, x_j) \) and by \( v^t(x_i, x_j) \) the pulse \( p_j(t) \) and value \( v_j(t) \) at node \( x_j \) at time \( t \). The quantities \( p^t(x, y) \) and \( v^t(x, y) \) are related to the signed number of sequences from \( x \) to \( y \) of length \( t \), i.e., the difference between the number of positive sequences from \( x \) to \( y \) of length \( t \) and the number of negative sequences from \( x \) to \( y \) of length \( t \). Specifically, by Roberts [9], we have

**THEOREM 2.1:** The quantities \( p^t(x, y) \) and \( v^t(x, y) \)

are given by the signed number of sequences from \( x \) to \( y \)
of length = \( t \) and length \( \leq t \), respectively.

The signed adjacency matrix of the signed digraph \( D \)
is the matrix \( A = (a_{ij}) \) with \( a_{ij} = \text{sgn}(x_i, x_j) \). Again by
Roberts [9], we have

*The reader should note that in [9], \( P^0(t) \) was called
the pulse vector.
THEOREM 2.2: \( p^t(x_i, x_j) \) is given by the \( i, j \) entry of \( A^t \), while \( v^t(x_i, x_j) \) is given by the \( i, j \) entry of \( A + A^2 + \ldots + A^t \).

In the next section, we apply the classical theory of linear operators to calculate pulse vectors. If the vector \( P(t) \) is considered a column vector, and \( S \) denotes the transpose of the signed adjacency matrix \( A \), then \( S \) may be considered a linear operator on pulse vectors, and in particular we have as a corollary of Theorem 2.2,

THEOREM 2.3: Under autonomous pulse processes, if \( t > T > 0 \), then

\[
P(t) = S^{t-T}P(T).
\]

There are various possible definitions of stability in pulse processes. We shall adopt the following definition, which captures the idea that no node gets arbitrarily large in value (or in pulse). A node \( x_j \) is pulse-stable under a pulse process if \( p_j(t) \) is bounded (in absolute value) and value-stable if \( v_j(t) \) is bounded (in absolute value). The signed digraph is pulse- (value-) stable under the pulse process if each node is. If we simply use the term stability, we refer to value stability. If we use the terms pulse-stable or value-stable without reference to a pulse
process, we mean pulse-stable or value-stable under all simple pulse processes.

Under any pulse process, value stability (at \( x_j \)) implies pulse stability (at \( x_j \)), since

\[
|p_j(t)| = |v_j(t) - v_j(t-1)| \leq |v_j(t)| + |v_j(t-1)|.
\]

On the other hand, pulse stability does not imply value stability: consider, for example, the positive 2-cycle D (Fig. 2.1).

![Fig. 2.1]
3. A MATRIX THEORY APPROACH

The purpose of this section is to point out how the classical theory of the structure of linear operators in an n-dimensional space can be applied to the theory of pulse processes in signed digraphs. In the next section we state and prove some specific theorems about pulse processes using this theory. The theory of the structure of linear operators in n-dimensional space is found in numerous textbooks (such as [2]), but in our opinion the best discussion for our purposes is found in Gantmacher [3]. Computational aspects are thoroughly discussed in Wilkinson [10].

A typical signed digraph is shown in Fig. 3.1 (which is taken from p. 12 of Roberts [9]). The transpose S of the signed adjacency matrix of this signed digraph is given in Fig. 3.2. For S, as for any square matrix, one can find a nonsingular square matrix R such that

$$S = RJR^{-1},$$

where J is a Jordan canonical form. The reader will recall that a Jordan canonical form is a matrix consisting entirely of zeros, except for submatrices down the main diagonal of the form
Fig. 3.1--Sample signed digraph


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Fig. 3.2—The matrix $S$ corresponding to sample signed digraph
\[
\begin{bmatrix}
|\lambda| & \lambda 1 & \lambda 0 & \lambda 0 0 \\
0 \lambda & 0 \lambda 1 & 0 \lambda 0 & 0 0 \lambda \\
0 0 \lambda & 0 0 \lambda 1 & 0 0 0 \lambda & \text{etc.}
\end{bmatrix}
\]

The \( \lambda \)'s are the eigenvalues of \( S \). If \( S \) has no multiple eigenvalues, then \( J \) is simply a diagonal matrix with the various eigenvalues down the main diagonal. \( R \), incidentally, has the corresponding eigenvectors as its columns. Note that by Theorem 2.3,

\[
P(t) = S^tP(0) = RW^tR^{-1}P(0).
\]

This suggests that \( P(t) \) will be a bounded function of \( t \) if all of its eigenvalues are less than 1 in absolute value.* Conversely \( P(t) \) will be unbounded for some initial conditions if any of its eigenvalues are greater than 1 in absolute value (see Theorem 4.4 in the next section for a rigorous proof of this fact). If all eigenvalues are less than or equal to 1 in absolute value, then the presence of repeated eigenvalues of magnitude 1 may make \( P(t) \) unbounded. For example, see Fig. 4.2 in the next section.

Now let us apply these concepts to the example of Fig. 3.1. The characteristic equation of the matrix \( S \) is

*It will be shown in Section 4 that in this case all eigenvalues are zero and \( P(t) \) becomes zero in finite time.
\[ \lambda^6 - \lambda^4 + \lambda^3 - \lambda^2 + \lambda = 0. \]

This equation has the following roots:

\[
\begin{align*}
\lambda_1 &= 0 \\
\lambda_2 &= -1.53416..
\end{align*}
\]
\[
\begin{align*}
\lambda_{3,4} &= -.08010.. \pm i.84530..
\end{align*}
\]
\[
\begin{align*}
\lambda_{5,6} &= .84718.. \pm i.43176..
\end{align*}
\]

Since we are dealing basically with real matrices it would be convenient to find some way of coping with the conjugate pairs of complex eigenvalues without going outside the real field. If the complex eigenvalues are free of Jordan chains (i.e., if those which are repeated are not linked by off-diagonal 1's in the Jordan canonical form) then the following device is natural and appropriate.

Consider the matrices \( Q, J \) given by

\[
Q = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}
\end{pmatrix}
\]

\[
J = \begin{pmatrix}
\alpha + i\beta & 0 \\
0 & \alpha - i\beta
\end{pmatrix}
\]
Then

\[ Q J Q^{-1} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \]

The reader will immediately note that

\[
\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}^n = Q J^n Q^{-1} = Q \begin{pmatrix} \gamma^n \cos \theta & \gamma^n \sin \theta \\ 0 & \gamma^n \cos \theta \end{pmatrix} Q^{-1}
\]

\[
= \begin{pmatrix} \gamma^n \cos \theta & \gamma^n \sin \theta \\ -\gamma^n \sin \theta & \gamma^n \cos \theta \end{pmatrix}
\]

where \( \gamma^2 = \alpha^2 + \beta^2 \)

\( \theta = \arctan \left( \frac{\beta}{\alpha} \right) \).

By similar transformations we may change any Jordan canonical form (provided complex eigenvalues are not "linked") into a "realized" canonical form. For example, the matrix of Fig. 3.2 (whose eigenvalues are given above) may be transformed into the realized canonical form of Fig. 3.3.

The real matrix \( RQ^{-1} \) which relates \( S \) to its realized Jordan canonical form has the eigenvectors of \( S \) as its columns, except for complex eigenvectors. It has the real and imaginary parts (separately) of complex eigenvectors
0     0     0     0     0     0     0     0
0     -1.53416... 0     0     0     0     0     0
0     0     -.08010...  .84530...  0     0     0     0
0     0     -.84530...  -.08010...  0     0     0     0
0     0     0     0     0     .84718...  .43176...  0
0     0     0     0     0     -.43176...  .84718...  0

Fig. 3.3--"Realized" Jordan canonical form of matrix S
for columns. The matrix $QR^{-1}$ corresponding to the $S$ of Fig. 3.2 is shown in Fig. 3.4. The inverse of $QR^{-1}$ is shown in Fig. 3.5. It is obvious how the matrices shown in Figs. 3.3, 3.4, and 3.5 can be used to calculate, with a minimum of effort, any $P(t)$ given a value for $P(0)$. Note that in this particular case the only eigenvalue whose magnitude exceeds unity is the second one, $-1.53416\ldots$. Thus, over the long term this eigenvalue and its associated eigenvector will come to dominate the behavior of the system. For example, if

$$P(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

then

$$P(30) = (1.53416)^{30} \times 0.02906 \times \begin{pmatrix} 8.07 \\ 7.49 \\ 3.18 \\ -2.07 \\ -3.43 \\ -12.38 \end{pmatrix}.$$

Knowledge of the matrix $RQ^{-1}$ also enables you to choose countervailing pulses wisely, in order to damp out unstable eigenvectors. For example, if you add to the pulse $P(0)$ above the following pulse


\[
\begin{matrix}
1 & 8.07378 & 1.25128 & -1.43302 & .28707 & .41739 \\
0 & 7.49865 & -1 & 0 & -1 & 0 \\
0 & 3.18598 & 1.36238 & -.26053 & -.64995 & .89493 \\
0 & -2.07670 & -.45683 & -1.56843 & -.18164 & 1.14894 \\
1 & -3.43034 & 1.33138 & -2.27832 & -.56011 & -.01437 \\
0 & -12.38645 & 1.11110 & 1.17249 & .06298 & .47754 \\
\end{matrix}
\]

\[
\text{real} \quad \text{imag.} \quad \text{real} \quad \text{imag.}
\]

\[
\begin{align*}
0 & 1.53416 & .08010 & + i.84530 & .84718 & + i.43176
\end{align*}
\]

Fig. 3.4--The matrix QR\(^{-1}\) for the matrix S
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</table>

Fig. 3.5--The matrix $RQ^{-1}$ for the matrix $S$
\[ X(0) = \begin{pmatrix} 0.02906 \cdots \\ 0.03934 \cdots \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

then a glance at \( RQ^{-1} \) shows you that the resulting pulse pattern will be stable. Of course, in real systems there are enough random pulses entering the system that constant correction will be required to prevent a buildup in the modes corresponding to unstable eigenvalues.
4. STABILITY AND THE EIGENSSPACE OF S

Many results which follow easily from classical matrix theory have direct interpretations as theorems about signed digraphs.

**Theorem 4.1:** Let \( \sum_{i=0}^{n} a_i \lambda^i = \det(\lambda I - S) \) be the characteristic polynomial of the matrix \( S \). Let \( p_j(t) \) denote the pulse at node \( j \) at time \( t \), and assume the digraph is subject to no external pulses over the period \( T - n \leq t \leq T \). Then

\[
p_j(T) = - \sum_{i=0}^{n-1} a_i p_j(T+n+i).
\]

**Proof:** \( p_j(t) \) is the \( j \)th component of \( P(t) \). By Theorem 2.3,

\[
P(t) = S^{t-(T-n)}P(t-n) \text{ for } T - n \leq t \leq T.
\]

But it is a well-known theorem that every matrix satisfies its own characteristic equation. Therefore

\[
0 = \left[ \sum_{i=0}^{n} a_i (S)^i \right] P(T-n) = \sum_{i=0}^{n} a_i P(T-n+i).
\]

It is obvious that \( a_n = 1 \), and the desired result follows. Q.E.D.
Put heuristically, this theorem says that if a system is not subject to external pulses, after a certain time you can predict the future pulses at a node strictly on the basis of the past $n$ pulses at that node; everything you have to know about the rest of the system (for this purpose) is included in those $n$ pulses. Furthermore, the same forecasting rule works at every node in the system! Of course, there may be simpler rules which work at particular nodes and not at others, but it still seems somewhat surprising that the characteristic equation gives you a "universal" rule which works at every node. Note that Theorem 4.1 is valid for digraphs with real numbers (rather than just $\pm 1$) attached to the arcs.

**THEOREM 4.2**: If a signed digraph has no cycles, then all of its eigenvalues are zero.

**Proof**: If the digraph corresponding to $S$ has no cycles, then there is no nontrivial permutation $i_j (j=1,2,\ldots,n)$ such that $\prod_{j=1}^n a_{ij} \neq 0$, where $a_{ij}$ is an element from the matrix $\lambda I - S$. Furthermore, $a_{ii} = \lambda$ (since otherwise the arc $i$, $i$ would be a cycle). Therefore

$$\det(\lambda I - S) = \lambda^n,$$

and the desired result follows. Q.E.D.
The converse of this theorem is not true. For example, the digraph shown in Fig. 4.1 has all eigenvalues zero but also has cycles.

It is an immediate consequence of Theorem 4.1 that if all eigenvalues are zero then the digraph is pulse-stable. Indeed, \( P(t) = 0 \) for \( t > n \). Thus the example of Fig. 4.1 also shows that the existence of cycles does not necessarily imply either pulse or value instability.

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix}
\]

Jordan canonical form =

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Char. equation: \( \lambda^4 \)
Eigenvalues: 0, 0, 0, 0
Pulse-stable

Fig. 4.1
THEOREM 4.3: If a signed digraph has any nonzero eigenvalues, it must have at least one whose magnitude is greater than or equal to unity.

Proof: All coefficients in the characteristic equation \[ \sum_{i=0}^{n} a_i \lambda^i \] are integers. The coefficient \( a_i \) (where \( i \) is the least integer such that \( a_i \neq 0 \)) will be the product of all the nonzero eigenvalues (times \( \pm 1 \)), provided that \( i < n \). If \( i = n \) then all eigenvalues are zero. If \( i < n \), then the product of the magnitudes of all the nonzero eigenvalues will be an integer, and thus at least one of them must have magnitude greater than or equal to 1. Q.E.D.

THEOREM 4.4: If a signed digraph has an eigenvalue greater than unity in absolute value, then it is pulse-unstable under some simple pulse process.

Proof: Let \( U \) be an eigenvector corresponding to an eigenvalue \( \lambda \) such that

\[ |\lambda| > 1, \]
\[ ||U|| = 1. \]

Write \( U = \sum_{i=1}^{n} a_i E_i \), where \( E_i \) is the vector with 1 in the \( i \)th component and 0 elsewhere. Then, for any
integer $N > 0$,

$$|\lambda|^N = \|S^N U\| = \| \sum_{i=1}^{n} a_i S^N E_i \| \leq \sum_{i=1}^{n} |a_i| \|S^N E_i\|.$$

Since $|a_i| \leq 1$, it follows that at least one of the $\|S^N E_i\| \geq \frac{1}{n} |\lambda|^N$; since there are only a finite number of $E_i$, it follows that for at least one of them, $\|S^N E_i\| \geq \frac{1}{n} |\lambda|^N$ for arbitrarily large $N$; thus the signed digraph is pulse-unstable. Q.E.D.

Theorem 4.4 also holds if the edges have weights other than $\pm 1$.

**THEOREM 4.5:** If all the nonzero eigenvalues of a signed digraph are distinct, and the largest magnitude of an eigenvalue is unity, then the digraph is pulse-stable under all autonomous pulse processes.

**Proof:** The pulse vector $P(0)$ can be written as a linear combination of the columns $U_i$ of $R$ (which are, the reader will recall, eigenvectors or elements in a Jordan chain leading to a zero eigenvector). Thus

$$S^N P(0) = \sum_{i=1}^{n} a_i \lambda_i^N U_i.$$ 

But

$$\|S^N P(0)\| \leq \sum_{i=1}^{n} |a_i| \lambda_i^N \|U_i\| \leq \sum_{i=1}^{n} |a_i| \|U_i\|. $$
So the pulses set off by any initial pulse vectors are bounded. Q.E.D.

It is easy to construct examples of pulse-unstable signed digraphs with multiple eigenvalues of magnitude less than or equal to unity; the digraph of Fig. 4.2 is one such example; both of the eigenvalues are +1. However, if you delete the arc (1,2) from this example, then you have a pulse-stable (but not value-stable) digraph. Deleting the arc (1,2) does not change the characteristic polynomial of the digraph. This shows very clearly that the characteristic polynomial does not tell the whole story about the asymptotic behavior of signed digraphs. In particular, it tells you nothing about the structure of the condensation of the digraph, a topic we shall discuss in the next section.

\[ S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

Char. equation: \( \lambda^2 - 2\lambda + 1 \)
Eigenvalues: +1, +1
Pulse-unstable

Fig. 4.2
5. CONDENSATION AND CHARACTERISTIC POLYNOMIALS

As we saw in the previous section, much information about the stability of a signed digraph under a pulse process is contained in the characteristic polynomial of its signed adjacency matrix. Further information is contained in the condensation by strong components of the underlying digraph. In this section we shall prove an interesting relation between the characteristic polynomial of the signed digraph and those of its strong components.

To introduce this relation, let us call a collection of nodes C in a digraph D strongly connected if for all distinct nodes x and y in C, there are paths from x to y and from y to x using only nodes in C. A strong component of the digraph is a maximal strongly connected set of nodes. For example, a single node is a strong component if and only if it is on no cycles. The condensation (by strong components) of D is the digraph $D^*$ whose nodes are the strong components of D, and such that there is an edge from component C to component $C'$ if and only if there is an edge in D from some node of C to some node of $C'$. The notion of condensation is discussed at length in Harary, Norman, and Cartwright [4]. One particularly important observation to make is that the condensation of any digraph is acyclic, i.e., it has no cycles.
THEOREM 5.1: The characteristic polynomial
of a signed digraph is the product of the charac-
teristic polynomials of its strong components.

Proof: If the digraph in question consists of just
one strong component, there is nothing to prove. We
proceed by induction on the number of strong components.
Suppose the theorem has been proved for all signed digraphs
with \( n \) strong components; let \( S \) be the transpose of the
signed adjacency matrix of a signed digraph with \( n+1 \) strong
components. Choose a strong component \( S_1 \) whose node \( S_1^* \)
in the condensation \( S^* \) of \( S \) has the property that no edge
comes from any other node in \( S^* \) to \( S_1^* \). Let \( S_2 \) denote the
signed digraph generated by the nodes of \( S \) which are not
in \( S_1 \). If we number the nodes of \( S \) in such a way that
the nodes of \( S_1 \) are \( \{1, 2, \ldots, k\} \), and the nodes of \( S_2 \)
are \( \{k+1, \ldots, m\} \), then the matrix \( S \) may be written

\[
S = \begin{bmatrix}
  k & m-k \\
  S_1 & 0 \\
  0 & S_2
\end{bmatrix}
\]

*We will "abuse the language" by using the same symbol for
a signed digraph and the transpose of its signed adjacency matrix.
where the k by m-k submatrix in the upper right-hand corner consists entirely of zeros. We see, therefore, that

\[ \det(\lambda I - S) = \det(\lambda I - S_1) \cdot \det(\lambda I - S_2) \]

from which it follows that the characteristic equation of S is the product of the characteristic equation of S_1 and the characteristic equation of S_2. But the characteristic equation of S_2 is simply the product of the characteristic equations of its strong components, which concludes the proof.

Since the characteristic equation of a signed digraph is completely determined by the characteristic equations of its strong components, it follows that the characteristic equation tells us nothing about how the individual strong components are related to one another (i.e., it tells us nothing about the structure of the condensation digraph).

Aside from its theoretical interest, Theorem 5.1 is often a great practical convenience in calculating the characteristic polynomial of a signed digraph.
6. PULSE PROCESSES FOR ROSETTES: AN APPLICATION OF THE
THEORY OF LINEAR RECURSION SEQUENCES

6.1. The Notion of a Rosette

Sequences $p_j(t)$ satisfying an equation like that of
Theorem 4.1 are called linear recursion sequences. These
sequences have been the object of some study. As a result
of Theorem 4.1, the theory of linear recursion sequences
can be profitably applied to answer questions about pulse
processes in signed digraphs. We shall illustrate this
point in the present section for a special class of signed
digraphs called rosettes (and advanced rosettes), where
the linear recursion of Theorem 4.1 takes a particularly
simple form. As we remarked in the introduction, although
it may appear to the reader that this class is rather
special, it has turned out that many signed digraphs arising
in the Rand Energy Study are reducible to members of this
class. For example, the signed digraph constructed in
Roberts [8] has as its basic (and only nontrivial) strong
component an advanced rosette. Thus, the techniques devel-
oped here for studying stability in advanced rosettes
enabled Roberts to determine exactly which "strategies"
would stabilize that signed digraph.

Modifying a notion of Berge [1], let us say a digraph
$D = (N, R)$ is a rosette if $|N| > 1$ and $D$ is either a single
cycle or is strongly connected and has exactly one node $x$, 
the central node, with more than two edges incident* to it. Rosettes are easily seen to be exactly those digraphs with a central node \( x \) and nonintersecting cycles leading out of \( x \), as shown in Fig. 6.1. As usual, a signed digraph will be called a rosette if its underlying digraph is a rosette.

Stability in rosettes reduces to stability at the central node \( x \). In fact, the situation is even simpler than that. Let us say a node \( x \) of a signed digraph is self-stable in value or pulse if the sequence \( v^t(x,x) \) or \( p^t(x,x) \) is bounded (in absolute value). We have

**Theorem 6.1.1:** A rosette with central node \( x \) is pulse-or value-stable if and only if \( x \) is self-stable in pulse or value respectively.

**Proof:** We use Theorem 2.1 to calculate \( v^t(y,z) \). For example, suppose first that every path from \( y \) to \( z \) goes through \( x \). Suppose that the unique path \( P_{yx} \) from \( y \) to \( x \) has length \( r \) and the unique path \( P_{xz} \) from \( x \) to \( z \) has length \( s \). Then \( v^t(y,z) = \pm v^{t-r-s}(x,x) \), since every sequence from \( y \) to \( z \) consists of \( P_{yx} \), a sequence from \( x \) to \( x \), plus \( P_{xz} \). If there is a path from \( y \) to \( z \) not going through \( x \), let

*An edge \( x, y \) is incident to the nodes \( x \) and \( y \) and no others.*
this path have length $r$ and sign $\sigma$. Then $v^t(y,z) = (\sigma)1 + v^{t-r}(z,z)$, which is bounded by the above argument. Q.E.D.

This result generalizes to what we shall call advanced rosettes. A digraph $D = (N,R)$ is an advanced rosette if $|N| > 1$ and $D$ is either a single cycle or is strongly connected and there is one node $x$, the central node, which is on all cycles in $D$.

**THEOREM 6.1.2:** An advanced rosette $D$ with central node $x$ is pulse- or value-stable if and only if $x$ is self-stable in pulse or value, respectively.

**Proof:** The result follows from Theorem 6.1 if $D$ is a single cycle. If $D$ is not a single cycle, note that for all $y$ and $z$ in $D$, there are only finitely many sequences from $y$ to $z$ not going through $x$. Then, to go from $y$ to $z$, except for finitely many sequences, one first goes from $y$ to $x$ (in finitely many ways), then $x$ to $x$, and then $x$ to $z$ (in finitely many ways). Q.E.D.

It should be remarked that the notion of local self-stability is not in general sufficient even for local stability, though it is in advanced rosettes. To give
a counterexample, note that in the signed digraph of Fig. 6.2a, node x is self-stable but $v^t(y,x)$ tends to $+\infty$. Even global self-stability (each node self-stable) is not sufficient for stability. For consider the signed digraph of Fig. 6.2b. Each node is self-stable. But node x is not stable. It is not even pulse-stable, for $p^{3+2k}(y,x) = (-1)^k(k+1)$. This follows since to get from y to x in $3+2k$ steps, one goes through node a m times and then node b k-m times, and each choice of m = 0, 1, ..., k gives a different sequence. It might be conjectured that the additional assumption that D be strongly connected is sufficient for global (local) self-stability to imply global (local) stability. But this conjecture is also false. Consider the signed digraph of Fig. 6.2c. Here each point is self-stable. For to get from any point $\alpha \neq u, v$ to $\alpha$, one first goes from $\alpha$ to x and then, after hitting x for the last time, one returns to $\alpha$. But for each such sequence whose last return goes through u, there is a corresponding one of equal length and opposite sign whose last return goes through v. Thus, $\alpha$ is stable. To get from u to u, one first goes from u to y and then y to u. The first trip to y can be either through i or through j. Thus there is again a one-to-one correspondence between these sequences, and so u is self-stable. Similarly, v is self-stable. Finally, the node x is not stable, or even pulse-stable. The sequences from y to x which go
through x more than once divide into those using u and those using v, and these cancel out as before. The calculation of $p^t(y, x)$ thus reduces to considering sequences which go from y to x without using x more than once, and this calculation is analogous to that for the signed digraph of Fig. 6.2b.
Fig. 6.1

Fig. 6.2
6.2. The Linear Recursion Sequence for a Rosette

We now want to analyze pulse processes in (advanced) rosettes. As we have seen, a complete evaluation of the numbers \( p^t(x, y) \) is dependent in an elementary way on the values \( p^t(C, C) \), where \( C \) is the center of the rosette. We set

\[
(6.2.1) \quad p(t) = p^t(C, C) \quad \text{and} \quad v(t) = v^t(C, C)
\]

and restrict our attention to calculation of these values. We assume that the initial value at \( C \) is zero. Then

\[
(6.2.2) \quad v(t) = \sum_{s=1}^{t} p(s).
\]

Let \( a_i \) denote the sum of the signs of the cycles of length \( i \). By calculating the characteristic polynomial of the rosette and applying Theorem 4.1, one can show that for \( t \geq n \),

\[
(6.2.3) \quad p(t) = \sum_{i=1}^{s} a_i p(t - i),
\]

where \( s \) is the maximal integer such that \( a_s \neq 0 \). (The main part of the proof consists in showing that the characteristic polynomial \( C(x) \) is given by \( x^{n-s} R(x) \), where \( R(x) \) is the inverted rosette polynomial defined below.) We say \( s \) is the order of the rosette and \( (a_1, \ldots, a_s) \) is the rosette
sequence. The rosette sequence determines the pulse process. Actually Eq. (6.2.3) follows quite directly without recourse to Theorem 4.1., for a pulse at C at time t must have come through a cycle and have been at C at time t - i. Eq. (6.2.3) actually holds for all \( t \geq 0 \) if we assume the initial conditions

\[
p(t) = 0, \ -s + 1 \leq t < 0
\]

(6.2.4)

\[
p(0) = 1.
\]

Actually, we may assume \( p(t) = 0 \) for all \( t < 0 \). However, Eq. (6.2.4) is sufficient to determine \( p \) and is more convenient for general analysis. In a certain sense, Eqs. (6.2.3) and (6.2.4) solve the Forecasting Problem defined in [9] (i.e., the problem of calculating the pulse and value of node C at time t) for the rosette. The function \( p \), and therefore \( v \), is easily calculable.

Since the rosette sequence determines the pulse process, all the results we get below for rosettes can be applied to advanced rosettes.
6.3. Generating Polynomials

Generating polynomials are an important tool in studying linear recursion sequences (LRS). Given an LRS \( p(t) \), the generating polynomial is the formal power series

\[
G(x) = \sum_{t=0}^{\infty} p(t)x^t.
\]

If \( p(t) \) is given by Eqs. (6.2.3) and (6.2.4) and has rosette sequence \( (a_1, \ldots, a_s) \), then

\[
G(x) = \frac{1}{1 - \sum_{i=1}^{s} a_i x^i}.
\]

To justify Eq. (6.3.2), it suffices to show

\[
G(x) (1 - \sum_{i=1}^{s} a_i x^i) = 1.
\]

(We omit the logical justification of manipulation of formal power series.) The coefficient of \( x^n \) on the left-hand side of Eq. (6.3.3) is

\[
p(n) - \sum_{i=1}^{s} a_i p(n - i) = \begin{cases} 0, & n > 0 \\ 1, & n = 0. \end{cases}
\]

We call \( Q(x) = 1 - \sum_{i=1}^{s} a_i x^i \) the **rosette polynomial**. By
the fundamental theorem of algebra we may factor

\[(6.3.5) \quad Q(x) = \prod_{i=1}^{u} (1 - \alpha_i x)^{d_i},\]

where \(\alpha_1, \ldots, \alpha_u\) are distinct complex numbers and \(d_i\) are positive integers, \(\sum_{i=1}^{u} d_i = s\). There exists a partial fraction decomposition

\[(6.3.6) \quad G(x) = \frac{1}{Q(x)} = \sum_{i=1}^{u} \frac{P_i(x)}{(1 - \alpha_i x)^{d_i}},\]

where \(P_i(x)\) are polynomials, \(\exists P_i < d_i\) (here \(\exists P = \text{degree } (P)\)).

Also \(1 - \alpha_i x \nmid P_i(x)\). For if it did, multiplying Eq. (6.3.6) by \(Q(x)\), we would obtain \(l = A(x)\), where \(1 - \alpha_i x | A(x)\), a contradiction. We have

\[(6.3.7) \quad (1 - \alpha_i x)^{-1} = \sum a_{i}^{t} x^{t}\]

\[(6.3.7) \quad (1 - \alpha_i x)^{-2} = \sum (t+1)a_{i}^{t} x^{t}\]

\[\vdots \]

\[(6.3.7) \quad (1 - \alpha_i x)^{-k} = \sum_{k-1}^{t+k-1} a_{i}^{t} x^{t}.\]

Suppose \(P_i(x) = \sum_{j} b_{ij} x^{j}\). Then using Eqs. (6.3.6) and (6.3.7), we find
\[ G(x) = \sum_t \left[ \sum_i \sum_j b_{ij} \left( \frac{t-j+d_i-1}{d_i-1} \right) a_i^{t-j} \right] x^t \]
\[ = \sum_t \left[ \sum_i a_i^t \sum_j b_{ij} \left( \frac{t-j+d_i-1}{d_i-1} \right) a_i^{-j} \right] x^t. \]

Thus, by Eq. (6.3.1),

\[ (6.3.8) \quad p(t) = \sum_{i=1}^{u} \xi_i(t) a_i^t, \]

where the \( \xi_i \) are polynomials. Moreover, \( \partial \xi_i = d_i-1 \). For the highest possible power of \( t \) in \( \xi_i(t) \) is \( d_i-1 \). Its coefficient is

\[ (6.3.9) \quad \frac{1}{(d_i-1)!} \sum_j b_{ij} a_i^{-j} = \frac{1}{(d_i-1)!} p_i(a_i^{-1}). \]

The latter is \( \neq 0 \) since \( 1-a_i x \not\mid P_i(x) \).

It is convenient to define

\[ (6.3.10) \quad R(x) = x^S Q(x^{-1}) = x^S - \sum_{i=1}^{s} a_i x^{s-i} \]

as the **inverted rosette polynomial**. Then

\[ (6.3.11) \quad R(x) = \prod_{i=1}^{u} (x - a_i)^{d_i}. \]

The roots of \( R \) are the inverses of the roots of \( Q \).

It should be noted that for a rosette, the characteristic polynomial \( C(x) \) of the signed adjacency matrix is
closely related to the inverted rosette polynomial. In particular, it is possible to prove that $C(x) = x^{n-S}R(x)$.

The main results about stability in rosettes, Theorems 6.5.1 and 6.5.2, are stated in terms of $R$. The next two sections present results which are used in the proof of Theorems 6.5.1 and 6.5.2.
6.4. Pulse Stability

**THEOREM 6.4.1**: Let \( p(t) \) be given by Eq. (6.3.8).

Then \( p(t) \) is bounded iff

\[
(*) \quad |a_i| \leq 1 \text{ for all } i \text{ and whenever } |a_i| = 1, \exists \xi_i = 0.
\]

**Proof**: By Eq. (6.3.8),

\[
(6.4.1) \quad |p(t)| \leq \sum_{i=1}^{u} |\xi_i(t)||a_i|^t.
\]

Assume (*) holds. If \( |a_i| < 1 \), \( \lim |\xi_i(t)||a_i|^t = 0 \) so

\( |\xi_i(t)||a_i|^t \) is bounded. If \( |a_i| = 1 \), \( |\xi_i(t)||a_i|^t = |\xi_i| \)

is constant. By Eq. (6.4.1) \( |p(t)| \) is bounded.

Assume (*) does not hold. Set \( D = \max |a_i| \), \( a = \max \exists \xi_i \) taken over those \( i \) with \( |a_i| = D \). Renumber so that

\( |a_i| = D \), \( a = \exists \xi_i \) iff \( 1 \leq i \leq v \). For \( 1 \leq i \leq v \) let \( \gamma_i \) be

the coefficient of \( t^a \) in \( \xi_i(t) \). Set \( \beta_i = a_i D^{-1} \).

\[
(6.4.2) \quad p(t) = t^a D^t \left[ \sum_{i=1}^{v} \gamma_i \beta_i^t \right] + o(t^a D^t).
\]

For \( k \geq 0 \) set

\[
(6.4.3) \quad B_k = \begin{bmatrix}
\beta_1^k & \cdots & \beta_v^k \\
\vdots & & \vdots \\
\beta_1^{k+(v-1)} & \cdots & \beta_v^{k+(v-1)}
\end{bmatrix}.
\]
Set

\[(6.4.4) \quad C_t = \sum_{i=1}^{\nu} \gamma_i \beta_i^t.\]

Then

\[(6.4.5) \quad B_k \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_v \end{bmatrix} = \begin{bmatrix} C_k \\ \vdots \\ C_{k+v-1} \end{bmatrix} \]

Applying the appropriate column multiplication to Eq. (6.4.3),

\[(6.4.6) \quad \det (B_k) = (\prod_{i=1}^{\nu} \beta_i^k) \det (B_0).\]

The matrix \(B_0\) is the Van der Monde matrix, hence by a well-known result \(\det (B_0) \neq 0\). Since \(|\beta_i| = 1\),

\[(6.4.7) \quad |\det (B_k)| = b \neq 0\]

is independent of \(k\). Left-multiplying Eq. (6.4.5) by \(B_k^{-1}\) we have

\[(6.4.8) \quad \gamma_i = \sum_{j=0}^{\nu-1} b_{ij} (k) C_{j+k},\]

where \(b_{ij}^{(k)}\) is the \((i,j)\) coefficient of \(B_k^{-1}\). By Cramer's Rule, \(b_{ij}^{(k)} = \pm \det [i,j \text{ minor of } B_k]/\det [B_k]\). Since all coefficients of \(B_k\) have absolute value 1, \(|\det(\text{minor})| \leq \nu!\). So \(|b_{ij}^{(k)}| \leq \nu!/b\), independent of \(k\). Since the
\( \gamma_i \) are fixed and some \( \gamma_i \neq 0 \), there is an \( \epsilon > 0 \) such that for all \( k \), \( |C_{j+k}| \geq \epsilon \) for some \( 0 \leq j < v - 1 \). Therefore, for infinitely many \( t \), \( |C_t| \geq \epsilon \), and

\[
(6.4.9) \quad |p(t)| \geq t^{\alpha D} c_t - o(t^{\alpha D}) \geq \frac{1}{2} \epsilon t^{\alpha D}.
\]

We conclude \( p(t) \) is unbounded. Q.E.D.

Note the connections between Theorems 6.4.1 and 4.4. Both essentially say that if the signed digraph contains an eigenvalue \( \alpha \), \( |\alpha| > 1 \), the pulse process is unstable. Theorem 4.4 clearly has the shorter of the two proofs. The proof of Theorem 6.4.1 gives the additional information that in every \( n \) consecutive time periods, \( p(t) \) is large at least once (since \( n < v \)). An extremal example of this is given by a rosette consisting of two plus cycles of size \( n \). Then \( p(t) = 0 \) unless \( n \) divides \( t \), at which time \( p(t) = 2^{t/n} \). This additional information in Theorem 6.4.1 will be useful in examining value stability.

We now apply the stability condition given by Theorem 6.4.1 when the \( \alpha_i \) are roots of the inverted rosette polynomial \( R(x) \). The constant coefficient \( a_s \) of \( R \) is a non-zero integer. But

\[
(6.4.10) \quad \frac{u}{n} \prod_{i=1}^{u} \alpha_i^{d_i} = (-1)^n a_s,
\]

where \( \alpha_i, \ldots, \alpha_n \) are the roots of \( R \), with multiplicities \( d_i \).
If \( a_s \neq \pm 1 \), some \( |\alpha_i| > 1 \), and the signed digraph is not pulse-stable. If \( |a_s| = 1 \),

\[
\sum_{i=1}^{u} |\alpha_i|^{d_i} = 1.
\]  

(6.4.11)

If the digraph is pulse-stable, all \( |\alpha_i| \leq 1 \). Hence all \( |\alpha_i| = 1 \). Furthermore, each \( \delta_i = 0 \), and \( R \) has no multiple roots. To summarize, we have

**THEOREM 6.4.2:** If the rosette with inverted rosette polynomial

\[
R(x) = x^s - \sum_{i=1}^{u} a_i x^{s-i} = \sum_{i=1}^{u} \frac{d_i}{(1-\alpha_i)}
\]

is pulse-stable, then

1. \( a_s = \pm 1 \)
2. \( |\alpha_i| = 1 \) \( i = 1, 2, \ldots, u \)
3. \( d_i = 1 \), \( i = 1, 2, \ldots, u \).
6.5. Cyclotomic Polynomials

By Theorem 6.4.2, the pulse stability of a rosette implies that all roots $\alpha$ of the rosette polynomial have $|\alpha| = 1$. We now use, without proof, a classical number theoretic result [5]:

**Kronecker's Theorem:** Let $R$ be a monic polynomial with integral coefficients. Suppose all roots $\alpha$ of $R$ have $|\alpha| = 1$. Then all roots $\alpha$ of $R$ are roots of unity. That is, if $R(\alpha) = 0$, there exists $n$ such that $\alpha^n = 1$.

We require a quick review of cyclotomic polynomials. The $n$th cyclotomic polynomial, denoted by $\Phi_n(x)$, is defined as the unique irreducible monic polynomial whose roots are the primitive $n$th roots of unity. These polynomials can also be defined by

$$
\Phi_1(x) = x - 1 \\
\Phi_n(x) = (x^n - 1) \prod_{d|n, d<n} \Phi_d(x).
$$

(6.5.1)

Fig. 6.3 gives all cyclotomic polynomials of degree $\leq 10$. The $\varphi(\Phi_n)$ = the number of primitive $n$th roots of unity = $\phi(n)$, where $\phi$ is Euler's function. If $n = \prod_{i}^{e_i} p_i^{e_i}$, $p_i$ distinct primes, $e_i > 0$, then $\phi(n) = n\prod_{i}(1-p_i^{-1})$. 
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<td>20</td>
<td>1 0 -1 0 1 0 -1 0 1</td>
</tr>
<tr>
<td>22</td>
<td>1 -1 1 -1 1 -1 1 -1 1</td>
</tr>
<tr>
<td>24</td>
<td>1 0 0 0 -1 0 0 0 1</td>
</tr>
<tr>
<td>30</td>
<td>1 1 0 -1 -1 -1 0 1 1</td>
</tr>
</tbody>
</table>

Fig. 6.3-- Cyclotomic polynomials
THEOREM 6.5.1: A rosette is pulse-stable iff
the inverted rosette polynomial \( R(x) \) is the product
of distinct cyclotomic polynomials.

Proof: Assume the rosette is pulse-stable. Then by
Theorem 6.4.2 and Kronecker's Theorem, \( R(x) = \pi (x-a_i) \),
where the \( a_i \) are distinct roots of unity. We factor
\[ R(x) = \prod_{j=1}^{t} P_j(x) \]
into irreducible factors \( P_j \). Each \( P_j(x) \)
has some root \( a \). But then \( P_j \) is determined as the unique
irreducible polynomial with \( a \) as a root. If \( a \) is a \( t_j \)-root
of unity, \( P_j = \hat{\phi}_{t_j} \). Then \( \hat{\phi}_{t_j} \) are distinct since \( R \) may have
no multiple roots.

Conversely, if \( R = \pi \hat{\phi}_{t_j} \), the roots \( a_1, \ldots, a_n \)
are distinct roots of unity, so the rosette is pulse-
stable. Q.E.D.
Let us say that a polynomial $R$ has PCP if it is expressible as the product of distinct cyclotomic polynomials. If $R$ has PCP,

(6.5.2) \[ R(0) = \pm 1 \]
(6.5.3) \[ R(\alpha) = 0 \text{ iff } R(\alpha^{-1}) = 0. \text{ Hence } R(x) = R(0) x^{\deg R(x^{-1})}. \]

Since $R(0) = -a_s$, we deduce

**THEOREM 6.5.2:** Let $R$ be defined by Eq. 6.3.10. If the rosette is pulse-stable, then

$$a_s = \pm 1$$
$$a_i = (-a_s) a_{s-i}, \ 1 \leq i \leq s-1.$$  

This result may be used to show many rosettes are not pulse-stable.

We now discuss algorithms for determining if $R$ has PCP. Given $R$ by Eq. (6.3.10), consider the pulse function $p(t)$ defined by Eqs. (6.2.3) and (6.2.4). Because $p(t)$ only takes integer values, $R$ has PCP if and only if $p(t)$ is periodic. If $R = \prod_{i=1}^{m} n_i$ then $p$ has period $\text{lcm}(n_1, \ldots, n_m)$. Let

$$W(n) = \max \ \text{lcm}(n_1, \ldots, n_m), \ \text{taken over } n_1, \ldots, n_m$$

satisfying $\sum_{i=1}^{m} \phi(n_i) = n$. Table 1 gives $W(n)$ for $n \leq 20$. The odd values are given by $W(2k + 1) = W(2k)$. 

Table 1

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>W(n)</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>60</td>
<td>168</td>
<td>360</td>
<td>840</td>
<td>1360</td>
<td>2720</td>
</tr>
</tbody>
</table>

**THEOREM 6.5.3:** \( R \) has PCP if and only if \( p \) has period \( \leq W(\partial R) \).

This theorem gives a necessary and sufficient condition for pulse stability. Given a rosette with largest cycle of size \( s \) the pulse process is stable if and only if it is periodic and the pulse process is periodic if and only if it has period \( \leq W(s) \). Thus it is relatively simple, by computer, to check if a polynomial \( R \), \( \partial R \leq 20 \) (pulse process, \( s \leq 20 \)) has PCP (respectively, is stable). For large \( n \),

\[
(6.5.4) \quad W(n) > e^{\sqrt{n \log n}}
\]

so the above method rapidly becomes impractical.

For larger \( \partial R \) we may program an algorithm to calculate \((A(x), B(x))\) for any polynomials \( A, B \). \((x, y)\) is defined as \( \gcd(x, y) \). This may be done by Euclid's algorithm in \((\partial A) \cdot (\partial B)\) steps.*

---

*All calculations of number of steps are, of course, subject to multiplication by a constant factor which depends on the structure of the computer used.
We first check if

\[(6.5.5) \quad \gcd(R(x), R'(x)) = 1.\]

If not, \(R\) has repeated roots and hence cannot have PCP.
For the remainder of this section we assume Eq. (6.5.5).
Now let \(n\) run through those integers \(m, \phi(m) \leq \partial R\).
For these \(n\) set

\[(6.5.6) \quad R_n = (R, x^n - 1)\]

and reset

\[(6.5.7) \quad R = R/R_n.\]

Then the original \(R\) has PCP iff the final \(R = \pm 1\). This algorithm takes \(s^3\) steps, where \(s = \partial R\).

A more sophisticated algorithm may be used when \(\partial R\)
becomes even larger (say \(\partial R > 1000\)). For any polynomial
\(R\) (with no multiple roots) set

\[(6.5.8) \quad R^{(p)}(x) = (R(x), R(x^p))\]

and set
(6.5.9) \[ R_p(x) = R(x)/R^{(p)}(x). \]

Then

(6.5.10) \[ R_p(a) = 0 \iff R(a) = 0 \text{ and } R(a^p) \neq 0. \]

Let \( p_i \) denote the \( i \)th prime, and let \( k \) be the minimal integer such that \( \prod_{i=2}^k (p_i - 1) > s. \)

**THEOREM 6.5.4:** \( R \) has PCP iff

(6.5.11) \[ \delta(R_{p_2 p_3 \ldots p_k}) = 0 \]

where by \( R_{p_2 \ldots p_k} \) we mean \( \cdot \cdot \cdot ((R_{p_2 p_3}) \ldots p_k). \)

**Proof:** Fix \( R \), let \( A \) be the set of roots of \( R \), and let \( A_i(1 \leq i \leq k - 1) \) be the roots of \( R_{p_2 \ldots p_{i+1}}. \) Then

\[ A_i = \{ \alpha : \alpha \in A_{i-1}, \alpha^{p_{i+1}} \not\in A_{i-1} \}. \]

We need show \( R \) has PCP iff \( A_{k-1} = \emptyset. \)

Assume \( R \) does not have PCP. Fix \( \alpha \in A \), \( |\alpha| \) maximal. By Theorems 6.4.1 and 6.5.1, \( |\alpha| > 1. \) Then \( |\alpha^{p_{i+1}}| = |\alpha|^{p_{i+1}} > |\alpha|, \) so \( \alpha^{p_{i+1}} \not\in A. \) By induction, \( \alpha \in A_j, \) some \( j \), so \( \alpha \in A_{k-1}. \)
Assume $R$ does have PCP. Fix $\alpha \in A$ and let $j$ be the minimal integer such that $\alpha^j = 1$. Then $\phi_j|_R$ so $\partial_R > \phi(j)$. Hence there exists $i$, $2 \leq i \leq k$, $p_i \not| j$. Then $\alpha$ and $\alpha^{p_i}$ are conjugates. Hence $\alpha \in A_{i-2}$ iff $\alpha^{p_i} \in A_{i-2}$, thus $\alpha \not\in A_{i-1}$, $\alpha \not\in A_{k-1}$. Since $A_{k-1} \subseteq A$, $A_{k-1} = \emptyset$. Q.E.D.

The above method gives an algorithm for checking if $R$ has PCP in $C s^2 (\log s)^2 / (\log \log s)$ steps, where $s = \partial_R$. 
6.6. Value Stability

We turn now to determination of value stability.
If a rosette is not pulse-stable, it is not value-stable.
We restrict our attention to those rosettes which are pulse-stable. Then

\[(6.6.1) \quad p(t) = \sum_{i=1}^{s} \xi_i \alpha_i^t\]

where the \( \alpha_i \) are roots of unity. There exists \( N \) such that
\( \alpha_i^N = 1 \) for \( 1 \leq i \leq N \). For any \( a \),

\[(6.6.2) \quad \sum_{t=a}^{a+N-1} p(t) = \sum_{i=1}^{s} \xi_i \alpha_i^a \sum_{j=0}^{N-1} \alpha_i^j.\]

Since \( \alpha_i^N = 1 \),

\[(6.6.3) \quad \sum_{j=0}^{N-1} \alpha_i^j = \begin{cases} 0 & \text{if } \alpha_i \neq 1 \\ N & \text{if } \alpha_i = 1. \end{cases}\]

If all \( \alpha_i \neq 1 \), \( v(a + N - 1) = v(a) \) for all \( a \) so the rosette is value-stable. If some \( \alpha_i = 1 \), say \( i = 1 \), \( v(a + N - 1) = v(a) + \xi_1 N \). Hence \( v(a + kN - 1) = v(a) + k\xi_1 N \), \( \xi_1 \neq 0 \), so the rosette is not value stable. Note that \( \alpha = 1 \) is a root of the inverted rosette polynomial iff the number of +cycles is one more than the number of -cycles. To restate:
THEOREM 6.6.1: A pulse-stable rosette is value-stable iff the number of +cycles is not exactly one more than the number of -cycles.
6.7. Adding External Pulses

Fix a rosette with rosette sequence \((a_1, \ldots, a_s)\).
(Throughout this section we refer to a rosette by its rosette sequence.) Let us activate a pulse process by adding pulses of size \(b_t\) at time \(t\). Then the pulse sequence satisfies

\[
(6.7.1) \quad p(t) = \sum_{i=1}^{s} a_i p(t-i) + b_t.
\]

Call

\[
(6.7.2) \quad I(x) = \sum_i b_i x^i
\]

the \textit{initial pulse}. Then the generating function is

\[
(6.7.3) \quad \sum p(t) x^t = I(x)/Q(x),
\]

where \(Q(x)\) is the rosette polynomial. We shall be concerned with finite initial pulses, i.e., \(b_i = 0\) for \(i\) sufficiently large. First assume \((I(x), Q(x)) = 1\). We factor \(Q\) by Eq. (6.3.5) and Eq. (6.3.6) becomes

\[
(6.7.4) \quad G(x) = \frac{I(x)}{Q(x)} = \sum_{i=1}^{u} \frac{P_i(x)}{(1-a_i x) d_i}.
\]

The derivations in Sections 6.3 to 6.5 hold since
1 - \alpha_i x \not\in P_i(x). \text{ Hence this pulse function has the same instability or stability as with the simple processes.}

Now let I(x), Q(x) be any two polynomials. We set \[ I_1(x) = I(x)/\gcd(I(x), Q(x)), \quad Q_1(x) = Q(x)/\gcd(I(x), Q(x)). \]
Then \( \gcd(I_1(x), Q_1(x)) = 1 \) and \( p(t) \) is given by

\[ (6.7.5) \quad \sum p(t)x^t = I_1(x)/Q_1(x). \]

By the previous paragraph: \textbf{The stability of the pulse process depends only on the function } \( Q_1(x) \).

Example: Rosette = (1,2), \( Q(x) = 1 - x - 2x^2 = (1+x)(1-2x) \).
In the simple process \( p(t) = [2^{t+1} + (-1)^t]/3 \). We may cancel the \( 1-2x \) factor with an initial pulse \( I(x) = 1-2x \).
That is, we introduce a pulse at \( t = 0 \) and two negative pulses at \( t = 1 \). Then \( p(0) = 1 \), \( p(1) = 1 \) (from the one cycle) - 2(added) = -1. Then the pulse process oscillates, \( p(t) = (-1)^t \).
Note that if \( I(x) = Q(x), I(x)/Q(x) = 1 \) so all pulses cancel after \( t = 0 \). In general: To make a pulse process stable it is necessary and sufficient that \( Q(x)/\gcd(I(x), Q(x)) \) be a PCP.

Example: Rosette = (-2, -1), \( Q(x) = 1 + 2x + x^2 = (1+x)^2 \).
With \( I(x) = 1 \) (simple process) \( p(t) = (-1)^t(t+1) \). By adding a second positive pulse at \( t = 1 \), \( I(x) = 1 + x \), \( G(x) = (1+x)^{-1} \) so \( p(t) = (-1)^t \).
Example: Rosette = (5, -6), \( Q(x) = 1 - 5x + 6x^2 = (1-2x)(1-3x) \). With \( I(x) = 1 \) (simple process) \( p(t) = 3^{t+1} - 2^{t+1} \), unstable. For the pulse process to be stable we need \( I(x) | Q(x) \). For example, if \( I(x) = Q(x) \), then \( p(t) = 0 \), \( t > 0 \), \( p(1) = 1 \). We can reduce the size of \( p \) by setting \( I(x) = 1 - 3x \) (adding three negative pulses at \( t = 1 \)). Then \( G(x) = (1-2x)^{-1} \) and \( p(t) = 2^t \).
7. REAL-WEIGHTED DIGRAPHS

7.1. A Natural Generalization

If the nodes of a digraph denote different quantities, such as air pollution, gasoline consumption, population in Los Angeles, and so on, then the edges and signs may be taken to denote whether an increase in node A results in an increase or decrease at node B. The pulse process in a signed digraph assumes that increases and decreases are all in unit steps, and that if a unit pulse at node A has any direct effect at B at all, it is either to increase or decrease B by exactly one unit. Clearly, a slightly more general model may give a significantly closer approximation to the behavior of the real world. If we assume a unit change at A induces a change r at B one time unit later, then we may think of the real number r being attached to the edge (A,B) instead of simply +1. If we rewrite equation (*) on page 4 by replacing \( \text{sgn}(x_j, x_i) \) by a real-valued function \( f(x_j, x_i) \), then we have a definition of a pulse process for real-weighted digraphs.* The definitions for value stability, pulse stability, and so on are unchanged. But to what extent

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*The mathematical objects called "real-weighted digraphs" in this Report were called "weighted signed digraphs" in R-756-NSF [9].
does the theory developed in the preceding pages depend on the assumption that the digraphs have only +1 instead of arbitrary real numbers attached to their edges?

7.2. The Classical Approach

For a real-weighted digraph, the object corresponding to the transpose of the signed adjacency matrix $S$ of a signed digraph is simply a real matrix. Such a matrix can be reduced to a Jordan canonical form, and the general discussion of Section 3 applies unchanged. The pulses at any node will "eventually" satisfy a linear recursion sequence (Theorem 4.1), and in the absence of cycles all eigenvalues must be zero (Theorem 4.2). Theorem 4.3 depends critically on all coefficients of the characteristic equation being integers; this is not true for all real-weighted digraphs. The rest of the theorems in Section 4 are true for real-weighted digraphs as well as for signed ones. The discussion of Section 5 ("Condensation and Characteristic Polynomials") is just as applicable to real-weighted digraphs as it is to signed digraphs.

7.3. Real-Weighted Rosettes

If the edges of a rosette have real numbers attached to them, then the coefficients $a_i$ of Eq. (6.2.3) may be any real number ($a_s \neq 0$). The results of Sections 6.2 ("The Linear Recursion Sequence for a Rosette"), 6.3 ("Generating Polynomials"), and 6.4 ("Pulse Stability") still hold with
the exception of Theorem 6.4.2. Section 6.5 ("Cyclotomic Polynomials") clearly depends in an essential way on having only $+1$'s assigned to the edges of the rosette, and so it does not apply to real-weighted rosettes. Results analogous to those of Section 6.6 ("Value Stability") still hold, although the proof must be adjusted somewhat. Section 6.7 ("Adding External Pulses") still applies.
REFERENCES


ADDENDUM TO R-926-NSF

by

T. A. Brown and F. S. Roberts*

The idea of studying energy demand and related environmental problems by means of signed digraphs was introduced in Roberts [2] and carried on in Brown, Roberts, and Spencer [1] and Roberts [3,4]. In this note, we state extensions of the theoretical results of [1] and [4]. The new theorems give criteria for pulse and value stability even when some eigenvalues have multiplicity greater than one, thus going significantly beyond the earlier results.

Theorem 4.4 of [1] states that if a signed digraph is pulse stable under all simple pulse processes, then every nonzero eigenvalue has magnitude less than or equal to unity. In fact, it follows that every nonzero eigenvalue has magnitude equal to unity.

Theorem 1. If a signed digraph is pulse stable under all simple pulse processes, then every nonzero eigenvalue has magnitude equal to unity.

Proof. By Theorem 4.4 of [1], each nonzero eigenvalue has magnitude at most unity. However, using the notation of the proof of Theorem 4.3 of [1], one sees that the product of all the nonzero eigenvalues is \((?)\) times the coefficient \(a_i\). Since \(a_i\) is an integer, each eigenvalue must have magnitude unity. Q.E.D.

*Dr. Roberts' work was partially supported by NSF Grant Number NSF-GI-34895 to Rutgers University.
Theorem 2. If a signed digraph is pulse stable under all simple pulse processes, then every nonzero eigenvalue is a root of unity.

Proof. Let \( P(\lambda) \) be the characteristic polynomial of the signed digraph divided by the highest power of \( \lambda \) which divides the characteristic polynomial. By Kronecker's Theorem (see Sec. 6.5 of [1]), and by Theorem 1, every root of \( P(\lambda) \) is a root of unity. Q.E.D.

According to Theorem 4.5 of [1], the converse of Theorem 2 is true provided all nonzero eigenvalues are distinct. Actually, Theorem 4.5 holds under a more general hypothesis. To introduce this hypothesis, let us suppose that \( S \) is the signed adjacency matrix of the signed digraph and \( J \) is its Jordan Canonical Form. We say \( J \) is chain-free if there are no nonzero diagonal entries in \( J \) linked by off-diagonal 1's. Put another way, \( J \) is chain-free if and only if every nonzero off-diagonal entry occurs in a row and a column in which all other entries are 0. If \( J \) is chain-free, then there are no nontrivial Jordan chains. The proof of Theorem 4.5 of [1] applies almost verbatim if the hypothesis that the nonzero eigenvalues are distinct is replaced by the hypothesis that \( J \) is chain-free. Indeed, the only change necessary is that the first equation should read

\[
S^t P(0) = \sum_{i=1}^{n} \alpha_i \lambda_i^t U_i, \quad \text{all } t \geq n.
\]
(This equation should replace the first equation even in [1].)
We formalize the results in the following theorem.

**Theorem 3.** Suppose $S$ is the transpose of the signed adjacency matrix of a signed digraph and $J$ is the Jordan Canonical Form corresponding to $S$. If $J$ is chain-free and $S$ has no eigenvalue of magnitude greater than unity, then the signed digraph is pulse stable under all autonomous pulse processes.

The converse of Theorem 3 is also true. To prove it we first need the following lemma.

**Lemma.** Let $S$ be the transpose of the signed adjacency matrix of a signed digraph and let $J$ be the Jordan Canonical Form corresponding to $S$. Then some entry of $\{S^t\}$ becomes unbounded if and only if some entry of $\{J^t\}$ becomes unbounded.

**Proof.** A similarity transformation is bicontinuous under the metric imposed by the usual norm on matrices, $||S|| = \max |s_{ij}|$. Thus, $\{S^t\}$ is contained in a sphere iff $\{J^t\}$ is contained in a sphere. Q.E.D.

**Theorem 4.** Suppose $S$ is the transpose of the signed adjacency matrix of a signed digraph which is pulse stable under all simple pulse processes. Then the Jordan Canonical Form $J$

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*Our thanks to Garrett Birkhoff for suggesting this lemma.

**Our thanks to Garrett Birkhoff for suggesting the proof of this theorem.
corresponding to $S$ is chain-free.

Proof. By the lemma, each entry of $\{J^t\}$ must remain bounded. Suppose $\lambda$ is a nonzero eigenvalue of $S$ which is linked by off-diagonal 1's in $J$. Now $J$ has the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Thus, it is easy to prove by induction that $J^t$ has the form

$$J^t = \begin{pmatrix} \lambda^t & t\lambda^{t-1} \\ 0 & \lambda^t \end{pmatrix}$$

We know by Theorem 2 that $|\lambda| = 1$. So the term $t\lambda^{t-1}$ gets larger and larger in magnitude as $t$ approaches $\infty$. Q.E.D.

To summarize, we have the following result characterizing pulse stability.
Theorem 5. Suppose $S$ is the transpose of the signed adjacency matrix of a signed digraph and $J$ is the Jordan Canonical Form corresponding to $S$. Then the following are equivalent:

1. The signed digraph is pulse stable under all autonomous pulse processes.
2. The signed digraph is pulse stable under all simple pulse processes.
3. Every eigenvalue has magnitude less than or equal to unity and $J$ is chain-free.
4. Every nonzero eigenvalue of $S$ is a root of unity and $J$ is chain-free.

Proof. Clearly, (1) $\rightarrow$ (2). That (2) $\rightarrow$ (3) follows by Theorem 4 and Theorem 4.4 of [1]. That (3) $\rightarrow$ (4) follows by Theorem 2. Finally, (4) $\rightarrow$ (1) by Theorem 3. Q.E.D.

This takes care of pulse stability. To characterize value stability, we shall prove the following theorem.

Theorem 6. A signed digraph is value stable under all autonomous pulse processes if and only if it is pulse stable under all autonomous pulse processes and unity is not an eigenvalue.

Proof. This theorem is proved in Roberts ([4], Appendix C), under the hypothesis that all nonzero eigenvalues are distinct. The same proof applies under the hypothesis that $J$ is chain-free, provided the first equation of the proof is asserted only for $t \geq n$. In that proof, as here, the $U_i$ are not all
eigenvectors, but are either eigenvectors or elements in
a Jordan chain leading to a zero eigenvalue. Finally, the
hypothesis that J is chain-free follows from Theorem 4. Q.E.D.

Note that Theorems 1 and 2 and hence 4, 5, and 6 hold
for integer-weighted digraphs. Theorem 3 holds for arbitrary
real-weighted digraphs.

Before closing, we pose the following question: what is
the graph-theoretical condition corresponding to J being
chain-free?
References


