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MATHEMATICAL THEORY OF ZERO-SUM
TWO-PERSON GAMES WITH A FINITE NUMBER
OR A CONTINUUM OF STRATEGIES

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The RAND *Corporation*

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MATHEMATICAL THEORY OF ZERO-SUM TWO-PERSON GAMES WITH A FINITE NUMBER OR A CONTINUUM OF STRATEGIES

INTRODUCTION

The mathematical theory of games of strategy has provided a new approach to the theory of competitive economic behavior. Many economic questions can be reduced mathematically to suitable games of strategy which may be played in optimal ways. Basic in this new approach is the recognition of the existence of opposing interests, each controlling only part of the variables. One group attempts to maximize a certain utility and the opposition attempts to minimize the same utility where control of only part of the variables is available to each group.

In the study of problems of national security and warfare, the competition of opposing forces is similar to a game of strategy between opposing players. Analogous to a game of strategy, national security depends not merely on our own actions but on the actions of our opponents as well. We have a system in which our objective is to maximize some function, such as defense or war power, of which we do not control all the variables. Simultaneously, the enemy aims to minimize this same function and he likewise does not control all the variables.

The problem of distributing the Armed Services budget among operations, procurement, and research is a major question of national security and might be considered a game of strategy in which the annual divisions of the budget money into the three categories are considered the successive moves against the enemy. These moves and successive moves of the enemy determine the probability of war. The tactical distribution of the field forces and industry is a part of the major problem.

Among tactical problems that can be solved by game theory methods is the area defense problem. In its simplest form it might be formulated as follows: Suppose a bomber force of a fixed strength is sent against two valuable areas. These areas

in turn are to be defended by a fixed total number of interceptors with no support between interceptor bases. What are the optimal ways for the offense to divide the bombers and for the defense to divide the interceptors between the two areas?

The mathematical theory of games with a finite number of strategies was developed by von Neumann in several successive stages in 1928 and 1941. The theory was extended to games with a continuum of strategies by Ville and Wald. More recently, several members of Project RAND have obtained further results in the theory of zero-sum two-person games. These results have not been published in the mathematical journals and are reproduced in these pages.

The contents of this report are drawn from published works and from Project RAND internal memoranda. The report presents a summary of zero-sum two-person games with a finite number of strategies as developed by von Neumann. However, Ville's proof of the fundamental theorem is given rather than the original proof of von Neumann. This is followed by the unpublished results on games with a finite number of strategies obtained by M.A. Girshick, O. Helmer, L.S. Shapley and R.N. Snow. Among the results on games with a continuum of strategies are those of H. Bohnenblust, M. Dresher, T.E. Harris and J.C.C. McKinsey.

Although several examples and solutions of games have been included, they were selected in order to demonstrate the techniques involved rather than to solve practical problems related to national security. It is hoped that the theory of mathematical games of strategy is now sufficiently developed to justify an attempt at application to major problems of national security. Efforts in this direction are now being undertaken.

I. GENERAL DESCRIPTION OF GAMES OF STRATEGY

In order to describe clearly the concept of a game, it will be necessary to clarify certain terminology associated with games. Everyday usage of some of these terms is frequently ambiguous. The terms will be used here as defined by von Neumann and Morgenstern in their "Theory of Games and Economic Behavior."

A *game* is the set of rules which describes it. These rules specify clearly what each person, called a *player*, is allowed to do under all possible circumstances. The rules stipulate, in particular, the time at which the game ends and the amount each player then pays or receives. If the game requires the use of chance devices the rules describe how the chance events shall be interpreted. A game of poker, for example, is the set of rules which describes the deals, draws, betting, etc.

Every instance in which a particular game is played, from beginning to end, is referred to as a *play* of that game. In the game of tennis a play is known as a set. In bridge a play is called a rubber.

A game is composed of a sequence of moves, where a move is the choice of one among several alternatives. This choice may be made by one of the players or by some chance device described by the rules of the game. Thus the first move in a game of chess is a chance move to determine which player shall be the first to move a chessman.

In the actual play of a game, each player instead of making his decision at each move may formulate a plan in advance to cover all possible contingencies. Such a plan is called a *strategy*. The plan will make use of the information available to the player in accordance with the rules of the game. No freedom of action is lost through the use of a strategy since the strategy specifies the choice as a function of the information available. A play of the game then consists in the choice of some strategy by each player without knowledge of that chosen by any other player. The strategies can be represented by numbers; and hence the totality of possible ways of playing a game, by a matrix.

Games can be classified according to the sum of all payments received by all players at the end of the game. If this sum is always zero — i.e., the players pay only to one other — the game is called a *zero-sum* game. In such a game there is no

production or destruction of wealth. Parlor games are examples of zero-sum games. If the sum of the payments is not zero, the game is referred to as a *non-zero-sum* game.

Games can also be classified according to the number of players as 2-person, 3-person, etc. If a game consists of more than two players, the players may form coalitions. A general n -person game can be treated as a new 2-person game, by considering a coalition and its complement as the two opposing players. Further, the n -person non-zero-sum game can be reduced to an $(n + 1)$ -person zero-sum game by the introduction of a so-called fictitious player.

This report will consider only 2-person zero-sum games.

II. GAMES WITH A FINITE NUMBER OF STRATEGIES

1. GENERAL DESCRIPTION

Consider a zero-sum game in which each of two players, A and B , chooses a number, each choice being made without any knowledge of the other. Let the numbers $1, 2, \dots, m$ and $1, 2, \dots, n$ represent the possible choices of A and B respectively. Let a_j^i represent the payment to A if A selects i and B selects j . Since the game is a zero-sum game, the payment to B will be $-a_j^i$. In playing this game, A wishes to maximize a_j^i , but he controls only the choice of i , while B wishes to minimize a_j^i , but he controls only the choice of j .

We can represent the game by a rectangular matrix of the payments with m rows (indicated by superscripts) and n columns (indicated by subscripts):

$$\begin{pmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^m & a_2^m & \dots & a_n^m \end{pmatrix}$$

If A chooses some i , then B may choose j so that a_j^i will be as small as possible, hence A will receive at least $\min_j a_j^i$. Since A can select i he can choose it so that he will receive at least $\max_i \min_j a_j^i$.

Similarly, if B chooses some j , then A may choose i so that the payment to B will be as small as possible, that is, $\min_i (-a_j^i)$. Further, B can choose j so that he receives at least $\max_j \min_i (-a_j^i)$, hence the payment to A is at most $-\max_j \min_i (-a_j^i)$, which is identical with $\min_j \max_i a_j^i$.

THEOREM 2.1. For every matrix $M = \| a_j^i \|$, $\max_i \min_j a_j^i \leq \min_j \max_i a_j^i$.

PROOF. Given any i , then

$$\min_j a_j^i \leq a_{j'}^i \text{ for all } j',$$

and given any j , then

$$a_j^{i'} \leq \max_i a_j^i \text{ for all } i'.$$

Hence,

$$\min_j a_j^{i'} \leq \max_i a_j^{i'}.$$

From the above, the theorem follows.

The theorem can be stated more generally as follows: For any function of two variables, $\phi(x, y)$,

$$\max_x \min_y \phi(x, y) \leq \min_y \max_x \phi(x, y).$$

2. SADDLE POINTS

If the matrix $M = \| a_j^i \|$ is such that $\max_i \min_j a_j^i = \min_j \max_i a_j^i = v$, then player A can choose some strategy i' so that he receives at least v , and player B can choose some strategy j' so that player A receives at most v . If there exists an (i', j') such that $\min_j a_j^{i'} = \max_i a_j^{i'}$, we say that the matrix of the game has a *saddle point* at (i', j') and i', j' are called *optimum strategies*. For example, the matrix

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 7 & 9 \\ -1 & 6 & 5 \end{pmatrix}$$

has a saddle point at $(2, 1)$ and its value is 4.

For a saddle point to exist it is necessary that there exist an element of the matrix which is at the same time the maximum of its row and the minimum of its column. The matrix may have several saddle points, but all will have the same value; for suppose (i', j') and (i'', j'') are two saddle points, then

$$a_{j'}^{i'} = \max_i a_{j'}^i \geq a_{j'}^{i''} \geq \min_j a_j^{i''} = a_j^{i''}.$$

or

$$a_{j'}^{i'} \geq a_j^{i''}.$$

In a similar way it can be shown that

$$a_j^{i''} \geq a_{j'}^{i'}.$$

Hence

$$a_{j'}^{i'} = a_j^{i''}.$$

3. MIXED STRATEGIES

There exist games whose matrices have no saddle points. For example, the game of "matching coins", whose matrix is

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

has no saddle point, since $\max_i \min_j a_j^i = -1$ and $\min_j \max_i a_j^i = +1$.

However, there exists some optimum way to play this game provided the players are not restricted to choosing a definite strategy but may choose several strategies with definite probabilities. In effect, each player is allowed to play all available strategies but must choose only the relative frequencies with which to play the respective strategies. These frequencies, with which the strategies are played, form a vector, which is called a *mixed strategy*.

Consider such a game in which A and B , instead of playing always the same strategies, play all strategies with certain probabilities chosen in advance. Let A play strategies $1, 2, \dots, m$ with probabilities $\xi_1, \xi_2, \dots, \xi_m$, where $\sum_{i=1}^m \xi_i = 1$ and let B play strategies $1, 2, \dots, n$ with probabilities $\eta_1, \eta_2, \dots, \eta_n$ for which $\sum_{j=1}^n \eta_j = 1$. The vectors $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ and $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ are the mixed strategies of the two

players. The mathematical expectation of A is

$$\phi(\xi, \eta) = \sum_{j=1}^n \sum_{i=1}^m a_j^i \xi_i \eta_j.$$

It is evident that for each choice of ξ , A can expect to win at least $\min_{\eta} \phi(\xi, \eta)$. Suppose A chooses $\xi^{(0)}$ so that

$$\min_{\eta} \phi(\xi^{(0)}, \eta) = \max_{\xi} \min_{\eta} \phi(\xi, \eta).$$

Then regardless of B 's choice, A can expect to win at least $\max_{\xi} \min_{\eta} \phi(\xi, \eta)$. Similarly, B can choose a mixed strategy $\eta^{(0)}$ so as to win at least

$$\min_{\xi} -\phi(\xi, \eta^{(0)}) = \max_{\eta} \min_{\xi} -\phi(\xi, \eta).$$

But

$$\max_{\eta} \min_{\xi} -\phi(\xi, \eta) = -\min_{\eta} \max_{\xi} \phi(\xi, \eta);$$

thus by choosing a proper strategy, B can expect to win at least $-\min_{\eta} \max_{\xi} \phi(\xi, \eta)$, or A can expect to win at most $\min_{\eta} \max_{\xi} \phi(\xi, \eta)$. Thus, if both A and B choose their optimum strategies, the expected gain of A will lie between $\min_{\eta} \max_{\xi} \phi(\xi, \eta)$ and $\max_{\xi} \min_{\eta} \phi(\xi, \eta)$.

The fundamental theorem of the theory of games states that $\max_{\xi} \min_{\eta} \phi(\xi, \eta) = \min_{\eta} \max_{\xi} \phi(\xi, \eta)$. It is frequently referred to as the minimax theorem and was first proved by von Neumann. We shall use Ville's method to prove this theorem.

Let S_r be the set of r -dimensional vectors $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ for which $\alpha_i \geq 0$ and $\sum_{i=1}^r \alpha_i = 1$. S_r is a closed, convex polyhedron having unit vectors for vertices.

THEOREM 2.2. Let $A_1, A_2, \dots, A_m$ be m arbitrary numbers, then

$$\max_{\alpha \in S_m} \sum_{i=1}^m A_i \alpha_i = \max_i A_i.$$

PROOF. For all i ,

$$A_i \leq \max_i A_i,$$

so that

$$A_i \alpha_i \leq \alpha_i \max_i A_i.$$

Summing, we get

$$\sum_{i=1}^n A_i \alpha_i \leq \max_i A_i \sum \alpha_i = \max_i A_i$$

and hence

$$(1) \quad \max_{\alpha \in S_m} \sum_{i=1}^n A_i \alpha_i \leq \max_i A_i.$$

Let k be defined by $A_k = \max_i A_i$ and consider the vector

$$\beta = (\beta_1, \beta_2, \dots, \beta_m) \text{ where } \beta_i = 0 \text{ if } i \neq k, \beta_k = 1.$$

Then

$$(2) \quad \max_{\alpha} \sum_{i=1}^n A_i \alpha_i \geq \sum_{i=1}^n A_i \beta_i = A_k = \max_i A_i$$

The theorem follows from (1) and (2).

We are now in a position to prove the main theorem of the theory of games.

THEOREM 2.3. *If $\phi(\xi, \eta)$ is the expectation of the first player if he uses mixed strategies ξ and the second player uses mixed strategies η , then*

$$\max_{\xi \in S_n} \min_{\eta \in S_m} \phi(\xi, \eta) = \min_{\eta \in S_m} \max_{\xi \in S_n} \phi(\xi, \eta).$$

PROOF. Let

$$H_i(\eta) = \sum_{j=1}^n a_j^i \eta_j \text{ for } i = 1, 2, \dots, m,$$

which represents the expectation of the first player if he plays a pure strategy i and the second player uses a mixed strategy η .

Then

$$(3) \quad \phi(\xi, \eta) = \sum_{i=1}^m H_i(\eta) \xi_i$$

and from Theorem 2.2

$$(4) \quad \max_{\xi} \phi(\xi, \eta) = \max_i H_i(\eta).$$

Let

$$t = \min_{\eta} \max_{\xi} \phi(\xi, \eta).$$

Then for any η in S_n

$$\max_{\xi} \phi(\xi, \eta) \geq t,$$

and therefore from (4)

$$(5) \quad \max_i H_i(\eta) \geq t.$$

Hence for every η in S_n , there exists some i such that

$$H_i(\eta) \geq t = t(\eta_1 + \eta_2 + \dots + \eta_n) = t\eta_1 + t\eta_2 + \dots + t\eta_n.$$

We thus have a set of homogeneous linear forms $H_1, H_2, \dots, H_m$, t in the n variables $\eta_1, \eta_2, \dots, \eta_n$ such that whenever $\eta \in S_n$, there exists an i for which $H_i(\eta) \geq t$. These are the hypotheses of Corollary 2 to the theorem proven in the Appendix. Hence there exists some $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in S_m$ such that

$$(6) \quad \sum_{i=1}^m H_i(\eta) \xi_i \geq t$$

for all η in S_n . Using (3) and (6) it follows that for every $\eta \in S_n$ there exists a $\xi \in S_m$ such that

$$(7) \quad \phi(\xi, \eta) \geq t = \min_{\eta} \max_{\xi} \phi(\xi, \eta).$$

In particular, there exists some $\xi \in S_m$ such that

$$(8) \quad \min_{\eta} \phi(\xi, \eta) \geq t.$$

In general, we have

$$(9) \quad \max_{\xi} \min_{\eta} \phi(\xi, \eta) \leq \min_{\eta} \max_{\xi} \phi(\xi, \eta) \geq t.$$

Hence, from (9) and (7),

$$(10) \quad \max_{\xi} \min_{\eta} \phi(\xi, \eta) \geq \min_{\eta} \max_{\xi} \phi(\xi, \eta).$$

But, as was shown in Section 1,

$$(11) \quad \max_{\xi} \min_{\eta} \phi(\xi, \eta) \leq \min_{\eta} \max_{\xi} \phi(\xi, \eta).$$

Our theorem follows from (10) and (11).

A pair of vectors (mixed strategies) x, y is a solution if

$$(12) \quad \max_{\xi \in S_m} \phi(\xi, y) = \min_{\eta \in S_n} \phi(x, \eta) = \phi(x, y).$$

In terms of the original matrix M , (12) is equivalent to

$$(13) \quad \max_j a^i \cdot y = \min_j a_j \cdot x = \phi(x, y)$$

where a^i and a_j are row and column vectors obtained from M . We shall call $\phi(x, y) = v$ the value of the game to the first player. It is obvious that for every solution x, y , $\phi(x, y)$ must have the same value. If a constant, b , is added to each element of M , it is apparent that the new game has the same set of solutions as the original game but the value of the game is changed to $v+b$. If there exists a pair of vectors x, y such that

$$(14) \quad a^i \cdot y = a_j \cdot x = \phi(x, y) = v$$

for all i and j , then the pair x, y is a simple solution.

A mixed strategy ξ or η is called optimal if it belongs to some solution. From (12) it follows that any pair of optimal mixed strategies is a solution. Let $X \subseteq S_m$ and $Y \subseteq S_n$ be the sets of optimal mixed strategies for the two players; they are bounded since S_r has a diameter $\sqrt{2}$. By theorem 2.3, X and Y are non-null.

THEOREM 2.4. *The sets of optimal mixed strategies, X and Y , are convex and closed.*

PROOF. Consider the vector defined by $\xi^{(0)} = \sum_{i=1}^N a_i x^{(i)}$ where $a_i \in S_n$, and $x^{(i)} \in X \subseteq S_m$. We have to show that $\xi^{(0)} \in X$. Now S_m is convex and so $\xi^{(0)} \in S_m$. Let $y \in Y$, then

$$(15) \quad \max_{\xi} \phi(\xi, y) = v.$$

Using the fact that ϕ is a linear function of ξ and η , we have

$$(16) \quad \min_{\eta} \phi(\xi^{(0)}, \eta) \leq \phi(\xi^{(0)}, y) = \sum a_i \phi(x^{(i)}, y) = v.$$

Further,

$$(17) \quad \min_{\eta} \phi(\xi^{(0)}, \eta) = \min_{\eta} \sum a_i \phi(x^{(i)}, \eta) \geq \sum a_i \min_{\eta} \phi(x^{(i)}, \eta) = v.$$

From (15), (16) and (17) it follows that

$$\max_{\xi} \phi(\xi, y) = \min_{\eta} \phi(\xi^{(0)}, \eta) = v,$$

and so $(\xi^{(0)}, y)$ is a solution, or $\xi^{(0)} \in X$. A similar argument will demonstrate that Y is convex. The above also shows that if more than one solution exists, then infinitely many must exist.

To show that X is closed, let

$$\xi^{(0)} = \lim_{i \rightarrow \infty} \xi^{(i)} \text{ where } \xi^{(i)} \in X \subseteq S_{\mathbf{m}}.$$

Now $S_{\mathbf{m}}$ is closed and so $\xi^{(0)} \in S_{\mathbf{m}}$. Let $y \in Y$, then

$$(18) \quad \max_{\xi} \phi(\xi, y) = v,$$

and

$$(19) \quad \min_{\eta} \phi(\xi^{(0)}, \eta) \leq \max_{\xi} \min_{\eta} \phi(\xi, \eta) = v.$$

Since ϕ is linear and therefore continuous, we can write

$$\min_{\eta} \phi(\xi^{(0)}, \eta) = \min_{\eta} \lim_{i \rightarrow \infty} \phi(\xi^{(i)}, \eta).$$

For all i and $\eta \in S_{\mathbf{n}}$, we have

$$\phi(\xi^{(i)}, \eta) \geq v,$$

and so

$$(20) \quad \min_{\eta} \lim_{i \rightarrow \infty} \phi(\xi^{(i)}, \eta) \geq v.$$

From (18), (19) and (20)

$$\max_{\xi} (\xi, \gamma) = \min_{\eta} \phi(\xi^{(0)}, \eta) = v.$$

Therefore according to (12), $\xi^{(0)}$, γ is a solution, that is,

$$\xi^{(0)} \in X.$$

4. SOLUTIONS OF GAME

In Section 3 we showed that every zero-sum two-person game with a finite number of pure strategies has a solution. This section will deal with the determination of all solutions of such games.

Let R be a bounded, non-null, convex and closed set in n -space. Define R^* as that subset of R which contains the vector r^* if and only if there do not exist two distinct vectors r' and r'' of R such that

$$(21) \quad r^* = \frac{1}{2}(r' + r'').$$

It follows that R^* is non-null and is the smallest set that spans R in the sense of convex linear combinations. A *basic solution* is a vector pair x^*, y^* where $x^* \in X^* \subseteq X$ and $y^* \in Y^* \subseteq Y$. Thus ξ, η is a solution of M if and only if ξ and η are convex linear combinations of the sets of basic optimal mixed strategies X^* and Y^* respectively.

Define $\dot{M}$ as a submatrix of M , $\dot{x}$ and $\dot{a}_j$ ($\dot{y}$ and $\dot{a}^i$) as the contractions of the vectors x and a_j (y and a^i) obtained by deleting those components corresponding to the rows (columns) of M which are absent in $\dot{M}$. $\dot{\Sigma}$ will denote a similar contraction of Σ or $\dot{\Sigma}$. $\dot{i}$ will denote a contracted column vector with each component equal to 1. We shall denote by $\dot{i}^T$ and $\dot{x}^T$ the transpose of $\dot{i}$ and $\dot{x}$ respectively.

The fundamental theorem on solutions is the following:

THEOREM 4.1. If a pair of mixed strategies x, y is a solution of a game whose value is different from zero, then a necessary and sufficient condition that x, y be a basic solution is that there exist a non-singular submatrix $\dot{M}$ of M such that

$$(22) \quad \begin{aligned} \dot{x}^T &= \frac{\dot{1}^T \dot{M}^{-1}}{\dot{1}^T \dot{M}^{-1} \dot{1}} \\ \dot{y} &= \frac{\dot{M}^{-1} \dot{1}}{\dot{1}^T \dot{M}^{-1} \dot{1}} \\ v &= \frac{1}{\dot{1}^T \dot{M}^{-1} \dot{1}} \end{aligned}$$

PROOF. Delete from M those rows and columns that correspond to the components of x and y which are zero. Let M_1 be the resulting submatrix. Any $\dot{M}$ satisfying (22) must contain M_1 . Further, x, y is a simple solution of M , the two games having the same value. Consider the matrix M_2 which is obtained from M by deleting those rows and columns of M corresponding to the components of $H = My$ and $K = M^T x$ which are not equal to v . Then $\dot{M}$ can satisfy (22) only if $M_1 \subseteq \dot{M} \subseteq M_2$.

We shall now show that condition (22) is sufficient. Suppose x is not an element of X^* and let $\dot{M}$ satisfy (22). Then from (22)

$$(23) \quad \dot{x}^T \dot{M} = v \dot{1}^T.$$

By our definition of X^* , there exist two distinct optimal mixed strategies x' and x'' such that $x = \frac{1}{2}(x' + x'')$. Now $\dot{x} = \frac{1}{2}(\dot{x}' + \dot{x}'')$ where no non-zero components have been deleted from x' or x'' . Further, for every column $\dot{a}_j$ in $\dot{M}$, we have

$$(24) \quad \begin{aligned} \dot{x}' \cdot \dot{a}_j &= x' \cdot a_j \geq v, \\ \dot{x}'' \cdot \dot{a}_j &= x'' \cdot a_j \geq v. \end{aligned}$$

From (23) we have

$$\frac{1}{2}(\dot{x}' + \dot{x}'') \cdot \dot{a}_j = v.$$

Hence

$$(25) \quad \begin{pmatrix} \dot{x}' \\ \dot{x}'' \end{pmatrix}^T \dot{M} = \begin{pmatrix} \dot{x}' \\ \dot{x}'' \end{pmatrix}^T \dot{M} = v \dot{1}^T.$$

Now since $x' \neq x''$, this implies that $\dot{M}$ is singular. Similarly, if $y \notin Y^*$ we can show that $\dot{M}$ is singular. The existence of a non-singular submatrix thus requires that x, y be basic.

To show that the condition is necessary we shall construct a submatrix with the given property. Let $x \in X^*$ and $y \in Y^*$. Expand M_1 by adjoining certain rows and columns as follows: every row, of the same length as the rows of M_1 , contained in M_2 but not in M_1 , is considered in turn and is adjoined or not according as it is or is not linearly independent of the set consisting of the rows of M_1 and the previously adjoined rows. Similarly we adjoin columns. We shall refer to the smallest matrix containing M_1 and the adjoined rows and columns as a *kernel* of the basic solution x, y . This kernel will not necessarily be unique.

Let $\dot{M}_0$ be a kernel of x, y and assume that $\dot{M}_0$ is singular. We may assume that the rows of $\dot{M}_0$ form a dependent set. Hence there exists a constant non-zero vector $c = (c_1, c_2, \dots, c_m)$ such that

$$\sum_i c_i \dot{a}^i = \dot{0}$$

with

$$c_i = 0 \text{ when } x_i = 0.$$

Now, if $\dot{a}^i$ is a row of $\dot{M}_0$, then

$$\dot{y} \cdot \dot{a}^i = v.$$

Therefore,

$$\sum_i c_i v = \sum_i c_i \dot{y} \cdot \dot{a}^i = \dot{y} \cdot \left(\sum_i c_i \dot{a}^i \right) = \dot{y} \cdot \dot{0} = 0$$

or, since $v \neq 0$,

$$\sum_i c_i = \sum_1^m c_i = 0.$$

Now consider the vectors $x \pm \epsilon c$. Then, for ϵ sufficiently small, we have $(x \pm \epsilon c) \in S_x$. Further, by definition of a solution, $(x \pm \epsilon c) \in X$ if

$$(26) \quad \min_j (x \pm \epsilon c) \cdot a_j = v.$$

Now $c \cdot a_j = \dot{c} \cdot \dot{a}_j = 0$ for a_j in $\dot{M}_0$. Hence $c \cdot a_j = 0$ for all $\dot{a}_j$ outside $\dot{M}_0$ but in M_2 . Thus, for $\dot{a}_j$ in M_2 ,

$$(x \pm \epsilon c) \cdot a_j = x \cdot a_j \pm \epsilon c \cdot a_j = v.$$

But, for every $\dot{a}_j$ outside M_2 ,

$$x \cdot a_j > v.$$

Therefore, there exists some ϵ such that (26) holds. However, the existence in X of the vectors $x + \epsilon c$ and $x - \epsilon c$ contradicts the hypothesis that $x \in X^*$. Thus $\dot{M}_0$ cannot be singular.

Now, by construction

$$\dot{x}^T \dot{M}_0 = v \dot{l}^T, \quad \dot{M}_0 \dot{y} = v \dot{l}.$$

Since M_0 is non-singular

$$\dot{x}^T = v \dot{l}^T \dot{M}_0^{-1}, \quad \dot{y} = v \dot{M}_0^{-1} \dot{l}.$$

But since

$$v \dot{l}^T \dot{M}_0^{-1} \dot{l} = \dot{x}^T \dot{l} = 1,$$

the theorem follows.

Using the preceding theorem and some elementary properties of determinants we can prove the following corollaries:

COROLLARY 1. A necessary and sufficient condition that a solution x, y of M be basic is that there exist a square submatrix $\dot{M}$ of M such that

$$(27) \quad \dot{x}^T = \frac{\dot{l}^T \text{adj } \dot{M}}{\dot{l}^T \text{adj } \dot{M} \dot{l}},$$

$$\dot{y} = \frac{\text{adj } \dot{M} \dot{l}}{\dot{l}^T \text{adj } \dot{M} \dot{l}},$$

$$v = \frac{|\dot{M}|}{\dot{l}^T \text{adj } \dot{M} \dot{l}}.$$

COROLLARY 2. Every game has a finite number of basic solutions, and the sets X and Y of optimal mixed strategies are polyhedra.

5. METHOD OF DETERMINING ALL SOLUTIONS

As has been pointed out, every basic solution is associated with at least one square submatrix (basic kernel). Therefore, to find all basic solutions it is sufficient to consider all square submatrices in the light of Corollary 1. A systematic procedure would be to consider all $r \times r$ submatrices for $r = 1, 2, \dots, \min(m, n)$. To each $r \times r$ submatrix $\dot{M}$, apply formulas (33) and determine whether x and y belong to S_r . If not, that submatrix cannot be a kernel; if so, determine by (13) whether the extended vectors x and y belong to X and Y respectively. This will be the case if and only if x, y is a basic solution. This procedure must find every basic solution and will thus determine the set of all solutions.

6. SOLUTION OF A GAME

Consider the game defined by the matrix

$$M = \begin{pmatrix} 0 & 3 & 8 \\ 8 & 5 & 4 \end{pmatrix}.$$

Since there is no saddle point, there must exist at least one 2×2 submatrix $\dot{M}$ which has a simple solution. Of the three possible 2×2 submatrices, $\begin{pmatrix} 0 & 3 \\ 8 & 5 \end{pmatrix}$ has no simple solution, $\begin{pmatrix} 0 & 8 \\ 8 & 4 \end{pmatrix}$ has one but it is no solution of M as is seen by substituting in (13). Therefore $M = \begin{pmatrix} 3 & 8 \\ 5 & 4 \end{pmatrix}$ yields the solution. We have

$$\dot{M}^{-1} = -\frac{1}{28} \begin{pmatrix} 4 & -8 \\ -5 & 3 \end{pmatrix} = 1.$$

The solutions are

$$x = \left(\frac{1}{6}, \frac{5}{6} \right), \quad y = \left(0, \frac{2}{3}, \frac{1}{3} \right).$$

The value of the game is $\frac{14}{3}$.

III. GAMES WITH A CONTINUUM OF STRATEGIES

1. GENERAL DESCRIPTION

It is possible to generalize the theory of two-person games by allowing each player to have a continuum of strategies. Consider a game in which each of two players, A and B , chooses a number between 0 and 1 inclusive. Suppose A chooses x and B chooses y , let the pay-off to A be designated by $M(x, y)$. The payment to B will be $-M(x, y)$.

If there exists a pair of numbers (x', y') such that

$$M(x', y') = \max_x M(x, y') = \min_y (x', y)$$

then $M(x, y)$ possesses a saddle point and

$$\max_x \min_y M(x, y) = \min_y \max_x M(x, y) = M(x', y').$$

The pair (x', y') will be optimal pure strategies.

If $M(x,y)$ has no saddle point, players A and B use mixed strategies, $F(x)$ and $G(y)$, where $F(x)$ is the probability that A will choose a number less than or equal to x and $G(y)$ is the probability that B will choose a number less than or equal to y . The expectation of A will be given by

$$(1) \quad \phi(F,G) = \int_0^1 \int_0^1 M(x,y) dF(x) dG(y).$$

We shall show that if $M(x,y)$ is a continuous function of x and y in the closed domain $0 \leq x \leq 1$, $0 \leq y \leq 1$, then there exist optimal strategies F^* and G^* such that

$$\max_F \phi(F, G^*) = \min_G (F^*, G) = \phi(F^*, G^*)$$

with the property that

$$\max_F \min_G \phi(F, G) = \min_G \max_F \phi(F, G) = \phi(F^*, G^*).$$

We call F^* , G^* a solution of the game and $v = \phi(F^*, G^*)$ its value.

2. THE MINIMAX THEOREM

We shall show that a minimax theorem holds for continuous $M(x,y)$. First, we demonstrate some relationships that hold for $\phi(F,G)$.

LEMMA 3.1. *If $F(x)$ and $G(y)$ are mixed strategies of two players, then*

$$(2) \quad \max_F \int_0^1 \int_0^1 M(x,y) dF(x) dG(y) = \max_x \int_0^1 M(x,y) dG(y).$$

$$\min_G \int_0^1 \int_0^1 M(x,y) dF(x) dG(y) = \min_y \int_0^1 M(x,y) dF(x).$$

PROOF. Take a fixed $G(y)$ and let

$$H(x) = \int_0^1 M(x,y) dG(y)$$

and define $\bar{x}$ by

$$\max_x H(x) = H(\bar{x}).$$

It follows that

$$(3) \quad \max_F \int_0^1 H(x) dF(x) \geq \int_0^1 H(x) dI_{\bar{x}}(x) = H(\bar{x})$$

where $I_{\bar{x}}(x)$ is a step-function with a single step of size 1 at $\bar{x}$. Further,

$$\int_0^1 H(x) dF(x) \leq \int_0^1 H(\bar{x}) dF(x) = H(\bar{x})$$

or

$$(4) \quad \max_F \int_0^1 H(x) dF(x) \leq H(\bar{x}).$$

The first equation of (2) follows from (3) and (4). In a similar manner we can demonstrate the second equation of (2).

Let B choose a strategy $G(y)$, then

$$\int_0^1 M(x, y) dG(y)$$

is a continuous function of x having a maximum, $\max_F \phi(F, G)$, at $\bar{x}$. Now let $G(y)$ vary, then $\max_F \phi(F, G)$ has a greatest lower bound, for

$$\begin{aligned} \max_F \phi(F, G) &= \max_x \int_0^1 M(x, y) dG(y) = \int_0^1 M(\bar{x}, y) dG(y) \\ &\geq \int_0^1 \min_y M(\bar{x}, y) dG(y) = \min_y M(\bar{x}, y), \end{aligned}$$

which means that $\max_F \phi(F, G)$ has a g.l.b. with respect to $G(y)$:

$$(5) \quad \mu = \inf_G \max_F \phi(F, G).$$

We can show that μ is actually assumed for some G . For let $\{G_n(y)\}$ be a sequence of distribution functions. Then this sequence contains a convergent subsequence which converges to a distribution function $G_0(y)$ at every point of continuity of $G_0(y)$.* Let $\bar{x}$ and t be defined by

$$\max_x \int_0^1 M(x, y) dG_0(y) = \int_0^1 M(\bar{x}, y) dG_0(y) = t.$$

Now since $M(x, y)$ is continuous in x and y , we have for any subsequence $\{G_n(y)\}$ converging to $G_0(y)$,

$$\lim_{n \rightarrow \infty} \int_0^1 M(\bar{x}, y) dG_n(y) = \int_0^1 M(\bar{x}, y) dG_0(y) = t$$

at every point of continuity. Also

$$\lim_{n \rightarrow \infty} \int_0^1 M(\bar{x}, y) dG_n(y) \leq \lim_{n \rightarrow \infty} \max_x \int_0^1 M(x, y) dG_n(y) = \mu.$$

Thus

$$t \leq \mu.$$

However,

$$\mu = \inf_G \max_F \phi(F, G) \leq \max_F \phi(F, G_0) = t.$$

Hence $\mu = t$, and the min-max therefore exists. Similarly, we can show that the max-min exists.

LEMMA 3.2.

(i) For any $G(y)$ there exists some x such that

$$(6) \quad \int_0^1 M(x, y) dG(y) \geq \min_G \max_F \phi(F, G).$$

*Cramer, Harold, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

(ii) For any $F(x)$ there exists a $G_0(y)$ such that

$$(7) \quad \int_0^1 \int_0^1 M(x, y) dF(x) dG_0(y) \leq \min_G \max_F \phi(F, G).$$

PROOF. We have

$$\mu = \min_G \max_F \phi(F, G) \leq \max_F \phi(F, G) = \max_x \int_0^1 M(x, y) dG(y).$$

By taking x such that $\int_0^1 M(x, y) dG(y)$ is a maximum we get (i).

To show (ii) we have

$$\max_F \phi(F, G) \geq \int_0^1 \int_0^1 M(x, y) dF(x) dG(y).$$

Then for some $G_0(y)$ we have

$$\begin{aligned} \mu = \min_G \max_F \phi(F, G) &\geq \min_G \int_0^1 \int_0^1 M(x, y) dF(x) dG(y) \\ &= \int_0^1 \int_0^1 M(x, y) dF(x) dG_0(y), \end{aligned}$$

which completes the proof of the lemma.

Let ϵ be an arbitrary small number, then we can find an integer n sufficiently large so that for

$$|x' - x''| < \frac{1}{n} \text{ and } |y' - y''| < \frac{1}{n}$$

we have

$$(8) \quad |M(x', y') - M(x'', y'')| < \epsilon.$$

Having chosen n , let

$$a_{ij} = M\left(\frac{i}{n}, \frac{j}{n}\right), \quad (i, j = 1, 2, \dots, n).$$

By (8), for every x there exists an i such that

$$(9) \quad \left| \int_0^1 M(x, y) dG(y) - \int_0^1 M\left(\frac{i}{n}, y\right) dG(y) \right| \leq \epsilon.$$

Define $G(y)$ as follows:

$$\begin{aligned} G(y) &= 0, & 0 \leq y < \frac{1}{n} \\ G(y) &= p_1, & \frac{1}{n} \leq y < \frac{2}{n} \\ G(y) &= p_1 + p_2, & \frac{2}{n} \leq y < \frac{3}{n} \\ &\dots\dots\dots & \dots\dots\dots \\ G(y) &= p_1 + p_2 + \dots + p_{n-1}, & \frac{n-1}{n} \leq y < 1 \\ G(y) &= p_1 + p_2 + \dots + p_n = 1, & y = 1 \end{aligned}$$

where

$$p_i \geq 0.$$

LEMMA 3.3. If $p_1, p_2, \dots, p_n$ are non-negative numbers whose sum is 1, there exists an i such that

$$(10) \quad \sum_{j=1}^n a_{ij} p_j \geq \min_G \max_F \phi(F, G) - \epsilon.$$

PROOF. From (9) we have that for any $\epsilon > 0$ and any x there exists some i such that

$$\int_0^1 M\left(\frac{i}{n}, y\right) dG(y) \geq \int_0^1 M(x, y) dG(y) - \epsilon.$$

From (i) of Lemma 3.2 it follows that there exists an i such that

$$\int_0^1 M\left(\frac{i}{n}, y\right) dG(y) \geq \min_G \max_F \phi(F, G) - \epsilon.$$

Evaluating the above integral we obtain

$$p_1 M\left(\frac{i}{n}, \frac{1}{n}\right) + p_2 M\left(\frac{i}{n}, \frac{2}{n}\right) + \dots + p_n M\left(\frac{1}{n}, 1\right)$$

or for some i

$$\sum_{i=1}^n a_{ij} p_j \geq \min_G \max_F \phi(F, G) - \epsilon.$$

The preceding lemma implies the following: Given a homogeneous linear form $g(p)$ and a set of homogeneous linear forms $f_i(p)$ in the n variables $p_1, p_2, \dots, p_n$ such that for every set $\{p_i\}$ for which $p_i \geq 0$ and $\sum p_i = 1$ there exists one linear form $f_2(p)$ such that $f_2(p) \geq g(p)$. These are the hypotheses of Corollary 2 to the theorem proven in the Appendix, hence there exists a set $\{q_j\}$, where $q_j \geq 0$ and $\sum q_j = 1$, such that for every set $\{p_i\}$

$$(11) \quad \sum_{i=1}^n \sum_{j=1}^n a_{ij} p_j q_i \geq \min_G \max_F \phi(F, G) - \epsilon.$$

In terms of the set $\{q_i\}$ define $F_0(x)$ as follows:

$$\begin{array}{ll} F_0(x) = 0, & 0 \leq x < \frac{1}{n} \\ F_0(x) = q_1, & \frac{1}{n} \leq x < \frac{2}{n} \\ F_0(x) = q_1 + q_2, & \frac{2}{n} \leq x < \frac{3}{n} \\ \dots & \dots \\ F_0(x) = q_1 + q_2 + \dots + q_{n-1}, & \frac{n-1}{n} \leq x < 1 \\ F_0(x) = q_1 + q_2 + \dots + q_n = 1, & x = 1 \end{array}$$

Hence, for all j

$$(12) \quad \int_0^1 M\left(x, \frac{j}{n}\right) dF_0(x) = \sum_{i=1}^n M\left(\frac{i}{n}, \frac{j}{n}\right) q_i.$$

Multiplying by p_j and summing, (12) yields

$$(13) \quad \sum_{j=1}^n p_j \int_0^1 M\left(x, \frac{j}{n}\right) dF_0(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} p_j q_i \geq \min_G \max_F \phi(F, G) - \epsilon.$$

Now (13) is valid for all sets $\{p_j\}$, where $p_j \geq 0$ and $\sum p_j = 1$. In particular we can let $p_j = 1$ and $p_k = 0$ for $k \neq j$. Hence there exists some $F_0(x)$ such that

$$(14) \quad \int_0^1 M\left(x, \frac{j}{n}\right) dF_0(x) \geq \min_G \max_F \phi(F, G) - \epsilon \text{ for } j = 1, 2, \dots, n.$$

THEOREM 3.1 *If $M(x, y)$ is continuous in the unit square, then*

$$\max_F \min_G \phi(F, G) = \min_G \max_F \phi(F, G).$$

PROOF. $M(x, y)$ is continuous, hence for every y there exists some j such that

$$(15) \quad \int_0^1 M(x, y) dF_0(x) \geq \int_0^1 M\left(x, \frac{j}{n}\right) dF_0(x) - \epsilon.$$

From (14) and (15) it follows that there exists some $F_0(x)$ such that for every y

$$(16) \quad \int_0^1 M(x, y) dF_0(x) \geq \min_G \max_F \phi(F, G) - 2\epsilon.$$

and

$$(17) \quad \int_0^1 \int_0^1 M(x, y) dF_0(x) dG(y) \geq \min_G \max_F \phi(F, G) - 2\epsilon.$$

It is evident that

$$\max_F \min_G \phi(F, G) \geq \min_G \int_0^1 \int_0^1 M(x, y) dF_0(x) dG(y) \geq \min_G \max_F \phi(F, G) - 2\epsilon.$$

If we let ϵ approach 0, we have

$$\max_F \min_G \phi(F, G) \geq \min_G \max_F \phi(F, G).$$

But

$$\max_F \min_G \phi(F, G) \leq \min_G \max_F \phi(F, G),$$

hence the equality must hold and the theorem is proven.

3. EXAMPLE AND SOLUTION OF GAME WITH A CONTINUUM OF STRATEGIES

Consider the game in which each player picks a number between 0 and 1 inclusive. Let the pay-off to the first player be $M(x, y) = x^2 y^2 - x - y$ if the first player picks x and the second player picks y . We wish to find the optimal strategies of the two players and the value of the game.

Let

$$H(x) = \int_0^1 (x^2 y^2 - x - y) dG(y) = x^2 g_2 - x - g_1$$

where

$$g_i = \int_0^1 y^i dG(y) \leq g_{i-1} \leq 1.$$

Then

$$v = \min_G \max_F \int_0^1 H(x) dG(x) = \min_G \max_x H(x).$$

Since $H(x)$ is a parabola, convex downward, we have

$$\max_x H(x) = \max [H(0), H(1)] = -g_1.$$

Therefore, the value of the game is $v = \min_G (-g_1) = -1$, and the optimal strategy of the second player is $G^*(y) = I_1(y)$.

To determine the optimal strategy of the first player, let

$$K(y) = \int_0^1 M(x, y) dF(x) = f_2 y^2 - f_1 - y$$

where

$$f_i = \int_0^1 x^i dF(x).$$

Then

$$v = \max_F \min_G \phi(F, G) = \max_G \min_y K(y).$$

Now $K(y)$ is a parabola, convex downward and having a minimum of $\frac{-1}{4f_2} - f_1$ at $\frac{1}{2f_2}$. Since we are interested in the interval $(0, 1)$ we need to consider the location of $\frac{1}{2f_2}$ as follows:

$$(i) \quad \frac{1}{2f_2} \leq 1, \text{ whereupon } \min_y K(y) = -\frac{1}{4f_2} - f_1.$$

$$(ii) \quad \frac{1}{2f_2} \geq 1, \text{ when } \min_y K(y) = K(1) = f_2 - f_1 - 1.$$

Therefore

$$v = \max_F \min_y K(y) = \max \left[\max_{f_2 \geq \frac{1}{2}} \left(-\frac{1}{4f_2} - f_1 \right), \max_{f_2 \leq \frac{1}{2}} (f_2 - f_1 - 1) \right].$$

Now since for all distribution functions

$$f_1^2 \leq f_2 \leq f_1,$$

it follows that

$$\max_{f_2 \geq \frac{1}{2}} \left(-\frac{1}{4f_2} - f_1 \right) = \max_{f_1 \geq \frac{1}{2}} \left(-\frac{1}{4f_1} - f_1 \right) = -1$$

and this value is assumed for $f_1 = f_2 = \frac{1}{2}$. Similarly,

$$\max_{f_2 \leq \frac{1}{2}} (f_2 - f_1 - 1) = -1$$

and this value is assumed for $f_1 = f_2 \leq \frac{1}{2}$.

Therefore the set of optimal strategies for the first player will be given by

$$F^*(x) = aI_0 + (1-a)I_1, \quad a \geq \frac{1}{2}.$$

4. A PROPERTY OF SOLUTIONS

In determining the solution of games with a continuum of strategies, the following theorem is frequently of value.

THEOREM 5.1. Any solution $F^*(x)$, $G^*(y)$ of an arbitrary game has the properties:

(i) If $H^*(x) = \int_0^1 M(x, y) dG^*(y) < v$ throughout an interval, then $F^*(x)$ is constant in that interval.

(ii) If $H^*(x) < v$ at $x = x_0$, then $F^*(x)$ is continuous at x_0 .

PROOF.

$$v = \max_F \int_0^1 H^*(x) dF(x).$$

Hence

$$v \geq \int_0^1 H^*(x) dI_{x_0}(x) = H^*(x_0) \text{ for all } x_0,$$

or

$$v - H^*(x) \geq 0 \text{ for all } x.$$

But

$$\int_0^1 [v - H^*(x)] dF^*(x) = v - \int_0^1 H^*(x) dF^*(x) = v - v = 0.$$

From the last equation, it is evident that if $H^*(x) < v$ in an interval, then $F^*(x)$ must be constant there; if $H^*(x) < v$ at some point, then $F^*(x)$ cannot have a jump there.

In a similar manner we can show that if $K^*(y) = \int_0^1 M(x, y) dF^*(x) > v$ throughout an interval, then $G^*(y)$ is constant in that interval, and if $K^*(y) > v$ at some point then $G^*(y)$ cannot have a jump at that point.

The previous theorem is particularly applicable to polynomial pay-off functions.

THEOREM 5.2. If $M(x, y)$ is a polynomial in x and y of degree m and n respectively, and if $H^*(x_0) < v$ for some x_0 , then every $F^*(x)$ is step-function having at most $\left[\frac{m}{2}\right] + 1$ jumps. If $K^*(y_0) > v$ for some y_0 then every $G^*(y)$ is a step-function having at most $\left[\frac{n}{2}\right] + 1$ jumps.

PROOF. Since $H^*(x_0) < v$ for some x_0 , there exists an interval (a, b) such that $H^*(x) < v$ for $a \leq x \leq b$. By the previous theorem, $F^*(x)$ is constant in this interval. Now $H^*(x)$ is a polynomial of degree m , hence there will be at most $\left[\frac{m}{2}\right] + 1$ intervals in which $H^*(x) < v$. Therefore there will be at most $\left[\frac{m}{2}\right] + 1$ points at which jumps may occur in $F^*(x)$.

The jumps in $F^*(x)$ can occur only at points x_i for which

$$H^*(x_i) = v = \max_x H^*(x).$$

5. GAME WITH CONVEX PAY-OFF FUNCTION

A special class of games for which the solution is completely known is that where the pay-off function depends on the difference $y - x$ and is twice differentiable and convex; i.e., $M(x, y) = f(y - x)$ and $f'' \geq 0$. We shall derive in detail the value of the game and the best strategies for the two players.

Let $u = y - x$ where $-1 \leq u \leq 1$. Since $f(u)$ is convex, we have

$$f(\lambda_1 u_1 + \lambda_2 u_2) \leq \lambda_1 f(u_1) + \lambda_2 f(u_2) \quad \text{if } \lambda_1 + \lambda_2 = 1.$$

For a closed interval α on $0 \leq u \leq 1$, the function G is said to belong to α if $G = 0$ to the left of α and $G = 1$ to the right of α . I_α denotes the function which belongs to the closed interval consisting of one point $u = \alpha$. Define $\phi(u)$ in $0 \leq u \leq 1$ as follows:

$$(18) \quad \phi(u) = \max [f(u), f(u-1)].$$

This function is convex and reaches its minimum over a closed interval denoted by Y , with end points y_0 and y_1 , where $y_0 \leq y_1$.

Let us consider the case where $f''(u) > 0$, $f(0) < f(-1)$, and $f(0) < f(1)$. Then Y reduces to one point $y_0 = y_1 = a$ in the interior of $0 \leq u \leq 1$ and its value is given by the solution of the equation $f(u) = f(u-1)$. To demonstrate this fact, we note that $f(1) - f(0) > 0$ and $f(0) - f(-1) < 0$, hence $f(u) - f(u-1) = 0$ has solutions and any solution is different from 0 and 1. The solution is unique, for let $0 < u_1 < u_2 < 1$ be two solutions. By strict convexity, $f(u_1) = f(u_1-1)$ and $u_1-1 < u_2-1 < u_1$ imply $f(u_2) = f(u_2-1) < f(u_1-1) = f(u_1)$, while $f(u_2) = f(u_2-1)$ and $u_2-1 < u_1 < u_2$ imply the contradictory statement $f(u_1) < f(u_2)$.

Let a be the unique solution of $f(u) = f(u-1)$, then $\phi(a) = \max [f(a), f(a-1)] = f(a)$. Further, $\phi(a)$ is the minimum of $\phi(u)$, for if $u > a$, $\phi(u) = f(u) > f(a)$, and if $u < a$, then $\phi(u) = f(u-1) > f(a)$. Let this minimum be denoted by ϕ_0 .

We shall now show that the value of the game is ϕ_0 . Since for $a-1 < u < a$ we have $f(u) < f(a)$, and since $f'' > 0$ it follows that $f'(a) > 0$ and $f'(a-1) < 0$. Further,

$$(19) \quad \begin{aligned} f(y) &\geq f(a) + f'(a)(y-a), \\ f(y-1) &\geq f(a) + f'(a-1)(y-a), \end{aligned}$$

where the equality sign holds if $y = a$.

Let

$$(20) \quad q_0 = \frac{-f'(a-1)}{f'(a) - f'(a-1)}, \quad q_1 = \frac{f'(a)}{f'(a) - f'(a-1)}$$

then $q_0, q_1 > 0$ and $q_0 + q_1 = 1$. Therefore, by (19),

$$(21) \quad q_0 f(y) + q_1 f(y-1) \geq f(a) = \phi_0.$$

The equality holds only for $y = a$.

Conversely, if

$$(22) \quad \lambda_0 f(y) + \lambda_1 f(y-1) \geq f(a)$$

for all y , and $\lambda_0, \lambda_1 \geq 0$, $\lambda_0 + \lambda_1 = 1$, then

$$(23) \quad \lambda_0 = q_0, \quad \lambda_1 = q_1$$

for the left side of (22) is differentiable and has a minimum at a , hence

$$\lambda_0 f'(a) + \lambda_1 f'(a-1) = 0$$

which, by virtue of (20) implies (23).

Let

$$F_0 = q_0 I_0 + q_1 I_1;$$

then for any G

$$\begin{aligned} \max_F \int_0^1 \int_0^1 f(y-x) dF dG &\geq \int_0^1 \int_0^1 f(y-x) dF_0 dG \\ &= \int_0^1 [q_0 f(y) + q_1 f(y-1)] dG \geq f(a) = \phi_0. \end{aligned}$$

Hence,

$$\min_G \max_F \int_0^1 \int_0^1 f(u) dF dG \geq \phi_0.$$

If $G_0 = I_a$ then for any F

$$\begin{aligned} \min_G \int_0^1 \int_0^1 f(y-x) dF dG &\leq \int_0^1 \int_0^1 f(y-x) dF dG_0 \\ &= \int_0^1 f(a-x) dF \leq f(a) = \phi_0, \end{aligned}$$

or

$$\max_F \min_G \leq \phi_0.$$

From these two results and Theorem 3.1 it follows that

$$\max_F \min_G = \min_G \max_F = \phi_0 = v.$$

To determine the optimal strategies of the two players we note that for the second player the strategies have the property that

$$\max_F \int_0^1 \int_0^1 f(y-x) dF dG = \phi_0.$$

But for

$$F = F_0 = q_0 I_0 + q_1 I_1$$

we have, by (21),

$$\int_0^1 \int_0^1 f(y-x) dF dG = \int_0^1 [q_0 f(y) + q_1 f(y-1)] dG \geq f(a) = \phi_0$$

and the equality can hold only if $G = I_a$, which therefore must be the optimal strategy.

For a best strategy of the first player we have

$$\min_G \int_0^1 \int_0^1 f(y-x) dF dG = \phi_0.$$

For $G = I_a$, we get

$$\int_0^1 \int_0^1 f(y-x) dF dI_a = \int_0^1 f(a-x) dF \leq f(a)$$

and the equality can hold only if

$$F = \lambda_0 I_0 + \lambda_1 I_1, \quad \lambda_0, \lambda_1 \geq 0, \quad \lambda_0 + \lambda_1 = 1.$$

To determine λ_0, λ_1 set $G = I_u$, in which case we obtain

$$\int_0^1 \int_0^1 f(y-x) dF dI_u = \lambda_0 f(u) + \lambda_1 f(u-1)$$

which should also be $\geq f(a)$. Therefore by a previous result $\lambda_0 = q_0, \lambda_1 = q_1$, so that $F = q_0 I_0 + q_1 I_1$, is optimal.

In all cases it can be shown that the value of the game is given by the minimum ϕ_0 of the function $\phi(u)$. This minimum will not always be given by the equation $f(u) = f(u-1)$. The best strategy of the second player is any function G belonging to the interval Y over which ϕ reaches its minimum. If this interval does not reduce to a point then the first player can use any strategy, since in this case $f(u) > \phi_0$.

IV. GAMES WITH A CONTINUUM OF STRATEGIES AND DISCONTINUOUS PAY-OFF FUNCTION

1. GENERAL DESCRIPTION

In Part III we showed that a game with a continuum of strategies and continuous pay-off function has a value and optimal strategies. A possible direction of investigation is to consider discontinuous pay-off functions. We can demonstrate by an example that there are games with a continuum of strategies which do not have solutions.

2. EXAMPLE OF A GAME WITHOUT SOLUTION

Consider the two-person game in which each player chooses a number in the closed interval $(0,1)$. Let x and y represent the choices of the first and second players respectively. Define the payment to the first player as follows:

$$M(x,y) = \begin{cases} 0, & x = y; \\ -1, & x = y, \quad y < 1 \text{ and } x < y < 1; \\ +1, & y = 1, \quad x < 1 \text{ and } y < x < 1. \end{cases}$$

Let $\phi(F,G)$ represent the expectation of the first player if the first player uses strategy $F(x)$ and the second player uses strategy $G(y)$. Write $\phi(x,G)$ for $\phi(I_x,G)$ and $\phi(F,y)$ for $\phi(F,I_y)$. Then $\phi(x,G)$ is the expectation of the first player if he uses a pure strategy x and the second player uses a mixed strategy $G(y)$.

We have then the following:

(i) If $0 < y < 1$ then

$$\phi(F,y) = -F(y-0) + 2F(1-0) - F(y) - 1.$$

(ii) If $y = 0$

$$\phi(F, y) = -1 + 2F(1-0) - F(0).$$

(iii) If $y = 1$ then

$$\phi(F, y) = +F(1-0).$$

From (i) it follows that if $0 < y < 1$, then

$$\phi(F, y) \leq -1 + 2[F(1-0) - F(y-0)].$$

Now $F(y-0)$ is continuous from the left, hence for any $\epsilon (0 < \epsilon < 1)$ there exists some y such that $\phi(F, y) \leq -1 + \epsilon$.

We have demonstrated that for any F and any $\epsilon (0 < \epsilon < 1)$ there exists some G , namely I_y , such that $\phi(F, G) \leq -1 + \epsilon$. In a similar manner we can show that for any G and any $\epsilon (0 < \epsilon < 1)$ there exists some F such that $\phi(F, G) \geq +1 - \epsilon$. Therefore,

$$\inf_G \phi(F, G) \leq -1 + \epsilon,$$

$$\sup_F \phi(F, G) \geq 1 - \epsilon.$$

It follows that

$$\sup_F \inf_G \phi(F, G) \leq -1 + \epsilon,$$

$$\inf_G \sup_F \phi(F, G) \geq +1 - \epsilon.$$

From the above it is obvious that

$$\sup_F \inf_G \phi(F, G) \neq \inf_G \sup_F \phi(F, G).$$

3. GAME WITH DISCONTINUOUS PAY-OFF FUNCTION HAVING A SOLUTION

The following is an example of a game with a continuum of strategies and a discontinuous pay-off function which possesses a solution. Consider the game for which the pay-off function in the interval $(0, 1)$ is defined as follows:

$$M(x, y) = x + y \quad \text{if } y \geq x,$$

$$M(x, y) = x - y \quad \text{if } y < x.$$

The expectation of the first player will be given by

$$\phi(F, G) = \int_0^1 \int_0^1 M(x, y) dF(x) dG(y).$$

Integrating with respect to x we obtain

$$\phi(F, G) = \int_0^1 x dF + \int_0^1 y [2F(y) - 1] dG(y).$$

For any F , the function $y[2F(y) - 1]$ is continuous from the right. Let $-p$ be its inf. At $y = 0$, the function is 0, hence $p \geq 0$. Further, since $F \geq 0$, $y[2F(y) - 1] \geq -1$ and hence $p \leq 1$. For any F ,

$$\inf_G \left\{ \int_0^1 x dF + \int_0^1 y [2F(y) - 1] dG \right\} = \int_0^1 x dF - p.$$

Therefore

$$\sup_F \inf_G = \sup_F \left(\int_0^1 x dF - p \right) = \sup_{0 \leq p \leq 1} \sup_{F \in p} \left(\int_0^1 x dF - p \right)$$

where $F \in p$ means that

$$-p = \inf_{0 \leq y \leq 1} y [2F(y) - 1].$$

For a given p , we have

$$\sup_{F \in p} \left(\int_0^1 x dF - p \right) = -p + \sup_{F \in p} \int_0^1 x dF.$$

To calculate the value of the preceding expression we note that, for $y > 0$,

$$-p \leq y [2F(y) - 1], \quad F(y) \geq \frac{1}{2} \left(1 - \frac{p}{y} \right).$$

If $p = 0$, then $F(y) \geq \frac{1}{2}$, and therefore

$$\int_0^1 x dF \leq 0 + \int_{0+0}^1 dF = 1 - F(0) \leq \frac{1}{2}.$$

The value $\frac{1}{2}$ is reached for $F = \frac{1}{2} I_0 + \frac{1}{2} I_1$, or

$$\sup_{F \in \mathcal{F}_0} \left(\int_0^1 x dF - p \right) = \frac{1}{2}.$$

If $p > 0$, then $F(x) \geq 0$ in $0 \leq x \leq p$ and $F(x) \geq \frac{1}{2} \left(1 - \frac{p}{x}\right)$ in $p < x \leq 1$. Therefore,

$$\int_0^1 x dF = F(1) - \int_0^1 F dx \leq 1 - \frac{1}{2} \int_p^1 \left(1 - \frac{p}{x}\right) dx = 1 - \frac{1}{2}(1-p) + \frac{1}{2} p (0 - \log p) = \frac{1}{2} + \frac{p}{2} - \frac{p}{2} \log p.$$

The equality is reached for the function

$$F(x) = \begin{cases} 0, & 0 \leq x \leq p, \\ \frac{1}{2} \left(1 - \frac{p}{x}\right), & p < x < 1, \\ 1, & x = 1. \end{cases}$$

Therefore,

$$\sup_{F \in \mathcal{F}_p} \left(\int_0^1 x dF - p \right) = \frac{1}{2} - \frac{p}{2} - \frac{p}{2} \log p,$$

and this is also the maximum value.

As p approaches 0, the sup assumes the value $\frac{1}{2}$ which is the value computed previously for $\phi = \frac{1}{2}$. Hence the value of the game is

$$v = \max \left(\frac{1}{2} - \frac{p}{2} - \frac{p}{2} \log p \right) = \frac{1}{2} (1 + e^{-2})$$

for $p = e^{-2}$.

Therefore an optimal strategy for the first player is

$$F(x) = \begin{cases} 0, & 0 \leq x \leq e^{-2}, \\ \frac{1}{2} \left(1 - \frac{1}{e^2 x}\right), & e^{-2} \leq x < 1, \\ 1, & x = 1. \end{cases}$$

To compute an optimal strategy for the second player, consider the function

$$G(y) = \begin{cases} 0, & 0 \leq y \leq e^{-2}, \\ \frac{1}{2} \log y + 1, & e^{-2} \leq y \leq 1. \end{cases}$$

We shall show that this is an optimal strategy in terms of game theory. Now for the above $G(y)$ and any $F(x)$ the expectation of the first player is

$$\begin{aligned} & \int_0^1 x dF + \frac{1}{2} \int_{e^{-2}}^1 [y 2F(y) - 1] \frac{dy}{y} \\ &= \int_0^1 x dF + \int_{e^{-2}}^1 F(x) dx - \frac{1}{2}(1 - e^{-2}) \\ &= 1 - \int_0^1 F(x) dx + \int_{e^{-2}}^1 F(x) dx - \frac{1}{2}(1 - e^{-2}) \\ &= 1 - \int_0^{e^{-2}} F(x) dx - \frac{1}{2}(1 - e^{-2}) \\ &= \frac{1}{2}(1 + e^{-2}) - \int_0^{e^{-2}} F(x) dx \\ &\leq \frac{1}{2}(1 + e^{-2}). \end{aligned}$$

Thus if the second player uses the above strategy, he can lose no more than the value of the game. Hence $G(y)$ is an optimal strategy.

APPENDIX

THEOREM ON LINEAR FORMS

In order to prove the basic theorem in game theory, the minimax theorem, we required a general relationship between linear forms. We shall derive this theorem here. First, it will be necessary to prove three algebraic lemmas.

LEMMA 1. Let m be a positive integer, and let $x_1, \dots, x_m$ and y be non-negative real numbers such that:

$$x_1 + x_2 + \dots + x_m \geq y.$$

Then there exist non-negative real numbers $u_1, u_2, \dots, u_m$ such that:

$$(i) \quad y = u_1 + u_2 + \dots + u_m;$$

$$(ii) \quad x_1 \geq u_1, x_2 \geq u_2, \dots, x_m \geq u_m.$$

PROOF. By induction on m .

LEMMA 2. Let m and n be positive integers, and let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be non-negative real numbers such that:

$$x_1 + \dots + x_m \geq y_1 + \dots + y_n.$$

Then there exist non-negative real numbers

$$u_1^{(1)}, \dots, u_1^{(n)}, u_2^{(1)}, \dots, u_2^{(n)}, \dots, u_m^{(1)}, \dots, u_m^{(n)}, u_1, \dots, u_m$$

such that:

$$x_i = u_i^{(1)} + u_i^{(2)} + \dots + u_i^{(n)} + u_i \quad \text{for } i = 1, 2, \dots, m,$$

$$y_j = u_1^{(j)} + u_2^{(j)} + \dots + u_m^{(j)} \quad \text{for } j = 1, 2, \dots, n.$$

PROOF. By an induction on n .

If $n=1$, then we have numbers $x_1, \dots, x_m$, and y_1 , such that:

$$(1) \quad x_1 + \dots + x_m \geq y_1.$$

Let $u_1^{(1)}, u_2^{(1)}, \dots, u_m^{(1)}$ be non-negative real numbers (which exist by Lemma 1) such that:

$$(2) \quad y_1 = u_1^{(1)} + u_2^{(1)} + \dots + u_m^{(1)}$$

and

$$(3) \quad x_1 \geq u_1^{(1)}, x_2 \geq u_2^{(1)}, \dots, x_m \geq u_m^{(1)}.$$

We set

$$(4) \quad \begin{cases} u_1 = x_1 - u_1^{(1)}, \\ u_2 = x_2 - u_2^{(1)}, \\ \vdots \\ u_m = x_m - u_m^{(1)}. \end{cases}$$

From (3) we see that the numbers $u_1, u_2, \dots, u_m$ are non-negative. From (4) and (2) we see that

$$x_1 = u_1^{(1)} + u_1$$

$$x_2 = u_2^{(1)} + u_2$$

$$\vdots$$

$$x_m = u_m^{(1)} + u_m$$

$$y_1 = u_1^{(1)} + \dots + u_m^{(1)},$$

as was to be shown.

Now suppose the lemma is true for $n = k$, and let $x_1, \dots, x_m$ and $y_1, \dots, y_{k-1}$ be non-negative real numbers such that

$$(5) \quad x_1 + \dots + x_m \geq y_1 + \dots + y_k.$$

From (5) we conclude, *a fortiori*, that

$$x_1 + \dots + x_m \geq y_k.$$

Hence, by Lemma 1, there exist non-negative real numbers $u_1^{(k+1)}, \dots, u_m^{(k+1)}$ such that:

$$(6) \quad y_{k+1} = u_1^{(k+1)} + \dots + u_m^{(k+1)}$$

and

$$(7) \quad x_1 \geq u_1^{(k+1)}, x_2 \geq u_2^{(k+1)}, \dots, x_m \geq u_m^{(k+1)}.$$

We now set

$$(8) \quad \left\{ \begin{array}{l} \bar{x}_1 = x_1 - u_1^{(k+1)} \\ \bar{x}_2 = x_2 - u_2^{(k+1)} \\ \vdots \\ \bar{x}_m = x_m - u_m^{(k+1)}. \end{array} \right.$$

From (7) we see that the numbers $\bar{x}_1, \dots, \bar{x}_m$ are non-negative.

Substituting (6) in (5), and transposing, we have

$$(9) \quad \bar{x}_1 + \dots + \bar{x}_m \geq y_1 + \dots + y_k.$$

Hence, by the induction hypothesis, there are non-negative real numbers

$$u_1^{(1)}, \dots, u_1^{(k)}, u_2^{(1)}, \dots, u_2^{(k)}, \dots, u_m^{(1)}, \dots, u_m^{(k)}, u_1, \dots, u_m$$

such that:

$$\bar{x}_i = u_i^{(1)} + u_i^{(2)} + \dots + u_i^{(k)} + u_i \quad \text{for } i = 1, 2, \dots, m$$

$$y_j = u_1^{(j)} + u_2^{(j)} + \dots + u_m^{(j)} \quad \text{for } j = 1, 2, \dots, k.$$

From (8), (9), and (6), we see that our lemma also holds for $n = k+1$, as was to be shown.

LEMMA 3. Let $a_1, a_2, \dots, a_n$ be arbitrary real numbers, and let D consist of those points of n -dimensional space whose coordinates $x_1, \dots, x_n$ are all non-negative and in addition satisfy the condition

$$a_1 x_1 + \dots + a_n x_n \geq 0.$$

Then there exist an integer s , and n linear homogeneous forms $f_1, \dots, f_n$ in s variables with non-negative coefficients, such that

$$(x_1, x_2, \dots, x_n) \in D$$

if and only if there exist non-negative $u_1, u_2, \dots, u_s$ for which

$$x_1 = f_1(u_1, u_2, \dots, u_s)$$

$$\vdots$$

$$x_n = f_n(u_1, u_2, \dots, u_s)$$

PROOF. In case all the a 's are non-negative, the set D consists of all points $x_1, \dots, x_n$ whose coordinates are all non-negative. Hence we take $s = n$, and set

$$f_i(u_1, u_2, \dots, u_n) = u_i \quad (i = 1, 2, \dots, n)$$

In case all the a 's are non-positive, the set D consists simply of the origin $(0, \dots, 0)$. Hence we can take $s = 1$, and set

$$f_i(u_1) = 0 \cdot u_1 \quad i = 1, 2, \dots, n$$

Hence we shall suppose that r of the a 's are positive, that s are negative, and that $t = n - r - s$ are zero. Without loss of generality we can suppose that $a_1, \dots, a_r$ are positive, that $a_{r+1}, \dots, a_{r+s}$ are negative, and that $a_{r+s+1}, \dots, a_n = 0$. Setting

$$b_1 = -a_{r+1}, b_2 = -a_{r+2}, \dots, b_s = -a_{r+s},$$

we see that D consists of all points

$$(x_1, \dots, x_r, y_1, \dots, y_s, z_1, \dots, z_t)$$

whose coordinates are non-negative and satisfy the inequality

$$a_1 x_1 + \dots + a_r x_r \geq b_1 y_1 + \dots + b_s y_s$$

(where $a_1, \dots, a_r$ and $b_1, \dots, b_s$ are positive).

Now we set

$$f_i = \frac{1}{a_i} [u_i^{(1)} + u_i^{(2)} + \dots + u_i^{(s)} + u_i] \quad (i = 1, 2, \dots, r),$$

$$f_{r+j} = \frac{1}{b_j} [u_1^{(j)} + u_2^{(j)} + \dots + u_r^{(j)}] \quad (j = 1, 2, \dots, s),$$

$$f_{r+s+k} = u_{r+k} \quad (k = 1, 2, \dots, t).$$

It is now easily verified from the above that if the u 's are any non-negative numbers, then $f_1, \dots, f_{r+s+t}$ are non-negative, and

$$a_1 f_1 + \dots + a_r f_r \geq b_1 f_{r+1} + \dots + b_s f_{r+s},$$

or

$$a_1 f_1 + \dots + a_n f_n = a_1 x_1 + \dots + a_n x_n \geq 0.$$

Conversely, if $(x_1, x_2, \dots, x_n) \in D$, it can be shown using Lemma 2 that there exist non-negative $u_1, u_2, \dots, u_s$ such that, for the above f_i , we have

$$x_i = f_i(u_1, u_2, \dots, u_s).$$

We can now state and prove the general theorem on linear forms.

THEOREM. Suppose that p homogeneous linear forms in n variables,

$$f_1(x_1, \dots, x_n) = a_1^{(1)}x_1 + a_1^{(2)}x_2 + \dots + a_1^{(n)}x_n,$$

$$f_2(x_1, \dots, x_n) = a_2^{(1)}x_1 + a_2^{(2)}x_2 + \dots + a_2^{(n)}x_n,$$

$$\vdots$$

$$f_p(x_1, \dots, x_n) = a_p^{(1)}x_1 + \dots + a_p^{(2)}x_2 + \dots + a_p^{(n)}x_n,$$

are such that, for every set of non-negative numbers $x_1, \dots, x_n$, there exists a $j \leq p$ such that $f_j(x_1, \dots, x_n) \geq 0$. Then there exists a set $\xi_1, \xi_2, \dots, \xi_p$ of non-negative numbers, whose sum is equal to 1, such that the linear form

$$\sum_{j=1}^p \xi_j f_j(x_1, \dots, x_n)$$

is non-negative for every set of non-negative values of $x_1, \dots, x_n$.

PROOF. We first deal with the case where $p = 2$ and $n = 2$. Suppose, then, that

$$a_1x_1 + a_2x_2, \quad b_1x_1 + b_2x_2$$

are two forms such that, whenever $x_1 \geq 0$ and $x_2 \geq 0$, we have either $a_1x_1 + a_2x_2 \geq 0$ or $b_1x_1 + b_2x_2 \geq 0$; we are to show that there are non-negative numbers ξ and $1 - \xi$ such that

$$\xi(a_1x_1 + a_2x_2) + (1 - \xi)(b_1x_1 + b_2x_2) \geq 0$$

whenever $x_1 \geq 0$ and $x_2 \geq 0$. This is trivial unless a_1, a_2, b_1 , and b_2 are all different from zero, a_1 and a_2 are of different signs, b_1 and b_2 are of different signs, and a_1 and b_1 are of different signs. Without loss of generality we can suppose $a_1 > 0$, $a_2 < 0$, $b_1 < 0$, and $b_2 > 0$. Setting $\alpha_1 = -a_1$, $\alpha_2 = -a_2$, $\beta_1 = -b_1$, and $\beta_2 = b_2$, we have the two forms

$$\alpha_1 x_1 - \alpha_2 x_2, \quad -\beta_1 x_1 + \beta_2 x_2,$$

where $\alpha_1, \alpha_2, \beta_1$, and β_2 are all positive. If we substitute α_2 for x_1 and $\alpha_1 + \epsilon$ for x_2 , then the first of these forms is negative for positive ϵ ; hence, for every positive ϵ , this substitution makes the second of the forms non-negative; so

$$-\beta_1(\alpha_2) + \beta_2(\alpha_1 + \epsilon) \geq 0$$

for every positive ϵ . Since the left member of this last inequality is a continuous function of ϵ , we conclude that the inequality holds also for $\epsilon = 0$; hence

$$(10) \quad -\beta_1 \alpha_2 + \beta_2 \alpha_1 \geq 0.$$

Now if we set $\xi = \frac{\beta_1}{\alpha_1 + \beta_1}$, we have

$$\xi [\alpha_1 x_1 - \alpha_2 x_2] + (1 - \xi) [-\beta_1 x_1 + \beta_2 x_2] = x_2 \frac{[-\beta_1 \alpha_2 + \beta_2 \alpha_1]}{\alpha_1 + \beta_1},$$

which by (10) is non-negative for non-negative x_2 . This completes the proof of our theorem for the case $p = 2$, $n = 2$.

We now pass to the case of two forms in n variables,

$$f_1(x_1, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$f_2(x_1, \dots, x_n) = b_1 x_1 + b_2 x_2 + \dots + b_n x_n$$

which satisfy the hypothesis of our theorem. We notice that, for every i and j , the forms

$$a_i x_i + a_j x_j, \quad b_i x_i + b_j x_j$$

satisfy the hypothesis of our theorem; to convince ourselves of this it suffices to replace all the other x 's by zero. Let us consider two corresponding coefficients, say a_i and b_i . The set of numbers x such that

$$0 \leq x \leq 1, \quad xa_i + (1-x)b_i \geq 0$$

constitutes a closed interval; this can be seen by solving the second inequality for x (considering separately the cases $a_i > b_i$, $a_i < b_i$, and $a_i = b_i$). We shall call this closed interval A_i .

To prove our theorem for this case (of two forms) it now suffices to show that the intersection of the intervals A_i is not empty. We notice first that none of the intervals in question is empty; A_i , indeed, can be empty only if $a_i < 0$ and $b_i < 0$, which is incompatible with the hypothesis. Since each of the intervals A_i contains either the point 0 or the point 1, the intersection of all the intervals can be empty only if there exists a pair of intervals, say A_i and A_j , which have no common point; but if this case arose it would be impossible to find a linear combination

$$x(a_i x_i + a_j x_j) + (1-x)(b_i x_i + b_j x_j)$$

(where $0 \leq x \leq 1$) with non-negative coefficients. Since these two forms in two variables satisfy the hypothesis of the theorem, however, we see by the first part of the proof that it leads to a contradiction to suppose that the intervals A_i have an empty intersection.

We now pass to the general case of p forms. We shall reason by induction and, supposing the theorem true for $p - 1$ forms, show that it is also true for p forms. Let, then, the p forms $f_1, f_2, \dots, f_p$ satisfy the conditions of the hypothesis. We shall suppose that the form f_p is not always non-negative for non-negative values of the arguments; for in that case the demonstration would be immediate by taking $p = 1$.

Let $x_1, x_2, \dots, x_n$ be the variables which occur in the forms f_i , and let D be the set of points whose coordinates are non-negative and satisfy the condition

$$f_p(x_1, \dots, x_n) \leq 0.$$

By Lemma 3 we can then define D parametrically by expressions of the form

$$x_i = \sum_{j=1}^p b_{ij} u_j, \quad b_{ij} \geq 0$$

in such a way that, when the point defined by $(u_1, \dots, u_s)$ describes the domain $u_1 \geq 0, u_2 \geq 0, \dots, u_s \geq 0$, the point $(x_1, \dots, x_n)$ describes the domain D (the correspondence not necessarily being one-to-one).

At every point of D , one at least of the forms $f_1, \dots, f_{p-1}$ is non-negative; the forms $f_1, f_2, \dots, f_{p-1}$ in the variables u_j satisfy, then, the conditions of the hypothesis, so there exists a linear combination

$$\phi = N_1 f_1 + \dots + N_{p-1} f_{p-1}, \quad N_i \geq 0, \quad \sum_{i=1}^{p-1} N_i = 1$$

such that the coefficients of the u 's are non-negative.

The form ϕ in the variables $x_1, x_2, \dots, x_n$ is then non-negative at every point of D . Thus the two forms, f_p and ϕ , satisfy the conditions of the hypothesis; we can therefore find a combination

$$\lambda \phi + \mu f_p \quad \lambda \geq 0; \mu \geq 0; \lambda + \mu = 1$$

which is always non-negative for non-negative values of $x_1, \dots, x_n$. Thus our theorem is also true for p forms; for we can take

$$\xi_1 = \lambda \eta_1$$

$$\xi_2 = \lambda \eta_2$$

$$\vdots$$

$$\xi_{p-1} = \lambda \eta_{p-1}$$

$$\xi_p = \mu$$

COROLLARY 1. Let $f_1, \dots, f_p$ and ϕ be homogeneous linear forms in the n variables $x_1, \dots, x_n$ such that, whenever $x_1 \geq 0, \dots, x_n \geq 0$, there exists an i such that

$$f_i(x_1, \dots, x_n) \geq \phi(x_1, \dots, x_n).$$

Then there exist non-negative numbers $\xi_1, \dots, \xi_p$, whose sum is equal to 1, such that, for all non-negative values of $x_1, \dots, x_n$, we have

$$\xi_1 f_1(x_1, \dots, x_n) + \dots + \xi_p f_p(x_1, \dots, x_n) \geq \phi(x_1, \dots, x_n).$$

PROOF. The forms $f_1 - \phi, f_2 - \phi, \dots, f_p - \phi$ satisfy the hypothesis of the Theorem.

DEFINITION. Let S_n be the set of all vectors

$$x = (x_1, \dots, x_n)$$

such that $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ and $x_1 + x_2 + \dots + x_n = 1$.

COROLLARY 2. Let $f_1, \dots, f_p$ and ϕ be homogeneous linear forms in the n variables $x_1, \dots, x_n$ such that, for every $x_1, \dots, x_n$ in S_n , there exists an i such that

$$f_i(x_1, \dots, x_n) \geq \phi(x_1, \dots, x_n).$$

Then there exist non-negative numbers $\xi_1, \dots, \xi_p$, whose sum is equal to 1, such that, for all non-negative $x_1, \dots, x_n$,

$$\xi_1 f_1(x_1, \dots, x_n) + \dots + \xi_p f_p(x_1, \dots, x_n) \geq \phi(x_1, \dots, x_n).$$

PROOF. The hypothesis of Corollary 2 implies the hypothesis of Corollary 1. For suppose the hypothesis of Corollary 2 is satisfied and let $z_1, \dots, z_n$ be non-negative numbers. Then if we set

$$x_1 = \frac{z_1}{z_1 + \dots + z_n}, \dots, x_n = \frac{z_n}{z_1 + \dots + z_n},$$

we see that $(x_1, \dots, x_n)$ is in S_n , and hence there is an i such that

$$f_i(x_1, \dots, x_n) \geq \phi(x_1, \dots, x_n).$$

Multiplying this equation by $z_1 + \dots + z_n$, we conclude, since f_i and ϕ are homogenous, that

$$f_i(z_1, \dots, z_n) \geq \phi(z_1, \dots, z_n).$$

Thus the hypothesis of Corollary 1 is true under the assumption that $z_1 + \dots + z_n \neq 0$. But if $z_1 + \dots + z_n = 0$, then $z_1 = \dots = z_n = 0$, and hence both

$$f_i(z_1, \dots, z_n)$$

and

$$\phi(z_1, \dots, z_n)$$

are equal to zero.

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