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# The Effects of Unsteady Potential Flow on Heated Laminar Boundary Layers in Water: Flow Properties and Stability

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A report prepared for  
DEFENSE ADVANCED RESEARCH PROJECTS AGENCY

**Rand**  
SANTA MONICA, CA. 90406

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PREFACE

This report documents an investigation at Rand of the laminar boundary layer over heated bodies undergoing either acceleration or deceleration in water. The research is part of a larger Rand research project, sponsored by the Defense Advanced Research Projects Agency, which is engaged in developing hydrodynamic design techniques that exploit various methods of boundary-layer control. The report presents a method of predicting laminar boundary-layer characteristics when heating, pressure gradients, and degrees of unsteadiness are acting in concert. Using these results and boundary-layer stability theory, the enhanced boundary-layer stability produced by an accelerating flow is illustrated.

The report should be useful to hydrodynamicists, designers of submersibles, and others engaged in fluid mechanics research. Other related Rand publications include:

R-1752-ARPA/ONR, *Low-Speed Boundary-Layer Transition Workshop*, June 1975.

R-1789-ARPA, *Controlling the Separation of Laminar Boundary Layers in Water: Heating and Suction*, September 1975.

R-1863-ARPA, *The Effects of Wall Temperature and Suction on Laminar Boundary-Layer Stability*, April 1976.

R-1898-ARPA, " $e^9$ ": *Stability Theory and Boundary-Layer Transition*, February 1977.

R-1966-ARPA, *The Buoyancy and Variable Viscosity Effects on a Water Laminar Boundary Layer Along a Heated Longitudinal Horizontal Cylinder*, February 1977.

R-2111-ARPA, *Entry Flow in a Heated Tube*, June 1977.

R-2165-ARPA, *Approximate Methods for Calculating the Properties of Laminar Boundary Layers in Water* (to be published).



### SUMMARY

The laminar boundary layer over heated bodies undergoing acceleration or deceleration in water has been investigated to determine the combined influence of heating, pressure gradient, and unsteadiness on boundary-layer stability. For simplicity, the boundary-layer flow over accelerating wedges or cones has been analyzed for values of the acceleration parameters,  $\epsilon_1 = (x/U_1^2)(dU_1/dt)$ , which are small but not negligible. It is shown that the boundary-layer stability analysis, in this range of  $\epsilon_1$ , requires solution of the unsteady boundary-layer equations, but that stability characteristics may be determined using the classic Orr-Sommerfeld approach.

Velocity profiles have been calculated by a perturbation expansion in  $\epsilon_1$  about the quasi-steady flow, and the stability of the resulting profiles has been calculated by an accurate and convenient modification of the Dunn-Lin theory, specialized to constant density, variable property flow. The results confirm that the acceleration is stabilizing, while deceleration is destabilizing, and show that the stability enhancement of acceleration is less dramatic when both positive velocity gradient and heating are present. A value of  $\epsilon_1$  of only .1 produces a four-fold increase in the critical Reynolds number for a flat plate and a two-fold increase for a highly favorable pressure gradient. Heating reduces these changes by as much as a factor of two. The probable explanation of the effect is discussed.





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## I. INTRODUCTION

Unsteady laminar boundary layers are found in a wide variety of practical circumstances, but the quantitative evaluation of the unsteady effects is not yet widely considered. The effects of unsteadiness are usually assessed by employing the quasi-steady approximation. However, for a slightly more vigorous unsteadiness, the calculation procedure has been to consider small departures from the quasi-steady state. This theory is well developed and is discussed in Refs. 1 through 3 for arbitrary magnitudes of slowly varying unsteadiness. The accuracy of this theory is presently estimated from order of magnitude estimates, or from the calculation of further terms in the solution sequence. However, a recent theoretical study of fluctuating free-stream perturbations to flat-plate flow via matched asymptotic methods suggests strongly that the quasi-steady approach is uniformly valid when the frequency or flow length is small in comparison to a reduced frequency or a body length, respectively (Ref. 4). Nevertheless, it appears that an assessment of the accuracy of the technique must await ultimate comparison with results from the newly fledged numerical schemes for unsteady flow. Until this is done, the quasi-steady approximation sequence used here must be considered the practical method of choice for realistic mild unsteadiness (Ref. 5).

This report discusses the effects of unsteadiness on laminar boundary-layer flows and their stability as governed by the Orr-Sommerfeld equation.\* It presents the required conditions for the validity of the quasi-steady approximation to be a valid approximation of the governing flow equations. It is well known that a critical parameter for boundary-layer flow is the ratio of diffusion time to flow time; the criterion for nearly quasi-steady flow is that this parameter should be small. It is also shown that the criteria for

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\* C. Von Kerczek of Naval Ship Research and Development Center (NSRDC) discussed the effect on transition in a presentation at the Rand Workshop on Low Speed Boundary-Layer Transition, Santa Monica, Calif., September 1976.

validity of the Orr-Sommerfeld equation are based on modifications of this ratio, and are not nearly as stringent. Examples of heated wedge flows in water are presented and discussed that show the profound effect of even slowly varying unsteadiness on both laminar boundary-layer flow and its stability.

## II. ANALYSIS

When the flow external to the boundary layer varies with time, variations in boundary-layer momentum are produced, diffused to the wall, and dissipated. It is through this mechanism that the boundary-layer flow accommodates variation in free-stream velocity. If the boundary layer accommodates instantaneously, the flow may be considered quasi-steady. If, on the other hand, the boundary layer accommodates slowly, the flow is unsteady and the rate of accommodation is important. A measure of how rapidly the boundary layer accommodates is the ratio of diffusion time to flow time. In the following discussion this is the critical parameter, and all comparisons of unsteadiness in boundary-layer flow to unsteadiness in boundary-layer stability are based on its magnitude.

The analysis will first focus on boundary-layer flow in that the quasi-steady flow parameters will be derived and numerical examples will be presented, and next on boundary-layer stability in that a criterion for nearly quasi-steady flow will be developed and numerical examples of the effect of unsteadiness on critical Reynolds numbers will be shown. The specific flows that are used as examples are two-dimensional Falkner-Skan flows with wall heating, variable viscosity, and Prandtl number corresponding to water.

### BOUNDARY-LAYER FLOW

The unsteady equations for a two-dimensional laminar boundary-layer flow at an incompressible fluid with heat transfer and variable viscosity are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left[ \mu(T) \frac{\partial u}{\partial y} \right] \quad (1)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{v_e}{Pr} \frac{\partial^2 T}{\partial y^2}, \quad (2)$$

where  $Pr$  is based on freestream properties.

To assess the effect of unsteadiness on boundary-layer flow, the relative order of magnitude of the first term on the left-hand side of the momentum equation and the viscous term on the right side must be determined. The order of magnitude of the viscous term is

$$\nu \frac{\partial^2 u}{\partial y^2} = O \left( \frac{\nu U_1}{\delta^2} \right),$$

where  $U_1$  is the external velocity with an arbitrary time dependence, and  $\delta$  is the momentum layer thickness.

The order of magnitude of the unsteadiness is estimated by the variation in time of the freestream velocity,

$$\frac{\partial u}{\partial t} = O \left( \frac{\partial U_1}{\partial t} \right).$$

Then, the ratio of magnitude of the two terms is

$$\frac{\delta^2}{\nu} \frac{\partial U_1}{\partial t} = \frac{\text{unsteady effect}}{\text{viscous effect}}.$$

A more physical interpretation consistent with that described in the Introduction is obtained by noting that

$$\frac{\partial U_1}{\partial t} \sim \frac{1}{t_f},$$

where  $t_f$  is a characteristic flow time, i.e., the time it takes the external flow to change from one state to another. Further,  $\nu/\delta$  is a diffusion velocity and  $\delta^2/\nu$  is the time it takes the momentum to diffuse to the wall. Therefore, the ratio of the magnitude of the unsteady effect to the viscous effect is related to the ratio of characteristic times,

$$\frac{\delta^2}{\nu} \frac{\partial U_1}{\partial t U_1} \sim \frac{t_d}{t_f} .$$

If the boundary-layer thickness is related to the local Reynolds number  $[\delta \sim x(\text{Re})^{-1/2}]$ , the quasi-steady parameter of Refs. 1 through 3 is obtained:

$$\frac{\delta^2}{\nu} \frac{\partial U_1}{\partial t U_1} \sim \frac{x}{U_1^2} \frac{\partial U_1}{\partial t} = \epsilon_1 . \quad (3)$$

When this parameter is small, or  $t_d/t_f \ll 1$ , the flow in the boundary layer is quasi-steady. A similar parameter is discussed in Ref. 4. For flows that are nearly quasi-steady, a systematic series expansion procedure can be developed. A detailed discussion of this procedure as it applies to general unsteady flow is given in Ref. 2. However, a special version of nearly quasi-steady flow and this technique are illustrated in the example discussed below.

#### Unsteady Boundary-Layer Flow over Heated Bodies

The theory developed in Refs. 2 and 3 will be extended to examine the unsteady effect on a flow over heated bodies in water with constant freestream and wall temperatures. Although the general theory will be developed, this discussion is concerned with nearly quasi-steady flows where the time variation is limited to slowly varying functions and wedge flows. The attractive attribute of this theory is that for nearly quasi-steady flows the unsteady component of the freestream velocity can have arbitrary magnitude.

Following Ref. 3, a boundary-layer scaling will be employed that allows the quasi-steady component to be determined as the zero-order solution. The scaling to transform Eqs. (1) and (2) is

$$\eta = \sqrt{\frac{U_1}{2\nu x}} y ,$$

$$\psi = \sqrt{2\nu x U_1} f(\eta, x, t) , \quad (4)$$

$$\theta = \frac{T - T_e}{T_w - T_e} ,$$

where  $U_1$ , the inviscid unsteady velocity along the surface, is given by  $U_1 = U_e(x) A(t)$ .

Note that this form of the time-dependent inviscid surface velocity corresponds to unsteady irrotational flow, in the absence of displacement effects.  $U_e(x)$  corresponds to the surface velocity when a body translates with unit velocity into a fluid at rest and  $A(t)$  is the actual time-dependent velocity of translation. Thus,  $U_e(x)$  is to be determined from a conventional potential flow solution corresponding to unit body velocity. The transformed equations for the reduced stream function,  $f(x, \eta, t)$ , and temperature,  $\theta(x, \eta, t)$ , are

$$\begin{aligned} (Nf_{\eta\eta})_{\eta} + (1 - M(x)) ff_{\eta\eta} + 2M(x)(1 - f_{\eta}^2) \\ = 2x(f_{\eta}f_{\eta x} - f_{\eta\eta}f_x) + \frac{2U_1 t x}{U_1^2} (f_{\eta} + \frac{\eta}{2} f_{\eta\eta} - 1) \\ + \frac{2x}{U_1} f_{\eta t} , \end{aligned} \quad (5)$$

and

$$\begin{aligned} \frac{\theta_{\eta\eta}}{Pr} + (1 + M(x)) f\theta_{\eta} = 2x(f_{\eta}\theta_x - \theta_{\eta}f_x) \\ + \frac{2U_1 t x}{U_1^2} (\frac{\eta}{2} \theta_{\eta}) + \frac{2x}{U_1} \theta_t , \end{aligned} \quad (6)$$



where  $M(x) = U_{1x}x/U_1$ , and  $N$  is the ratio of boundary layer to free-stream viscosity.  $N$  can be approximated by the empirical expression for water used in Ref. 6:

$$N = \mu/\mu_e = (a + br + cr^2 + dr^3 + er^4)^{-1} \frac{\mu_{ref}}{\mu_e}, \quad (7)$$

where  $r = T/491.69^\circ R$ , and  $\mu_{ref}/\mu_e = 4.339 \text{ lb/hrft}/\mu_e$ . The appropriate boundary conditions are

$$\begin{aligned} t &\geq 0, \\ \eta &\rightarrow \infty, \quad f_\eta \rightarrow 1, \quad \theta \rightarrow 0, \\ \eta &= 0, \quad f = f_\eta = 0, \quad \theta = 1. \end{aligned} \quad (8)$$

The general velocity and temperature distributions are functions of three independent variables, i.e.,

$$f = f(\eta, x, t). \quad (9)$$

However, for special cases where  $f$  is a slowly varying function of time, one will find it convenient to express the velocity distribution as

$$f = f(\eta, x; \epsilon_n), \quad (10)$$

where now  $f$  depends on a sequence of general unsteady parameters

$$\epsilon_n = \left( \frac{x}{U_1} \right)^n \frac{\partial^n U_1}{U_1 \partial t^n}, \quad n = 1, 2, 3 \dots \quad (11)$$

The first parameter,  $\epsilon_1$ , is an obvious expansion parameter (the ratio of diffusion time to flow adjustment times) that is explicitly displayed

in the equations. For  $n > 1$ , the parameters are ratios of various diffusion times to various flow times and a result of substituting the expansion of  $f$  into the governing equations and requiring a self-consistent set of differential equations.

When a body has constant acceleration, the parameters  $\epsilon_n$  for  $n > 1$  are precisely zero. For flows that have nearly constant acceleration, some consideration must be provided for  $\epsilon_n$  when  $n > 1$ . It is assumed here that  $|\epsilon_1|^n > \epsilon_n$  for  $n > 1$ . This allows the velocity and temperature distributions to be expressed to first order as

$$f(\eta, x, t) = f_0(\eta, x) + \epsilon_1 f_1(\eta, x) , \quad (12)$$

$$\theta(\eta, x, t) = \theta_0(\eta, x) + \epsilon_1 \theta_1(\eta, x) .$$

After substituting Eq. (12) into Eqs. (5) and (6), one obtains the set of differential equations representing the zero and first-order approximations to a nearly quasi-steady flow:\*

Zero Order:

$$\begin{aligned} (N_0 f_{0\eta\eta})_\eta + (1 + M(x)) f_0 f_{0\eta\eta} + 2M(x)(1 - f_{0\eta}^2) \\ = 2x(f_{0\eta} f_{0\eta x} - f_{0\eta\eta} f_{0x}) \end{aligned} \quad (13)$$

$$\frac{\theta_{0\eta\eta}}{\text{Pr}} + (1 + M(x)) f_0 \theta_{0\eta} = 2x(f_{0\eta} \theta_{0x} - \theta_{0\eta} f_{0x}) . \quad (14)$$

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\* Note that the viscosity ratio,  $N$ , is also expanded in the form  $N = N_0 + \epsilon_1 N_1$ .

First Order:

$$\begin{aligned}
 (N_0 f_{1\eta\eta})_\eta + (1 + M(x)) f_0 f_{1\eta\eta} - 2 (1 + M(x)) f_{0\eta} f_{1\eta} + (3 - M(x)) f_{0\eta\eta} f_1 \\
 = 2x(f_{0\eta} f_{1\eta x} + f_{1\eta} f_{0\eta x} - f_{0\eta\eta} f_{1x} - f_{1\eta\eta} f_{0x}) \\
 + 2(f_{0\eta} + \frac{\eta}{2} f_{0\eta\eta} - 1) - (N_1 f_{0\eta\eta})_\eta ;
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 \frac{\theta_{1\eta\eta}}{\text{Pr}} + (1 + M(x)) f_0 \theta_{1\eta} - 2 (1 - M(x)) f_{0\eta} \theta_1 \\
 = -f_1 \theta_{0\eta} (3 - M(x)) + 2x(f_{0\eta} \theta_{1x} + f_{1\eta} \theta_{0x} - \theta_{0\eta} f_{1x} - \theta_{1\eta} f_{0x}) \\
 + \eta \theta_{0\eta} .
 \end{aligned} \tag{16}$$

The boundary conditions are

$$\begin{aligned}
 \eta = 0: \quad f_0 = f_{0\eta} = f_1 = f_{1\eta} = 0; \quad \theta_0 = 1; \quad \theta_1 = 0 , \\
 \eta \rightarrow \infty: \quad f_{0\eta} \rightarrow 1; \quad f_{1\eta} \rightarrow 0; \quad \theta_0 \rightarrow 0; \quad \theta_1 \rightarrow 0 .
 \end{aligned} \tag{17}$$

In a later section a special version of these equations will be specialized to wedge flows where  $U_e \sim x^m$ .

#### BOUNDARY-LAYER STABILITY

At present there is no theory to predict the hydrodynamic stability of an unsteady flow when unsteady effects are large. For cases where the time-dependent component of the freestream velocity varies slowly, one could develop a theory involving multiple time scales, but this approach has not yet been carried out for the spatial stability problem. In this subsection, characteristic times for the Orr-Sommerfeld equation will be derived and used to establish a heuristic criterion to define the region where unsteadiness modifies the stability analysis.

Using this criterion, it will be shown that the stability of an unsteady boundary layer can be studied by using an approach that employs unsteady boundary-layer flow theory and quasi-steady stability analysis. This approximation, together with an approximate solution of the Orr-Sommerfeld equation, will be employed to discuss the stability of the flows presented in the previous section.

Referring to Ref. 7 for a heuristic discussion of the Orr-Sommerfeld equation for an arbitrary time-dependent mean flow, one will find that the constant property two-dimensional version of the equation is

$$\begin{aligned} \frac{\delta}{U_1 i \alpha} \frac{\partial}{\partial t} (\phi'' - \alpha^2 \phi) + (u - c)(\phi'' - \alpha^2 \phi) - u'' \phi \\ = \left( \frac{1}{i \alpha R} \right) [\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi] , \end{aligned} \quad (18)$$

where the superscript prime denotes a derivative with respect to  $(y/\delta)$ . All velocities are scaled with  $U_1$ ;  $R$ , the Reynolds number, is based on boundary-layer thickness; and  $\alpha$  is the wave number.

The approach that will be used in this subsection will be to investigate the Orr-Sommerfeld equation in three regimes: the outer inviscid region, the inner viscous region near the wall, and the critical layer. These are the important regions for a uniformly valid solution (Ref. 8), when matched asymptotic expansion methods are applied.

#### The Inviscid Stability Equations

In the limit of large  $\alpha R$ , the equation reduces to

$$\frac{\delta}{U_1 i \alpha} \frac{\partial}{\partial t} (\phi'' - \alpha^2 \phi) + (u - c)(\phi'' - \alpha^2 \phi) - u'' \phi \cong 0 . \quad (19)$$

To determine the relative effect of an unsteady freestream velocity, the magnitude of the first two terms must be compared. The magnitude of the ratio of the unsteady term to a convection term is at most

$$\frac{(\delta/U_1 i\alpha)/(\partial/\partial t) \phi''}{(u - c) \phi''} = 0 \left( \frac{\delta}{\alpha} \frac{\partial U_1}{\partial t U_1^2} \right), \quad (20)$$

where the magnitude of the convection term is approximated by

$$(u - c)(\phi'' - \alpha^2 \phi) = 0(1). \quad (21)$$

For the purpose of comparison with the boundary-layer quasi-steady parameter, it is convenient to reduce Eq. (20) to a multiple of the ratio of diffusion time to flow time. The new parameter is

$$\frac{\delta}{\alpha} \frac{\partial U_1}{\partial t U_1^2} = \frac{\delta^2}{\alpha R \nu} \frac{\partial U_1}{\partial t U_1} = \frac{t_d}{t_f} \cdot \frac{1}{\alpha R}, \quad (22)$$

and for quasi-steady flow in the inviscid stability equation this parameter must be small. Note that this restriction is less stringent than the boundary-layer condition since  $\alpha R \sim 10^3$  for a flat-plate boundary layer.

#### The Wall Viscous Region

Following Ref. 8, a scaling appropriate for the viscous region near the wall is

$$\tilde{\eta} = y(R\alpha)^{1/2}; \quad \frac{F(\tilde{\eta}, t)}{(R\alpha)} = \phi(y, t). \quad (23)$$

Employing the new dependent and independent variables to evaluate the unsteady term and the first viscous term which is the dominant viscous term of Eq. (18), it can be shown that since

$$\frac{\delta}{U_1 \alpha} \frac{\partial}{\partial t} (\phi'' - \alpha^2 \phi) = \frac{\delta}{U_1 \alpha} \frac{\partial}{\partial t} \left( F_{\tilde{\eta}\tilde{\eta}} - \frac{\alpha^2 F}{R\alpha} \right) \quad \text{and} \quad \frac{\phi''''}{\alpha R} = F_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}}. \quad (24)$$

If one estimates that  $O(\partial F_{\eta\eta}/U_1 \partial t) = O(\partial/\partial t)(\ln U_1)$ , it can be concluded that the reduced viscous term is of unit order and the relative magnitude of the unsteady term is at most

$$\frac{\delta}{U_1^2 \alpha} \frac{\partial U_1}{\partial t} = \frac{t_d}{t_f} \cdot \frac{1}{\alpha R} . \quad (25)$$

Thus the unsteady contribution for mild unsteadiness is the same order of magnitude in the wall viscous region as it is in the outer inviscid region.

#### The Region around the Critical Layer

The next region to investigate is the region near the critical layer. The most convenient way to analyze this region is to use the following change in dependent and independent variables:

$$\eta = (y - y_c)(\alpha R)^{1/3} ; \quad \phi(y, t) = G \frac{(\eta, t)}{(\alpha R)^{1/3}} . \quad (26)$$

These definitions scale the unsteady term and the first viscous term. The results are

$$\frac{\delta}{U_1 \alpha} \frac{\partial}{\partial t} (\phi'' - \alpha^2 \phi) = \frac{\delta}{U_1 \alpha} (\alpha R)^{1/3} \frac{\partial}{\partial t} \left( G_{\eta\eta} - \frac{\alpha^2 G}{(\alpha R)^{2/3}} \right) ; \quad (27)$$

$$\frac{\phi''''}{\alpha R} = G_{\eta\eta\eta\eta} .$$

The resulting order of magnitude of the ratio of these two terms is

$$\frac{\delta}{\alpha} \frac{(\alpha R)^{1/3}}{U_1^2} \frac{dU_1}{dt} \cong (t_d/t_f) (\alpha R)^{-2/3} . \quad (28)$$

This parameter must be small for the quasi-steady approximation to be accurate in the critical layer. From this sequence of estimates it is seen that when  $t_d/t_f$  is small, then the unsteady laminar velocity profile can be obtained as a correction to the quasi-steady flow (nearly quasi-steady), and the stability of the resulting profiles may be studied using classical Orr-Sommerfeld analysis.

#### Simplified Stability Analysis of Unsteady Flow over Heated Surfaces

Ultimately, a complete analysis of the stability characteristics of specific unsteady flows, via numerical solution of the Orr-Sommerfeld equation, would be useful. For this preliminary study, stability analysis is restricted to the estimation of minimum critical Reynolds number. Lin's original simplified analysis (Ref. 9), and Dunn's subsequent modification for compressible flow (Ref. 10) have been widely used in simplified stability analyses. Recently, it was shown that a modified version of this theory and formulae is appropriate for water boundary layers with variable viscosity (Ref. 11). In addition, certain numerical constants in the original Dunn-Lin formulae have been revised on the basis of accurate numerical results which were not yet available to Dunn and Lin. While the original Dunn-Lin theory is now known to be inappropriate for compressible flow, its treatment of property variations is satisfactory. For completeness, the modified Dunn-Lin relations of Ref. 11 are

$$v(c) \left( 1 - 2\lambda(c) \right) = .58 , \quad (29a)$$

where

$$v(c) = - \frac{\pi u_c'' u_w' c}{u_c'^3} ; \quad (29b)$$

$$\lambda = \frac{u'_w}{c} \left[ \frac{v_w/v_e}{c} \right]^{\frac{1}{2}} \left( \frac{3}{2} \int_0^{\eta_c} \sqrt{\frac{c-u}{v/v_e}} d\eta \right) \quad (30)$$

$$-1 \approx .4 \left\{ \left( 1 - \frac{u'_c}{u'_w} \right) + .5 \left( 1 - \frac{v(c)}{v_w} \right) \right\}.$$

$$Re_{crit} \approx 28 \frac{u'_w}{c^4} \left( \frac{v_w}{v_e} \right). \quad (31)$$

The equations, together with the boundary-layer velocity profiles, will be used to recalculate  $Re_{crit} = U_1 \delta_1 / v_e$  evaluated at the minimum point of neutral stability.



### III. UNSTEADY BOUNDARY-LAYER FLOW OVER HEATED WEDGE FLOWS

The approach developed and discussed in Section II has been used to investigate the stability of heated wedge flows under conditions of positive or negative acceleration. This section describes the application of the general method and numerical results to this class of flows. In this way, the combined effects of heating, pressure gradient, and unsteadiness can be studied.

#### UNSTEADY BOUNDARY-LAYER FLOW

When the flow is steady and the external velocity is represented by  $U_e \sim x^m$  (wedge flow), the boundary-layer flow is self-similar. The boundary-layer equations become ordinary differential equations, and the variable  $M(x)$  in Eq. (5) is replaced by the constant parameter  $\beta$ , that is equal to  $2m/(1+m)$ . For nearly quasi-steady flow and  $U_e \sim x^m$ , the departure from a self-similar flow is governed by the departure from a quasi-steady flow, and this is controlled by the magnitude of the parameter  $\epsilon_1$ . Thus, for the special case of wedge flows, Eq. (12) reduces to

$$\begin{aligned} f(\eta, x, t) &= \left( \bar{f}_0(\bar{\eta}) + \epsilon_1 f_1(\bar{\eta}) \right) / \left( \sqrt{1+m} \right), \\ \theta(\eta, x, t) &= \theta_0(\bar{\eta}) + \epsilon_1 \theta_1(\bar{\eta}), \end{aligned} \quad (32)$$

where  $\bar{\eta} = \eta \sqrt{1+m}$ . The new scaling by the constant  $\sqrt{1+m}$  introduced into Eq. (32) is to facilitate comparison to related works, i.e., Refs. 2 and 12. If Eq. (32) is substituted into Eqs. (13) through (16), the following sets of ordinary differential equations are obtained:

Momentum:

$$\begin{aligned} (N_0 \bar{f}_0'')' + \bar{f}_0 \bar{f}_0'' + \beta [1 - (\bar{f}_0')^2] &= 0, \\ (N_0 \bar{f}_1'')' + \bar{f}_0 \bar{f}_1'' - 2 \bar{f}_0' \bar{f}_1' + (3 - 2\beta) \bar{f}_0'' \bar{f}_1 &= (2 - \beta) \left[ \bar{f}_0' + \eta \frac{\bar{f}_0''}{2} - 1 \right] + (\alpha N_0^2 \theta_1 \bar{f}_0'')'. \end{aligned} \quad (33)$$

Energy:

$$\frac{\theta_0''}{Pr} + \bar{f}_0 \theta_0' = 0 , \quad (34)$$

$$\frac{\theta_1''}{Pr} + \bar{f}_0 \theta_1' - 2(1 - \beta) \bar{f}_0' \theta_1 = - (3 - 2\beta) \theta_0' \bar{f}_1 + (2 - \beta) \frac{\eta \theta_0'}{2} .$$

The boundary conditions are the same as those shown in Eq. (17). The first-order equations for  $\bar{f}_1$  and  $\theta_1$  have homogeneous boundary conditions and would have a trivial solution if the right-hand sides of Eqs. (33) and (34) were zero. From this it is implied that the effects of unsteadiness decrease as the pressure gradient becomes more favorable because the magnitude of the inhomogeneous term decreases. The impact of heating is not clear.

Equations (33) and (34) are solved numerically, and the resulting parameters are given in Table 1 for a heated water boundary layer with an ambient temperature of 520°R. Moreover, the skin friction coefficient,  $C_f$ , the Nusselt number,  $Nu$ , the displacement thickness,  $\delta_1$ , and the momentum thickness,  $\delta_2$ , are written as

$$\begin{aligned} \frac{\delta_2}{x} \sqrt{\frac{Re_x(1+m)}{2}} &= \int_0^\infty (\bar{f}' - \bar{f}'^2) d\bar{\eta} \\ &= \delta_{2,0} + \epsilon \delta_{2,1} ; \end{aligned} \quad (35)$$

$$C_f \frac{\mu_e}{\mu_w} \sqrt{\frac{Re_x}{2(1+m)}} = \bar{f}_0''(0) + \epsilon_1 \bar{f}_1''(0) ; \quad (36)$$

$$Nu = \sqrt{\frac{Re_x(1+m)}{2}} \theta_0'(0) + \epsilon_1 \theta_1'(0) ; \quad (37)$$

$$\frac{\delta_1}{x} \sqrt{\frac{Re_x(1+m)}{2}} = \int_0^\infty (1 - \bar{f}') d\bar{\eta} = \delta_{1,0} + \epsilon \delta_{1,1} . \quad (38)$$

## RESULTS

In view of the previous discussion and that of Ref. 2, the results presented in Table 1 are not surprising. In fact, the results for  $\beta = 0$  and  $T_w - T_e = 0$  are identical to those given in Ref. 2. The new results are those for  $\beta \neq 0$  and  $T_w - T_e \neq 0$ . In general, we can conclude that accelerating flows demonstrate increased skin friction coefficient and decreased Nusselt number. These conclusions are consistent with those presented in Ref. 2. Further, acceleration results in a decreased displacement thickness and a small increase in momentum thickness. The physical explanation for this is that acceleration causes an increase in the convection of momentum, and the wall shear must increase to dissipate this excess momentum.

Similarly, for a nearly quasi-steady accelerating flow, the unsteady contribution to the heat transfer is affected principally by the thermal inertia, which resists heat transfer and leads to a decrease in Nusselt number. Similar findings have been reported in Ref. 13. There is a diminution of these trends as the pressure gradient becomes more favorable. Naturally, the contrary is true for decelerating flows.

A word of caution regarding flows with adverse pressure gradient: Note the relatively large magnitude of  $\bar{f}_1''(0)$  in comparison to that of a flat plate. This implies that the unsteadiness correction is larger, and the region of validity of the two-term expansion may be more limited for adverse gradients.

The stability results were obtained by applying Eqs. (29) through (31) to determine the critical Reynolds number (based on displacement thickness,  $Re_{crit}$ ). The effect of flow unsteadiness in the range  $-.05 < \epsilon < .1$  was considered for a variety of pressure gradients and surface overheat. These results are summarized in Table 2 and are illustrated in Figs. 1 through 6.

Using  $Re_{crit}$  as an indication of stability, it can be seen that for  $\beta = 0$ , and  $T_w - T_e = 0$ , the effects of unsteadiness are most significant. This parameter is reduced by 35 percent when  $\epsilon$  is  $-.05$ , increased by 200 percent when  $\epsilon$  is  $+.05$ , and increases by almost 600 percent when  $\epsilon$  doubles from  $.05$  to  $1$ .

Table 1

BOUNDARY-LAYER CHARACTERISTICS

| $\beta$ | $T_w - T_e$ | $\bar{f}_0''(0)$ | $\theta_0'(0)$ | $\bar{f}_1''(0)$ | $\theta_{1,0}'$ | $\delta_{1,0}$ | $-\delta_{1,1}$ | $\delta_{2,0}$ | $-\delta_{2,2}$ |
|---------|-------------|------------------|----------------|------------------|-----------------|----------------|-----------------|----------------|-----------------|
| 0       | 0           | 0.46947          |                | 1.1999           |                 | 1.218          | .7273           | .46917         | .0060           |
|         | 30          | 0.65699          | -1.0161        | 1.4275           | .31512          | 1.070          | .6232           | .4459          | .0338           |
|         | 40          | 0.72072          | -1.0394        | 1.4919           | .31156          | 1.028          | .59127          | .4378          | .0406           |
|         | 60          | 0.84796          | -1.0326        | 1.6063           | .30943          | 0.9525         | .53284          | .4216          | .0513           |
| 0.1     | 0           | 0.58694          |                | 0.97274          |                 | 1.080          | .44490          | .4354          | .00302          |
|         | 30          | 0.8062           | -1.072         | 1.1533           | .45502          | 0.9491         | .39775          | .4111          | .0273           |
|         | 40          | 0.8797           | -1.0948        | 1.2046           | .44761          | .9115          | .381542         | .4029          | .0331           |
|         | 60          | 1.0252           | -1.1371        | 1.2959           | .43780          | .8441          | .35034          | .3866          | .0423           |
| 0.2     | 0           | 0.6866           |                | 0.8208           |                 | .9838          | .30339          | .4082          | .0014           |
|         | 30          | 0.932558         | -1.11379       | 0.96722          | .52867          | .8632          | .28058          | .3836          | .0230           |
|         | 40          | 1.0143           | -1.1365        | 1.0087           | .5200           | .8287          | .2714           | .3753          | .0281           |
|         | 60          | 1.1752           | -1.178         | 1.0825           | .507496         | .7668          | .2527           | .3591          | .0362           |
| 0.5     | 0           | .92776           |                | .57146           |                 | .8048          | .1277           | .3502          | .0127           |
|         | 30          | 1.2366           | -1.2002        | .62156           | .62266          | .7022          | .12988          | .3255          | .0150           |
|         | 40          | 1.3378           | -1.2222        | .64356           | .61370          | .6732          | .12828          | .3174          | .0199           |
|         | 60          | 1.53511          | -1.2630        | .68228           | .59963          | .6215          | .1234           | .3016          | .0261           |
| -0.05   | 0           | .400123          |                | 1.3729           |                 | 1.313          | 1.0028          | .4903          | .0077           |

Table 2

CRITICAL REYNOLDS NUMBER AND SHAPE FACTOR AS A FUNCTION OF  
UNSTEADINESS PARAMETER  $\epsilon$ , SURFACE OVERHEAT,  $T_w - T_e$ ,  
AND PRESSURE GRADIENT PARAMETER  $\beta$

| $\beta$ | Critical Reynolds Number, $(U_1 \delta_1 / \nu_e)_{crit}$ |                |                   |                    |                  | Shape Factor, $H = \delta_1 / \delta_2$ |                   |                    |                  |
|---------|---|----------------|-------------------|--------------------|------------------|---|-------------------|--------------------|------------------|
|         | $T_w - T_e$   | $\epsilon = 0$ | $\epsilon = 0.05$ | $\epsilon = -0.05$ | $\epsilon = 0.1$ | $\epsilon = 0$                          | $\epsilon = 0.05$ | $\epsilon = -0.05$ | $\epsilon = 0.1$ |
| 0       | 0   | 514            | 1040              | 332                | 2940             | 2.59                                    | 2.51              | 2.67               | 2.44             |
|         | 30  | 4770           | 10240             | 2050               | 18100            | 2.40                                    | 2.34              | .246               | 2.28             |
|         | 40  | 7270           | 13680             | 3450               | 21800            | 2.35                                    | 2.29              | 2.41               | 2.23             |
|         | 60  | 11100          | 17900             | 6190               | 25700            | 2.26                                    | 2.21              | 2.31               | 2.16             |
| 0.1     | 0   | 1380           | 2750              | 794                | 5490             | 2.48                                    | 2.43              | 2.53               | 2.38             |
|         | 30  | 9460           | 14490             | 5690               | 20300            | 2.31                                    | 2.27              | 2.36               | 2.23             |
|         | 40  | 12270          | 17520             | 7980               | 23200            | 2.26                                    | 2.22              | 2.30               | 2.19             |
|         | 60  | 15570          | 20470             | 11170              | 25710            | 2.18                                    | 2.15              | 2.22               | 2.12             |
| 0.2     | 0   | 2880           | 4800              | 1750               | 7610             | 2.41                                    | 2.37              | 2.45               | 2.34             |
|         | 30  | 13100          | 17200             | 9540               | 21500            | 2.25                                    | 2.22              | 2.28               | 2.19             |
|         | 40  | 15700          | 19700             | 12100              | 23800            | 2.21                                    | 2.18              | 2.24               | 2.15             |
|         | 60  | 18300          | 21700             | 14800              | 25200            | 2.14                                    | 2.11              | 2.16               | 2.09             |
| 0.5     | 0   | 7860           | 9670              | 6300               | 11700            | 2.30                                    | 2.28              | 2.32               | 2.25             |
|         | 30  | 18900          | 20800             | 16900              | 22700            | 2.16                                    | 2.14              | 2.17               | 2.13             |
|         | 40  | 20600          | 22300             | 18800              | 24000            | 2.12                                    | 2.11              | 2.14               | 2.09             |
|         | 60  | 21400          | 22800             | 19900              | 24100            | 2.06                                    | 2.05              | 2.07               | 2.04             |
| -0.05   | 0   | 311            | 555               | 216                | 1681             | 2.68                                    | 2.58              | 2.78               | 2.48             |

From Figs. 1 through 5, we observe that for those situations in which pressure gradient and surface overheat is small, even mild unsteadiness ( $|\varepsilon| \leq .05$ ) can have powerful effects on the value of the critical Reynolds number and the location of the point of neutral stability. However, larger values of favorable pressure gradient and surface overheat which are already characterized by increased stability, exhibit less sensitivity to unsteadiness. From the viewpoint of the boundary-layer wall compatibility condition alone, this result is surprising. However, further reflection and reconsideration of the known effects of combined heating and pressure gradient on stability (Ref. 14) suggest that the situation may not be very different here. For combined heating and pressure gradient, it is known that the relative effect of additional surface overheat on stability decreases with increasing favorable pressure gradient or surface overheat. In fact, it has been shown that large values of overheat can decrease the stability of flat-plate and adverse pressure gradient flows.

Note that the shape of the curves changes when  $\Delta T$  increases. As suggested above, the larger the value of  $\Delta T$ , the less sensitive  $Re_{crit}$  is to further heating. At  $\beta = .5$ , for example,  $Re_{crit}$  exhibits only weak dependence on further surface overheat and unsteadiness when  $\Delta T > 30^\circ$ .

Figure 5 shows an adverse pressure gradient case ( $\beta = -.05$ ) in which the effects of unsteadiness are again significant. This arises because unsteadiness dominates the wall region, and stability is primarily controlled by details of the velocity profile between the critical layer and wall.

Figure 6 shows a collection of the results for the various pressure gradients, wall heating, and degree of unsteadiness, and a function of the shape factor  $H = \delta_1/\delta_2$ . A fair degree of correlation is demonstrated, indicating that  $H$  is, to a rough approximation, a universal parameter that can be used to estimate the combined effects of these several parameters on stability. The line on this figure is the result of exact numerical computations (Ref. 15) for steady isothermal wedge flows. Similar computations for steady heated wedge flows in water (Ref. 14) have already demonstrated that  $H$  is a useful correlation parameter for the combined effects of pressure gradient and heating under conditions of moderate heating.

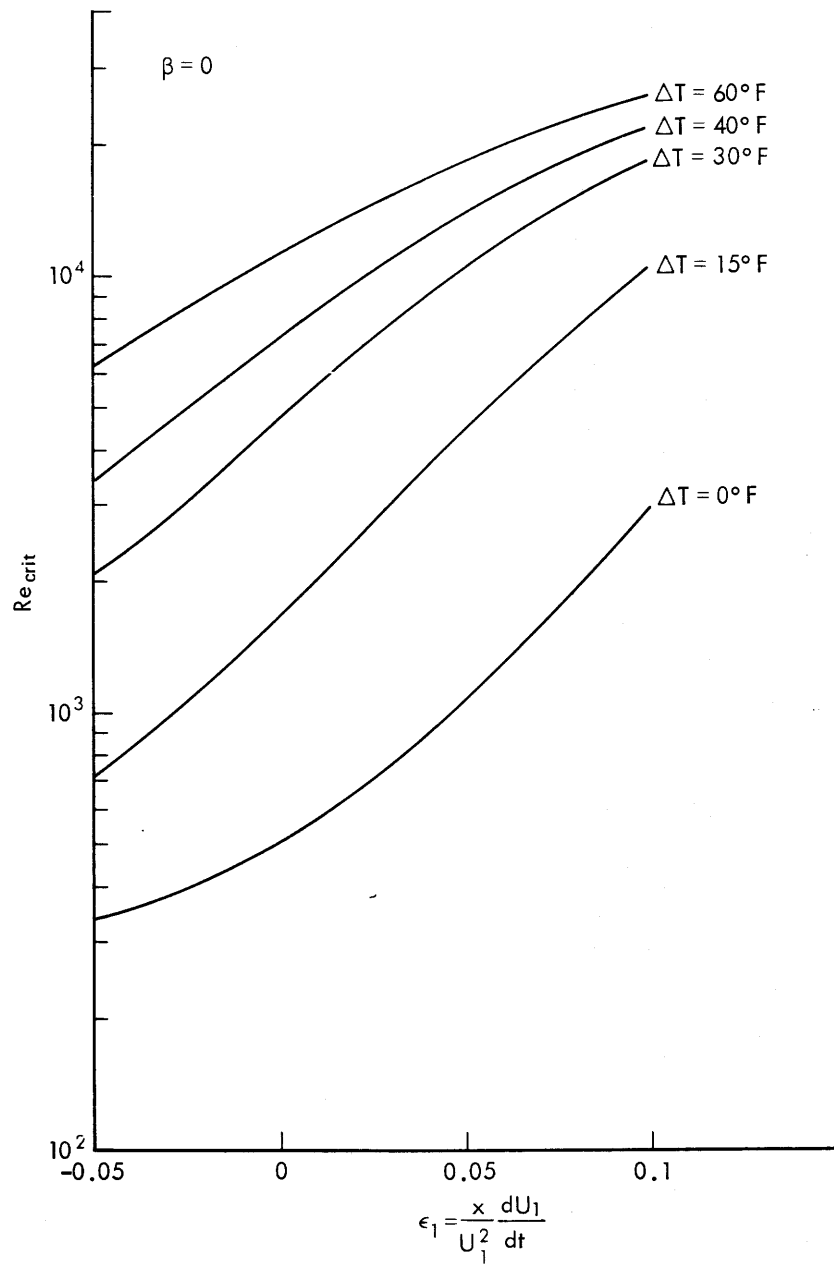


Fig. 1—Critical Reynolds number as a function of the quasi-steady parameter with wall heating and zero pressure gradient ( $\beta = 0$ )

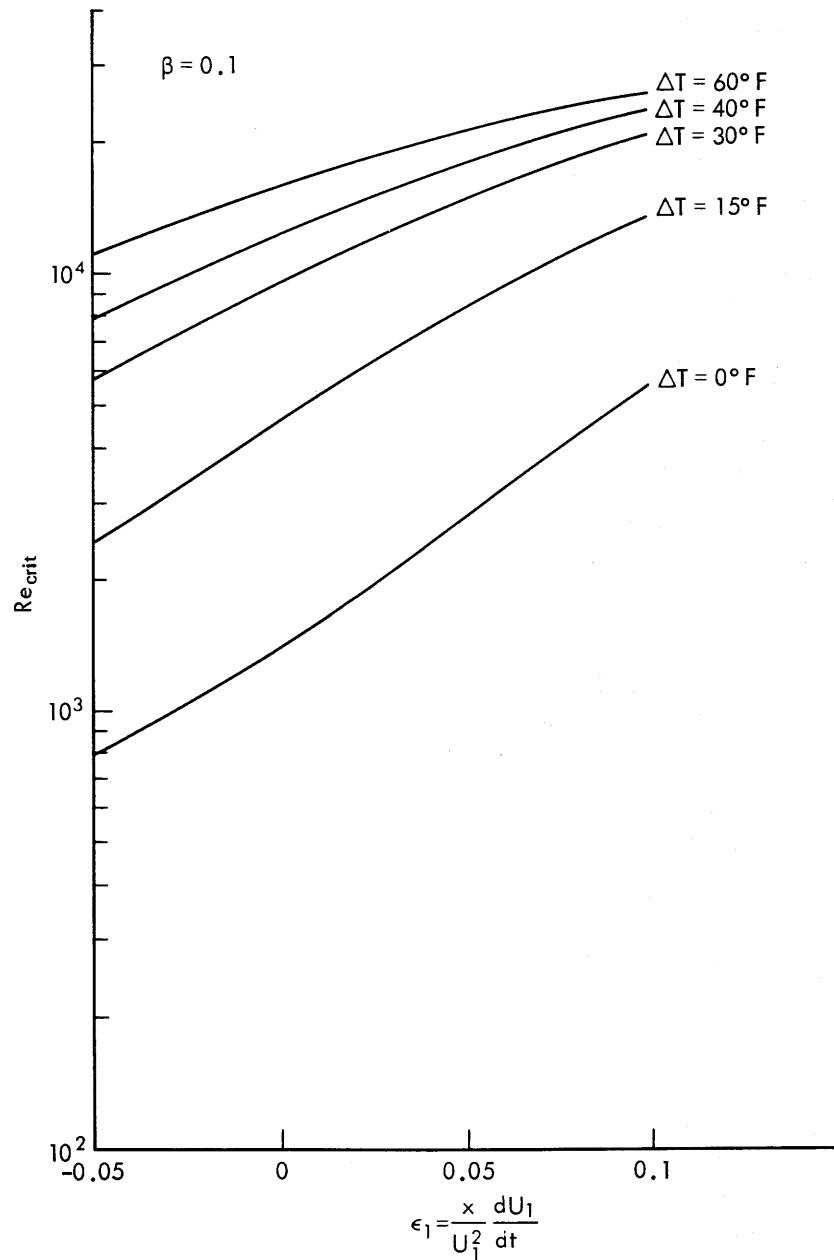


Fig. 2— Critical Reynolds number as a function of the quasi-steady parameter with wall heat and pressure gradient  $\beta = 0.1$

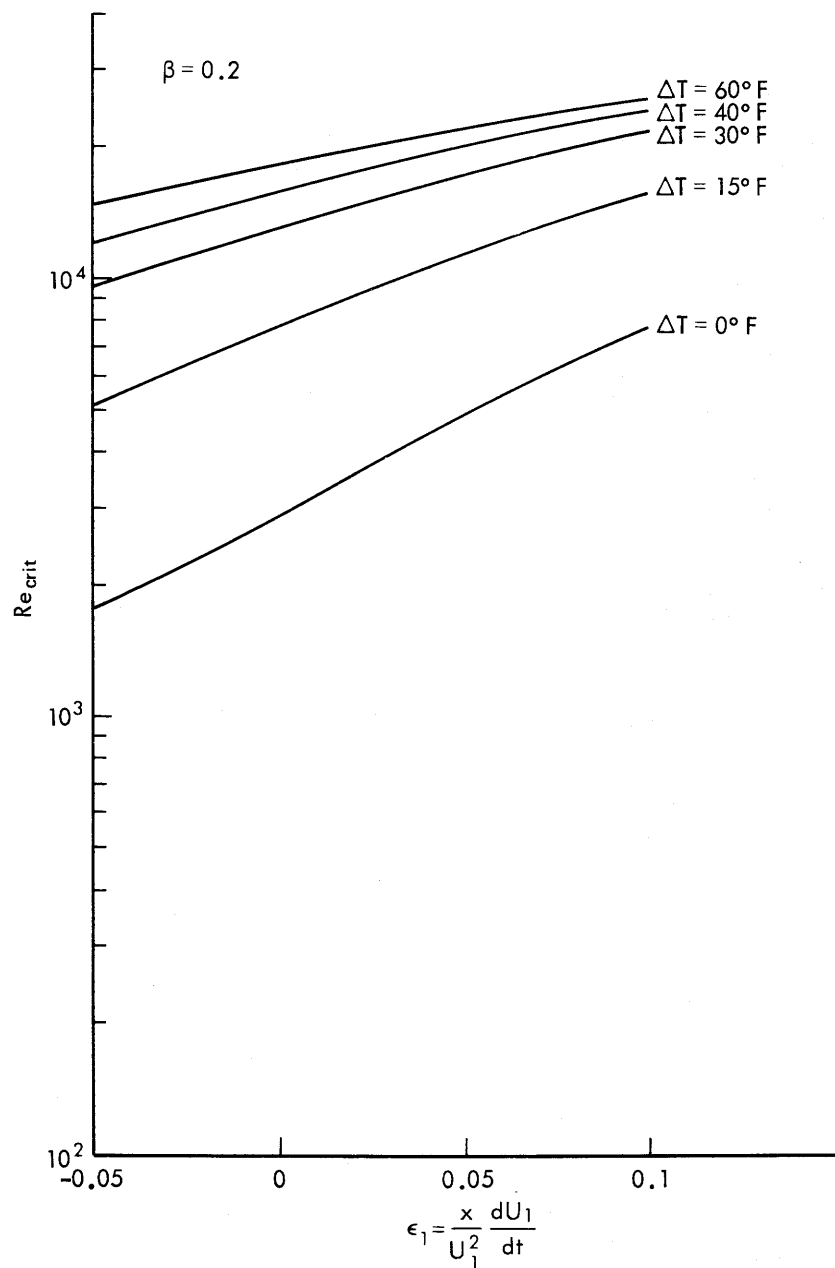


Fig. 3—Critical Reynolds number as a function of the quasi-steady parameter with wall heat and pressure gradient  $\beta = 0.2$



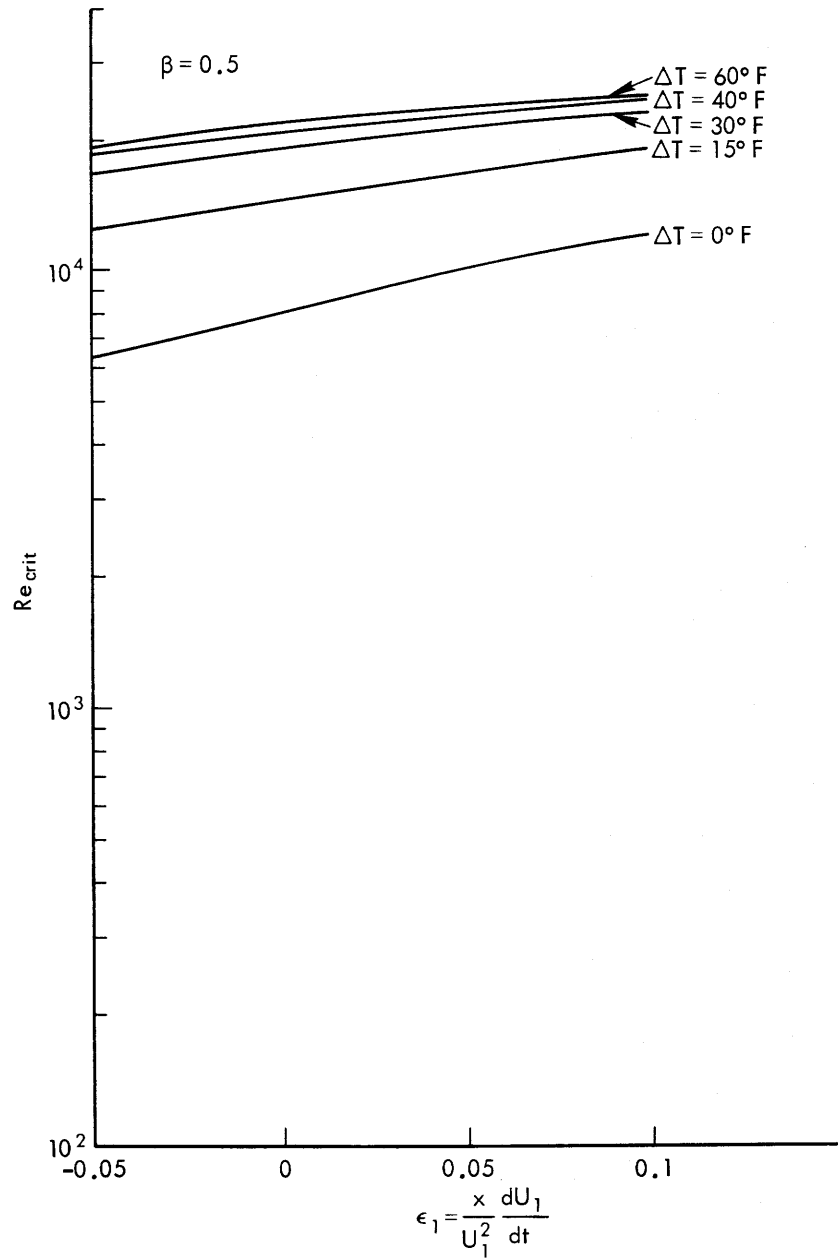


Fig. 4— Critical Reynolds number as a function of the quasi-steady parameter with wall heat and pressure gradient  $\beta = 0.5$

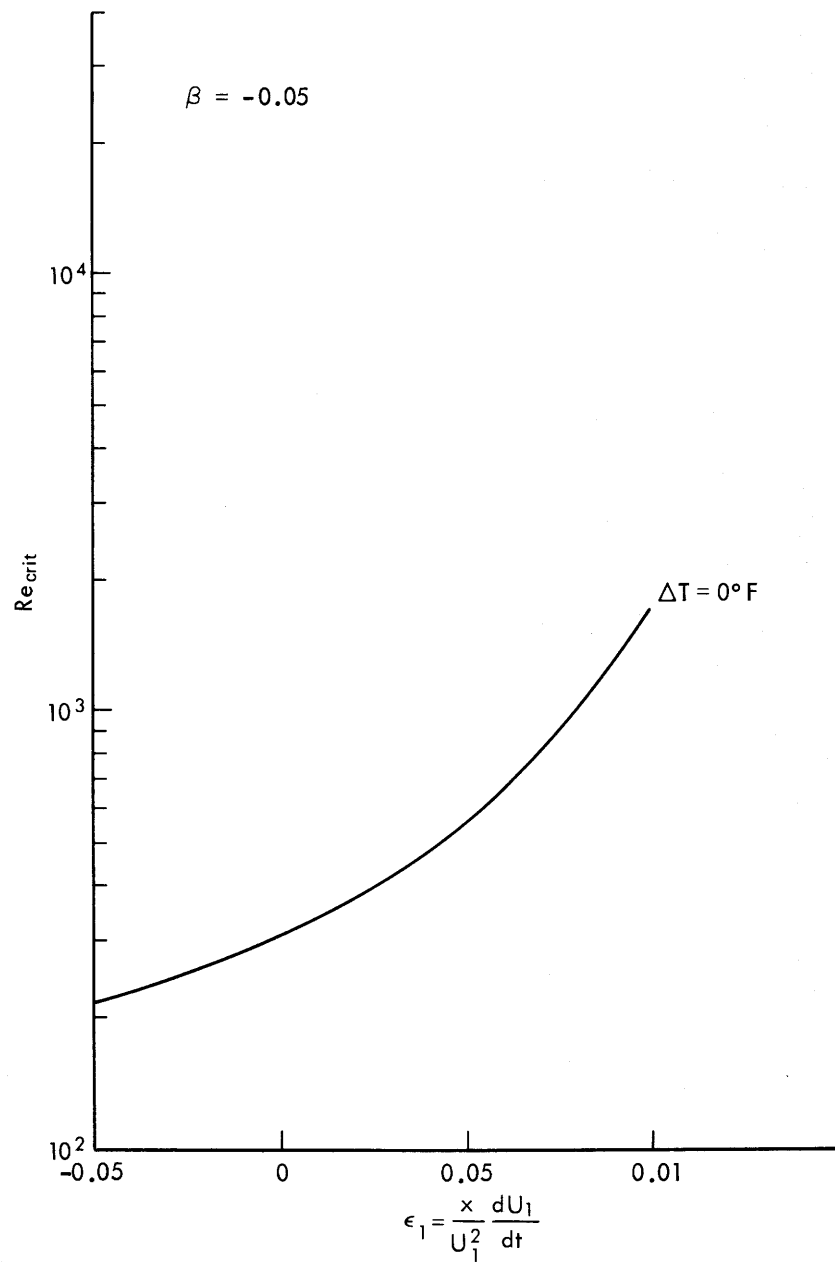


Fig. 5—Critical Reynolds number as a function of the quasi-steady parameter with wall heat and pressure gradient  $\beta = -0.05$

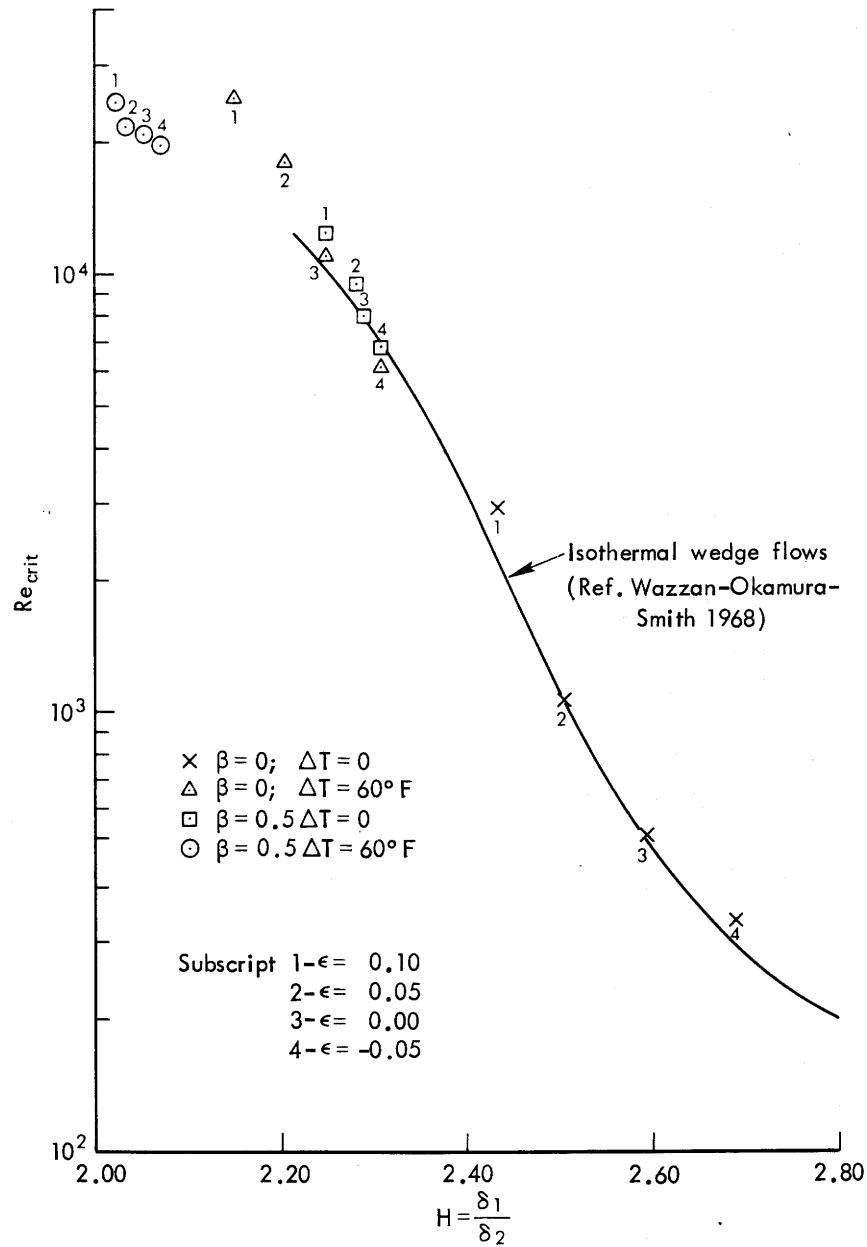


Fig. 6—Critical Reynolds number as a function of the shape parameter

#### IV. CONCLUSION

Under conditions that are likely to occur in practice, the present nearly quasi-steady approach to the calculation of unsteady laminar boundary-layer velocity profiles should be highly accurate. Our analysis of the stability of unsteady flow is not yet as precise as it will be after deeper study involving multiple time scales. However, the heuristic arguments presented here suggest that there is an important practical flow regime where classical Orr-Sommerfeld analysis can still be used to define the stability characteristics of nearly quasi-steady velocity profiles. Our analysis and computation, performed on this basis, infer that even mild unsteadiness can have a powerful impact on flow stability. This impact is particularly strong at realistic levels of pressure gradient and heating in water.

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