AN EXAMPLE OF A SLOW-CONVERGING CORE

PREPARED FOR THE NATIONAL SCIENCE FOUNDATION

LLOYD S. SHAPLEY

R-1476-NSF
JANUARY 1974

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This report was sponsored by the National Science Foundation under Grant GS-31253. Reports of The Rand Corporation do not necessarily reflect the opinions or policies of the sponsors of Rand research.

Published by The Rand Corporation

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PREFACE

This Report presents a result in basic economic theory obtained under a grant from the National Science Foundation for investigations into mathematical economics using methods of game theory.

Previous Rand publications in the immediate subject area include R-1056-NSF: On Cores and Indivisibility (Scarf and Shapley), RM-4917-PR: Pure Competition, Coalitional Power, and Fair Division (Shapley and Shubik), and RM-4248-PR: A General Exchange Economy With Money (Shapley). For a broader overview of the subject, the reader is referred to R-904-NSF: Game Theory in Economics (Shapley and Shubik), of which Chapters 1, 2, 3, and 6 are now available.

The author wishes to thank Dr. Robert Shishko for several useful suggestions.

SUMMARY

This Report concerns the shrinking of the core of a trading economy as the number of traders is increased without limit. The core is a "cooperative" construct, depending as it does on the capabilities of coalitions, but the limiting solution that it approaches has been shown (under suitably general conditions) to consist of the purely "competitive" outcomes that arise from impersonal price schedules that equilibrate supply and demand in the market place. The rate at which the core shrinks therefore provides a means of studying the lessening role of collusion in the face of increased competition.

Experience with replicated market models (i.e., increasing sequences of economies in which each trader type appears in a fixed proportion) suggests that the "normal" asymptotic rate of convergence of the core is inversely proportional to the number of traders. Our present finding, however, is that neither this nor any other rate can be guaranteed; indeed, one can construct cores that converge as slowly as one pleases. Our proof uses just two commodities, two types of trader, and a single concave, homothetic utility function that is continuously differentiable, though not twice-differentiable.

Our result seems to indicate that even in a "large" economic system, with many perfectly substitutable traders

of every type, it is still possible for significant departures from pure competition to occur if the kind of collusion embodied in the "core" concept is not enjoined.

CONTENTS

1. INTRODUCTION

In considering the asymptotic properties of economies with many traders and their limiting solutions (of various kinds), it is important not to lose sight of the manner in which the limits are approached. The ideal, limiting solutions of a large economic model will not have much relevance to economics--either practical or theoretical--unless there is reason to believe that corresponding real systems exist or could exist that are sufficiently large for the limiting results to apply.* The study of "error terms" and rates of convergence is therefore crucial to the question of model validation.

This note concerns the shrinking of the core in k-replicated market models. Experience with particular cases** would suggest a typical or "normal" rate of convergence of the order of 1/k. But our present construction will show that neither this nor any other rate can be guaranteed, even for markets with continuously differentiable utility functions. If there is a general theorem that promises a rate of O(1/k), it will apparently have to include a condition

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*In a recent study (unpublished) of the convergence of the bargaining set for k-replicated markets, our best estimate for the rate of convergence, even with favorable differentiability assumptions, proved to be only $O(k^{-1/4})$. At that rate, to assure another decimal place of accuracy would require 10,000 times as many traders of each type!

**See [3] for some examples and for a general discussion of replicated economies and their solutions.

that keeps the traders' indifference surfaces from bending too sharply--for example, a condition that bounds the <u>second</u> derivatives of the utility functions. As we shall see, if we allow arbitrarily short radii of curvature, such as are found in the graph of the function $|t|^{3/2}$ near $t = 0$, then rates of convergence slower than $O(1/k)$ can occur; indeed, the rate can be made as slow as we please.*

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*Actual kinks or corners, however, do not retard the the convergence; their effect is instead to enlarge the limit core.

2. THE MODEL

The following simple k-replicated "Edgeworth market game" will provide the setting for our examples of slow core convergence. There are two goods and two types of traders, with k traders of each type. Initially, the goods are distributed according to the allocation

$$<(1, 0)^k, (0, 1)^k>.$$

(The superscript "k" on a bundle of goods indicates it is to be repeated k times in the allocation.) We are not concerned here with the details of the market transactions, but we do assume that every redistribution of goods among a consenting set of traders is possible. The traders have identical tastes, represented by the utility function $u(x, y)$, applied by each trader to his final holding. We shall particularize this function later; for the present it is just assumed to be continuous, increasing, strictly concave, and homothetic.* It follows that the indifference curves are multiples of each other, are strictly convex, and have tangents (or support lines) with strictly positive normals.

The core of the market may be defined as the set of final allocations** that are coalitionally stable, in that

*That is, there is an increasing function f such that $f(u,(x, y))$ is positively homogeneous of degree 1.

**In general game theory, the core is usually defined no submarket could have "improved" on the result, i.e., could have obtained more utility for each of its members using just their original resources. Since the market as a whole is one possible submarket, core allocations are automatically Pareto optimal. We are interested in the behavior of the core as a function of k.

Let E(k) denote the set of allocations of the form

$$\langle (x,\ y)^k,\ (1 - x,\ 1 - y)^k \rangle; \tag{1}$$

these are sometimes called the "equal treatment" allocations. It is a relatively elementary result that the core (in this context) consists entirely of equal treatment allocations; the reason is that any other allocation would be subject to "improvement" by a coalition consisting of one least-favored trader of each type. The two-dimensional set E(k) is a subset of the (4k - 2)-dimensional set of all feasible allocations.

We can of course make an identification between points (1) in E(k) and points (x, y) in the unit square, $Q = [0,\ 1]^2$. Indeed, Q is the familiar Edgeworth box. For clarity, however, a distinction between E(k) and Q must be main-

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in terms of *payoffs* (utility vectors) rather than *outcomes* (allocations), since the space of unevaluated outcomes need not have any particular structure. It would not be difficult to adapt our present treatment of convergence to the case of cores defined in utility terms.

tained, since we shall be concerned with comparing distances for different values of k, and E(k) is in a space whose dimension changes with k while Q is not.* We shall call the image of the core in Q the "Q-core."

In the present setting, the equal-treatment Pareto allocations are just those of the form**

$$(2) \qquad \langle (x, x)^k, (1 - x, 1 - x)^k \rangle, \qquad 0 \leqq x \leqq 1.$$

The Pareto set therefore corresponds to the diagonal of Q and the core to a subset of that diagonal. Similarly, in any submarket having "profile" (ℓ, m), i.e., having ℓ traders of the first type and m of the second, the Pareto allocations are given by

$$(3) \qquad \langle (x, \tfrac{m}{\ell}x)^{\ell}, (\tfrac{\ell}{m}(1 - x), 1 - x)^m \rangle, \qquad 0 \leqq x \leqq 1.$$

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*For example, in the Euclidean or ℓ_2 norm it is perfectly possible for the diameter of the core to go to infinity even as the diameter of the Q-core goes to zero. In the ℓ_∞ norm, which is used in [1], this does not happen. The issue, however, is not in what type of norm or distance function is used, but rather in how they are scaled for different values of k. The simplest scaling for ℓ_2 happens to have a different effect from the simplest scaling for ℓ_∞.

**Since all traders have identical utility functions and Pareto optimality requires the same marginal rate of substitution for all traders, homotheticity and strict concavity of u implies that the final consumption bundles will differ from each other only by a factor of proportionality, It is not difficult to show (see [2] or [3]) that the submarkets that have the best chance of "improving" on an allocation of the form (2) are those that omit just one trader, i.e., the submarkets having profile either (k, k - 1) or (k - 1, k). This means that the endpoints of the Q-core--call them (c_1, c_1) and (c_2, c_2), $c_1 \leq c_2$--are determined by the equations

$$(4) \quad \begin{cases} u(x, \frac{k-1}{k}x) = u(c_1, c_1) \\ u(\frac{k}{k-1}(1-x), 1-x) = u(1-c_1, 1-c_1), \end{cases}$$

with a similar pair of equations for c_2.*

It can be shown that c_1 and c_2 are nondecreasing and nonincreasing functions of k, respectively, so that the Q-core can only shrink as k increases. If u is differentiable then c_1 and c_2 converge to a common limit, making the limiting Q-core a single point.** Without differen-

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that is, the final bundles can only lie along a ray. Equal treatment and the material balance equations then yield (2). The argument for (3) is the same.

*The interpretation of (4) is that a coalition with profile $(\ell, m) = (k, k-1)$ can, by using (3) with a suitable value of x, just barely achieve the utility levels they get at the allocation $\langle (c_1, c_1)^k, (1-c_1, 1-c_1)^k$.

**As the reader is no doubt aware, this common limit represents the unique <u>competitive</u> solution to the underlying market. (This solution is the same for all k.) Uniqueness here depends on homotheticity as well as differentiability.

tiability, however, the limits may differ, making the limiting Q-core a closed interval.

3. RATE OF CONVERGENCE

We shall now particularize $u(x, y)$ to a class of continuously differentiable* utility functions for which the diameter of the Q-core goes to zero arbitrarily slowly. That is, given any arbitrary, preassigned rate of convergence, there can be found a function in the class that yields something slower.

Let $F(t)$ be any function on $(0, \infty)$ that converges to 0 more slowly than $O(1/t)$, as $t \to \infty$; an example to help fix ideas is $F(t) \equiv 1/\sqrt{t}$. Without loss of generality, we may take $F(t)$ to be continuous and monotonically decreasing, but with $tF(t)$ monotonically increasing without limit.

<u>THEOREM</u>. <u>There exists a continuously differentiable,* strictly concave, strictly monotonic, homothetic utility function</u> $u(x, y)$ <u>for the</u> k-<u>replicated Edgeworth market (as described above) for which</u> $|c_2 - c_1|$ <u>goes to zero at least as slowly as</u> $F(k)$, <u>as</u> $k \to \infty$; <u>in other words, we have</u>

$$\liminf_{k\to\infty} \frac{|c_2 - c_1|}{F(k)} > 0. \tag{5}$$

<u>Proof</u>. Let $f(t) \equiv F(1/t)$. Without loss of generality, $f(1) = F(1) = 1$. Define (see Fig. 1)

*Except at the origin.

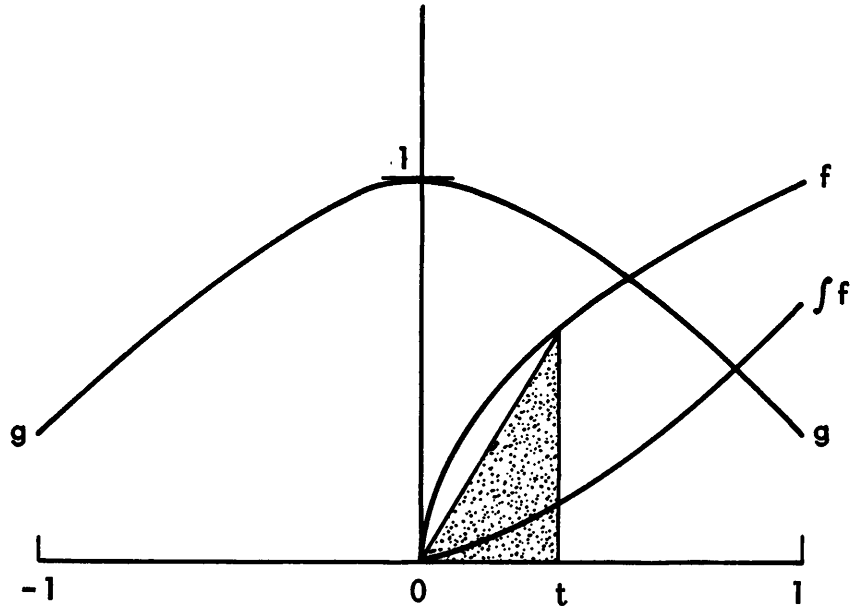

Fig. 1

(6) $$g(t) = 1 - \int_0^{|t|} f(s)ds, \qquad t \in [-1, 1].$$

This is an even function: $g(t) \equiv g(-t)$; it is also positive and strictly concave and continuously differentiable (even at $t = 0$). Note that for all $t \in (0, 1]$ we have

(7) $$1 - g(t) > tf(t)/2;$$

this may be seen by comparing the area under the graph of f with that of the shaded triangle--note that the chord from (0, 0) to (t, f(t)) lies beneath the graph of f because $f(t)/t \equiv (1/t)F(1/t)$ is a decreasing function of t.

Now define $u(0, 0) = 0$ and, for all other $(x, y) \geqq 0$,

(8) $$u(x, y) = (x + y)g(\frac{x - y}{x + y}).$$

This function is continuously differentiable except at the origin (where it is continuous), and it is concave, strictly monotonic, and homogeneous of degree 1. Indeed, of all the properties called for in the theorem we lack only strict concavity, and that property fails only along rays from the origin. But that is easily taken care of: let h be a strictly concave, increasing, continuously differentiable function; then the function $u_h(x, y) \equiv h(u(x, y))$ will have <u>all</u> the required properties. Moreover, we can just as well use u as u_h in our analysis since the order-preserving transformations h, h^{-1} make no difference to the core, which is strictly an ordinal concept.

Applying (8) to (4) we obtain

$$(9)\qquad \begin{cases} \frac{2k-1}{k} x g(\frac{1}{2k-1}) = 2c_1 \\ \frac{2k-1}{k-1}(1-x) g(\frac{1}{2k-1}) = 2(1-c_1), \end{cases}$$

since u(1, 1) = 2. To eliminate x we multiply by k and k - 1 respectively and add, obtaining

$$\begin{aligned}(2k-1)g &= 2kc_1 + 2(k-1)(1-c_1) \\ &= 2c_1 + 2k - 2,\end{aligned}$$

where we have written g for g(1/(2k - 1)), and so

$$c_1 = \frac{1}{2} - \frac{2k-1}{2}(1-g).$$

From $1 - g(t) \leq |t|$ it follows that c_1 and x are both in [0, 1], so the solution is legitimate. By symmetry, $c_2 = 1 - c_1$, and so we obtain the explicit formula:

$$(10)\qquad |c_2 - c_1| = (2k-1)(1 - g(\frac{1}{2k-1})).$$

To complete the proof, we apply (7) to (10) and obtain

$$|c_2 - c_1| > \frac{1}{2} f(\frac{1}{2k - 1}) = \frac{1}{2} F(2k - 1) > \frac{1}{2} F(2k) > \frac{1}{4} F(k),$$

using the properties of F. This shows that the limit in (5) is in fact $\geq 1/4$. Q.E.D.

To illustrate the result, set $F(t) \equiv t^{-\gamma}$ for γ between 0 and 1. Then $f(t) = t^{\gamma}$ and $g(t) = 1 - |t|^{1+\gamma}/(1 + \gamma)$, whereupon (8) and (10) yield

$$u(x, y) = (x + y)[1 - \frac{1}{1 + \gamma}(\frac{|x - y|}{x + y})^{1+\gamma}] \tag{11}$$

and

$$|c_2 - c_1| = \frac{1}{1 + \gamma}(2k - 1)^{-\gamma}. \tag{12}$$

Figure 2 illustrates the indifference curves of the utility function (11) for several values of γ, and Table 1 illustrates the rate of convergence of the core, as measured by the difference $|c_2 - c_1|$.

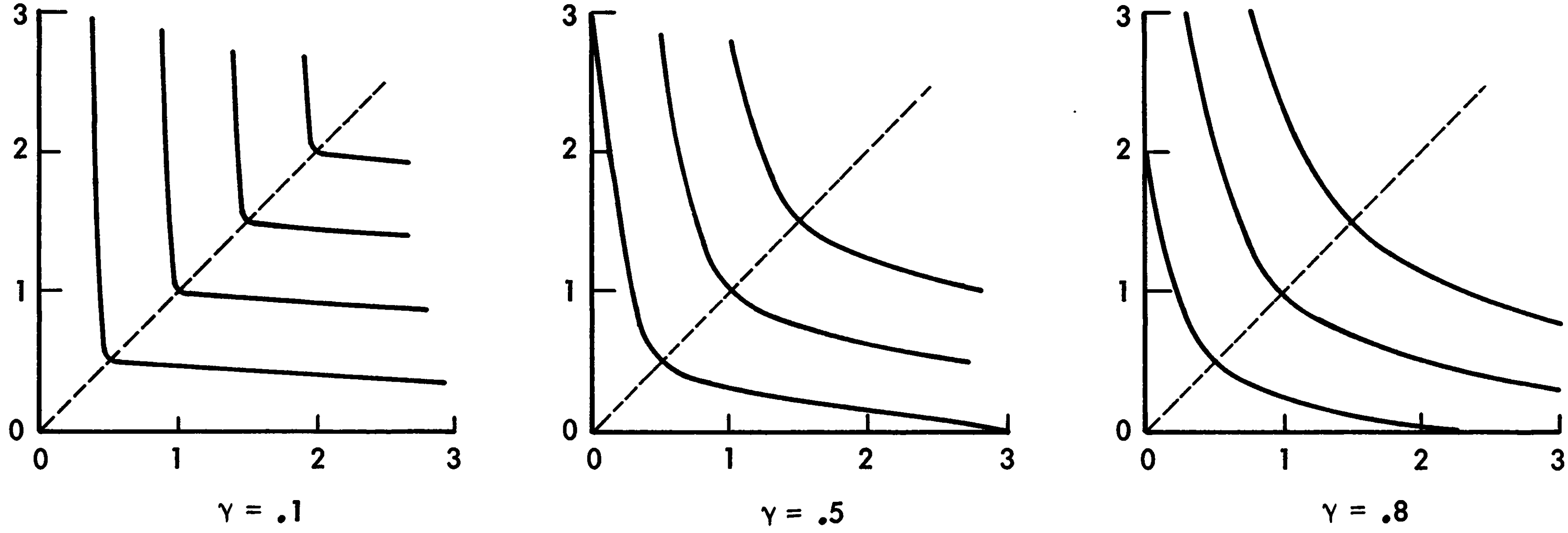

Fig. 2

Table 1

CONVERGENCE OF THE Q-CORE FOR $F(k) = k^{-\gamma}$

	$\gamma = .1$		$\gamma = .5$		$\gamma = .8$	
	$\lvert c_2 - c \rvert$	F(k)	$\lvert c_2 - c_1 \rvert$	F(k)	$\lvert c_2 - c_1 \rvert$	F(k)
k=10	.6772	.7943	.1529	.3162	.0527	.1585
$k=10^2$	.5355	.6310	.0473	.1000	.0081	.0251
$k=10^3$	.4251	.5012	.0149	.0316	1.27×10^{-3}	3.98×10^{-3}
$k=10^4$	.3377	.3981	4.71×10^{-3}	.0100	2.01×10^{-4}	6.31×10^{-4}
$k=10^5$	.2682	.3162	1.49×10^{-3}	3.16×10^{-3}	3.19×10^{-5}	1.00×10^{-4}
$k=10^6$	.2131	.2512	4.71×10^{-4}	1.00×10^{-3}	5.06×10^{-6}	1.59×10^{-5}

4. REMARKS

Our assumption of homotheticity is hardly more than a convenience; it is not a special condition without which the example would fail. It enables us to be explicit at (2) and (3), and hence at (4) and eventually at (10). Like the other conditions on u, it helps to show how "nice" the utility function can be and still produce unusual behavior.

In another sense, however, our example is not robust. It should be clear that a small perturbation of the initial bundles would, in general, move the scene of action away from the critical diagonal, the ray $x = y$, where the unboundedness of the second derivatives of $u(x, y)$ is what makes the example work. Indeed, away from that ray the convergence is at the "normal" rate of $O(1/k)$. It would be possible to construct a utility function having more such critical rays--even countably many of them. But it seems reasonable to conjecture that, for any fixed concave utility function, only a set of initial allocations of measure zero will yield Q-cores that converge more slowly than $O(1/k)$.

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