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A BOMBER FIGHTER DUEL (II)

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0. Summary. This memorandum completes the study of the fighter–bomber duel described in RM-165. The duel is one in which a fighter fires a single rocket burst at a bomber, which has limited ammunition, and defends itself by intermittent firing. It turns out that, for fixed accuracies and values of the bomber and fighter, the nature of the strategies depends on the amount δ of ammunition at the disposal of the bomber. There are two critical amounts δ_0, δ_1, $\delta_0 \geq \delta_1$. If $\delta < \delta_1$, the bomber starts firing at a specified range at less than maximum intensity and continues firing, with decreasing intensity, until the end of the engagement. If $\delta_1 < \delta < \delta_0$, the bomber starts firing at a specified range, greater than in the preceding case, fires at full intensity for a specified time, and with decreasing intensity for the rest of the engagement. In both cases the fighter uses a mixed strategy, spread over the entire time the bomber is firing at less than maximum intensity, with a positive probability of firing at range 0. This probability decreases as δ increases. For $\delta > \delta_0$, the bomber fires at full intensity from the time the fighter comes within range until a specified time, and decreasing intensity thereafter. There is a certain time, during the period in which the bomber is firing at full intensity, at which the fighter should always fire. Ammunition in excess of δ_0 is useful to the bomber only in case the fighter makes the mistake of waiting too long before firing.

1. <u>Preliminaries</u>. The game considered has pay-off

$$V(x, p) = A(x)\Phi(x, p)$$

where
$$\Phi(x, p) = e^{-\int_0^x p(y)r(y)dy} .$$

I chooses a number x, $0 \leq x \leq 1$, while II chooses a function p(y) subject to $0 \leq p(y) \leq 1$ and $\int_0^1 p(y)dy = \delta$. The member δ and the functions A(x), r(y) are given, and we suppose that A(x), r(y) are non-negative, continuous, and strictly increasing functions on the unit interval. It is known that the best strategy for II is a pure one, while for I it is mixed. Let F(x) be a distribution function in the unit interval (F(x) is monotonically increasing with F(0) = 0, F(1) = 1) describing a possible mixed strategy of I. The expected value of the game is then

$$E[F(x), p(y)] = \int_0^1 A(x)\, e^{-\int_0^x p(y)r(y)dy}\, dF(x) .$$

The solution given here for this game applies in the case that $\frac{A'(x)}{A(x)r(x)}$ is a decreasing function of x; this will be true for example whenever log A(x) is concave. The extension to the general case can be made, but it is more tedious and seems of little practical importance.

2. <u>The good strategy for the bomber</u>. The good strategy for the bomber can be obtained as in RM-165, using the following lemma proved there:

<u>Lemma</u>. For any two strategies $p_1(y)$, $p_2(y)$, $V(x, p_1) \geq V(x, p_2)$ for all x implies $p_1 = p_2$ almost everywhere (so that $V(x, p_1) = V(x, p_2)$ for all x).

It is also obtained independently on the considerations of §3 below.

Define $m(y) = \frac{A'(y)}{A(y)(y)}$, $w(y) = \min(m(y), 1)$, $\delta_0 = \int_0^1 w(y)dy$.

We are supposing that m(y) is decreasing, so that w(y) is also decreasing.

<u>CASE I</u>. $\delta \leq \delta_0$. Define $c \ni \int_c^1 w(y)dy = \delta$, $p_0(y) = 0$, $y < c$, $p_0(y) = w(y)$, $c \leq y \leq 1$. <u>We now prove that p_0 is a good strategy for the bomber</u>.

There is a number $d \ni p_0 = 1$, $c \leq y \leq d$, $p_0 = m(y)$, $d \leq y \leq 1$. Concerning d, we see that $d = c$ if $m(c) \leq 1$, $d = 1$ if $m(1) \geq 1$, while d is the unique point where $m(d) = 1$ if $m(c) > 1$ and $m(1) < 1$. We have

$$V(x, p_0) = \begin{cases} A(x), & 0 \leq x \leq c \\ A(x)\, e^{-\int_c^x r(y)dy}, & c \leq x \leq d \\ V(d, p_0), & d \leq x \leq 1 \,. \end{cases}$$

For $c \leq x \leq d$, $V' = V\left[\frac{A'}{A} - r\right]$. Since $m = \frac{A'}{Ar} \geq 1$ in this interval,

$V \geq 0$ and V is increasing. Thus $\max_x V(x, p_0) = v(d, p_0)$. Now for any $p \neq p_0$ a.e., the lemma asserts that there is an x^* with $V(x^*, p) > V(x^*, p_0)$. This x^* must exceed c, since no $V(x, p)$ can exceed $A(x)$ for any x. If $x^* \geq d$, $\max_x V(x, p) > V(d, p_0)$. If $c \leq x^* \leq d$, we shall show that $V(d, p) \geq V(d, p_0)$. We have

$$V(d, p) = \frac{A(d)}{A(x)} V(x^*, p) e^{-\int_{x^*}^{d} p(y) r(y) dy}$$

$$V(d, p_0) = \frac{A(d)}{A(x)} V(x^*, p_0) e^{-\int_{x^*}^{d} r(y) dy} .$$

Comparison shows $V(d, p) \geq V(d, p_0)$. Thus for any p, $\max_x V(x, p) \geq V(d, p_0)$; p_0 is a good strategy, and the value of the game is $V(d, p_0)$.

<u>Case II</u>. $\delta > \delta_0$. Any $p_0 \ni p_0(y) \geq w(y)$ for all y turns out to be a good strategy in this case. Clearly, for any such p_0, $V(x, p_0) \leq V(x, w)$ for all x so that $\max_x V(x, p_0) \leq \max_x V(x, w)$. We show that, for any p, $\max_x V(x, p) \geq \max_x V(x, w)$. We have

$$V(x, w) = \begin{cases} A(x) e^{-\int_{0}^{x} r(y) dy} & 0 \leq x \leq d \\ v(d, w) & d \leq x \leq 1 \end{cases} .$$

As in Case I, $\max_x V(x, w) = V(x, d)$. Moreover, for any p,

$$V(d,\ p) = A(d)e^{-\int_0^d r(y)p(y)dy} \geq V(d,\ w)\ .$$

Thus the value of the game is V(d, w), and any $p_o \geq w$ is a good strategy.

3. <u>A good strategy for the fighter</u>. The fighter will select a mixed strategy as its good strategy. This will be obtained by using the following well-known inequality:

$$\int_0^1 e^{-\phi(x)}\ dF(x) \geq e^{-\int_0^1 \phi(x)dF(x)}$$

valid for any distribution function F(x) and Borel measurable function $\phi(x)$, with the equality sign holding if and only if $\phi(x)$ is constant except on a set of F-measure 0 (this is a general inequality valid for an arbitrary convex function in place of the exponential).

In the expression for $\mathbf{E}[F(x),\ p(y)]$ write the integrand as

$$e^{-\left[\int_0^x p(y)r(y)dy - \log A(x)\right]}$$

and apply the inequality, there results

$$\text{(1)} \qquad \mathbf{E}[F(x),\ p(y)] \geq \mathbf{h}[F(x),\ p(y)]$$

where

$$\text{(2)} \qquad \mathbf{h}[F,\ p] = e^{-\int_0^1\left[\int_0^x p(y)r(y)dy - \log A(x)\right]dF(x)}\ .$$

Integrating by parts in the exponential, we obtain

$$(3) \qquad \log h[F,\, p] = -\int_0^1 p(y)r(y)dy + \log A(1) + \int_0^1 F(x)\left\{p(x)r(x) - \frac{A'(x)}{A(x)}\right\}dx$$

$$= \int_0^1 \left\{F(x) - 1\right\} p(x)r(x)dx + \log A(1) - \int_0^1 F(x)\,\frac{A'(x)}{A(x)}\,dx\,.$$

(We will apply the above formulas to a particular distribution which vanishes identically near $x = 0$, so there is no difficulty in the integration by parts concerning the term $\log A(x)$ in case $A(0) = 0$). In this expression for $\log h$, the only term containing $p(x)$ is $\int_0^1 \left\{F(x) - 1\right\} p(x)r(x)dx$, while $p(x)$ must satisfy the condition $\int_0^1 p(x)dx = \delta$. This suggests selecting a distribution $F(x)$ for which $\left(F(x) - 1\right)r(x) = \text{constant}$. This suggestion will provide us with a good strategy for the fighter.

Case I. (See discussion of Case I in § 2.) Define c and d as in the discussion of § 2.

Define the distribution function $F^*(x)$ by

$$(4) \qquad F^*(x) = \begin{cases} 0 & \text{for } x < d \\ 1 - \dfrac{r(c)}{r(x)} & \text{for } d < x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

From (3), we have

$$\log h[F^*, p] = -\int_0^d p(x)r(x)dx - r(c)\int_d^1 p(x)dx$$

$$+ \log A(1) - \log \frac{A(1)}{A(d)} + r(c)\int_d^1 \frac{A'(x)}{A(x)r(x)}\, dx .$$

But $\int_d^1 p(x)dx = \delta - \int_0^d p(x)dx$,

and $\int_d^1 \frac{A'(x)}{A(x)r(x)}\, dx + \int_c^d 1 \cdot dx = \delta .$

Thus

$$\log h[F^*, p] = -\int_0^d p(x)r(x)dx + r(c)\int_0^d p(x)dx$$

$$+ \log A(d) - r(c) \cdot \int_c^d 1 \cdot dx$$

$$= \log A(d) + \int_0^c \{r(c) - r(x)\}\, p(x)dx$$

$$+ \int_c^d \{1 - p(x)\}\{r(x) - r(c)\}\, dx - \int_c^d r(x)dx$$

$$= \log A(d) - \int_c^d r(x)dx + \int_0^c \{r(c) - r(x)\}\, p(x)dx$$

$$+ \int_c^d \{1 - p(x)\}\{r(x) - r(c)\}\, dx$$

$$\geq \log A(d) - \int_c^d r(x)dx$$

with equality holding only if $p(x) = 0$ in $0 < x < c$, $p(x) = 1$ in $c < x < d$. Thus

$$(5) \qquad h[F^*, p] \geq A(d)e^{-\int_c^d r(x)dx}$$

with equality holding if and only if $p(x) = 0$ in $0 < x < c$, and $p(x) = 1$ in $c < x < d$.

For the distribution $F^*(x)$, the equality sign in (1) holds if and only if $\int_0^x p(y)r(y)dy - \log A(x) = \text{constant}$ in $d < x < 1$, or $p(x) = \frac{A'(x)}{A(x)r(x)}$ in this interval. We have therefore finally obtained, as a result of (1) and (5),

$$(6) \qquad A(d)e^{-\int_c^d r(x)dx} \leq E[F^*(x), p(y)]$$

with equality holding if and only if $p(y) = p_0(y)$.

On the other hand, an elementary discussion for the case $p_0(y)$, which is of course included in § 2, shows that

$$(7) \qquad E[F(x), p_0(y)] \leq A(d)e^{-\int_c^d r(x)dx} .$$

The combination of these two inequalities, (6) and (7), shows that $F^*(x)$ is a good strategy for the fighter, and $p_0(y)$ the only good strategy for the bomber, with the value of the game being

$$A(d)e^{-\int_c^d r(x)dx} .$$

Case II. In this case, set

$$F^*(x) = \begin{cases} 0 & x < d \\ 1 & d < x < 1 \\ 1 & x = 1 \end{cases} .$$

Then

$$E\left[F^*(x),\ p(y)\right] = A(d)e^{-\int_0^d p(y)r(y)dy} \geq A(d)e^{-\int_0^d r(y)dy}$$

with equality holding if and only if $p(y) = 1$ in $0 < y < d$. But this is the value of the game obtained in §2 for this case, so that $F^*(x)$ is a good strategy for the fighter.