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SOLUTIONS OF A SPECIAL RECONNAISSANCE GAME

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R. L. Belzer[1]

1. Introduction:

In this paper, solutions are given for a special case of a general reconnaissance model previously investigated by S. Sherman in [1].[2] The same notation will be used here in so far as possible. Some examples are given in detail illustrating how the theorems established in [2] insure that all of the extreme points of the sets of optimal strategies have been found.

2. Summary of Results:

The game investigated here permits player I to reconnoiter at a cost and player II to take certain countermeasures for which he also pays a price. It is interesting to notice that the strategies of player I do not depend on the price of reconnaissance and the strategies of player II do not depend on the cost of countermeasures. We conjecture that this is true for the 2m x 2n matrix, R. We also suspect that if a chance mechanism were used to determine if, and how much, information should be given to player I about player II's move, in the event both reconnaissance and countermeasures were used, the disparity between the numbers of optimal strategies available to each player, which is so noticeable here, might be removed. Except for a small range of values of the costs of employment of reconnaissance

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[1] The author utilized many helpful suggestions of L. S. Shapley and S. Sherman in the solution of this problem.

[2] The numbers in square brackets refer to the bibliography at the end of the paper.

and countermeasures, the value of the game with reconnaissance does not depend on the cost of countermeasures. This perhaps indicates that the solutions of the game are not significant outside of these ranges from the point of view of interest in the model. In these ranges, both players have a unique optimal strategy and the value of the game depends both upon the cost of reconnaissance and the cost of countermeasures. The optimal strategies and values of the game for the various cases are summarized in tabular form at the end of this paper.

3. The Problem:

In [1], let $m = 2$ and $n = 2$. Then A becomes

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} .$$

We shall consider only two cases which will be determined by conditions on the a_{ij}. In neither case do we allow a saddle point to exist in A. The cases are:

I. $a_{11} > a_{22} > a_{12} \geq a_{21} > 0$,

II. $a_{11} = a_{22} > a_{12} > a_{21} > 0$.

The requirement that the a_{ij} be positive involves, of course, no loss of generality. The other conditions limit, somewhat, the multitude of case distinctions which might otherwise arise. In case there would ever arise numerical examples which would not fall into one of these cases, the changes would be slight indeed, probably necessitating only a little labor to find one or two additional extreme points of the convex sets of optimal strategies. If a saddle point did exist in A, then the associated matrix, R, would become a 2 x 2 matrix and the formulae given below for A could be applied to determine the optimal strategies and the value of the game.

If the conditions postulated above hold, then

$$x' = \left(\frac{a_{22} - a_{21}}{k}, \frac{a_{11} - a_{12}}{k}\right),$$

$$y' = \left(\frac{a_{22} - a_{12}}{k}, \frac{a_{11} - a_{21}}{k}\right),$$

$$v = \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{k},$$

where the prime indicates the transposed of the column matrix, and $k = a_{11} + a_{22} - a_{12} - a_{21}$.

For our special case the matrix R is

$$\begin{pmatrix} a_{11} - c + d & a_{12} - c + d & a_{11} - c & a_{22} - c \\ a_{21} - c + d & a_{22} - c + d & a_{11} - c & a_{22} - c \\ a_{11} + d & a_{12} + d & a_{11} & a_{12} \\ a_{21} + d & a_{22} + d & a_{21} & a_{22} \end{pmatrix},$$

where c and d are the costs of reconnaissance and countermeasures to the first and second players, respectively. We shall require that

$$c \neq \frac{a_{22}(a_{11} - a_{21})}{a_{22} - a_{21}} = c_1 .$$

If $c = c_1$, then some essential matrices, in the sense of [2], degenerate. Thus new extreme points in the sets of optimal strategies must be sought in order to reconcile the conditions and theorems stated and proved in [2]. These theorems are precisely the insurance that we have located every extreme point of the sets of optimal strategies. As the reader will see, this requirement involves the loss of very little generality since c_1 lies above the critical value, c_0, in the range of c.

Write

$$c_0 = \frac{(a_{11} - a_{21})(a_{22} - a_{12})}{k} ,$$

$$d_0 = \frac{(a_{22} - a_{21})(a_{22} - a_{12})}{k} ,$$

$$d_1 = \frac{(a_{22} - a_{21})(a_{22} - a_{12})}{(a_{11} - a_{21})} ,$$

$$d_2 = a_{22} - a_{21} .$$

We always have

$$c_0 < c_1 ,$$

$$d_0 < d_1 < d_2 ,$$

$$d_0 \leq c_0 < d_2 .$$

Instead of writing $v(c, d)$ for the value of P as in [1] we shall use v_1, since with one exception $v_1 = v_1(c)$ or $v_1 = v$. In addition it will be convenient to define

$$h_j^r = a_j' x^r , \quad j = 1, 2, 3, 4, \quad r = 1, 2, \cdots,$$

$$k_j^r = a_i y^r , \quad i, j = 1, 2, 3, 4, \quad r = 1, 2, \cdots.$$

4. <u>Case I</u>:

4.01. Assume that $c < c_0$ and $d_1 \geq d_2$. Under these conditions

$$v_1 = c\left(\frac{a_{21} - a_{22}}{a_{11} - a_{21}}\right) + a_{22} ,$$

$$y' = \left(0, 0, \frac{c}{a_{11} - a_{21}} , 1 - \frac{c}{a_{11} - a_{21}}\right),$$

and X is a one-dimensional set contained between the extreme points

$$x^{1'} = \left(\frac{a_{22} - a_{21}}{a_{11} - a_{21}} , 0, 0, \frac{a_{11} - a_{22}}{a_{11} - a_{21}}\right), ^*$$

$$x^{2'} = \left(0, \frac{a_{22} - a_{21}}{a_{11} - a_{21}} \ 0, \frac{a_{11} - a_{22}}{a_{11} - a_{21}}\right).$$

One may readily verify that

$$v_1 > v.$$

Since $k_1 = k_2 = k_4 = v_1$ and $k_3 < v_1$, we have

$$I_1 = I_2 .$$

*The superscripts are necessary to denote particular optimal strategies. The index in the superscript has no connection with r as used in [2].

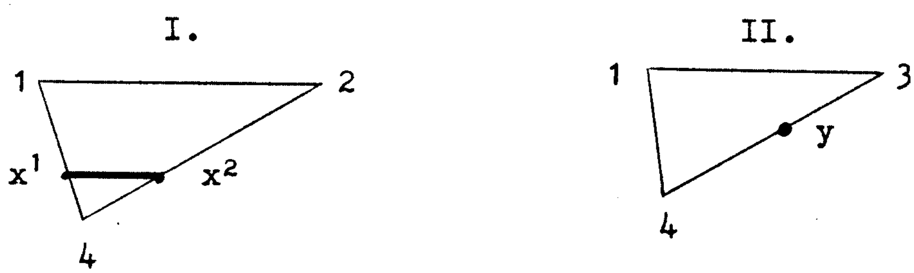
Similarly, since $h_3^1 = h_4^1 = h_3^2 = h_4^2$ and $h_1^1, h_2^1, h_1^2, h_2^2, > v_1$,

$$J_2 = \{3, 4\} \cdot \{3, 4\} = \{3, 4\} = J_1 .$$

If $d = d_2$, then $h_1^2 = v_1$, but since J_2 is the logical product of the sets $\{h_j^1\}$ and $\{h_j^2\}$ we still have

$$J_1 = J_2 .$$

Schematically, the simplices for the two players are exhibited below. The numbers refer to the x_i and y_i.



If we consider the matrix composed of the rows for which the corresponding $x_i > 0$ and the columns for which the corresponding y_i (of y') > 0, and denote this matrix by R_ε , we see that R_ε is

$$\begin{pmatrix} a_{11} - c & a_{22} - c \\ a_{11} - c & a_{22} - c \\ a_{21} & a_{22} \end{pmatrix} .$$

Since $v_1 \neq 0$, the theorems in [2] demand that

$$\dim X_1 - \dim X = 2 - 1 = \dim Y_1 - \dim Y = 1 - 0$$

and

$$\dim X_1 - \dim X = 1 = \text{Rank } R_\varepsilon - 1 = 2 - 1 = 1.$$

Moreover, since dim X, dim Y < 2, these theorems are sufficient to insure that all of the extreme points of both sets of optimal strategies have been found. In what follows we shall only exhibit the details of the application of the necessary theorems of [2] in some of the cases where either dim X = 2 or dim Y = 2. However, the necessary information will be recorded here to allow the reader to verify that the conditions of the theorems are fulfilled in the other cases.

4.02. Let $c < c_0$ and $d_1 \leq d \leq d_2$. The results are the same as in the previous case with the exception that

$$x^{2'} = \left\{\frac{a_{22} - a_{21} - d}{a_{11} - a_{21}},\ \frac{d}{a_{11} - a_{21}},\ 0,\ \frac{a_{11} - a_{22}}{a_{11} - a_{21}}\right\},$$

$$h_1^2 = v_1 .$$

4.03. If $c < c_0$ and $d_0 < d \leq d_1$, then except for the fact that

$$x^{1'} = \left\{\frac{d}{a_{22} - a_{12}},\ \frac{a_{22} - a_{21}}{a_{11} - a_{21}} - \frac{d}{a_{22} - a_{12}},\ 0,\ \frac{a_{11} - a_{22}}{a_{11} - a_{21}}\right\},$$

$$h_2^1 = v_1 ,$$

we obtain the same solution as in 4.02.

4.04. Again, if $c < c_0$ and $d = d_0$, then

$$x' = \left(\frac{a_{22}-a_{21}}{k}, \frac{d_0}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$$

$$y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$$

$$y^{2'} = \left(\frac{a_{22}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{22}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{22}-a_{12}}\right);$$

$$v_1 = v + d\left(1 - \frac{c}{c_0}\right);$$

$$h_1 = h_2 = h_3 = h_4 = v_1;$$

$$k_1^1 = k_2^1 = k_4^1 = k_1^2 = k_2^2 = k_3^2 = k_4^2 = v_1, \; k_3^1 < v_1.$$

4.05. Let $c < c_0$ and $d < d_0$. In this range the solutions are

$$x' = \left(\frac{d}{a_{22}-a_{12}}, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}}{k} - \frac{d}{a_{22}-a_{12}}, \frac{a_{11}-a_{12}}{a_{11}-a_{21}} - \frac{d}{a_{11}-a_{21}}\right);$$

$$y' = \left(\frac{a_{22}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{22}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{22}-a_{12}}\right);$$

$$v_1 = v + d\left(1 - \frac{c}{c_0}\right);$$

$h_j = k_j = v_1$, for $j = 1, 2, 3, 4$.

4.06. Now let $c = c_0$ and $d \geq d_2$, and then

$$x^{2'} = \left(\frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$$

$$x^{2'} = \left(0, \frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$$

$$x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$$

$$y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$$

$$v_1 = v;$$

$$k_j = v, \quad \text{for } j = 1, 2, 3, 4;$$

$$h_3^1 = h_4^1 = h_2^2 = h_3^2 = h_4^2 = h_3^3 = h_4^3 = v;$$

$$h_1^1, h_2^1, h_1^2, h_1^3, h_2^3 > v; \quad (\text{if } d = d_2, \text{ then } h_1^2 = v).$$

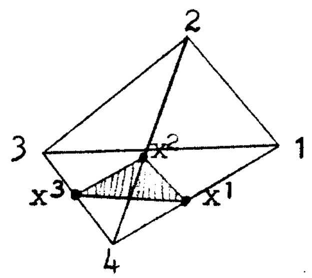
Thus

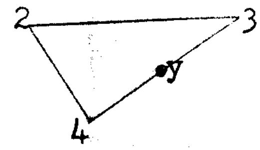
$$I_1 = \{1, 2, 3, 4\} = I_2 ,$$

and

$$J_1 = \{3, 4\} = J_2 .$$

In the diagrams, the numbers refer to the elements of strategy, x_i and y_i.

I. 

II. 

We can observe from the diagrams that

$$\dim X_1 - \dim X = 3 - 2 = \dim Y_1 - \dim Y = 1 - 0 = 1,$$

and that the essential submatrix of R to consider is a 4 x 2 matrix of all four rows and the last two columns. So that

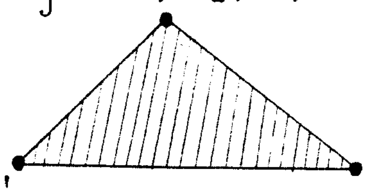
$$\text{Rank } R_\varepsilon - 1 = 2 - 1 = \dim X_1 - \dim X = 1.$$

Since dim X = 2, we must insure that the lines connecting x^1 to x^2, x^2 to x^3, x^3 to x^1 are accounted for either by the fact that an element, <u>identical in position</u>, in each of the three pairs of the x^r is zero or else that corresponding $h_j^r \notin J_2$ are both equal to v. In our case the former fact accounts for the faces. That is, $x_3 = 0$ in both x^1 and x^2, $x_1 = 0$ in both x^2 and x^3, and $x_2 = 0$ in both x^3 and x^1.

4.07. Stipulate that $c = c_0$ and $d_1 \leq d \leq d_2$ and we might believe that except for the change in x^2, namely,

$$x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}} , \frac{d}{a_{11}-a_{21}} , 0 \; \frac{a_{11}-a_{22}}{a_{11}-a_{21}} \right) ,$$

the solution would be the same as in 4.06. When we examine the convex set, X_1 and the h^r_j, however, we see that this is not the case. Glance below at the diagram of the X with h^r_j displayed.



Observe that now the face between x^2 and x^3 cannot be accounted for since there is neither a common zero element in x^2 and x^3, nor a common h^r_j ($r = 2, 3$; $j = 1, 2$), which is not a member of J_2, that is equal to v. Hence we must seek an additional strategy which is an extreme point of X or perhaps more than one. In 4.06 we stated that $h^2_1 > v$ except when $d = d_2$ and then the remark was that $h^2_1 = v$. It is clear then that in 4.06

$$h^2_1 = v + d - d_2 ,$$

and that for this reason x^2 in 4.06 fails to be an extreme point, in 4.07, of X, or even contained in X. This fact, however, is important, for using it we can determine one additional extreme point of X in 4.07. It can be readily verified that

$$h^3_1 = v + d,$$

so to find another extreme point of X we may form

$$\alpha h_1^2 + (1-\alpha)h_1^3 = v = \alpha(v + d - d_2) + (1-\alpha)(v + d).$$

Solving for α we find that

$$\alpha = \frac{d}{a_{22} - a_{21}},$$

and

$$1 - \alpha = \frac{a_{22} - a_{21} - d}{a_{22} - a_{21}} \ .$$

Thus another extreme point, x^4, can be found by using the $x^{2'}$ from 4.06 and writing

$$x^{4'} = \alpha x^{2'} + (1-\alpha)x^{3'}$$

$$= \left(0,\ \frac{d}{a_{11}-a_{21}},\ \frac{a_{22}-a_{21}-d}{k},\ \frac{a_{11}-a_{12}}{k} - \frac{d}{k} \cdot \frac{(a_{22}-a_{12})}{(a_{11}-a_{21})}\right).$$

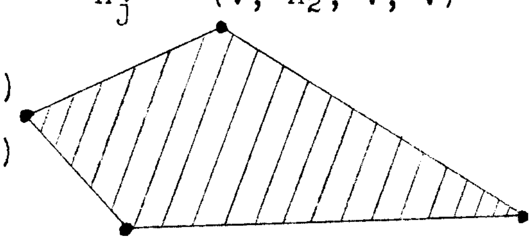
Computing

$$h_j^4 = (v,\ h_2^4,\ v,\ v),$$

our previous diagram now becomes, schematically,

$x^{2'} = (\bar{x}_1,\ x_2,\ 0,\ x_4)$
$h_j^2 = (v,\ h_2^2,\ v,\ v)$

$x^{4'} = (0, x_2, \bar{x}_1, \bar{x}_4)$
$h_j^4 = (v,\ h_2^4,\ v, v)$



$x^{1'} = (x_1,\ 0,\ 0,\ x_4)$
$h_j^1 = (h_1^1,\ h_2^1,\ v,\ v)$;
except that
$h_2^1 = v$ for
$d = d_2$

$x^{3'} = (0, 0, x_3, x_4)$
$h_j^3 = (h_1^3, h_2^3, v, v)$

where

$$\bar{x}_1 \neq x_1, \ \bar{x}_4 \neq x_4; \ h_1^1, \ h_2^1, \ h_2^2, \ h_1^3, \ h_2^3, \ h_2^4 > v, \text{ except as noted.}$$

Now x^4 and x^3 have a common first element equal to zero and x^4 and x^2 are such that

$$h_1^4 = h_1^2 = v \ ,$$

and all of the faces of X are accounted for. There is no change in the dimension of X_1 or X due to this additional extreme point and the equalities

$$I_1 = I_2, \quad J_1 = J_2 \ ,$$

$$\text{Rank } R_\varepsilon - 1 = \dim X_1 - \dim X = \dim Y_1 - \dim Y = 1.$$

still hold.

4.08. Suppose that $c = c_0$ and $d_0 < d \leq d_1$, then X has the extreme points

$$x^{1'} = \left(\frac{d}{a_{22}-a_{12}}, \frac{a_{22}-a_{21}}{a_{11}-a_{21}} - \frac{d}{a_{22}-a_{12}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right),$$

$$x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right),$$

$$x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$$

$$x^{4'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\frac{(a_{22}-a_{12})}{(a_{11}-a_{21})}\right),$$

$$x^{5'} = \left(\frac{d}{a_{22}-a_{12}}, 0, \frac{a_{22}-a_{21}}{k} - \frac{d}{k} \cdot \frac{(a_{11}-a_{21})}{(a_{22}-a_{12})}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\right).$$

As before, player II uses

$$y' = \left(0,\ 0,\ \frac{a_{22}-a_{12}}{k},\ \frac{a_{11}-a_{21}}{k}\right),$$

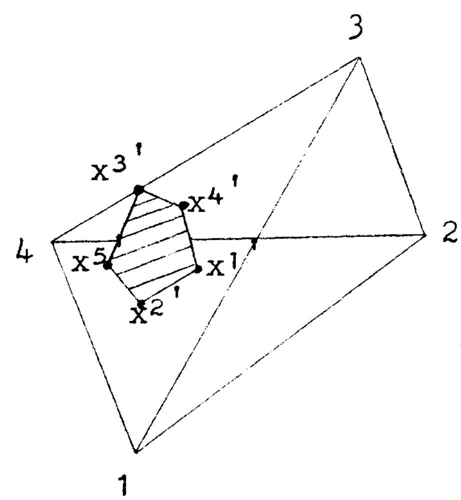
and so

$$v_1 = v,$$

$$h_j^r = v, \text{ except for } \begin{cases} r = 1 \text{ and } j = 1,\ r=2 \text{ and } j=2, \\ r = 3 \text{ and } j = 1 \text{ or } 2,\ r=4 \text{ and } j=2, \\ r = 5 \text{ and } j = 1; \end{cases}$$

$$k_j = v, \text{ for } j = 1, 2, 3, 4.$$

X can be represented as



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where the numbers, as usual, represent the elements of strategy. We leave it to the reader to verify that

$$I_1 = I_2,\quad J_1 = J_2,$$

$$\dim X_1 - \dim X = \dim Y_1 - \dim Y = \text{Rank } R_\varepsilon - 1.$$

4.09. If $c = c_0$ and $d = d_0$, then in 4.08, $x^2 = x^1$, and $h_1^1 = v$ and everything else remains unchanged.

4.10. For $c = c_0$ and $d < d_0$, the extreme points of X are

$$x^{1'} = \left(\frac{d}{a_{22}-a_{12}}, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}}{k} - \frac{d}{a_{22}-a_{12}}, \frac{a_{11}-a_{12}}{k} - \frac{d}{a_{11}-a_{21}}\right),$$

$$x^{2'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right),$$

$$x^{3'} = \left(\frac{d}{a_{22}-a_{12}}, 0, \frac{a_{22}-a_{21}}{k} - \frac{d}{k}\frac{(a_{11}-a_{21})}{(a_{22}-a_{12})}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\right),$$

$$x^{4'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\frac{(a_{22}-a_{12})}{(a_{11}-a_{21})}\right).$$

Player II uses the same strategy as in the previous case and the value of the game is the same, too. The h_j^r and k_j are readily obtainable from the previous cases.

4.11. If we assume that $c > c_0$ and $d > 0$, then both players play Γ instead of P and the solutions are those listed at the top of page 3. For convenience we have tabulated the optimal strategies and values of the game below

Conditions on c and d	Extreme Points of X and Y	v_1
$c < c_o;\ d \geq d_2$	$x^{1'} = \left(\frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $y' = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $x^{2'} = \left(0, \frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$	$a_{22} - \frac{c(a_{22}-a_{21})}{(a_{11}-a_{21})}$
$c < c_o;\ d_1 \leq d \leq d_2$	$x^{1'} = \left(\frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $y' = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$	$a_{22} - \frac{c(a_{22}-a_{21})}{(a_{11}-a_{21})}$
$c < c_o;\ d_o < d \leq d_1$	$x^{1'} = \left(\frac{d}{a_{22}-a_{12}}, \frac{a_{22}-a_{21}}{a_{11}-a_{21}} - \frac{d}{a_{22}-a_{12}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $y' = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$	$a_{22} - \frac{c(a_{22}-a_{21})}{(a_{11}-a_{21})}$
$c < c_o;\ d = d_o$	$x' = \left(\frac{a_{22}-a_{21}}{k}, \frac{d_o}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $y^{2'} = \left(\frac{a_{22}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{22}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{22}-a_{12}}\right);$	$a_{22} - \frac{c(a_{22}-a_{21})}{(a_{11}-a_{21})}$

Conditions on c and d	Extreme Points of X and Y	v_1
$c < c_o;\ d < d_o$	$x' = \left(\frac{d}{a_{22}-a_{12}}, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}}{k} - \frac{d}{a_{22}-a_{12}}, \frac{a_{11}-a_{12}}{k} - \frac{d}{a_{11}-a_{21}}\right);$ $y' = \left(\frac{a_{22}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{22}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{22}-a_{12}}\right);$	$v + d\left(1 - \frac{c}{c_o}\right)$
$c = c_o;\ d \geq d_2$	$x^{1'} = \left(\frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);\quad y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(0, \frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	v
$c = c_o;\ d_1 \leq d \leq d_2$	$x^{1'} = \left(\frac{a_{22}-a_{21}}{a_{11}-a_{21}}, 0, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);\quad y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$ $x^{3'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\frac{(a_{22}-a_{12})}{(a_{11}-a_{21})}\right);$ $x^{4'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	v

Conditions on c and d	Extreme Points of X and Y	v_1
$c = c_o;\ d_o < d \leq d_1$	$x^{1'} = \left(\frac{d}{a_{22}-a_{12}}, \frac{a_{22}-a_{21}}{a_{11}-a_{21}} - \frac{d}{a_{22}-a_{12}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);\quad y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$	v
	$x^{2'} = \left(\frac{a_{22}-a_{21}-d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);$	
	$x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	
	$x^{4'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\frac{(a_{22}-a_{12})}{(a_{11}-a_{21})}\right);$	
	$x^{5'} = \left(\frac{d}{a_{22}-a_{12}}, 0, \frac{a_{22}-a_{21}}{k} - \frac{d}{k}\frac{(a_{11}-a_{21})}{(a_{22}-a_{12})}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\right);$	
$c = c_o;\ d = d_o$	$x^{1'} = \left(\frac{a_{22}-a_{21}}{k}, \frac{(a_{22}-a_{21})(a_{22}-a_{12})}{k(a_{11}-a_{21})}, 0, \frac{a_{11}-a_{22}}{a_{11}-a_{21}}\right);\quad y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$	v
	$x^{2'} = \left(0, \frac{(a_{22}-a_{21})(a_{22}-a_{12})}{k(a_{11}-a_{21})}, \frac{(a_{22}-a_{21})(a_{11}-a_{21})}{k^2}, \frac{a_{11}-a_{12}}{k} - \frac{(a_{22}-a_{12})^2(a_{22}-a_{21})}{k^2(a_{11}-a_{21})}\right);$	
	$x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	
	$x^{4'} = \left(\frac{a_{22}-a_{21}}{k}, 0, \frac{(a_{22}-a_{21})(a_{22}-a_{12})}{k^2}, \frac{a_{11}-a_{12}}{k} - \frac{(a_{22}-a_{21})(a_{22}-a_{12})}{k^2}\right);$	

Conditions on c and d	Extreme Points of X and Y	v_1
$c = c_o;\ d < d_o$	$x^{1'} = \left(\frac{d}{a_{22}-a_{12}}, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}}{k} - \frac{d}{a_{22}-a_{12}}, \frac{a_{11}-a_{12}}{k} - \frac{d}{a_{11}-a_{21}}\right);\ y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(\frac{d}{a_{22}-a_{12}}, 0, \frac{a_{22}-a_{21}}{k} - \frac{d}{k}\,\frac{(a_{11}-a_{21})}{(a_{22}-a_{12})}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\right);$ $x^{3'} = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$ $x^{4'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{22}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d(a_{22}-a_{12})}{k(a_{11}-a_{21})}\right);$	v
$c > c_o;\ d > 0$	$x' = \left(0, 0, \frac{a_{22}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);\quad y' = \left(0, 0, \frac{a_{22}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$	v

5. Case II.

In this case we omit the details that were given for case I and summarize by means of a table. Letting $a_{11} = a_{22}$ simply necessitates looking for additional extreme points of X and Y, and the method for doing this was described previously.

Conditions on c and d	Extreme Points of X and Y	v_1
$c < c_o;\ d \geq d_2$	$x^{1'} = (1, 0, 0, 0);$ $\quad y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $x^{2'} = (0, 1, 0, 0);$ $\quad y^{2'} = \left(0, 0, 1 - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{12}}\right);$	$v_1 = a_{11} - c;$
$c < c_o;\ d_1 \leq d \leq d_2$	$x^{1'} = (1, 0, 0, 0);$ $\quad y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $x^{2'} = \left(1 - \frac{d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, 0\right);$ $\quad y^{2'} = \left(0, 0, 1 - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{12}}\right);$	$v_1 = a_{11} - c;$
$c < c_o;\ d_o < d_2 < d_1$	$x^{1'} = \left(\frac{d}{a_{11}-a_{12}}, 1 - \frac{d}{a_{11}-a_{12}}, 0, 0\right);$ $\quad y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $x^{2'} = \left(1 - \frac{d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, 0\right);$ $\quad y^{2'} = \left(0, 0, 1 - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{12}}\right);$	$v_1 = a_{11} - c;$
$c < c_o;\ d = d_o$	$x' = \left(\frac{a_{11}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}, 0, 0\right);$ $\quad y^{1'} = \left(0, 0, \frac{c}{a_{11}-a_{21}}, 1 - \frac{c}{a_{11}-a_{21}}\right);$ $y^{2'} = \left(0, 0, 1 - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{12}}\right);$ $y^{3'} = \left(\frac{a_{11}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{11}-a_{12}}\right);$	$v_1 = a_{11} - c$

Conditions on c and d	Extreme Points of X and Y	v_1
$c < c_o;\ d < d_o$	$x' = \left(\frac{d}{a_{11}-a_{12}}, \frac{d}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{d}{a_{11}-a_{12}}, \frac{a_{11}-a_{12}}{k} - \frac{d}{a_{11}-a_{21}}\right);$ $y' = \left(\frac{a_{11}-a_{12}}{k} - \frac{c}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}}{k} - \frac{c}{a_{11}-a_{12}}, \frac{c}{a_{11}-a_{21}}, \frac{c}{a_{11}-a_{12}}\right);$	$v_1 = a_{11} - c$
$c = c_o;\ d \geq d_2$	$x^{1'} = (1, 0, 0, 0);$ $\quad y' = \left(0, 0, \frac{a_{11}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = (0, 1, 0, 0);$ $x^{3'} = \left(0, 0, \frac{a_{11}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	v
$c = c_o;\ d_1 \leq d < d_2$	$x^{1'} = (1, 0, 0, 0);$ $\quad y' = \left(0, 0, \frac{a_{11}-a_{12}}{k}, \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(1 - \frac{d}{a_{11}-a_{21}}, \frac{d}{a_{11}-a_{21}}, 0, 0\right);$ $x^{3'} = \left(0, \frac{d}{a_{11}-a_{21}}, \frac{a_{11}-a_{21}-d}{k}, \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\ \frac{(a_{11}-a_{12})}{(a_{11}-a_{21})}\right);$ $x^{4'} = \left(0, 0, \frac{a_{11}-a_{21}}{k}, \frac{a_{11}-a_{12}}{k}\right);$	v

Conditions on c and d	Extreme Points of X and Y	v_1
$c = c_o;\ d_o \le d \le d_1$	$x^{1'} = \left(\frac{d}{a_{11}-a_{12}},\ 1 - \frac{d}{a_{11}-a_{12}},\ 0,\ 0\right);\quad y' = \left(0,\ 0,\ \frac{a_{11}-a_{12}}{k},\ \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(1 - \frac{d}{a_{11}-a_{21}},\ \frac{d}{a_{11}-a_{21}},\ 0,\ 0\right);$ $x^{3'} = \left(0,\ \frac{d}{a_{11}-a_{21}},\ \frac{a_{11}-a_{21}-d}{k},\ \frac{a_{11}-a_{12}}{k} - \frac{d}{k}\ \frac{(a_{11}-a_{12})}{(a_{11}-a_{21})}\right);$ $x^{4'} = \left(\frac{d}{a_{11}-a_{12}},\ 0,\ \frac{a_{11}-a_{21}}{k} - \frac{d}{k}\ \frac{(a_{11}-a_{21})}{(a_{11}-a_{12})},\ \frac{a_{11}-a_{12}-d}{k}\right);$ $x^{5'} = \left(0,\ 0,\ \frac{a_{11}-a_{21}}{k},\ \frac{a_{11}-a_{12}}{k}\right);$	v
$c = c_o;\ d < d_o$	$x^{1'} = \left(\frac{a_{11}-a_{21}}{k},\ \frac{a_{11}-a_{12}}{k},\ 0,\ 0\right);\quad y' = \left(0,\ 0,\ \frac{a_{11}-a_{12}}{k},\ \frac{a_{11}-a_{21}}{k}\right);$ $x^{2'} = \left(0,\ \frac{d}{a_{11}-a_{21}},\ \frac{a_{11}-a_{21}-d}{k},\ \frac{a_{11}-a_{12}+d}{k} - \frac{d}{a_{11}-a_{21}}\right);$ $x^{3'} = \left(\frac{d}{a_{11}-a_{12}},\ 0,\ \frac{a_{11}-a_{21}-d}{k},\ \frac{a_{11}-a_{12}+d}{k} - \frac{d}{a_{11}-a_{12}}\right);$ $x^{4'} = \left(0,\ 0,\ \frac{a_{11}-a_{21}}{k},\ \frac{a_{11}-a_{12}}{k}\right);$	v
$c > c_o;\ d > 0$	$x' = \left(0,\ 0,\ \frac{a_{11}-a_{21}}{k}\ \ \frac{a_{11}-a_{12}}{k}\right);\quad y' = \left(0,\ 0,\ \frac{a_{11}-a_{12}}{k}\ \ \frac{a_{11}-a_{21}}{k}\right);$	v

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