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RESEARCH MEMORANDUM

RECONNAISSANCE IN GAME THEORY

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1. Introduction.

Reconnaissance in a game is an attempt by the players to obtain complete or partial information in the course of the game about the outcome either of previous chance moves or of previous moves made by the opponent. The players may adopt counter measures designed to make reconnaissance difficult or impossible. The costs of reconnaissance and of counter measures enter in the pay-off function of the game.

A formal definition of a reconnaissance game is proposed in the next section. Its ultimate justification can rest only on a study of examples, some of which will be presented in another report. The proposed definition is the outgrowth of suggestions made by J. von Neumann at a conference held July 7, 1949.

The present report is restricted to the case where one player uses a fixed type of reconnaissance, and where the second player attempts neither reconnaissance on his own nor counter measure. The influence of the reconnaissance on the strategies of the players and on the value of the game is studied.

2. The reconnaissance game.

A fixed game Γ_0 in normal form is assumed given, with the pay-off matrix a_{ij}; $i = 1, \ldots, M$; $j = 1, \ldots, N$. A reconnaissance game Γ is introduced whose moves consist of

(i) a choice of j by the second player, unknown to the first player,

(ii) a reconnaissance move (described precisely later), whose outcome is communicated to the first player,

(iii) a choice of i by the first player.

The pay-off is assumed to be the value a_{ij}. No attention need be paid to the cost of the reconnaissance as long as it is not a question of comparing different types of reconnaissance.

Let R be the set of all informations which the reconnaissance move may yield. No restriction is made about R, in particular it may be finite or infinite. Its elements are denoted by r. The reconnaissance move is taken to be a chance move played according to known probability densities $d\nu_j$, where j is the outcome of the first move. The pure strategies of the first player consist then of a partition of R into sets $R_1, \ldots, R_M$. He will play i, if $r \in R_i$.

Stated more precisely a reconnaissance game Γ involves the following elements:

(a) a game Γ_o, with the matrix a_{ij}

(b) a set R

(c) a Borel field of "measurable" subsets of R

(d) N completely additive measure theories ν_j such that for each j, $\nu_j(R) = 1$.

A pure strategy of the first player is a partition P of R into measurable sets R_i. A pure strategy of the second player is a choice of j. The pay-off is

$$A_j(P) = \sum_i a_{ij}\nu_j(R_i) .$$

The reconnaissance game is in general an infinite by N game. Theoretically its solution can of course be obtained by plotting for each partition P the point $A(P) = (A_j(P))$ in the N-dimensional space. The points A(P) form a set A whose convex closure is denoted by B. The value v of the game is given by

$$v = \underset{b \in B}{\text{Max}} \; \underset{j}{\text{Min}} \; b_j$$

where $b = (b_j)$ is an arbitrary point of B, or by

$$v = \underset{c_j}{\text{Min}}\ \underset{P}{\text{Sup}} \sum A_j(P) \cdot c_j = \underset{c_j}{\text{Min}}\ \underset{b}{\text{Max}} \sum b_j c_j$$

where $c_j \geq 0$, $\sum c_j = 1$.

The convex set B depends on the game Γ_o, the set R and the measures ν_j. It can be used as a gauge for the efficiency of the reconnaissance. It includes the convex set of the original game, which is spanned by the points $a_i = (a_{ij})$. On the other hand, it is included in the parallelipiped whose vertices are the 2^N points of coordinates max or min a_{ij} over i. This parallelipiped is the convex B of the game with complete information.

The convex set B can be utilized to partially order the different types of reconnaissance. A reconnaissance (R, ν_j) is more informative <u>with respect to</u> Γ_o than (R', ν_j') if its convex set B includes the convex set B' of the other one. If $B \supset B'$ no matter what the game Γ_o is (keeping N fixed) then the reconnaissance (R, ν_j) is said to be absolutely more informative than (R', ν_j'). If $B = B'$, then the two reconnaissances are equivalent, this will occur, if and only if, for each choice of c_j

$$\sup_P \sum_{i,j} c_j a_{ij} \nu_j(R_i)$$

are equal for both reconnaissance, no matter what the a_{ij} are. In the next two sections, it is shown that each reconnaissance game is equivalent to exactly one game, of a certain standard type so that in the sequel only such games need be considered.

<u>3. Standard reconnaissance games</u>. A reconnaissance game Γ is called standardized if

(a) the set R is the (N-1)-dimensional simplex Q. The points of Q are denoted by q, and their barycentric coordinates by q_j, $q_j \geq 0$, $\sum q_j = 1$.

(b) A completely additive measure function μ defined for the Borel sets in Q is given such that for each j

$$\int_Q q_j d\mu = 1$$

and therefore

$$\int_Q d\mu = N.$$

(c) The measure theories ν_j are given by

$$\nu_j(S) = \int_S q_j d\mu .$$

<u>Theorem 1</u>. Each reconnaissance game is equivalent to a standardized game.

Proof: The measure $\nu = \sum \nu_j$ of the given arbitrary reconnaissance game is completely additive, and if $\nu(S) = 0$ then each $\nu_j(S) = 0$. By the Nikodym-Radon theorem there exists N integrable functions $f_j(r)$ such that for each measurable set S

$$\nu_j(S) = \int_S f_j(r) d\nu .$$

Almost everywhere $f_j \geq 0$ and $\sum f_j = 1$. We may assume these relations true everywhere. These functions define a mapping Φ of R into Q by

$$q_j = f_j(r) .$$

A set S of Q is called measurable if it is a Borel set and if its counter image $\Phi^{-1}(S)$ is measurable in R. For example the set $S = \left\{q \,\middle|\, \sum c_j q_j > c\right\}$ is measurable since

$$\Phi^{-1}(S) = \left\{r \,\middle|\, \sum c_j f_j > c\right\} .$$

The measurable sets form a Borel field. A measure function μ, completely additive, is defined by

$$\mu(S) = \nu(\Phi^{-1}(S)) .$$

The functions q_j are integrable and for each measurable set S

$$\int_S q_j d\mu = \int_{\Phi^{-1}(S)} f_j d\nu = \nu_j(\Phi^{-1}(S)) .$$

In particular, μ satisfies all the conditions which were imposed on the standard reconnaissance game.

It remains to show that the given reconnaissance game and the deduced standard game are equivalent. For each choice of c_j, we must prove that

$$\sup_P \sum_{i,j} a_{ij} c_j \nu_j(R_i) = \sup_P \sum_{i,j} a_{ij} c_j \int_{S_i} q_j d\mu.$$

The left hand side is equal to

$$\sup_P \sum_{i,j} \int_{R_i} a_{ij} c_j f_j d\nu,$$

and is reached for any partition (R_i) such that, for each i_o, $r \in R_{i_o}$ implies

$\sum_j a_{ij}c_j f_j \leq \sum_j a_{i_o j}c_j f_j$. In other words, the supremum is reached for some partition (R_i) for which $\Phi^{-1}(\Phi(R_i)) = R_i$. If $S_i = \Phi(R_i)$, then the left hand side is equal to

$$\sum_{i,j} a_{ij}c_j \int_{S_i} q_j d\mu$$

so that

$$\sup \sum_{i,j} a_{ij}c_j \nu_j(R_i) \leq \sup \sum_{i,j} a_{ij}c_j \int_{S_i} q_j d\mu .$$

The converse inequality is obvious, since for any S_i

$$\sum_{i,j} a_{ij}c_j \int_{S_i} q_j d\mu = \sum_{i,j} a_{ij}c_j \nu_j(\Phi^{-1}(S_i)) .$$

The importance of theorem 1 lies in the fact that the intrinsic nature of information is shown to be represented by a point in a simplex. For example, if R is finite and the measure functions ν_j are described by $\nu_j(r_\alpha)$, $\alpha = 1, 2, \ldots$ then two r's can be identified if the two rows of ν_j are proportional. The combined point is given a weight equal to the sum of the previous weights.

This intrinsic interpretation is the natural one since, as will be proved in the next section, two standardized games are absolutely equivalent only if they are identical.

4. Comparison of reconnaissance games.

Let two standardized reconnaissance games over the same Γ_o be given; they are described, say, by the measures μ and ν. By definition,

$$\int_Q q_j d\mu = \int_Q q_j d\nu = 1.$$

<u>Theorem 2</u>. The measure μ is absolutely more informative than ν if and only if for each convex continuous function $C(q)$

$$(4.1) \qquad \int_Q C(q)\,d\nu \leq \int_Q C(q)\,d\mu \ .$$

Proof: Assume the condition (4.1) satisfied and let Γ_o be a given game. For any choice of c_j, let $L_i(q)$ be the linear function $\sum_j a_{ij}c_jq_j$, and let $L(q) = \underset{i}{\mathrm{Max}}\, L_i(q)$. The function $L(q)$ is convex and continuous and thus by assumption

$$\int_Q L(q)\,d\nu \leq \int_Q L(q)\,d\mu \ .$$

But $\int_Q L(q)\,d\mu = \sup_P \sum a_{ij}c_j \int_{S_i} q_j\,d\mu$ since the right hand side is is equal to $\sum_i \int_{S_i} L_i(q)\,d\mu \leq \sum_i \int_{S_i} L(q)\,d\mu = \int_Q L(q)\,d\mu$ and equality is reached for any subdivision S_i for which $q \in S_i$ implies $L_i = L$. A similar relation holds for ν .

But

$$\sup_P \sum_{i,j} a_{ij}c_j \int_{S_i} q_j\,d\nu \leq \sup_P \sum_{i,j} a_{ij}c_j \int_{S_i} q_j\,d\mu$$

states geometrically that the convex $B(\nu)$ lies on one side of any plane of support of the convex $B(\mu)$, i.e.,

$$B(\nu) \subset B(\mu) .$$

Conversely, if $B(\nu) \subset B(\mu)$, retracing the preceding steps, the inequality

$$\int_Q L(q)d\nu \leq \int_Q L(q)d\mu$$

must hold. But if $C(q)$ is any convex function, there exists linear functions $L_i(q)$ such that for $L(q) = \text{Max } L_i(q)$

$$|C(q) - L(q)| < \varepsilon.$$

These $L_i(q)$ determine a game Γ_o after c_j have been taken = 1. Hence

$$\int_Q C(q)d\nu \leq \int_Q C(q)d\mu + 2N\varepsilon$$

and this inequality must hold for every ε .

Theorem 3. Two absolutely equivalent standard games are identical.

Proof: Theorem 2 shows that for any convex function $C(q)$

$$\int_Q C(q)d\nu = \int_Q C(q)d\mu$$

If C_1 and C_2 are two convex functions then $\text{Max}(C_1, C_2)$ is also convex. But $\text{Min}(C_1, C_2) = C_1 + C_2 - \text{Max}(C_1, C_2)$ and hence

$$\int_Q \text{Min}(C_1, C_2)d\nu = \int_Q \text{Min}(C_1, C_2)d\mu$$

In particular, if C is any convex function then

$$\int_Q \text{Max}(-C,\ 0)\,d\nu = \int_Q \text{Max}(-C,\ 0)\,d\mu$$

and hence $\mu(S) = \nu(S)$ for any closed convex set with an interior point since the characteristic functions of such a set is the decreasing limit of functions of the type: Max(-C, 0). The relation is then extended to any measurable set S.

If N = 2, i.e., j either 1 or 2, theorem 2 can be reworded in a simpler form. In that case, the simplex Q is a segment, $0 \leq q \leq 1$ with $q_1 = q$ and $q_2 = 1 - q$. The measure is described by a non-decreasing function f(q). The assumption on μ becomes

$$(4.2) \qquad \int_0^1 df = 2 \qquad \text{and} \qquad \int_0^1 q\,df = 1.$$

The function f is normalized by f(0) = 0, then f(1) = 2.

The condition of theorem 2 can be applied to the special convex function

$$(4.3) \qquad C(q) = \begin{cases} \tau - q & \text{if } q \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

where τ is any number between 0 and 1. Then

$$\int_Q c(q)\,df = \int_0^\tau (t-q)\,df = \int_0^\tau f\cdot dq \ .$$

<u>Theorem 4</u>. If N = 2, and f is absolutely more informative than g then

$$\int_0^\tau f\,dq \geq \int_0^\tau g\cdot dq \ ,$$

and conversely.

Proof: The preceding considerations prove the first part of the theorem. The converse follows from the facts that by assumption $\int C(q)df \leq \int C(q)dg$ for any function of type (4.3), that by (4.2) $\int L(q)df = \int L(q)dg$ for any linear function L, that any convex polygonal line is a sum of such functions, and finally that any convex continuous function can be approximated uniformly by a convex polygonal line.

In the case N = 2 the different reconnaissance measures f form a lattice. Whether this is true in general has not been investigated yet.

5. Extreme cases. The two extreme cases of complete information and of no information are briefly mentioned as trivial illustrations of the general theory.

(1) Complete information. The measure function μ_1 which assigns the measure 1 to each vertex of the simplex Q defines a reconnaissance game. It is more informative than any other reconnaissance game since for any convex function C(q)

$$C(q) \leq \sum C_j \cdot q_j$$

where C_j is the value of C at the j^{th} vertex, and hence

$$\int C(q) \leq \sum C_j \int q_j d\mu = \sum C_j = \int C(q) d\mu_1 .$$

By theorem 3, μ_1 is the only measure function with this property. The chance move is removed in this case, in the sense that the first player is told exactly the move made by the second player.

(2) No information. The measure function μ_0 which assigns the full measure N at the centroid $(1/N, \ldots, 1/N)$ of the simplex Q also defines a reconnaissance game. It is the only measure which is less informative than any other one. For let $C(q)$ be any convex function, determine a linear function $L(q)$ such that $C(q) \geq L(q)$ and equality holds at the centroid q^0. Then

$$\int C(q)\,d\mu \geq \int L(q)\,d\mu = \int \sum \ell_j q_j \,d\mu = \sum \ell_j = NL(q^0) = NC(q^0) = \int C(q)\,d\mu .$$

This measure μ_0 is the case of no-information. The chance move is removed in the sense that irrespective of j the chance move always gives the same answer, q^0.

6. Analysis of the "general" standard game. The main feature of a reconnaissance game is that in general the first player has a unique best pure strategy. The occurrence of mixed strategies are due either to special relations between the elements of the matrix a_{ij} or to a partial concentration of the mass distribution at certain points, or lines etc., up to the dimension N - 2. To avoid these complications the present analysis of the reconnaissance games deals with the case where

(a) No two elements a_{ij} in a same column are equal.

(b) The mass distribution μ is such that the mass $\int_S d\mu$ of the sets $S = \left\{ q \mid \sum c_j q_j < c \right\}$ is a continuous function of c for any c_j.

Any reconnaissance game can of course be approximated arbitrarily closely by a game satisfying these two conditions. Using the same terminology as in section 2, for any partition P of the simplex Q into sets S_i, the point A(P) of coordinates

$$A_j(P) = \sum a_{ij} \int_{S_i} q_j \,d\mu$$

is plotted in the N-dimensional affine space. The coordinates of the points in this space will be denoted by u_j. The points A(P) form a set A whose convex closure is B.

Let $\sum c_j u_j = c$ be any plane of support of the set B. Of course not all c_j are zero. Put $L_i(q) = \sum a_{ij} c_j q_j$ and $L(q) = \text{Max } L_i(q)$. Then

$$c = \sup_P \sum_i \int_{S_i} L_i(q) d\mu = \int_Q L(q) d\mu .$$

The supremum is a maximum reached for any partition P^o into sets S_i^o such that $q \in S_i^o$ implies $L_i(q) = L(q)$. Denote by $u^o = (u_j^o)$, where

$$u_j^o = \sum_i a_{ij} \int_{S_i^o} q_j d\mu ,$$

the point of A which corresponds to a partition P^o.

<u>Lemma 1</u>. Any plane of support of B contains exactly one point of A.

Proof: The existence has already been proved. To prove the uniqueness we observe that if $L_i(q) = L_{i'}(q)$ for all q then in particular $c_j(a_{ij} - a_{i'j}) = 0$ for each j, and for some j, $a_{ij} = a_{i'j}$. Assumption (a) implies that $i = i'$. In other words if $i \neq i'$ then the set where $L_i = L_{i'}$ is at most (N-2)-dimensional and by assumption (b) its measure must be zero. For future use we remark that, by continuity, to any given $\eta > 0$ there exists a $\delta > 0$ such that

(6.1) the set S where $\left| L_i - L_{i'} \right| < \delta$, $(i \neq i')$, has measure $< \eta$.

Let P' be a partition into sets T_i which leads to a point on the plane of support. Then $q \in T_i$ implies $L_i(q) = L(q)$ a.e. and for $i \neq i'$, $L(q) = L_i(q) = L_{i'}(q)$ a.e. over the set $T_{i'} \cap S_i$. Hence this set has measure zero and $T_i = S_i$ up to a set of measure zero. Hence P^o and P^1 lead to the same point u^o.

Lemma 2. If $u^{(n)}$ is a sequence of points of A such that $\sum c_j u_j^{(n)}$ tends to c then $u^{(n)}$ tends to u^o.

Proof: Given $\varepsilon > o$, let $u^{(n)}$ be such that

$$\sum c_j u_j^{(n)} > c - \varepsilon = \int_Q L(q) d\mu - \varepsilon.$$

Let $P^{(n)}$ be a partition which yields $u^{(n)}$. Then $\sum_i \int_{S_i^{(n)}} (L - L_i) d\mu < \varepsilon$, that is $\int_{S_i^{(n)}} (L - L_i) d\mu < \varepsilon$ for each i. Let $i' \neq i$ and let R be the intersection of $S_{i'}^{(n)}$ with $S_i^{(o)}$, where $S_i^{(o)}$ is the partition which gives u^o. Split R into two parts R' and R'' where over R', $| L_{i'} - L | < \sqrt{\varepsilon}$ and over R'', $|L_{i'} - L| \geq \sqrt{\varepsilon}$. Then

$$\mu(R) \leq \delta(\sqrt{\varepsilon}) + N\sqrt{\varepsilon},$$

where $\delta(\sqrt{\varepsilon})$ is the δ of (6.1) corresponding to $\eta = \sqrt{\varepsilon}$. This shows that

$$\left| u_j^{(n)} - u_j^{(o)} \right| = O(\delta(\sqrt{\varepsilon}) + \sqrt{\varepsilon})$$

i.e., that $u^{(n)}$ tends to $u^{(o)}$ as $\varepsilon \to 0$.

Lemma 3. Any plane of support of B contains exactly one point of B and this point belongs to A.

This lemma is a simple consequence of the two preceding lemmas, since any point of B is the limit of centers of gravity of points of A.

Theorem 5. In a "general" game, i.e. a game satisfying conditions (a) and (b) the first player has a unique best pure strategy.

Proof: Let v be the value of the game. The set B and the "octant" $u_j \geq v$ are separated by some plane of support of B, which contains by lemma 3 the only point in common of B and the octant. This point belongs to A, it is unique and the proof of lemma 1 shows that the corresponding partition S_i^o is uniquely determined up to sets of measure zero.

If the coefficients c_j were known, the best partition is determined by solving a set of linear equations. Qualitatively, it is already clear that the best partition consist of sets, each of which is a connected, convex polyhedron. If N = 2, the best partition is a division of the segment into intervals. This result shows another advantage of the standardized reconnaissance games. The interpretation of the informations as points of a simplex arrange them in such a way that the best partition separates these points in as simple a way as possible, namely by planes.

For any choice of $c_j > o$, there is connected with it the partition $P = (S_i)$ which yields the point on the corresponding plane of support; namely $q \in S_i$ implies $L_i = L$. If the c_j are chosen equal to one then the points q at which L reaches its minimum are the mixed best strategies of the second player in the underlying non-reconnaissance game Γ_o. As the values c_j vary the partition P varies, but as in the expression of L_i only the combinations $c_j q_j$ occur any partition P is deducible from any other one by a projective transformation of the simplex Q onto itself which leaves each vertex fixed. If some c_j become zero then the transformation becomes degenerate and sends the whole interior of Q into a face or edge or vertex of Q. At any rate as the c vary the qualitative features of the partition, like neighboring S_i, remain the same.

The solution of the game can thus be obtained from the following construction:

For $c_j = 1$, the corresponding partition $S_i^{(1)}$ is determined. Let T be an arbitrary projective transformation of Q into Q leaving the vertices fixed:

$$Tq_j = \frac{t_j q_j}{\sum t_j q_j} \qquad t_j \geq 0, \quad \sum t_j = 1.$$

Let

$$F_j(T) = \sum_i a_{ij} \int_{T^{-1}(S_i^{(1)})} q_j d\mu$$

then the value v of the game is

$$v = \underset{T}{\text{Max}} \ \underset{j}{\text{Min}} \ F_j(T) \ ,$$

and if T_o is the T for which the maximum is reached then $T_o(S^{(1)})$ is the best pure strategy of the first player. Alternately the value v can be determined from

$$v = \underset{T}{\text{Min}} \sum t_j F_j(T) \ .$$

The $F_j(T)$ are perfectly well determined even if T is degenerate, and are continuous functions of T. Conjecture: If in the original game Γ_o, the second player has only mixed strategies for which each $y_j > 0$ then the transformation T which gives the solution of the reconnaissance game is non-degenerate and can be determined by solving the equations.

$$F_1(T) = \cdots = F_N(T) .$$

<u>7. The M x 2 game</u>. Let (a_i, b_i), $i = 1, \ldots, M$ be the matrix of the original game. The simplex Q is the segment $0 \leq q \leq 1$, $q_1 = q$, $1 - q = q_2$; the measure μ is described by a non-decreasing function f such that

$$f(0) = 0; \qquad f(1) = 2; \qquad \int_0^1 x df = 1.$$

Following the procedure outlined in the preceding section, let $L_i(q) = a_i q + b_i(1 - q) = (a_i - b_i) q + b_i$; and $L(q) = \text{Max } L_i(q)$. The interval $0 \leq q \leq 1$ is divided into intervals I_i over which $L_i = L$. [Assuming for simplicity, no two a and no two b equal, this decomposition is unambiguous] . Relabeling the i's, if necessary, we assume

$$I_1 = (0, \tau_1), \; I_2 = (\tau_1, \tau_2), \; \ldots, \; I_{i_o} = (\tau_{i_o - 1}, 1)$$

Of course $i_o \leq M$. The indices $i > i_o$ play no role. For simplicity we assume $i_o = M$. The value of τ_i is

$$\tau_i = (b_{i+1} - b_i)/(a_i - a_{i+1} - b_i + b_{i+1}),$$

and the a's and b's after the relabeling satisfy the conditions

$$a_1 < a_2 < \cdots < a_M ; \qquad b_1 > b_2 > \cdots > b_M .$$

Three cases must be distinguished

(1) $b_1 \leq a_1$ (2) $a_M \leq b_M$ (3) other cases.

The first two cases occur if in the original game the second player has a best pure strategy. These cases are uninteresting since no amount of information will increase the value of the game.

The third case $a_1 < b_1$; $b_M < a_M$ occurs if in the original game all best strategies of the second player are strictly mixed strategies.

The most general projective transformation T^{-1} leaving 0, 1 invariant can be written in the form

$$\tau' = \lambda\tau/\big(1 + (\lambda - 1)\tau\big) = T^{-1}(\tau).$$

where $0 \leq \lambda \leq \infty$. Its inverse is obtained by replacing λ by $1/\lambda$.

The F_j of the preceding section are

$$F_1 = \sum_i a_i \int_{\tau'_{i-1}}^{\tau'_i} q\,df; \qquad F_2 = \sum_i b_i \int_{\tau'_{i-1}}^{\tau'_i} (1 - q)\,df .$$

The function f is assumed continuous for simplicity, so that the above integrals have a unique value, without further specification of the limits of integration. The τ'_i are equal to $T^{-1}(\tau_i)$. It is easily seen that F_1 decreases from a_M to a_1 as λ increases from 0 to ∞. The value F_2 increases. There exists λ for which both are equal, this value is $0 < \lambda < \infty$ and the conjecture of the preceding section is verified in this special case. The values λ which maximize the min F_1, F_2 may not be unique, if $\lambda_1 \leq \lambda \leq \lambda_2$ is the interval of these λ then $\int_{\lambda_1}^{\lambda_2} df = 0$, so that within sets of measure zero (for the partition) any λ in this interval lead to the same solution. The common value F_1, F_2 is the value of the game, and the corresponding λ determine the strategies in an obvious way.

In particular, if also M = 2, and the matrix of the game is written as

$$\begin{vmatrix} a & -b \\ -c & d \end{vmatrix}$$

where a, b, c, d > 0 then the solution v of the problem is given by

$$v = a - (a+c) \int_0^{\tau} q\,df = -b + (b+d) \int_0^{\tau} (1-q)\,df$$

where τ is any solution in $0 < \tau < 1$ of the equation

$$\int_0^{\tau} \left\{ (b+d) + (a+c-b-d)q \right\} df = a + b.$$

If the function f is not continuous then this equation may not have a solution unless the notion of partition of the segment $0 \leq q \leq 1$ in intervals is extended to include the possibility of splitting a common end point of two intervals in any ratio and to assign the parts to the two intervals. If this is the case mixed strategies for the first player occur. This shows how the case where μ does not satisfy condition (b), page 11, must be handled. The details will be published elsewhere.

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