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RESEARCH MEMORANDUM

n-PERSON GAMES
- AN EXAMPLE AND A PROOF -

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n-Person Games

An Example and a Proof

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Summary: In part 1 a duopoly example is given, and in part 2 a brief proof of the existence of equilibrium points yields a simple proof of the main theorem of 2-person zero-sum games.

1. A duopoly example.

Let

γ_1, γ_2 be the production costs of producer I and II, respectively,

q_1, q_2 be the output of producer I and II, respectively.

Assume that the costs, as a function of the output, are given by

$$\gamma_1 = 4 - q_1 + q_1^2$$
$$\gamma_2 = 5 - q_2 + q_2^2 \quad .$$

Assume that the market price, p , is given by

$$p = 10 - 2(q_1 + q_2) \ .$$

The following table compares the various solution to this duopoly example, where P_1 and P_2 represent the profits of the two producers:

	Cournot	Koopmans' "Efficiency"	von Neumann Morgenstern	2-person Cooperative with side payments: Threat	Cooperation
q_1	.94	1.1716	.9161	1.1196	.9161
q_2	.74	.9411	.4125	1.0000	.4125
P_1	2.53	1.8437	—	1.8214	2.6299*
P_2	1.36	.7812	—	.7607	1.5692*
$P_1 + P_2$	3.89	2.6249	4.1991	2.5821	4.1991
$q_1 + q_2$	1.67	2.2127	1.3286	2.1196	1.3286

* Accomplished by side-payment of .5028 from I to II.

2. Existence of equilibrium points.

The following brief proof of the existence of equilibrium points also yields a simple proof of the "main theorem" of 2-person 0-sum games.

Let $P_i(s)$ be the payment to player i when $s = (s_1, s_2, \ldots, s_n)$ are the mixed strategies used by the n players. Let $P_{i\alpha}(s)$ be the payment to player i if he changes from s_i to his α-th pure strategy $\pi_{i\alpha}$.

Let

$$\phi_{i\alpha}(s) = \max\left[0, P_{i\alpha}(s) - P_i(s)\right] .$$

$$s_i' = \frac{s_i + \sum_\alpha \phi_{i\alpha}(s)\, \pi_i}{1 + \sum_\alpha \phi_{i\alpha}(s)} , \quad i = 1, 2, \ldots, n$$

$$s' = (s_1', s_2', \ldots, s_n') .$$

The transformation $s \to s'$ is continuous. Therefore there exists a fixed point, s. Each s_i in s contains a pure strategy $\pi_{i\alpha}$, with positive probability where $P_{i\alpha}(s) \le P_i(s)$. Therefore

$\phi_{i\alpha}(s) = 0$ for that i and α. Hence, $\phi_{i\beta}(s) = 0$ for that i and all β's since the proportion of $\pi_{i\alpha}$ used in s_i must not be decreased by the transformation. This is a condition for the existence of an equilibrium point.