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VALUES OF LARGE GAMES, VI: EVALUATING THE ELECTORAL COLLEGE EXACTLY

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PREFACE

This paper is part of a continuing study on the theory of games which is sponsored by Project RAND. It solves a problem which has been prominent for some time--the calculation of the power indices of players in a large game.

The applications are likely to be mostly theoretical in nature. But the technique used here is now part of our general knowledge.

SUMMARY

From a formal point of view, the electoral college provides an application for the theory of games and the use of the power index. It is a relatively large game and the exact calculation until recently has not been feasible. Now, due to a new idea, the calculation is quite easy.

Since the results have some sociological, as well as mathematical, interest, they are given for three different cases including the current one. One sees that there is a bias in favor of the large states as against the small of as much as five per cent. Of course, this bias is hardly significant as compared to many others not of a mathematical nature, which are on all sides.

We are now able to obtain these numbers for other games of a similar--and, in fact, somewhat larger--size. These may be used for further research into their nature, or for the intrinsic interest in the numbers themselves.

CONTENTS

I. THE CONTEXT

For some time it has been of interest to use the concept of a priori value in an application to the powers of the states in the electoral college. Until recently, we have not known how to calculate these values exactly in any feasible amount of time. In [3], we obtained montecarlo approximations of the values, but without the accuracy which would have allowed us to use the answers definitively for most purposes.

In this note we give the exact calculation. The key computational idea, which provided the breakthrough, is due to David G. Cantor.* This idea, together with its ramifications, has made it possible to calculate the exact power indices with no more effort than would be required for montecarlo estimates of the quality obtained in [3].

This note is intended primarily to present the new numerical results, but an attempt has been made to keep it self-contained, so that a casual reader may look at this question which is of both mathematical and political interest.

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*His suggestion was made to one of the authors in conversation, following a lecture at Princeton University in October 1960.

II. THE PROBLEM

A central notion in the theory of games is the "value" to a player in the game. Whatever this is, it connotes his expected winnings, what he might be willing to pay to play, or his share in some "fairly" arbitrated version of the game. The "value" is interesting, in fact crucial, because it gives a means of comparing the relative strengths of the players in a game.

For two-sided games with directly opposed interests, the values are derived from the principle of strategic optimization. For n-sided games, however, there are no clear-cut criteria for optimal methods of play, and the values must be described independently of any strategies that might be used to realize them. However, mathematical formulas giving well-defined (if not strategically enforceable) values can be derived from a set of properties which seem desirable in the context of the theory of games. These formulas may be applicable to a wide class of n-person games, including the "games" that arise in the procedural rules or constitutions of many political institutions.*

The question of this paper has to do with calculating the relative powers of the different states in the electoral college, when it is regarded as an n-person game. Of course, many historical, political, cultural, and other factors enter into this question which we do not pretend to capture in our game model; yet the latter, which does little

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*The mathematical foundation of the "value" which is described here will be found in [1]. The first description of its use in a political process is found in [2].

more than count votes, turns up some interesting points whose significance would not disappear in a more sophisticated analysis.

One elementary point is that power, in any reasonable sense of the word, is not automatically proportional to the voting strengths. A simple example will illustrate this: a hypothetical "electoral college" containing five states, having 12, 6, 6, 4, and 3 electoral votes. A majority requires 16 or more of the 31 votes. If we consider the 3-vote state, we see that it can never be an essential member of a winning coalition, since no coalition of the other states come to exactly 13, 14, or 15 votes. Hence that state's real power is nil. Likewise, it can be seen that the 6- and 4-vote states have equal power, in the same operational sense, because if one of these is switched for another one in a coalition, the winning or losing quality of the coalition does not change. Thus, in this example, the nominal voting strengths, represented by the numbers 12, 6, 6, 4, and 3, are far from an accurate guide to the relative abilities of the different states to influence the outcome.* It follows that any reasonable definition of "power" must take this phenomenon into account.

We shall calculate a particular kind of "value" for the electoral college, and will call it the "power index." It is normalized so that absolute power is represented by 1 and a total absence of power by 0. The indices for the states will then be numbers between 0 and 1, and will sum to 1.

The formula for the power index of the i^{th} state, denoted by ϕ_i, is simple in appearance:**

*The values in this five-person game, under our definition, are in the proportions 3:1:1:1:0.

**The basic description and derivation of this formula is in [1].

$$\phi_i = \sum \frac{(\bar{S} - 1)!\,(n - \bar{S})!}{n!} \qquad (1)$$

The summation is extended over all winning coalitions S in which the i^{th} state is essential. (The notation "$\bar{S}$" means the number of states in S.)

In our present application, with n as large as 51, the number of such coalitions is so large that the problem of enumerating them efficiently seemed, until recently, insurmountable. Methods of approximating the power indices were used instead.* The new technique circumvents this obstacle, and the exact values have now been obtained for several cases of interest.

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*These are discussed in Part II of Ref. 3.

III. THE METHOD

Let there be n players, with voting strengths $w_1, w_2, \ldots, w_n$. Let w be the total number of votes, and let q be the smallest number of votes required for a winning coalition. We wish to find the power index of some particular player I.

Let c_{jk} be the number of ways in which k players, other than I, can have a sum of votes equal to j. The indices j and k will be such that $0 \leq k \leq n - 1$ and $0 \leq j \leq w - w_I$. Then we may reformulate (1) as follows:

$$\phi_I = \sum_{k=0}^{n-1} \frac{k!\,(n-1-k)!}{n!} \sum_{j=q-w_I}^{q-1} c_{jk}, \tag{2}$$

since a player will be essential to a winning coalition only if the other players in it have insufficient votes to win without his help (j less than q), but sufficient votes to win with it ($j + w_I \geq q$). The number of terms to be summed in (2) is not large. The problem has always been to find a way of computing the numbers c_{jk}. Cantor's suggestion was to use the generating function:

$$f(x,y) = \prod (1 + x^{w_i} y).$$

This is a polynomial in x and y. The coefficient of $x^j y^k$ is precisely c_{jk}. The problem reduces then to multiplying out the polynomial (omitting just i = I) and determining the coefficients c_{jk}.

This can be carried out quite conveniently on a computing machine. One factor is taken at a time. A sequence of coefficient matrices $C^{(i)}$

is generated. $C^{(0)}$ is all 0 except for $c^{(0)}_{00} = 1$. The first factor is introduced. Then $c^{(1)}_{w_1 1} = 1$. The next factor produces $c_{w_1+w_2, 2} = 1$ and an addition in the "1" column. In fact, each $C^{(i)}$ is generated from the preceding one by the rule:

$$c^{(i)}_{jk} = c^{(i-1)}_{jk} + c^{(i-1)}_{j-w_i, k-1} \tag{3}$$

where the last term is understood to be 0 if either subscript is negative. The polynomial factors can be introduced in any order and the end result will be the same. If player I is left out, then $C^{(n-1)}$ is the matrix with elements equal to the c_{jk} given in (2).

The computer does not have to store the entire sequence of matrices. In fact, if the columns are taken in descending order, so that the weight of a player will not be counted twice in applying the algorithm (3), then each $C^{(i)}$ can simply be superimposed on $C^{(i-1)}$, obliterating it in the process. Furthermore, as a time saver, (3) need be applied only for $k \leq i$, since $c^{(i)}_{jk} = 0$ for $k > i$.

We now perform the summation given in (2). Here too there are some savings in operations which should be mentioned:

(a) The summation does not use values of j greater than q-1. Hence, only the upper half of the matrix need be determined.

(b) The matrix $C^{(n-1)}$ is symmetric by virtue of the identy

$$c_{jk} \equiv c_{w-w_I-j,\ n-1-k}.$$

Hence (2) can be rewritten using only values of k that are less than or equal to $(n-1)/2$. When this has been done, the largest value of j that appears is still q-1.

This means that we can restrict our attention throughout the sequence $C^{(i)}$ to the submatrix defined by $0 \leq j \leq q-1$, $0 \leq k \leq (n-1)/2$.

In obtaining the power index for another player J, it is not necessary to start from the beginning. One merely divides out the factor $(1 + x^{w_J} y)$, by "reversing" (3), and then multiplies by the factor $(1 + x^{w_I} y)$. This means that successive power indices can be obtained with little extra work, and is the source of a large saving in time. Of course, if $w_J = w_I$, no new computation is necessary at all.

In the three cases considered here, there are 18, 19, and 19 different indices to compute. Computation time on the IBM 7090 machine, after compiling about 60 FORTRAN statements, was about 70 seconds for each case.

IV. THE RESULTS

The states and their number of votes in the electoral college are given in Table 1. The first column gives the voting weights actually used in the 1960 presidential election. They are based on the 1950 census, with the addition of three votes each for Alaska and Hawaii. The second column gives the distribution of votes based on the 1960 census, with the further addition of three votes for the District of Columbia, as provided by the 23rd Admendment to the Constitution (ratified March 29, 1961). In our computation, we have determined the power indices for both cases, as well as for the interim case (50 "players" and 535 electoral votes) that existed for a few weeks early in 1961, after the reapportionment but before the inclusion of the District of Columbia.

It should be noted that the power indices are rational numbers by their nature, being some integral part of 51!. The computing machine, however, did not obtain this number, but only its decimal expansion to eight places. These power indices are given in Table 2.* The right-hand column is the current situation. Table 3 shows for each player the power indices rescaled to be directly comparable to the voting weights. Table 4, giving the ratios between the rescaled power indices and the voting weights, shows that these two measures of strength differ by little more than five per cent. There is, however, a systematic bias giving an

*Since the power indices must add to 1, there is an average deficiency of about 1 in the eighth place. This is due partly to systematic rounding down by the machine, and partly to various idiosyncrasies of the input-output equipment. This deficiency is also reflected into the next tables.

advantage to the larger states.* This effect is quite smooth and almost linear, as shown in Fig. 1. In particular, there are no sharp anomalies, such as in the toy example given on page 3.

The big-state bias had been revealed by our previous work,** but not the extremely high degree of regularity. (Parabolic fits can be made to the plots in Fig. 1 with a maximum error of the order of 0.0001.) This might have been expected, given the wide distribution of different voting weights, but until the calculations were made, the number-theoretical possibilities of the voting weights remained in doubt.

It is clear that the mathematical properties discussed here are not very important in the total consequences of the electoral college. But in any discussion of the electoral college and its implications, these results, though small in their effect, can meaningfully be included.

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*It is to be wondered what is the case in the multimillion-person game, when the voters, rather than the states, are considered the individual players. There is some intuitive evidence that the power indices would again be in favor of the voter in the large states, and that this bias quantitatively might be as much as double the one seen by treating the states as the players.

**See the discussion accompanying Fig. 1 in Ref. 3.

Table 1

ELECTORAL VOTES

November 8, 1960	
New York	45
California	32
Pennsylvania	32
Illinois	27
Ohio	25
Texas	24
Michigan	20
New Jersey	16
Massachusetts	16
North Carolina	14
Indiana	13
Missouri	13
Georgia	12
Virginia	12
Wisconsin	12
Tennessee	11
Alabama	11
Minnesota	11
Florida	10
Louisiana	10
Iowa	10
Kentucky	10
Maryland	9
Washington	9
Connecticut	8
Oklahoma	8
South Carolina	8
Kansas	8
Mississippi	8
West Virginia	8
Arkansas	8
Colorado	6
Oregon	6
Nebraska	6
Maine	5
Arizona	4
Idaho	4
Montana	4
New Hampshire	4
New Mexico	4
North Dakota	4
Rhode Island	4
South Dakota	4
Utah	4
Hawaii	3
Alaska	3
Delaware	3
Nevada	3
Vermont	3
Wyoming	3
	537

March 29, 1961	
New York	43
California	40
Pennsylvania	29
Illinois	26
Ohio	26
Texas	25
Michigan	21
New Jersey	17
Massachusetts	14
Florida	14
North Carolina	13
Indiana	13
Missouri	12
Georgia	12
Virginia	12
Wisconsin	12
Tennessee	11
Alabama	10
Minnesota	10
Louisiana	10
Maryland	10
Iowa	9
Kentucky	9
Washington	9
Connecticut	8
Oklahoma	8
South Carolina	8
Kansas	7
Mississippi	7
West Virginia	7
Arkansas	6
Colorado	6
Oregon	6
Nebraska	5
Arizona	5
Maine	4
Idaho	4
Montana	4
New Hampshire	4
New Mexico	4
North Dakota	4
Rhode Island	4
South Dakota	4
Utah	4
Hawaii	4
Alaska	3
Delaware	3
Nevada	3
Vermont	3
Wyoming	3
District of Columbia	3
	538

Table 2

POWER INDICES

Electoral Votes		November 8, 1960		Interim		March 29, 1961
45		.08862901				
43				.08454671		.08406425
40				.07811297		.07767063
32	(2)	.06123387				
29				.05530754		.05500222
27		.05113447				
26			(2)	.04928295	(2)	.04901260
25		.04715589		.04729199		.04703309
24		.04517923				
21				.03941121		.03919718
20		.03735415				
17				.03165826		.03148765
16	(2)	.02965467				
14		.02585012	(2)	.02592367	(2)	.02578474
13	(2)	.02395883	(2)	.02402692	(2)	.02389839
12	(3)	.02207476	(4)	.02213740	(4)	.02201920
11	(3)	.02019781		.02025506		.02014710
10	(4)	.01832791	(4)	.01837979	(4)	.01828200
9	(2)	.01646497	(3)	.01651153	(3)	.01642383
8	(7)	.01460893	(3)	.01465019	(3)	.01457252
7			(3)	.01279570	(3)	.01272798
6	(3)	.01091723	(3)	.01094799	(3)	.01089014
5		.00908142	(2)	.00910697	(2)	.00905894
4	(9)	.00725221	(10)	.00727259	(10)	.00723429
3	(6)	.00542953	(5)	.00544477	(6)	.00541615
Check Sum	(50)	.99999959	(50)	.99999950	(51)	.99999947

Table 3

RESCALED POWER INDICES

Electoral Votes	November 8, 1960 ∅ x 537		Interim ∅ x 535		March 29, 1961 ∅ x 538	
45		47.593780				
43				45.232495		45.226568
40				41.790440		41.786804
32	(2)	32.882591				
29				29.589537		29.591194
27		27.459216				
26			(2)	26.366383	(2)	26.368783
25		25.322717		25.301217		25.303807
24		24.261247				
21				21.084998		21.088085
20		20.059178				
17				16.937171		16.940359
16	(2)	15.924559				
14		13.881517	(2)	13.869164	(2)	13.872193
13	(2)	12.865895	(2)	12.854401	(2)	12.857334
12	(3)	11.854146	(4)	11.843514	(4)	11.846331
11	(3)	10.846224		10.836458		10.839140
10	(4)	9.842088	(4)	9.833191	(4)	9.835718
9	(2)	8.841693	(3)	8.833669	(3)	8.836024
8	(7)	7.844999	(3)	7.837854	(3)	7.840015
7			(3)	6.845701	(3)	6.847652
6	(3)	5.862553	(3)	5.857174	(3)	5.858896
5		4.876723	(2)	4.872232	(2)	4.873709
4	(9)	3.894437	(10)	3.890837	(10)	3.892051
3	(6)	2.915657	(5)	2.912953	(6)	2.913888
	(50)	536.999843	(50)	534.999825	(51)	537.999790

Table 4

POWER RATIOS

Electoral Votes	November 8, 1960	Interim	March 29, 1961
45	1.0576395		
43		1.0519185	1.0517806
40		1.0447610	1.0446701
32	1.0275810		
29		1.0203288	1.0203860
27	1.0170080		
26		1.0140916	1.0141839
25	1.0129087	1.0120486	1.0121523
24	1.0108853		
21		1.0040475	1.0041945
20	1.0029589		
17		.9963042	.9964917
16	.9952849		
14	.9915369	.9906546	.9908709
13	.9896843	.9888000	.9890257
12	.9878455	.9869595	.9871943
11	.9860204	.9851325	.9853764
10	.9842087	.9833191	.9835718
9	.9824103	.9815188	.9817804
8	.9806249	.9797318	.9800019
7		.9779573	.9782360
6	.9770922	.9761956	.9764827
5	.9753446	.9744464	.9747417
4	.9736092	.9727094	.9730128
3	.9718858	.9709844	.9712958

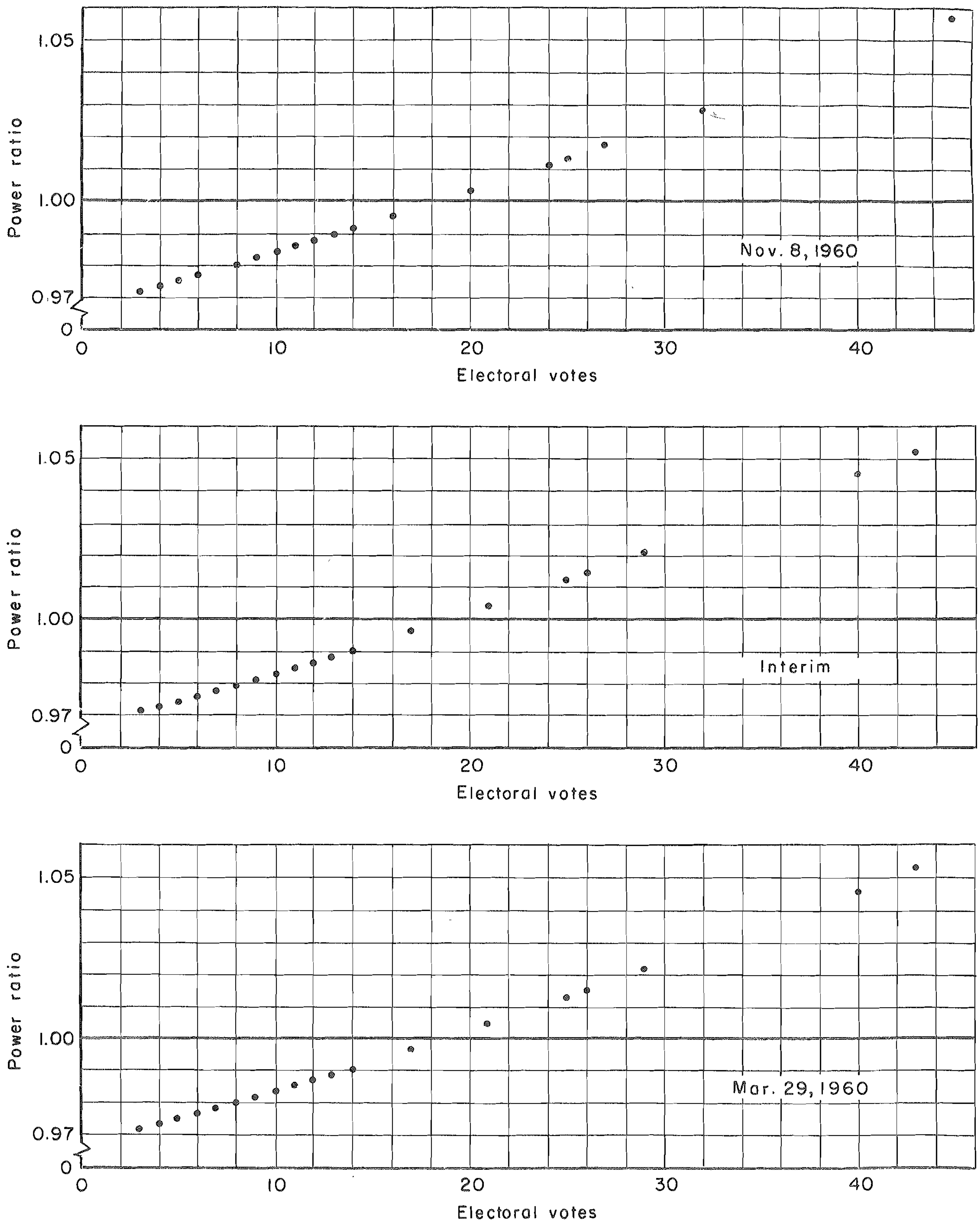

Fig. 1—Power ratio as a function of voting strength

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