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RESEARCH MEMORANDUM

A CONTINUOUS COLONEL BLOTTO GAME

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Colonel Blotto and the enemy are confronted with the following situation:

Colonel Blotto has at his disposal a total of $B (> 0)$ units of attack, capable of continuous partition; whereas the enemy has $E (> 0)$ units of like character. They are to attack simultaneously and in full force a set of n hills of different value, such that the payoff by the enemy to C. B. for the j^{th} hill is $+a_j (> 0)$ if Blotto's forces exceed his there, $-a_j$ if the enemy's exceed Blotto's, and 0 otherwise. How should they play?

Stated as a continuous two-person, zero-sum game, the foregoing becomes:

Let $B > 0$, $E > 0$, $a_j > 0$, $j = 1(1)n$ where n is a given positive integer.

1. The maximizing player (C.B.) chooses a point x in Euclidean space, having coordinates x_i , $i = 1(1)n$ subject to $x_i \geq 0$ and $\sum_{i=1}^n x_i = B$.

2. The minimizing player chooses a point y with similar properties, namely $y_i \geq 0$, $i = 1(1)n$ and $\sum_{i=1}^n y_i = E$.

3. For any pair (x,y) of pure strategies, the payoff to the maximizing player is given by $M(x,y)$ where

$$M(x,y) = \sum_{i=1}^n a_i \operatorname{sgn}(x_i - y_i).$$

Solutions for the above game have been obtained in the following cases, which we shall treat in order:

- I. $n = 2$ (all cases)
- II. $n = 3$ and $B = E$
- III. $a_i = a_j$; $i, j = 1(1)n \geq 3$ and $B = E$.

Case I (n = 2)

Without loss of generality, assume $a_2 \geq a_1$ and let $c = \frac{a_2}{a_1}$. Write $x_1 = x$ and $y_1 = y$.

In the symmetric case, $B = E$, the optimal strategy is

$$F^*(x) = I_0(x)$$

for which

$$\begin{aligned} K^*(y) &= \int_0^1 M(x, y) dF^*(x) \\ &= a_1 \operatorname{sgn}(-y) + a_2(y). \end{aligned}$$

$$\geq 0 = v.$$

If $B > E$, let $d = B - E$, and let m (an integer) and r be such that

$$B = m d + r. \quad (0 \leq r < d).$$

Take p in the interval $r < p < d$ and let $s = \sum_{j=0}^{m-1} c^j$.

Then,

$$v = (a_1/B)(c^m + 1) = \begin{cases} \frac{2a_1}{m} & \text{when } a_2 = a_1 \\ \frac{a_2^m + a_1^m}{a_2^m - a_1^m} (a_2 - a_1) & \text{when } a_2 > a_1 \end{cases}$$

and an optimal strategy for the maximizing player (B) is

$$F^*(x) = \frac{1}{s} \sum_{k=1}^m c^{m-k} I_{p+(k-1)d}(x),$$

for which

$$\begin{aligned} K^*(y) &= \frac{a_1}{s} \sum_{k=1}^m c^{m-k} \left\{ \operatorname{sgn} [p+(k-1)d-y] - c \operatorname{sgn} [p+(k-2)d-y] \right\} \\ &= \frac{a_1}{s} \left\{ \sum_{k=1}^m c^{m-k} \operatorname{sgn} [p+(k-1)d-y] - \sum_{k=1}^m c^{m-(k-1)} \operatorname{sgn} [p+(k-2)d-y] \right\} \\ &= \frac{a_1}{s} \left\{ \sum_{k=1}^m c^{m-k} \operatorname{sgn} [p+(k-1)d-y] - \sum_{k=0}^{m-1} c^{m-k} \operatorname{sgn} [p+(k-1)d-y] \right\} \\ &= \frac{a_1}{s} \left(c^0 \operatorname{sgn} [p + (m-1)d-y] - c^m \operatorname{sgn} [p-d-y] \right). \end{aligned}$$

Since

$$\begin{aligned} p + md &> r + md = B \\ p + (m-1)d &> B-d = E \\ p + (m-1)d - y &> E-y \geq 0, \end{aligned}$$

and since

$$p - d < 0,$$

then

$$K^*(y) = \frac{a_1}{s} (1+c^m) = v.$$

An optimal strategy for the minimizing player is

$$G^*(y) = \frac{1}{s} \sum_{j=0}^{m-1} c^j I_{jq}(y), \quad \left(q = \frac{E}{m-1}\right)$$

for which

$$H^*(x) = \frac{a_1}{s} \sum_{j=0}^{m-1} c^j [\text{sgn}(x-jq) - c \text{sgn}(x-jq-d)] .$$

Since

$$(m-1)d \leq B-d = E$$

$$d \leq q,$$

then

$$H^*(x) \leq \frac{a_1}{s} \left\{ \sum_{j=0}^{m-1} c^j \text{sgn}(x-jq) - \sum_{j=0}^{m-1} c^{j+1} \text{sgn}[x - (j+1)q] \right\}$$

$$\leq \frac{a_1}{s} \left\{ \sum_{j=0}^{m-1} c^j \text{sgn}(x-jq) - \sum_{j=1}^m c^j \text{sgn}(x-jq) \right\}$$

$$\leq \frac{a_1}{s} \left\{ c^0 \text{sgn}(x) - c^m \text{sgn}(x-mq) \right\}$$

$$\leq \frac{a_1}{s} (1+c^m) = v.$$

Case II

The game is clearly symmetric and hence $v = 0$. We shall show that the following strategy of Blotto's (and hence of his opponent) is an optimal one. (It appears to the writer that a geometric description of Blotto's behavior is handier and more interesting than an analytic formulation.)

If Blotto cannot form a non-degenerate triangle having sides of lengths a_1 , a_2 , and a_3 , he attacks the hill of greatest value with all of his forces (pure strategy). If otherwise, he constructs such a triangle (figure 1). He then inscribes a circle within it and erects a hemisphere upon this circle. He next chooses a point from a density uniformly distributed over the surface of the hemisphere and projects this point straight down into the plane of the triangle (point P in figure 1). He then divides his forces in respective proportion to the triangular areas subtended by P and the sides, i.e.

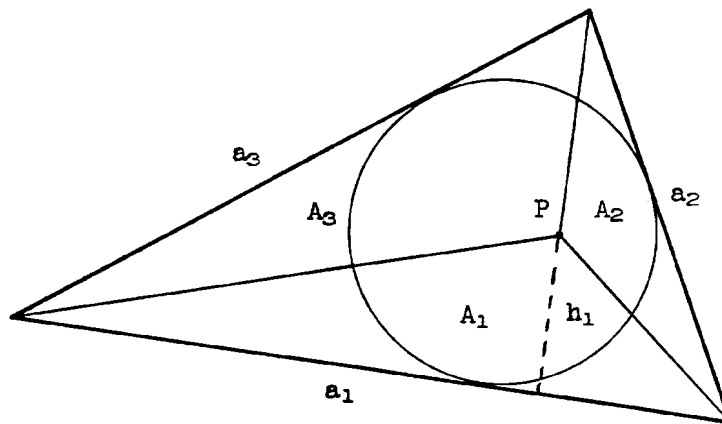


Figure 1.

$x_1: x_2: x_3 = A_1: A_2: A_3$. Call this strategy F^* .

Verification:

In the first instance it is unnecessary to show that the pure strategy is a solution and hence we confine ourselves to testing the optimality of F^* . Since the game is symmetric we need show only that $K^*(y) \geq 0$ for each y .

Let $F_1^*(x_1)$ denote the respective marginal d.f.'s of Blotto's continuous mixed strategy F^* . Blotto's expectation from e.g. hill No. 1 is then given by

$$a_1 [1 - F_1^*(y_1)] - a_1 F_1^*(y_1) = a_1 [1 - 2F_1^*(y)] \quad ,$$

and hence his total expectation is given by

$$K^*(y) = \sum_{i=1}^3 a_i [1 - 2F_i^*(y_i)] \quad (1)$$

Let h_1 denote the altitude of the triangle of area A_1 subtended by P (figure 1).

From a well-known property of the surface area of a sphere, we see that

h_1 is distributed uniformly over $(0, 2r)$, r being the radius of the sphere.

Now, $A_1 = \frac{1}{2} a_1 h_1$, and hence A_1 is distributed uniformly over $(0, a_1 r)$. Also,

since $x_1 + x_2 + x_3 = B$ and $x_1: x_2: x_3 = A_1: A_2: A_3$, it follows that

$x_1 = \frac{A_1}{\Delta} B$ where Δ is the area of the originally constructed triangle. Thus

x_1 is uniformly distributed over $(0, \frac{a_1 r B}{\Delta})$. But $\Delta = \frac{1}{2}(a_1 r + a_2 r + a_3 r)$,

hence x_1 is uniformly distributed over $(0, \frac{2a_1 B}{a_1 + a_2 + a_3})$, i.e.

$$F_1^*(x_1) = \min \left[1, \frac{a_1 + a_2 + a_3}{2a_1 B} x_1 \right] \quad x_1 \geq 0.$$

Consequently, (on substituting in (1))

$$K^*(y) = \sum_{i=1}^3 a_i \left\{ 1 - 2 \min \left[1, \frac{a_1 + a_2 + a_3}{2a_1 B} y_1 \right] \right\}.$$

Note, that if $\alpha > 0$, $y \geq 0$, then $\min(1, \alpha y) \leq \alpha y$, and hence

$$\begin{aligned} K^*(y) &\geq \sum_{i=1}^3 a_i \left\{ 1 - \frac{a_1 + a_2 + a_3}{a_1 B} y_1 \right\} \\ &= a_1 + a_2 + a_3 - (a_1 + a_2 + a_3) \frac{\sum y_1}{B} \\ &= 0 \quad (\text{since } \sum y_1 = B). \end{aligned}$$

This completes the verification.

Case III

For $n \geq 3$ ($a_i = a_j$) and $B = K$) an obvious solution consists in a special generalization of case II, since a circle can always be inscribed in a regular n -gon. In other words, an optimal strategy for Blotto in this case is to inscribe a circle in a regular n -gon the sides of which correspond to the hills to be attacked. He erects a hemisphere upon this circle and chooses a point on its surface from a density uniformly distributed over the surface of the hemisphere. He then projects this point down to the plane of

the n -gon and divides his forces in respective proportion to the distances of this projected point from the sides of the n -gon. We shall not treat this case in detail, but rather discuss some more solutions for the special case $n = 3$, $a_1 = a_2 = a_3$. An interesting result here is that given any $\epsilon > 0$, there exists a bivariate density function solution such that the simplicial sub-region which is played with positive probability has an area less than ϵ .

Consider figure 2.

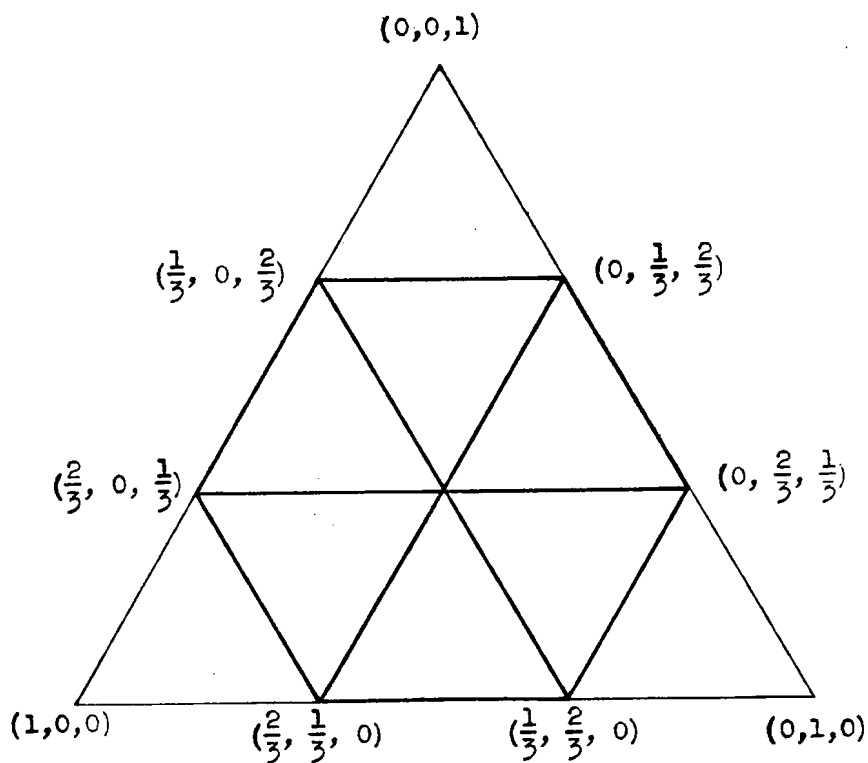


Figure 2.

There is no loss in generality in assuming that $B = E = 1$. Thus the equilateral triangle of unit altitude is the simplex of Blotto's pure strategies $x = (x_1, x_2, x_3)$. We shall describe an optimal strategy (differing from the hemispherical construction) and indicate the method of verification. Colonel Blotto might do the following: He plays a continuous density function p over the regular hexagon (of figure 2), i.e.

if $d\sigma$ denotes an element of area at point x within this hexagon, the probability that he chooses a point within $d\sigma$ is given by $p(x)d\sigma$. The density function p is determined as follows: p is constant¹ on the perimeter of the hexagon, zero at the center, and linear along any straight line segment joining the center with an arbitrary point on the perimeter.

In order to see that the foregoing is a solution, it suffices to show that, as in the case of the hemispherical construction, the marginals of p are uniform over $(0, \frac{2}{3})$, i.e. $F_1^*(x_1) = \min(1, \frac{3}{2}x_1)$. The reader is invited to do this.

One might now ask "Is every solution a linear combination of the hexagonal and hemispherical constructions?" From the ξ -remark made at the beginning of this section, one suspects a negative answer. Indeed, consider the following construction (figure 3):

¹ Normalization reveals the value of this constant to be $\frac{9}{4}\sqrt{3}$.

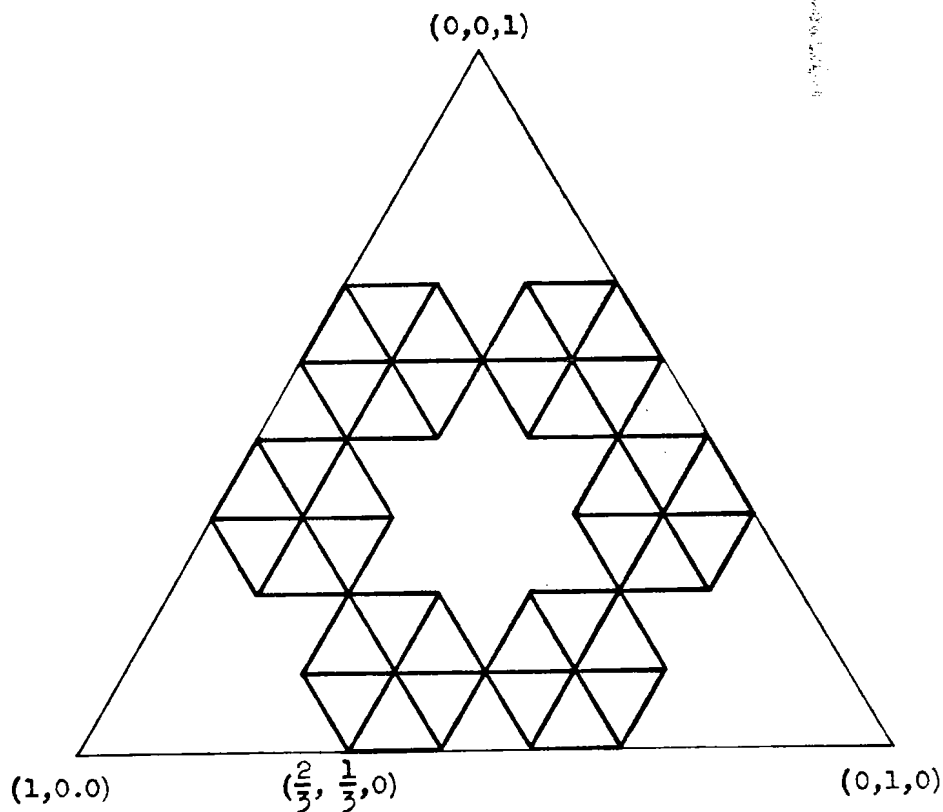


Figure 3.

Here, the wily Blotto's density is distributed over the six regular hexagons (heavy lines in figure 3), each hexagon being a replica of the one originally described and each having weight $\frac{1}{6}$. In other words, Blotto tosses an unbiased die to determine which of the hexagons in figure 3 he is to make his selection from. He then plays over this hexagon as in the original construction, dividing his forces according to the simplicial coordinates of the point finally chosen.

Here, also, the marginals are uniform. It is, in fact, believed (though the writer has not been able to show this) that the totality of solutions for the case under discussion is characterized by uniform marginal df's.

To see how it is possible to construct a solution such that the region over which Blotto plays with positive probability has an arbitrarily small area, note that every one of the six hexagons of figure 3 can be replaced by six smaller equally weighted hexagons similarly placed (as in figure 4) thereby yielding a valid solution. The area, however, has been decreased by a factor of $\frac{2}{3}$. This process can be repeated as often as desired, the successive "snowflakes" approaching a discontinuum of zero area.

Many more combinations are possible; for example, in a "snowflake" solution, any one of the hexagons can be replaced by an equally weighted hemisphere.

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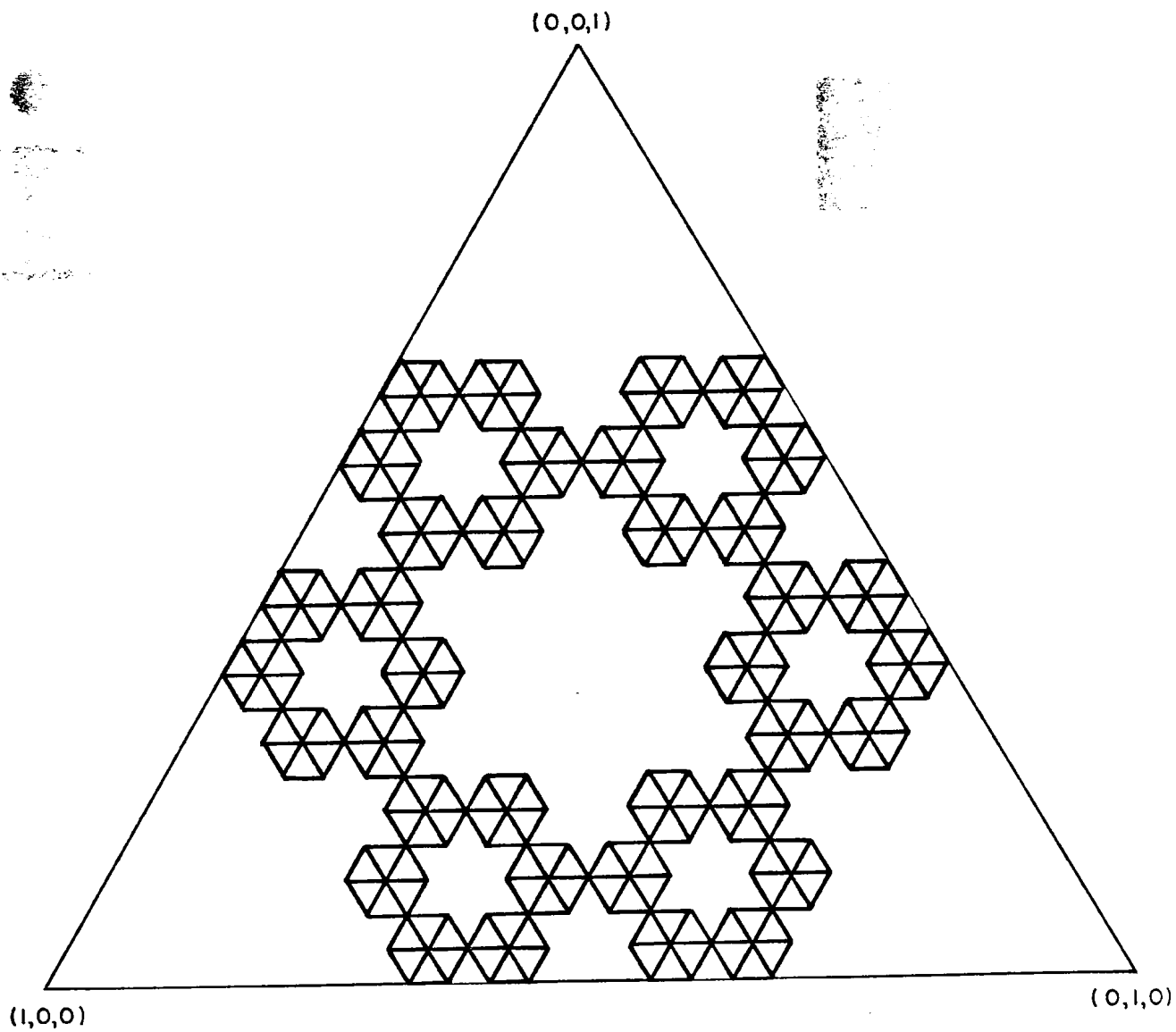


FIG. 4