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THE APPLICATION OF DYNAMIC PROGRAMMING TO SATELLITE INTERCEPT AND RENDEZVOUS PROBLEMS

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THE APPLICATION OF
DYNAMIC PROGRAMMING TO
SATELLITE INTERCEPT AND
RENDEZVOUS PROBLEMS

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Center (DDC).
This Memorandum is part of a study, currently in progress at The RAND Corporation, on the application of dynamic programming to satellite intercept and rendezvous problems. It should be of particular interest to the Flight Control Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, in connection with their studies of the application of dynamic programming to flight control problems.
SUMMARY

This Memorandum discusses the application of dynamic programming to a terminal guidance process for satellite intercept and rendezvous problems.

The mathematical equations describing the guidance process are derived and simplified by linearizing assumptions in order to reduce the complexity of the computation process. Some typical numerical data are then used to evaluate the effect of these linearizing assumptions on system performance.

It is concluded that these linearizing assumptions result in satisfactory system performance within practical tolerances as far as the optimal control system is concerned. However, the linearizing assumptions appear to have an adverse effect on the performance of the filter used to process observational data.
ACKNOWLEDGMENT

The author gratefully acknowledges the contributions of Mrs. Joy Jolissaint, who programmed the entire problem and prepared the computational results.
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<td>$A(r)$</td>
<td>$6 \times 6$ matrix in interceptor's differential equation of motion</td>
</tr>
<tr>
<td>$A_r$</td>
<td>$6 \times 6$ matrix in target's differential equation of motion</td>
</tr>
<tr>
<td>$a$</td>
<td>semi-major axis of elliptic orbit</td>
</tr>
<tr>
<td>$B$</td>
<td>$6 \times 3$ matrix in interceptor's differential equation of motion</td>
</tr>
<tr>
<td>$b$</td>
<td>angle locating line-of-sight from interceptor to target</td>
</tr>
<tr>
<td>$\dot{b}$</td>
<td>time derivative of angle $b$</td>
</tr>
<tr>
<td>$C(t,k)$</td>
<td>$2 \times 2$ matrix associated with computation of matrix $\dot{e}(t,k)$</td>
</tr>
<tr>
<td>$D(k,T)$</td>
<td>$2 \times 2$ matrix associated with computation of matrix $\dot{e}(t_k,T)$</td>
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<td>$E(t_{k+1},t_k)$</td>
<td>$6 \times 6$ matrix associated with the computation of target motion correction vector</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity of elliptic orbit</td>
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<tr>
<td>$F(x)$</td>
<td>$12 \times 12$ matrix in differential equation for combined state vector</td>
</tr>
<tr>
<td>$F(t)$</td>
<td>$6 \times 6$ matrix in relative state vector differential equation</td>
</tr>
<tr>
<td>$f(b,\Delta c;\tau)$</td>
<td>cost functional or minimized performance index</td>
</tr>
<tr>
<td>$G$</td>
<td>$12 \times 3$ matrix in combined state vector differential equation</td>
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<td>$G(t)$</td>
<td>$6 \times 6$ matrix in relative state vector differential equation</td>
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<tr>
<td>$g(nT,t_k)$</td>
<td>target motion correction vector</td>
</tr>
<tr>
<td>$H$</td>
<td>$m \times n$ matrix relating $x$ to $z$</td>
</tr>
<tr>
<td>$h$</td>
<td>infinitesimal interval of time</td>
</tr>
<tr>
<td>$J(\tau)$</td>
<td>performance index for control process of duration $\tau$</td>
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Jm
performance index for observation estimation process
K(ξ)
2 x 2 matrix associated with the computation of matrix
E(t_{k+1}, t_k)
ℓ
angle locating line-of-sight from interceptor to target
\dot{\ell}
time derivative of angle ℓ
M
mean anomaly
M(t_k)
6 x 6 Jacobian matrix of transformation from q(t_k) to
Δ\tilde{S}(t_k)
n
mean daily motion in elliptic orbit
nr
mean daily motion in target orbit
P_m
covariance matrix associated with errors in components
of x
Q_N
matrix weighting terminal errors in performance index
q(t_k)
6 x 1 vector of target observations
R_q
covariance matrix associated with errors in components
of q(t_k)
R(m)
covariance matrix associated with components of random
vector v(m)
R_S(t_k)
covariance matrix associated with errors in components
of Δ\tilde{S}(t_k)
r
distance of interceptor from center of earth
s(t)
interceptor state vector
s_r(t)
target state vector
T
time of perigee passage or duration of estimation inter-
val
U
matrix associated with solution of matrix Riccati equa-
tion
u(t)
control vector in interceptor equation of motion
V
matrix associated with solution of matrix Riccati equa-
tion
v(m)
m x 1 random vector
\( \mathbf{v}_q(t) \) 
random vector of errors associated with components of \( q(t) \)

\( \mathbf{v}_s(t) \) 
random vector of errors associated with components of \( \Delta \mathbf{s}(t) \)

\( \mathbf{w} \) 
parameter in performance index equal to zero for the intercept case and unity for the rendezvous case

\( \mathbf{x} \) 
\( n \times 1 \) parameter vector in observation estimation process

\( (\hat{x})_k \) 
k-th estimate of \( x \)

\( \mathbf{Y} \) 
matrix associated with solution of matrix Riccati differential equation

\( \mathbf{Z} \) 
matrix associated with solution of matrix Riccati differential equation

\( z(m) \) 
m \( n \) observation vector

\( z(t) \) 
combined state vector

\( \alpha(\tau) \) 
6 \( \times \) 6 matrix in cost functional

\( \alpha_k \) 
scalar associated with method of mean coefficients

\( \beta(\tau) \) 
6 \( \times \) 6 matrix in cost functional

\( \Gamma(\tau-t) \) 
6 \( \times \) 3 matrix determining optimal control vector

\( \gamma(\tau) \) 
6 \( \times \) 6 matrix in cost functional

\( \Delta \mathbf{s}(t) \) 
th true relative state vector

\( \Delta \hat{\mathbf{s}}(t) \) 
computed relative state vector

\( \hat{\Delta \hat{\mathbf{s}}}(t) \) 
estimated relative state vector

\( [\Delta \hat{\mathbf{s}}(T)]_k \) 
k-th estimate of relative state vector at time \( T \)

\( [\delta \Delta \hat{\mathbf{s}}(T)]_k \) 
correction to \( [\Delta \hat{\mathbf{s}}(T)]_k \)

\( \Delta t \) 
duration of observation interval

\( (\Delta \hat{x})_k \) 
correction to \( (\hat{x})_k \)

\( \Delta \rho \) 
relative range measurement between interceptor and target

\( \Delta \hat{\rho} \) 
relative range rate measurement between interceptor and target
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<th>Symbol</th>
<th>Description</th>
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<td>$\lambda$</td>
<td>Lagrange multiplier in performance index</td>
</tr>
<tr>
<td>$\mu$</td>
<td>gravitational constant</td>
</tr>
<tr>
<td>$\sigma_b^2$</td>
<td>variance of uncertainty in $b$</td>
</tr>
<tr>
<td>$\sigma_\dot{b}^2$</td>
<td>variance of uncertainty in $\dot{b}$</td>
</tr>
<tr>
<td>$\sigma_\ell^2$</td>
<td>variance of uncertainty in $\ell$</td>
</tr>
<tr>
<td>$\sigma_\dot{\ell}^2$</td>
<td>variance of uncertainty in $\dot{\ell}$</td>
</tr>
<tr>
<td>$\sigma_\rho^2$</td>
<td>variance of uncertainty in $\rho$</td>
</tr>
<tr>
<td>$\sigma_\dot{\rho}^2$</td>
<td>variance of uncertainty in $\dot{\rho}$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>duration of guidance process</td>
</tr>
<tr>
<td>$\Phi(t_k,T)$</td>
<td>inverse of fundamental solution matrix for relative state vector differential equation</td>
</tr>
<tr>
<td>$\omega_k$</td>
<td>parameter associated with method of mean coefficients</td>
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1.0 INTRODUCTION

1.1 GENERAL REMARKS

This Memorandum discusses the application of dynamic programming to a terminal guidance process for satellite intercept and rendezvous problems. More specifically, it attempts to ascertain how far the mathematical formulation of the problem might be simplified by the use of linearizing assumptions and still realize system performance meeting acceptable engineering tolerances.

The optimization of control processes by dynamic programming can take two approaches: a solution determined by numerical search techniques using a digital computer (machine dynamic programming), or an analytical approach where the control system equations are linearized and usually a quadratic performance index is used (analytical dynamic programming). Machine dynamic programming, while applicable in principle to a wide class of problems, has the well-known dimensionality limitation due to limited computer storage capacity. Analytical dynamic programming, while restricted in application to a much narrower class of problems, allows higher dimensional systems to be considered. This approach will be used in the investigation described in this Memorandum.

When a mathematician investigates an optimal control problem, he is concerned first with the existence and uniqueness of an optimal solution and then with how to compute it. When an engineer investigates an optimal control problem he generally begins with a particular physical situation, and an optimal solution is usually (but not always) known to exist. Further, the engineer desires the performance of his
system to satisfy a certain set of tolerances or specifications. This implies that not only the optimal solution, but also a certain number (possibly infinite) of sub-optimal solutions will satisfy the set of system tolerances. The problem then becomes how to formulate the simplest mathematical description of the system that will yield a sub-optimal solution satisfying the system tolerances.

Generally, the analytical dynamic programming approach involves some linearizing assumptions in formulating the equations defining system performance. These linearizing assumptions mean that any solution obtained must necessarily be sub-optimal. One question to be asked is, how far can the system equations be linearized before the sub-optimal solution no longer satisfies system performance tolerances? The answer to this question, of course, depends upon the particular system under consideration. One particular system configuration will be considered in this Memorandum, and some quantitative effects of the linearization assumptions made in formulating the system equations will be discussed.

There have been papers, too numerous to mention, dealing both directly and indirectly with satellite rendezvous problems. References 1-4 deal more or less directly with the problem. References 5 and 6 deal with the application of dynamic programming to orbit transfers. The discussion in this Memorandum differs from the first group of references in that it uses dynamic programming to synthesize the optimal control and optimal estimation problem. It presents a formulation of the dynamic programming approach that is different from the second group of references.
1.2 STATEMENT OF PROBLEM

Assume that the target is moving in a circular, geocentric orbit. If the rendezvous time is chosen, then it is possible to choose an infinite family of Keplerian orbits, in the plane of the target orbit, whose apogees all coincide with the desired rendezvous point. If the terminal guidance process is assumed to start at a point lying in the target orbit plane, then one of the above family of Keplerian orbits will pass through the initial interceptor position.

If the initial velocity vector of the interceptor is appropriately chosen, then the interceptor can travel along this elliptic orbit and collide with the target without further application of thrust (neglecting the effects of natural perturbations).

As far as the intercept problem is concerned, this Keplerian orbit can be considered as a true optimal trajectory since no fuel is consumed and zero terminal position errors result. As will be explained later, this orbit can be used to test the effects of the linearizing assumptions on the control and estimation processes.

For the rendezvous problem no such "zero fuel consumption" orbit exists. This follows from the uniqueness of the solution of the differential equations defining the target motion, i.e., the target orbit is the only possible one passing through the intercept point with zero position and velocity errors and zero fuel consumption.

Due to uncertainties in the knowledge of the target orbit and in the operation of the initial guidance system in establishing the state of the interceptor at the start of the terminal guidance process, it is unlikely that the interceptor will start out on one of these
"optimal intercept" orbits. If a rendezvous is contemplated, some expenditure of fuel will be required to match the target's velocity at apogee, even if the interceptor did travel along one of these "optimal intercept" orbits. Accordingly, it is assumed that some form of terminal guidance process is desirable.

The problem then is to investigate the application of dynamic programming to the synthesis of such a terminal guidance process, with interest directed toward how much the computational equations can be simplified before system performance ceases to satisfy performance tolerances.

1.3 DESCRIPTION OF A POSSIBLE GUIDANCE SYSTEM

Several possible guidance configurations can be adapted to the terminal guidance problem. The particular configuration to be discussed is not necessarily the best one, but it is straightforward in concept and convenient for a preliminary investigation such as discussed in this Memorandum.

A terminal guidance system requires relative measurements of position and velocity in some form between the intercepting and target vehicles. This implies a sensor system on board the interceptor. For the system discussed here it will be assumed that the sensor system is capable of making simultaneous measurements of six independent quantities consisting of range, range rate, two angles, and two angular rates.

A functional block diagram of the terminal guidance system to be discussed is shown in Fig. 1.

The observation transformation process represents the sensor
Fig. 1—Functional block diagram of guidance loop
system and the transformation of each set of sensor observations to the components of a relative state vector $\Delta \tilde{s}(t)$. The components of the relative state vector are the components of relative position and velocity between the interceptor and target. They will be defined more precisely later (Section 2.2).

The estimation process uses a Kalman filter (7-9) to filter all previous observational data to obtain an updated estimate of the relative state vector at a finite set of times equally distributed over the entire terminal guidance process.

The extrapolation process computes the relative state vector components at instants of time in between the times when the updated relative state vectors are available from the Kalman filter.

The optimal control process computes the optimal control vector components as linear functions of the relative state vector components as obtained from the extrapolation process.

The true orbit transfer dynamics represent the 12th-order system of nonlinear differential equations defining the motions of the target and interceptor vehicles.

An idealized control system is assumed to apply thrust to the interceptor vehicle in response to the optimal control vector. The optimal control vector is in units of force per unit mass, or acceleration. To mechanize such a system an accelerometer feedback loop which controls the magnitude and direction of the interceptor's thrust vector might be used. An analysis of such a mechanization is beyond the scope of a preliminary analysis such as this.
1.4 THE SEPARATION OF THE OPTIMAL CONTROL AND ESTIMATION PROBLEMS

It has been proved by a number of people that the optimal overall system can consist of the "best" deterministic controller cascaded with an optimal least squares filter (see, for example, Refs. 10 and 11). This approach will be used in the discussion to follow, and the optimal control process and the optimal estimation process will be synthesized separately.
2.0 THE OPTIMAL CONTROL PROBLEM

2.1 INTRODUCTORY REMARKS

The state variables of a point mass in a two-body orbit can be defined as the rectangular coordinates of position and velocity with respect to an inertial coordinate system. These state variables can be considered as the components of a system state vector.

\[ s(t) = \begin{bmatrix} x(t), \dot{x}(t), y(t), \dot{y}(t), z(t), \dot{z}(t) \end{bmatrix}^T \]

When an orbit transfer process is under consideration, there must be a perturbing force applied to the point mass to bring about the transfer. An orbit transfer process implies the existence of an initial two-body orbit specified by \( s(t_0) \) from which the orbit transfer process starts, and a desired terminal two-body orbit specified by \( s(t_0 + \tau) \) at which the transfer is to terminate. This orbit transfer process is usually required to satisfy some performance index by the suitable choice of the magnitude and direction of the perturbing force. When the perturbing force is continuously applied to the point mass, the orbit transfer process can be regarded as a continuous decision process, and the techniques of dynamic programming can be used to choose the time behavior of the perturbing force.

Before considering the use of dynamic programming, it is necessary to determine the equations of motion of the point mass and define a performance index which measures how closely the actual orbit transfer matches the desired transfer.

2.2 EQUATIONS OF MOTION

It will be assumed that the target vehicle is moving in an orbit
that can be, to a sufficient degree of approximation, represented by a circular geocentric orbit during the orbit transfer process. If we denote the state vector for the target by

\[ s_r(t) = [x_r(t), \dot{x}_r(t), y_r(t), \dot{y}_r(t), z_r(t), \dot{z}_r(t)]^T \]

then the two-body equations of motion for the target vehicle in vector-matrix form are given by

\[ \frac{d s_r}{dt} = A_r s_r, \quad s_r(t_0) = b \]

over the time interval \( t_0 \leq t \leq t_0 + \tau \). The matrix \( A_r \) is a constant 6 x 6 block diagonal matrix with identical 2 x 2 blocks given by

\[
\begin{bmatrix}
0 & 1 \\
-\frac{\mathbf{n}_r^2}{\mathbf{n}_r} & 0
\end{bmatrix}
\]

where \( \mathbf{n}_r \) is the mean daily motion of the target in the assumed circular orbit.

Similarly, the state vector for the interceptor vehicle will be denoted by

\[ s(t) = [x(t), \dot{x}(t), y(t), \dot{y}(t), z(t), \dot{z}(t)]^T \]

The components of both \( s_r(t) \) and \( s(t) \) are referred to an earth-centered, non-rotating coordinate system with the x and y axes in the
equatorial plane and the z axis along the earth's polar axis and positive toward the north pole. The x axis is assumed to lie along the line of nodes of the earth's equatorial plane and the ecliptic.

The equations of motion for the interceptor in vector-matrix form are then given by

\[ \frac{d\mathbf{s}}{dt} = A(r)s + B\mathbf{u}(t) , \quad s(t_0) = \mathbf{c} \]

where matrix \( A(r) \) is a 6 x 6 block diagonal matrix with identical 2 x 2 blocks given by

\[
\begin{bmatrix}
0 & 1 \\
-\frac{\mu}{r^3} & 0
\end{bmatrix}
\]

with

\[ r^2 = x^2 + y^2 + z^2 \]

Matrix \( B \) is a constant 6 x 3 matrix given by

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The 3 x 1 vector \( \mathbf{u}(t) \) is the control vector. Its components are the components of perturbing force per unit mass applied to the point mass to effect the orbit transfer. The perturbing force per unit mass is
the acceleration the point mass must experience, apart from that due to the inverse square gravitational field, to accomplish the orbit transfer.

Since the orbit transfer process is to be treated as a feedback control problem involving relative measurements of the target with respect to the interceptor, it is convenient to formulate the equations of motion for a relative system state vector defined by

\[ \Delta s(t) = s_r(t) - s(t) \]

By appropriate differentiation and substitution from the equations of motion, a differential equation for the relative system state vector is obtained

\[ \frac{d\Delta s}{dt} = A(r) \Delta s - \left[ A(r) - A_r \right] s_r - Bu(t) \]

\[ \Delta s(t_0) = b - c = \Delta c \]

2.3 THE SYSTEM PERFORMANCE INDEX

The primary objective of the orbit transfer process depends upon whether an intercept or a rendezvous case is being considered. For the rendezvous case, it is desired to make all of the components of \( s(t_0 + \tau) \) coincide as closely as possible with those of \( s_r(t_0 + \tau) \). For the intercept case, it is desired to make only the position components of the two vectors coincide. Since a quadratic performance index is to be assumed, the terminal performance criterion actually
used requires the minimization of the sum of the squares of all the components of $\Delta s(t_o + \tau)$ for the rendezvous case and the sum of the squares of the terminal position errors, i.e., the square of the miss distance, for the intercept case.

It will be further assumed that the orbit transfer is to take place over a specified time interval $t_o \leq t \leq t_o + \tau$, and that a constraint will be placed on the amount of rocket fuel consumed.

Such a performance index is defined by the expression

$$J(\tau) = \Delta s^T(t_o + \tau) Q_N \Delta s(t_o + \tau) + \lambda \int_{t_o}^{t_o + \tau} u^T(t) u(t)dt$$

The matrix $Q_N$ is a $6 \times 6$ block diagonal matrix with identical $2 \times 2$ blocks given by

$$
\begin{bmatrix}
1 & 0 \\
0 & w
\end{bmatrix}
$$

where parameter $w$ is set equal to zero for the intercept case, and equal to unity or some other constant for the rendezvous case. The parameter $\lambda$ is a Lagrange multiplier which weights the mass of the fuel consumed against the sum of the squares of the terminal errors. A lower limit on the value of $\lambda$ is set by computational considerations.

The continuous decision process referred to above consists of choosing vector $u(t)$ over the time interval $t_o \leq t \leq t_o + \tau$ such that $J(\tau)$ is minimized.
2.4 THE APPLICATION OF DYNAMIC PROGRAMMING

The application of dynamic programming to essentially this same problem is given by N. N. Krasovskii.\(^{(12)}\) Krasovskii assumes that matrices \(A(x), A_r,\) and \(B\) have time-varying elements. An alternate approach is given by the author\(^{(13)}\) and compared with Krasovskii's approach. The alternate approach is that used by Kalman and Koepcke\(^{(14)}\) for discrete systems and the author for continuous systems.\(^{(15)}\) It consists of defining a combined system state vector

\[
z(t) = \begin{bmatrix} s_r(t) \\ s(t) \end{bmatrix}^T
\]

where \(s_r(t)\) is considered as an input state vector and \(s(t)\) is considered as an output state vector. The combined system state vector then satisfies the vector-matrix differential equation

\[
\frac{dz}{dt} = F(x)z + G u(t), \quad z(t_0) = \begin{bmatrix} b \\ c \end{bmatrix}^T
\]

where

\[
F(x) = \begin{bmatrix} A_r & 0 \\ 0 & A(x) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ B \end{bmatrix}
\]

The performance index becomes

\[
J(\tau) = z^T(t_0 + \tau) Q z(t_0 + \tau) + \lambda \int_{t_0}^{t_0 + \tau} u^T(t) u(t) dt
\]
The matrix $Q_N$ is now given for the rendezvous case by

$$Q_N = \begin{bmatrix} I & -I \\
-I & I \end{bmatrix}, \quad I = 6 \times 6 \text{ unit matrix}$$

and with obvious modifications for the intercept problem. Details of this approach are covered in Refs. 13-15 and will not be considered further here.

The details of the derivation of Krasovskii's approach given below differ somewhat from Ref. 12. At appropriate points in the derivation, linearizing assumptions will be made in order to obtain an analytical solution for the optimal control vector and to simplify the computational process.

The derivation is begun by defining the minimum value of the performance index to be

$$f(b, \Delta c; \tau) = \text{cost of a control process of duration } \tau, \; \text{the initial states being } b \text{ and } \Delta c, \text{ and an optimal policy being used}$$

where

$$\Delta c = b - c$$

Consider first, the infinitesimal initial time interval $t_0 \leq t \leq t_0 + h$. Assume $u(t)$ is a constant $u$ over this interval. The cost of the process during this initial interval is then given by $lu^T u h$. At time $t = t_0 + h$, the relative system state vector is given approximately by

$$\Delta s(t_0 + h) = \Delta c + \frac{d\Delta s}{dt} h$$

$$= \Delta c + \left\{ A(r_o) \Delta c - [A(r_o) - A_x] b - Bu \right\} h$$
where

\[ r_o = \sqrt{[x(t_o)]^2 + [y(t_o)]^2 + [z(t_o)]^2} \]

The principle of optimality may be stated in the following form:

An optimal policy has the property that whatever the state and choice of control vector are at the initial instant of time, the remaining continuous choice of control vector must constitute an optimal policy with regard to the instantaneous state resulting from the choice of control vector at the initial instant.

From the principle of optimality we can form the recurrence relation

\[
f(b, \Delta c; \tau) = \min_u \left\{ \lambda u^T h + f(b + A_r b; \Delta c + \{A(r_o)\Delta c \\
- [A(r_o) - A_r] b - Bu\} h; \tau) \right\}
\]

By making the usual Taylor’s series expansion we can derive the partial differential equation

\[
\frac{\partial f}{\partial \tau} = \min_u \left\{ \lambda u^T u + \left[ \frac{\partial f}{\partial b} \right] A_r b + \left[ \frac{\partial f}{\partial \Delta c} \right]^T \{A(r_o) \Delta c - (A(r_o) - A_r) b - Bu\} \right\}
\]

Minimizing the terms inside the braces with respect to \( u \) gives

\[
u^* = \frac{1}{2\lambda} B^T \left[ \frac{\partial f}{\partial \Delta c} \right]
\]

for the optimal value of \( u \). Substituting for \( u^* \) in the partial differential equation yields

\[
\frac{\partial f}{\partial \tau} = \left[ \frac{\partial f}{\partial b} \right] A_r b + \left[ \frac{\partial f}{\partial \Delta c} \right]^T \{A(r_o) \Delta c - [A(r_o) - A_r] b\} - \frac{1}{4\lambda} B^T \left[ \frac{\partial f}{\partial \Delta c} \right] B
\]
Linearizing assumptions to be discussed in Section 4.1, will permit matrix $A(r_o)$ to be replaced by $A(t_o)$. This substitution is made at this point in order to permit variables to be separated, below. It is clear from the definition of $J(\tau)$ that $f(b, \Delta c; \tau)$ will be a quadratic form. Following Krasovskii, we assume

$$f(b, \Delta c; \tau) = \Delta c^T \alpha(\tau) \Delta c + 2 \Delta c^T \beta(\tau) b + b^T \gamma(\tau) b$$

where $\alpha(\tau)$, $\beta(\tau)$, and $\gamma(\tau)$ are symmetric matrices. Since the performance index does not depend on $s_\tau(t_o + \tau)$ explicitly the boundary conditions must be

$$\alpha(0) = I, \quad \beta(0) = 0, \quad \gamma(0) = 0$$

Forming the required partial derivatives of $f(b, \Delta c; \tau)$ we obtain

$$\frac{\partial f}{\partial \tau} = \Delta c^T \frac{d \alpha}{d \tau} \Delta c + 2 \Delta c^T \frac{d \beta}{d \tau} b + b^T \frac{d \gamma}{d \tau} b$$

$$\begin{bmatrix} \frac{\partial f}{\partial \beta} \\ \frac{\partial f}{\partial \gamma} \end{bmatrix}^T = 2 \Delta c^T \beta(\tau) + 2b^T \gamma(\tau)$$

$$\begin{bmatrix} \frac{\partial f}{\partial \Delta c} \\ \frac{\partial f}{\partial \gamma} \end{bmatrix}^T = 2 \Delta c^T \alpha(\tau) + 2b^T \beta(\tau)$$

Substituting for these derivatives in the partial differential equation for $f(b, \Delta c; \tau)$, collecting terms, and symmetrizing yields

$$\Delta c^T \frac{d \alpha}{d \tau} - \alpha A(t_o) - A^T(t_o) \alpha + \frac{1}{\lambda} \alpha B B^T \Delta c + 2 \Delta c^T \frac{d \beta}{d \tau} - \beta A_t$$

$$- [A^T(t_o) - \frac{1}{\lambda} \alpha(\tau) B B^T] \beta + \alpha(\tau) [A(t_o) - A_t] b$$

$$+ b^T \frac{d \gamma}{d \tau} - \gamma A_t - A_t^T \gamma + 2 \beta [A(t_o) - A_t] + \frac{1}{\lambda} \beta B B^T B] b = 0$$
In order that this relation vanish independently of vectors $\Delta c$ and $b$, the matrix expressions inside the braces must vanish identically, yielding the three matrix differential equations

\[
\frac{d\alpha}{dt} - \alpha A(t_o) - A^T(t_o) \alpha + \frac{1}{\lambda} \alpha B B^T \alpha = 0, \quad \alpha(0) = I
\]

\[
\frac{db}{dt} - \beta A_r - [A^T(t_o) - \frac{1}{\lambda} \alpha(\tau) B B^T] \beta = - \alpha(\tau) [A(t_o) - A_r]
\]

\[\beta(0) = 0\]

\[
\frac{dv}{dt} - \gamma A_r - A^T_r \gamma + 2\beta [A(t_o) - A_r] + \frac{1}{\lambda} \beta B B^T \beta = 0
\]

\[\gamma(0) = 0\]

The expression for the optimal control vector is given by

\[
u^* = \frac{1}{2\lambda} B^T \left[ \frac{\partial f}{\partial \Delta c} \right]
\]

\[
= \frac{1}{\lambda} B^T [\alpha(\tau) \Delta c + \beta(\tau) b]
\]

Time $\tau$ is the remaining time to go in the process in the above equation. For an arbitrary process of total duration $\tau$, starting at time $t_o$, we have

\[
u(t) = \frac{1}{\lambda} B^T [\alpha(\tau + t_o - t) \Delta s(t) + \beta(\tau + t_o - t) s_r(t)]
\]

This equation indicates that only the solutions for the first two matrix differential equations are required.
3.0 THE OPTIMAL ESTIMATION PROBLEM

3.1 INTRODUCTORY REMARKS

The components of the optimal control vector are linear combinations of the relative system state variables. Since the relative system state variables are not directly observable, they must be computed from physical quantities that can be directly observed. The observations of these physical quantities are made at discrete instants of time. These observations will involve measurement uncertainties masking the true values of the observed quantities.

Ideally, the optimal estimation process should generate the best possible estimates of the relative state variables as continuous functions of time from the discrete set of observations, improving these estimates each time a new set of observations is obtained.

There are a number of possible ways of constructing the optimal estimation process. The method to be discussed consists of three subprocesses:

- The transformation from observed quantities to relative state variables.
- A smoothing and prediction process.
- An extrapolation process.

The transformation from observed quantities to relative system state variables is a straightforward process and needs no further discussion at this moment.

The smoothing and prediction process will make use of a Kalman filter. This filter will smooth the available computations of the relative system state variables and estimate these variables at discrete instants of time equally spaced throughout the terminal guidance
process.

The extrapolation process uses a linearized solution to the differential equation for the optimal trajectory to compute the components of the relative system state vector in between the estimates from the Kalman filter. This furnishes the input to the optimal control process.

3.2 FORMULATION AS A MULTI-STAGE DECISION PROCESS

The Kalman filter will be formulated as a multi-stage decision process and the techniques of dynamic programming applied to determine the recurrence relations used for the computations. Other derivations of the Kalman filter equations have been published.\(7,8,16\) The alternate approach below is used because it seems both interesting and natural for this problem.\(9\)

A set of observations obtained at consecutive instants of time are to be combined sequentially to obtain optimal estimates of a finite set of parameters. It is assumed that each observation \(z_k, k = 1, 2, \ldots\) is corrupted with white Gaussian noise of zero mean. The noise associated with \(k\)-th observation is denoted by \(v_k, k = 1, 2, \ldots\).

With the first \(m\) observations we form the \(m\)-dimensional observation vector\(*\)

\[ z(m) = \begin{bmatrix} z_1, \ldots, z_m \end{bmatrix}^T \]

and the vector \(v(m)\) given by

\[ v(m) = \begin{bmatrix} v_1, \ldots, v_m \end{bmatrix}^T \]

\(*\)This particular notation \(z(m)\) is used because the dimension of vector \(z\) changes from stage to stage of the process.
Let the $n$ parameters, whose values are to be estimated, form the vector

$$x = \begin{bmatrix} x_1, \cdots, x_n \end{bmatrix}^T$$

and denote the $k$-th estimate of this vector by $(\hat{x})_k$.

Assume that the first $m$ observations are linearly related to the $n$ parameters by the vector matrix equation

$$v = z - Hx$$

where $H$ is an $m \times n$ matrix whose gram determinant $|H^T H|$ is non-zero.

The estimation problem consists of choosing an estimate of vector $x$ that minimizes the quadratic performance index

$$J_m = v^T(m) R^{-1}(m) v(m)$$

where $R(m)$ is the $m \times m$ covariance matrix of the Gaussian random processes represented by the vector $v(m)$. It is clear that the estimate of $x$ depends on the number and values of the components of $z(m)$, as will the value of $J_m$.

Instead of referring to the estimation process as a multi-stage decision process, we shall call it an $m$-observation estimation process and define the minimum value of $J_m$ as

$$f_m^*(z(m)) = \text{The cost of an } m \text{-observation estimation process based on an } m \text{-dimensional observation vector } z(m) \text{ and an optimal estimation policy.}$$

By an optimal estimation policy is meant the choice of successive estimates of vector $x$ such that the performance index

$$J_m = v^T(m) R^{-1}(m) v(m), \quad m = n, n+1, \cdots$$
is minimized. The reason for index \( m \) starting at \( n \) is that a minimum of \( n \) observations is required to obtain a useful estimate of the \( n \) components of vector \( x \).

It is well known from the theory of least squares that the best estimate of \( x \) in the least squares sense is given by

\[
\hat{x} = \left[ H^T(m) R^{-1}(m) H(m) \right]^{-1} H^T(m) R^{-1}(m) z(m)
\]

Substitution of this expression into \( J_m \) yields

\[
f_m[z(m)] = z^T(m) \left[ R^{-1}(m) - R^{-1}(m) H(m) \left[ H^T(m) R^{-1}(m) H(m) \right]^{-1} H^T(m) R^{-1}(m) \right] z(m)
\]

\[
= z^T(m) Q(m) z(m)
\]

When an additional observation \( z_{m+1} \) is utilized, the estimate of \( x \) will, in general, change. Consider now the effect on the value of \( J_m \) of a deviation from the optimal estimate of vector \( x \). We shall have for the value of \( J_m \)

\[
J_m = \{ z(m) - H(m) [\hat{x} + \Delta x] \}^T R^{-1}(m) \{ z(m) - H(m) [\hat{x} + \Delta x] \}
\]

If this quadratic form is expanded and the expression for \( \hat{x} \) substituted, it can be shown by straightforward algebraic manipulations...
that $J_m$ reduces to (9)

$$J_m = \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m[z(m)]$$

If we now consider the effect of observation $z_{m+1}$ with the associated uncertainty $v_{m+1}$ we must minimize the quadratic form

$$J_{m+1} = [v(m) \mid v_{m+1}]^T \begin{bmatrix} R^{-1}(m) & 0 \\ 0 & R_{m+1}^{-1} \end{bmatrix} \begin{bmatrix} v(m) \\ v_{m+1} \end{bmatrix}$$

$$= v_{m+1}^T R_{m+1}^{-1} v_{m+1} + J_m$$

where $R_{m+1}$ is the variance of the Gaussian process generating $v_{m+1}$.

Substituting for $v_{m+1}$ and $J_m$ and applying the principle of optimality we obtain the recurrence relation*

$$f_{m+1}[z(m+1)] = \min_{\Delta x} \left[ \left\{ z_{m+1} - H_{m+1} [\hat{x}]_{m-n} + \Delta x \right\}^T R_{m+1}^{-1} \left\{ z_{m+1} - H_{m+1} [\hat{x}]_{m-n} + \Delta x \right\} 
- H_{m+1} [\hat{x}]_{m-n} + \Delta x \right] + \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m[z(m)]$$

*$(\hat{x})_{m-n}$ is the $(m-n)$-th estimate of $x$.  

where

$$i_m^T[z(m)] = z^T(m) Q(m) z(m)$$

$$m = n, n+1, \ldots$$

The optimal policy is determined by successive choices of $\Delta x$ which minimize the right-hand side of the recurrence relation.

Differentiating with respect to the components of vector $\Delta x$, setting the result equal to zero, and solving for $\Delta x$ yields

$$(\Delta \hat{x})_{m-n} = \left[ H^T(m) R^{-1}(m) H(m) + H^T_{m+1} R^{-1}_{m+1} H_{m+1} \right]^{-1} H^T_{m+1} R^{-1}_{m+1} \left[ z_{m+1} - H_{m+1} (\hat{x})_{m-n} \right]$$

and the new estimate of $x$ is

$$(\hat{x})_{m-n+1} = (\hat{x})_{m-n} + (\Delta \hat{x})_{m-n}$$

To avoid the matrix inversion problem, we follow Ref. 8 and define

$$P^{-1}_m = H^T(m) R^{-1}(m) H(m)$$

$$P_{m+1}^{-1} = P^{-1}_m + H^T_{m+1} R^{-1}_{m+1} H_{m+1}$$

Then by the lemma in Ref. 8 we have

$$P_{m+1} = P_m - P_m H^T_{m+1} (H_{m+1} P_m H^T_{m+1} + R_{m+1})^{-1} H_{m+1} P_m$$
Letting \( k = m-n \) we can write

\[
(\hat{x})_{k+1} = (\hat{x})_k + (\Delta \hat{x})_k
\]

\[
(\Delta \hat{x})_k = p_{k+1} H_{k+1}^T \left[ z_{k+1} - H_{k+1} (\hat{x})_k \right]
\]

\[
p_{k+1} = p_k - p_k H_{k+1}^T (H_{k+1} p_k H_{k+1}^T + R_{k+1})^{-1} H_{k+1} p_k
\]

\[ k = 0, 1, 2, \ldots \]

These recurrence relations are the computational equations for the Kalman filter.

In order to use these recurrence relations we need an initial estimate of \( x \), the matrix \( P_0 \) and the set of variances associated with the Gaussian random processes generating \( \nu_k \), \( k = 1, 2, \ldots \).

The above derivation assumes that observations are processed one at a time. The advantage of doing this is that \( H_{k+1} p_k H_{k+1}^T + R_{k+1} \) and \( R_{k+1} \) are scalars, and matrix inversions are completely eliminated. However, the above recurrence relations are sufficiently general so that observations may be processed \( r \) at a time. Then, the quantity \( z_{k+1} \) becomes an \( r \times 1 \) vector, the matrix \( H_{k+1} \) becomes an \( r \times n \) matrix, \( R_{k+1} \) becomes an \( r \times r \) covariance matrix, and the inversion of two \( r \times r \) matrices is required.

### 3.3 The Estimation of the Relative System State Vector

The computation of the optimal control vector depends on a continuous knowledge of the relative system state vector. As pointed out above the components of the relative system state vector are not
directly observable and must be computed from quantities that can be observed.

Rather than a continuous estimation of the relative system state vector, it will be estimated at \( N \) distinct instants of time equally spaced throughout the duration of the guidance process. This is based on the requirement that the computations must be carried out by a digital computer. Thus, the relative system state variables will be estimated a finite number of times during the guidance process, and these estimates will be based on a finite set of observations which are accumulated as the guidance process unfolds.

Each estimation of a set of the relative system state variables will be preceded by an interval of time referred to as an estimation interval. During each estimation interval, \( n-1 \) sets of six observations will be made at equally spaced intervals of time throughout the estimation interval. It is assumed that these observations are made by a sensor system capable of measuring range, range rate, two angles, and two angular rates. Each set of observations of the target then consists of six independent measurements. By using suitable transformation equations a set of relative system state variables can be computed for the time of the observation. These computed relative system state variables make up the components of the computed relative system state vector

\[
\Delta \mathbf{s}(t_k) = \left[ \Delta \mathbf{x}(t_k), \Delta \mathbf{x}(t_k), \cdots, \Delta \mathbf{x}(t_k) \right]^T
\]

where \( t_k \) is the time of the observation. Because of the finite computation time needed to compute the values of the components of the optimal control vector, \( \Delta \mathbf{s}(t_k) \) must be updated in time. Further, it
is desired to combine all previous observations in order to smooth out uncertainties. This smoothing and updating process will be carried out using a Kalman filter.

The updating process implies the necessity for solving the differential equation defining the time behavior of the relative system state vector along the optimal trajectory. The linearized approximation to this differential equation is (Section 4.0)

\[
\frac{d\Delta s}{dt} = A(t)\Delta s - [A(t) - A_R] s_R(t) - B u(t), \quad \Delta s(t_0) = \Delta c
\]

The optimal control vector is given by

\[
u(t) = \frac{1}{\lambda} B^T \left[ \alpha(\tau + t_0 - t) \Delta s(t) + \beta(\tau + t_0 - t) s_R(t) \right]
\]

Substituting in the differential equation reduces it to

\[
\frac{d\Delta s}{dt} = F(t)\Delta s + G(t) s_R(t)
\]

where

\[
F(t) = A(t) - \frac{1}{\lambda} B B^T \alpha(\tau + t_0 - t)
\]

\[
G(t) = A_R - A(t) - \frac{1}{\lambda} B B^T \beta(\tau + t_0 - t)
\]

The solution can be expressed as

\[
\Delta s(t) = \delta(t, t_0) \Delta s(t_0) + \int_{t_0}^{t} \delta(t, \xi) G(\xi) s_R(\xi) d\xi
\]
where the matrix $\dot{\Phi}(t, t_0)$ satisfies the matrix differential equation

$$\frac{d\Phi}{dt} = P(t)\Phi, \quad \Phi(t_0, t_0) = I$$

We are interested in applying this equation to relate the computed relative system state vector based on observations to the estimated relative system state vector at the end of an estimation interval. If time $T$ is the end of the first estimation interval then

$$\Delta s(T) = \Phi(T, t_k) \Delta s(t_k) + g(T, t_k)$$

where vector $g(T, t_k)$, the explicit change in the relative system state vector over $t_k \leq t \leq T$ due to motion of the target, represents the integral term in the expression for $\Delta s(t)$. We want to refer the estimate of $\Delta s(T)$ back to the observation time in order to compare with $\Delta \hat{s}(t_k)$.

From the above equation we obtain

$$\Delta s(t_k) = \Phi(t_k, T) \left[ \Delta s(T) - g(T, t_k) \right]$$

where

$$\Phi(t_k, T) = \Phi^{-1}(T, t_k)$$

If we let $(\hat{\Delta})_k$ be $[\Delta \hat{s}(T)]_k$, $(\Delta \hat{s})_k$ be $[\delta \hat{s}(T)]_k$, and $H_{k+1}$ be $\Phi(t_{k+1}, T)$ we can write the Kalman filter equations as

$$[\Delta \hat{s}(T)]_{k+1} = [\Delta \hat{s}(T)]_k + [\delta \hat{s}(T)]_k$$
\[
[\delta \hat{z}(T)]_k = P_{k+1} \Phi^T(t_{k+1}, T) R_s^{-1}(t_{k+1}) \left[ \Delta \hat{z}(t_{k+1}) - \hat{z}(t_{k+1}, T) \right] \left[ \Delta \hat{z}(T) \right]_k \\
- \delta(T, t_{k+1}) \right]
\]

\[
P_{k+1} = P_k - P_k \Phi^T(t_{k+1}, T) \left[ \Phi(t_{k+1}, T) P_k \Phi^T(t_{k+1}, T) + R_s(t_{k+1}) \right]^{-1} \Phi(t_{k+1}, T) P_k
\]

\[
\delta(T, t_{k+1}) = \int_{t_{k+1}}^{T} \delta(T, \xi) G(\xi) s_r(\xi) d\xi
\]

Uncertainties associated with \( s_r(t) \) are neglected. In this case we are combining observations six at a time. This means there is a 6 x 6 matrix inversion process involved. It is certainly possible to process the computed state variables one at a time for each set of observations and to eliminate the matrix inversion process.

Matrix \( R_s(t_{k+1}) \) is the covariance matrix for the random processes generating the uncertainties associated with the computed relative system state variables. We shall assume that the levels of the uncertainties associated with the actual observations are small relative to the magnitudes of the observed quantities. Matrix \( R_s(t_k) \) can be approximately computed from

\[
R_s(t_k) = M(t_k) R_q M^T(t_k)
\]

where \( M(t_k) \) is the Jacobian matrix of the transformation equations
relating the state variables and the observed quantities at time $t_k$. Matrix $R_q$ is assumed to be a diagonal matrix whose diagonal elements are the variances associated with the observed quantities.

3.4 THE EXTRAPOLATION PROCESS

The optimal control process equations have been derived for a continuous process. Although the optimal control vector components are computed digitally, the values of the relative state variables are required much more frequently than furnished by the Kalman filter. The extrapolation process consists of the computational process necessary to extrapolate from the estimates furnished by the Kalman filter in order to provide values of the relative state variables when needed by the optimal control computation.

As indicated above, the relative system state vector can be expressed as

$$
\Delta \hat{s}(t) = \hat{s}(t, T) \Delta \hat{s}(T) + \int_{T}^{t} \hat{s}(t, \xi) \, \gamma(\xi) \, s_{x}(\xi) \, d\xi
$$

$$
= \hat{s}(t, T) \Delta \hat{s}(T) + g(t, T)
$$

where

$$
T \leq t \leq 2T
$$

where $\Delta \hat{s}(T)$ is the estimated relative system state vector determined by the Kalman filter. The values of the relative system state variables computed from this equation form the input to the optimal control computation.
3.5 SUCCESSIVE ESTIMATION INTERVALS

The complete terminal guidance process consists of \( N \) estimation intervals. At the end of each estimation interval an estimate is made by the Kalman filter of the relative system state vector. Each estimate is based on all preceding observations. Some provisions are necessary to transfer data between estimation intervals.

The matrices \( R_s(t_k) \) and \( P_k \) are dependent only on the time when observations are made. They are computed sequentially throughout the entire process without regard to estimation interval changes.

The estimate of the relative system state vector, however, must be transferred from estimation interval to estimation interval as the terminal guidance process unfolds. This is necessary if all previous observations are to contribute to the current estimate. This transfer is accomplished with the same equation utilized by the extrapolation process. We have

\[
\left\{ \Delta \hat{x}[(n+1)T] \right\}_l = \Phi[(n+1)T, nT] \left\{ \Delta \hat{x}[nT] \right\}_{m-1} + g[(n+1)T, nT]
\]

where

\[
\left\{ \Delta \hat{x}[(n+1)T] \right\}_l = \text{The first estimate of the } (n+1)\text{-th estimation interval}
\]
\[ \Delta^m[nT] = \text{The last estimate of the } n\text{-th estimation interval based on } n(m-1) \text{ observations} \]

\[ g[(n+1)T, nT] = \int_{nT}^{(n+1)T} \phi[(n+1)T, \xi] G(\xi) e(\xi) d\xi \]

\[ n = 0, 1, \ldots, N-1 \]
4.0 LINEARIZATION CONSIDERATIONS

4.1 APPROXIMATION BY A TIME-VARYING SYSTEM

The source of the nonlinearity is the element $-\mu/r^3$ which appears in three places in the matrix $A(r)$ in the differential equation for the interceptor state vector $s(t)$. The quantity $r^3$ is the cube of the distance of the interceptor from the center of the earth.

A terminal guidance process is being considered in which the interceptor passes through a point in space, on or close to the target orbit. For the orbits considered, the difference between the target's distance from the center of the earth and the interceptor's distance from the center of the earth will always be small relative to the interceptor's distance from the center of the earth. Further, for a specific set of initial conditions, i.e., set of values for the vectors $b$ and $\Delta c$, a family of two-body orbits can be chosen that passes through $s(t_0)$ and $s_r(t_0 + \tau)$. At least one of these two-body orbits can be selected such that the difference between it and the actual optimal trajectory followed by the interceptor is relatively small.* As a linearizing approximation, we shall accordingly consider the behavior of $r$ along the two-body orbit rather than along the actual optimal trajectory.

From the expansion techniques commonly used in celestial mechanics it is a straightforward procedure to expand $-\mu/r^3$ to the fourth power of $e$ as

$$-\frac{\mu}{r^3_k} = -\frac{\mu}{a^3} \left[ (1 + \frac{3}{2} e^2 + \frac{15}{8} e^4) + (3e + \frac{27}{8} e^3) \cos M_k + \cdots \right]$$

*The orbit actually chosen is the one whose apogee coincides with the desired rendezvous point.
where

\[ a = \text{semi-major axis of approximating two-body orbit} \]

\[ e = \text{eccentricity of approximating two-body orbit} \]

\[ M_k = \text{mean anomaly} \]

The mean anomaly is given by

\[ M_k = n(t_k - T) = n(t + t_o - t_k) + \pi \]

where

\[ n = \text{mean daily motion in the approximating two-body orbit} \]

\[ T = \text{time of perigee passage for the approximating two-body orbit} \]

By use of this series expansion it is possible to replace matrix \( A(\tau) \) by a time-varying matrix \( A(t) \), as was done in Section 2.4.

4.2 SOLUTION OF THE TIME-VARYING SYSTEM

The matrix differential equations for \( \alpha(\tau) \) and \( \beta(\tau) \) are made linear by replacing matrix \( A(\tau) \) by \( A(t) \) and become

\[
\frac{d\alpha}{dt} - \alpha A(t - \tau) - A'(t - \tau)\alpha + \frac{1}{\lambda} \alpha B B^T \alpha = 0, \quad \alpha(0) = I
\]

\[
\frac{d\beta}{dt} - \beta A - \left[ A'(t - \tau) + \frac{1}{\lambda} \alpha(t) B B^T \right] \beta = - \alpha(\tau) \left[ A(t - \tau) - A_r \right] \]

\[
\beta(0) = 0
\]

where \( t_r \) is the time of intercept \( t_o + \tau \) when \( \tau \) becomes fixed.

The differential equation for \( \alpha(\tau) \) is solved first to obtain an explicit representation of \( \alpha(\tau) \). This representation is then substituted
into the second differential equation, which is then solved to obtain an explicit representation of matrix $\beta(\tau)$.

The differential equation for $\alpha(\tau)$ is a matrix Riccati equation. W. T. Reid shows (17) that this equation may be solved by solving the associated system of linear, matrix differential equations

$$\frac{dY}{d\tau} = -A(t_\tau - \tau)Y + \frac{1}{\lambda} B B^T Z, \quad Y(0) = I$$

$$\frac{dZ}{d\tau} = A^T(t_\tau - \tau) Z, \quad Z(0) = I$$

where $\alpha(\tau) = Z(\tau) Y^{-1}(\tau)$

The matrix differential equation for $\beta(\tau)$ can also be solved using a similar associated system of linear, matrix differential equations:

$$\frac{dU}{d\tau} = A_x U, \quad U(0) = I$$

$$\frac{dV}{d\tau} = -\alpha(\tau) \left[ A(t_\tau - \tau) - A_x \right] U + \left[ A^T(t_\tau - \tau) - \frac{1}{\lambda} \alpha(\tau) B B^T \right] V, \quad V(0) = 0$$

where $\beta(\tau) = V(\tau) U^{-1}(\tau)$

In terms of these solutions, the optimal control vector is given by

$$u(t) = \frac{1}{\lambda} \beta^T \left[ Z(\tau + t_o - t) Y^{-1}(\tau + t_o - t) \Delta s(t) \right.$$

$$\left. + V(\tau + t_o - t) U^{-1}(\tau + t_o - t) s_2(t) \right]$$
where \( \tau \) is the total process duration and \( t_0 \) is the initial time.

The use of the method of mean coefficients to integrate the associated system of linear matrix differential equations is discussed in detail in Ref. 13.

4.3 APPROXIMATION BY A TIME-INVARIENT SYSTEM

It is possible to simplify the problem even further by taking only the constant term in the series for \(- \frac{\mu}{r^3}\), i.e.:

\[
- \frac{\mu}{r^3} \approx - \frac{\mu}{a^3} (1 + 3 \frac{r}{a} \epsilon^2)
\]

The matrix \( A(r) \) is then replaced by a constant matrix \( A \). Matrix \( A \) is a block diagonal matrix whose 2 x 2 identical blocks are given by

\[
\begin{bmatrix}
0 & 1 \\
-\bar{n}^2 & 0
\end{bmatrix}
\]

where \( \bar{n}^2 = \frac{\mu}{a^3} (1 + 3 \frac{r}{a} \epsilon^2) \)

The differential equation for the relative system state vector becomes

\[
\frac{d\Delta s}{dt} = A \Delta s - [A - A_r] s_r(t) - B u(t)
\]

\( \Delta s(0) = \Delta c, \quad s_r(0) = b \)

The associated system of linear equations can then be solved explicitly,
obtaining closed form expressions for the elements of matrices $U$, $V$, $Y$, and $Z$.

For the case where the transfer is from a nearly circular orbit to a circular one, matrix $A$ can be replaced by $A_r$. Then the correction term depending on $s_r(t)$ is omitted and the differential equation becomes

$$\frac{d\Delta s}{dt} = A_r \Delta s - B u(t)$$

$$\Delta s(t_0) = \Delta c$$

The expression for the optimal control vector is now given by

$$u(t) = \frac{1}{\lambda} B^T Z(\tau-t) Y^{-1}(\tau-t) \Delta s(t)$$

and the equation for $\beta(\tau)$ need not be solved at all.

In this last assumption the target and interceptor obviously have the same equations of motion except for the effect of the optimal control vector. It can be shown that the nonzero components of the vector $[A(r) - A_r] s_r(t)$, which is omitted, can have absolute values as much as three times those of the nonzero components of the vector $A_r \Delta s$, which are retained. However, its omission greatly simplifies the computational procedures and appears to have negligible practical effect on the optimal control vector computation, as indicated by the numerical data in Section 6.0.
5.0 COMPUTATIONAL DETAILS

5.1 DEFINITION OF TERMS AND SUBPROCESSES

The terminal guidance computational process is made up of four subprocesses

- The Optimal Control Process
- The Transformation of Observations
- The Estimation Process
- The Extrapolation Process

The interconnections of these processes are indicated in Fig. 1. Before discussing the computational details of these processes, certain terms associated with them will be defined. In addition, the time sequences involved in processing observational data and estimating state variables will be described.

The optimal control process operates on the components of the estimated relative state vector $\hat{\Delta s}(t)$ to obtain the components of the optimal control vector $u(t)$. The components of the optimal control vector are the components of force per unit mass due to rocket thrust projected along a rectangular, non-rotating coordinate system. The estimated relative state vector $\hat{\Delta s}(t)$ is the output of the extrapolation process.

In order to investigate system behavior it is necessary to integrate the actual nonlinear equations of motion of the target and interceptor. The difference between the vector solutions of these sets of differential equations is called the true relative state vector $\Delta s(t)$ (Section 2.2).
The process of estimating relative state variables begins with observations of the target vehicle by sensor equipment on board the interceptor. The set of observations made at a particular instant of time will be considered as components of an observation vector \( q(t) \). These components are called observation variables, and for the situation under consideration they will be six in number. By the use of suitable point transformations the relative state variables can be computed for the particular instants of time at which the observations were made. These values of the relative state variables form the components of a computed relative state vector \( \Delta \tilde{\mathbf{s}}(t) \).

The estimation process operates on the computed relative state variables in order to smooth and update their values. The updated sets of values are estimated at \( N \) instants of time equally distributed throughout the terminal guidance process. These sets of values form the components of estimated relative state vectors, \( \Delta \tilde{\mathbf{s}}(nT) \), \( n=0, 1, \ldots, N-1 \).

The extrapolation process uses the solution of the linearized optimal trajectory equations to extrapolate \( \Delta \tilde{\mathbf{s}}(nT) \) to \( \Delta \tilde{\mathbf{s}}(t) \) where \( nT < t \leq (n+1)T \).

The time duration of the overall terminal guidance process is assumed to be \( T \) minutes. This duration is divided into \( N \) equal intervals of \( T \) minutes (Fig. 2). These intervals are called estimation intervals. The instants of time at the end of each estimation interval \( nT \), \( n=1, \ldots, N \) are called estimation times.

Each estimation interval is subdivided into \( m \) equal subintervals of \( T/m = \Delta t \) minutes called observation intervals. The instants of
Process duration

Estimation interval

Observation interval

The process duration is divided into N estimation intervals.

Each estimation interval is divided into m observation intervals.

Each observation interval is divided into l extrapolation intervals.

Fig. 2—Time intervals
time at the end of each observation interval are called observation times. A set of observations of the target is made at each observation time \( t_1, t_2, \ldots, t_m \).

Each observation interval is subdivided into \( \lambda \) equal subintervals of \( \Delta t/\lambda \) minutes, called extrapolation intervals. The instants of time \( \Delta t/\lambda, 2\Delta t/\lambda, \ldots, \Delta t \), at the ends of the extrapolation intervals are called extrapolation times. These are the times at which the extrapolation process computes the estimated relative system state vector \( \Delta \hat{s}(t) \).

5.2 THE OPTIMAL CONTROL EQUATIONS

The optimal control equations to be used are based on the simplest form of the differential equation for the relative state vector (Section 4.3).

\[
\frac{d\Delta s}{dt} = A_r \Delta s - B u(t) , \quad \Delta s(t_0) = \Delta c
\]

The resulting expression for the optimal control vector is

\[ u(t) = \Gamma(t-t) \Delta \hat{s}(t) \]

where

\[
\Gamma(t-t) = \begin{bmatrix}
\gamma_1(t-t) & \gamma_2(t-t) & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma_1(t-t) & \gamma_2(t-t) & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1(t-t) & \gamma_2(t-t)
\end{bmatrix}
\]
\[ \Delta \hat{s} = \left[ \Delta \hat{x}, \Delta \hat{y}, \Delta \hat{z}, \Delta \hat{\eta}, \Delta \hat{\xi}, \Delta \hat{\zeta} \right]^T \]

\[ \nu_1(\tau-t) = -\frac{C Y n_r^2 \left[ 2n_r \lambda (1-n_r^2 \omega) \sin n_r(\tau-t) \cos n_r(\tau-t) + \omega \sin^2 n_r(\tau-t) \right]}{D} \]

\[ \nu_2(\tau-t) = -\frac{C Y n_r \left[ 2n_r \lambda \left( \sin^2 n_r(\tau-t) + \omega n_r^2 \cos^2 n_r(\tau-t) \right) \right]}{D} - \frac{C Y \left[ \omega n_r(\tau-t) - \sin n_r(\tau-t) \cos n_r(\tau-t) \right]}{D} \]

\[ D = \left[ 2n_r^3 + n_r(\tau-t) \right] \left[ 2n_r \lambda + \omega n_r(\tau-t) \right] \]

\[ + \left[ 2n_r \lambda \left( n_r^2 \omega - 1 \right) \sin n_r(\tau-t) \cos n_r(\tau-t) - \omega \sin^2 n_r(\tau-t) \right] \]

\[ u(t) = \left[ a_x(t), a_y(t), a_z(t) \right]^T \]

The matrix \( \Gamma(\tau-t) \) is

\[ \Gamma(\tau-t) = \frac{1}{\lambda} B^\pi Z(\tau-t) Y^{-1}(\tau-t) \]

The explicit solutions for the elements of matrix \( \Gamma(\tau-t) \) involve only the solutions of two linear time-invariant differential equations.

Most of the details are given in Ref. 15. The constant \( C_Y \) is normally set equal to 2. The parameter \( n_r \) is the mean daily motion of the target. The parameter \( \omega \) is set equal to zero for intercept problems and to unity for rendezvous problems.
The elements $\gamma_1(\tau-t)$ and $\gamma_2(\tau-t)$ are evaluated every $\Delta t/\ell$ minutes, i.e., at the end of each extrapolation interval.

5.3 THE TRANSFORMATION OF OBSERVATIONS

The relative system state variables are expressed with respect to a non-rotating coordinate system and are not directly observable. For the process to be considered they must be expressed in terms of quantities that can be observed by the sensor equipment.

It is assumed that the sensor equipment consists of a gimbaled tracking unit and the necessary auxiliary equipment to permit the measurement of range and range rate. It is also assumed to be possible to measure gimbal angles and angular rates with respect to an inertial reference unit. Uncertainties associated with these measurements can be introduced into the computational process.

From Fig. 3 it is possible to derive the following equations:

$$\Delta x = \Delta \rho \cos b \cos \ell$$

$$\Delta y = \Delta \rho \cos b \sin \ell$$

$$\Delta z = \Delta \rho \sin b$$

By differentiating with respect to time we obtain

$$\dot{\Delta x} = \dot{\Delta \rho} \cos b \cos \ell - \Delta \rho \dot{b} \sin b \cos \ell - \Delta \rho \ell \cos b \sin \ell$$

$$\dot{\Delta y} = \dot{\Delta \rho} \cos b \sin \ell - \Delta \rho \dot{b} \sin b \sin \ell + \Delta \rho \ell \cos b \cos \ell$$

$$\dot{\Delta z} = \dot{\Delta \rho} \sin b + \Delta \rho \dot{b} \cos b$$

These two sets of equations relate the components of the relative state vector to the observational variables $\Delta \rho$, $\Delta \rho$, $b$, $\dot{b}$, $\ell$, and $\dot{\ell}$. 
Fig. 3 — Coordinate system
These may be regarded as components of the observational vector

$$q(t) = [\Delta \rho, \Delta \phi, v, \dot{\rho}, \dot{\phi}, \dot{v}]^T$$

The components of the true relative state vector are obtained from integrating the nonlinear equations of motion of the interceptor and target.

Rather than simulating the observational process by computing the observational variables from the true relative state vector components, adding random variables to simulate observational uncertainties, and then recomputing the state variables, a random vector is added directly to the true relative state vector (Fig. 4).

$$\Delta \tilde{s}(t) = \Delta s(t) + v_s(t)$$

The vector $\Delta \tilde{s}(t)$ is then the input to the estimation process. The random vector $v_s(t)$ is computed for each set of observations from

$$v_s(t) = M(t) v_q(t)$$

where $M(t)$ is a $6 \times 6$ matrix and $v_q(t)$ is a random vector whose components are the uncertainties associated with the observational variables. The components of $v_q(t)$ are assumed to be independent, have zero means, and known Gaussian probability distributions.

Since the components of $v_q(t)$ are generally small relative to the values of the observational variables, the matrix $M(t)$ will be taken as the Jacobian matrix of the transformation equations with its elements evaluated from the components of the true relative state vector at the time of each observation.
Fig. 4—Introduction of noise into guidance loop
5.4 COVARIANCE MATRICES

The estimation process requires a knowledge of the covariance matrix for the components of the computed relative system state vector $\Delta \mathbf{x}(t)$. This covariance matrix depends on matrix $M(t)$ and on the covariance matrix for the observational variables. The two covariance matrices are related by the following matrix equation

$$
R_s(t) = M(t)^T R_q M(t)^T
$$

where $R_q$ is the covariance matrix for the observational variables. It is a diagonal matrix whose non-zero elements are the variances associated with the observational variables.

$$
\begin{align*}
\sigma_{\rho}^2 & \quad (1) \\
\sigma_{\rho}^2 & \quad (2) \\
\sigma_{\rho}^2 & \quad (3) \\
\sigma_{\rho}^2 & \quad (4) \\
\sigma_{\rho}^2 & \quad (5) \\
\sigma_{\rho}^2 & \quad (6)
\end{align*}
$$

The matrix $R_q$ is assumed to remain constant, but $R_s$ will change from observation to observation due to changes in the elements of $M(t)$. 
5.5 THE ESTIMATION PROCESS

It was pointed out in Section 3.3 that the estimation process requires the solution of the differential equation defining the time behavior of the relative state vector along the optimal trajectory. Initially, we shall neglect the effect of \( s_r(t) \), and the approximate differential equation for the relative state vector along the optimal trajectory becomes

\[
\frac{d\Delta s}{dt} = F(t)\Delta s, \quad \Delta s(t_0) = \Delta c
\]

where

\[
F(t) = A_r - B\Gamma(\tau-t)
\]

Thus, the same linearizing assumptions are made for the estimation process as for the control process.

This time-varying linear differential equation is solved by the method of mean coefficients. \(^{(18)}\) The observation times divide the duration of the terminal guidance process into \( mN \) equal time intervals

\[
t_0 < t_1 < \cdots < t_{m-1} < t_m = T < \cdots < t_{(m-1)N} t_{mN} = NT + t_0
\]

Over each observation interval matrix \( F(t) \) is replaced by a constant matrix \( F_k \), \( k=0, 1, \cdots, (m-1)N \). This constant matrix is obtained by replacing the time-varying elements in \( F(t) \) by their time averages over the observation interval under consideration, i.e., \( \gamma_r(\tau-t) \) is
replaced by $\gamma_{1k}$ and $\gamma_{2}(\tau-t)$ by $\gamma_{2k}$ where

$$\gamma_{1k} = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \gamma_1(\tau-t)\,dt$$

$$\gamma_{2k} = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \gamma_2(\tau-t)\,dt$$

and $t_{k+1} - t_k$ is the time interval between observations. The quantities $\gamma_{1k}$ and $\gamma_{2k}$ are the time averages over the $(k+1)$-th observation interval. Although explicit expressions are available for $\gamma_1(\tau-t)$ and $\gamma_2(\tau-t)$, it is simpler to evaluate $\gamma_{1k}$ and $\gamma_{2k}$ by numerical integration.

The solution of the approximate differential equation over the $(k+1)$-th observation interval becomes

$$\Delta s(t) = e^{P_k(t-t_k)} \Delta s(t_k)$$

$$= \hat{g}(t,t_k) \Delta s(t_k)$$

The relative state vector at each observation time is obtained from

$$\Delta s(t_{k+1}) = \hat{g}(t_{k+1}, t_k) \Delta s(t_k)$$

From the equations of motion given in Section 2.2 it is clear that
F(t) is a block diagonal matrix with three identical 2 x 2 blocks. It follows that $\psi(t, t_k)$ will also consist of three identical 2 x 2 blocks.

$$
\psi(t, t_k) = \begin{bmatrix}
C(t, k) & 0 & 0 \\
0 & C(t, k) & 0 \\
0 & 0 & C(t, k)
\end{bmatrix}
$$

where we can express $C(t, k)$ as the 2 x 2 matrix

$$
C(t, k) = e^{-\alpha_k (t-t_k)} \left[ \cos \omega_k (t-t_k) \right] I + \frac{1}{\omega_k} \sin \omega_k (t-t_k) \left[ P_k + \alpha_k I \right]
$$

where

$$
\alpha_k = -\frac{1}{2} \gamma_{2k}
$$

$$
\omega_k = \frac{1}{2} \sqrt{4 \gamma_k^2 + 4 \gamma_{1k} - \gamma_{2k}^2}
$$

Since we are interested in evaluating matrix $C(t, k)$ at each extrapolation time for the extrapolation process and at each observation time for the estimation process we shall replace $t-t_k$ by $r \Delta t / \ell$, $r=0, 1, \cdots, \ell$. The four elements for matrix $C(t, k)$ are then given by

$$
c_{ll} (k, r) = e^{-\alpha_k \left( \frac{r \Delta t}{\ell} \right)} \left[ \cos \omega_k \frac{r \Delta t}{\ell} + \frac{\alpha_k}{\omega_k} \sin \omega_k \frac{r \Delta t}{\ell} \right]
$$

*The matrix exponential can be represented explicitly as a square matrix by using Sylvester's interpolation formula (Ref. 15).*
\[ c_{12}(k,r) = \frac{1}{\mu_k} e^{-\alpha_k \frac{r \Delta t}{L}} \sin \mu_k \left( \frac{r \Delta t}{L} \right) \]

\[ c_{21}(k,r) = - (\mu_k^2 + \alpha_k^2) c_{12}(k,r) \]

\[ c_{22}(k,r) = e^{-\alpha_k \frac{r \Delta t}{L}} \left[ \cos \mu_k \left( \frac{r \Delta t}{L} \right) - \frac{\alpha_k}{\mu_k} \sin \mu_k \left( \frac{r \Delta t}{L} \right) \right] \]

For the estimation process the C matrix elements are needed only for \( r = \ell \) i.e., \( t = t_{k+1}, k = 0, 1, \ldots, m-1 \). Further, it is the inverse of the \( \phi \) matrix that is required.

The relative state vector at the end of the first estimation interval is related to the relative state vector at time \( t_k < T \) by

\[ \Delta s(T) = \phi(T,t_k) \Delta s(t_k) \]

\[ = \phi(T,t_{m-1}) \phi(t_{m-1}, t_{m-2}) \cdots \phi(t_{k+1}, t_k) \Delta s(t_k) \]

and

\[ \Delta s(t_k) = \phi^{-1}(T,t_k) \Delta s(T) \]

where

\[ \phi^{-1}(T,t_k) = \left[ \phi(T,t_{m-1}) \cdots \phi(t_{k+1}, t_k) \right]^{-1} \]

\[ = \phi^{-1}(t_{k+1}, t_k) \cdots \phi^{-1}(T, t_{m-1}) \]
\[ \phi(t_k, t_{k+1}) \cdots \phi(t_{m-1}, T) = \phi(t_k, T) \]

Since the \( \phi \) matrices are block diagonal in structure, it is clear that in forming the above matrix product we can restrict our attention to manipulations involving the \( 2 \times 2 \) \( C \) matrix. The matrix \( \phi(t_{m-1}, T) \) involves the matrix \( C^{-1}(T, t_{m-1}) \) which is obtained from \( C(T, t_{m-1}) \) by replacing \( \Delta t \) by \( -\Delta t \) in the expressions for its elements. We shall define the matrix \( \phi(t_k, T) \) by

\[
\phi(t_k, T) = \begin{bmatrix}
D(k, T) & 0 & 0 \\
0 & D(k, T) & 0 \\
0 & 0 & D(k, T)
\end{bmatrix}
\]

where the \( 2 \times 2 \) matrix \( D(k, T) \) is given by the recurrence relation

\[
D(k, T) = C^{-1}(k, \ell) D(k+1, T), \quad D(m, T) = I
\]

and

\[
C(k, \ell) = \begin{bmatrix}
c_{11}(k, \ell) & c_{12}(k, \ell) \\
c_{21}(k, \ell) & c_{22}(k, \ell)
\end{bmatrix}
\]
The sequence of computations is as follows:

- For each observation interval in the first estimation interval a pair of values of $\alpha_k, \omega_k$ are computed after averaging $\gamma_1(\tau-t)$ and $\gamma_2(\tau-t)$, $k=0, 1, \cdots, m-1$.
- Using the values for $\alpha_{m-1}, \omega_{m-1}$, the elements of matrix $C^{-1}(m-1, t)$ are computed.
- Then the elements of matrix $D(m-1, T)$ are computed and stored.
- Using the values for $\alpha_{m-2}, \omega_{m-2}$ the elements of $C^{-1}(m-2, T)$ are computed.
- The four elements of $D(m-2, T)$ are then computed and stored.
- The process continues until $D(1, T)$ has been computed and stored.
- The $D$ matrices are used, each in turn, as each observation is processed to form the required $\hat{V}$ matrix.

The Kalman filter equations, which involve matrix $\hat{V}(t_{k+1}, T)$, are given in Section 3.3. The correction term involving $g(t_{k+1})$ will be discussed in Section 5.7.

The same set of computations is repeated for each estimation interval. The interconnection of adjacent estimation intervals was discussed in Section 3.5.

5.6 THE EXTRAPOLATION PROCESS

The extrapolation process also depends on the integration of the linearized equations by the method of mean coefficients. The relative state vector at each extrapolation time is computed by using the matrix $\hat{V}(t, t_k)$ obtained by solving the approximate differential equation...
for the optimal trajectory

$$\Delta \hat{s}(t_r) = \hat{\phi}(t_r, t_k) \Delta \hat{s}(t_k), \quad r = 1, 2, \ldots, \ell$$

$$\Delta \hat{s}(t_o) = \Delta \hat{s}(nT), \quad n = 0, 1, \ldots, N-1$$

The dependence of matrix \( \hat{\phi}(t_r, t_k) \) on matrix \( C(t_r, k) \) is discussed in Section 5.4.

5.7 THE TARGET MOTION CORRECTION VECTOR

The linearizing assumptions involved in the estimation process have neglected the term in the differential equation for the relative state vector depending explicitly on vector \( s_r(t) \). It is, however, possible to modify the process discussed above to include the approximate effect of this neglected term, and this modification will be discussed next.

The linearized differential equation for the relative state vector is (See Section 3.3)

$$\frac{d\Delta s}{dt} = F(t)\Delta s + G(t)s_r(t), \quad \Delta s(t_o) = \Delta c$$

where

$$F(t) = A(t) - \frac{1}{\lambda} B B^T(\tau + t_o - t)$$

$$G(t) = A_r - A(t) - \frac{1}{\lambda} B B^T(\tau + t_o - t)$$
This equation is further simplified by letting

\[ F(t) = A_R - B\Gamma(t - t) \]

\[ G(t) = A_R - A(t) \]

The approximate solution to this differential equation is obtained by a slight modification of the method of mean coefficients. Over the time interval \( t_k < t \leq t_{k+1} \) matrix \( F(t) \) is replaced by \( F_k \) as in Section 5.4.

In order to partially compensate for the term \( G(t) s_r(t) \) in the above equation and not excessively complicate the computation procedure, the matrix \( G(t) \) will be replaced by a matrix \( G_k \) where

\[ G_k = A_R - A(t_k) \]

and \( A_R \) is as defined in Section 2.2. Matrix \( A(t_k) \) is \( A(r) \) of Section 2.2 with \( -\mu r^{-3} \) replaced by its expansion given in Section 4.1. The solution over \( t_k < t \leq t_{k+1} \) becomes

\[ \Delta s(t) \approx e^{F_k(t-t_k)} \Delta s(t_k) + \int_{t_k}^{t} e^{F_k(t-\xi)} G_k s_r(\xi) d\xi \]

Since the target is assumed to be in a circular orbit, \( s_r(\xi) \) is given by

\[ s_r(\xi) = e^{A_R(\xi-t_k)} s_r(t_k) \]
Further, we are interested in the solution only at the observation times, so we can write

$$\Delta s(t_{k+1}) = \delta(t_{k+1}, t_k) \Delta s(t_k) + g(t_{k+1}, t_k)$$

where

$$g(t_{k+1}, t_k) = E(t_{k+1}, t_k) s_r(t_k)$$

$$E(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} e_k(t_{k+1} - \xi) A_r(\xi - t_k) d\xi$$

The matrix $E(t_{k+1}, t_k)$ will be block diagonal due to the block diagonal structure of the matrix exponentials and matrix $G_k$. Further, the diagonal blocks of $E(t_{k+1}, t_k)$ will be identical, and only four time-varying elements need be integrated to obtain matrix $E(t_{k+1}, t_k)$.

If we define the matrix integrand as

$$F_k(t_{k+1} - \xi) e_k(t_{k+1} - \xi) A_r(\xi - t_k) G_k e_r(\xi - t_k) = \begin{bmatrix} K(\xi) & 0 & 0 \\ 0 & K(\xi) & 0 \\ 0 & 0 & K(\xi) \end{bmatrix}$$
it is easily shown that

\[
K(\xi) = (n_r^2 - n_k^2) \begin{bmatrix}
    c_{12}(\xi) \cos n_r(\xi-t_k) & \frac{c_{12}(\xi)}{n_r} \sin n_r(\xi-t_k) \\
    c_{22}(\xi) \cos n_r(\xi-t_k) & \frac{c_{22}(\xi)}{n_r} \sin n_r(\xi-t_k)
\end{bmatrix}
\]

where

\[
n_r^2 = \frac{\mu}{a_r^3} = \text{Expansion in Section 4.1}
\]

\[
n_k^2 = \frac{\mu}{r_k^3} = \text{Expansion in Section 4.1}
\]

\[
c_{12}(\xi) = \frac{1}{u_k} e^{-\alpha_k(t_{k+1} - \xi)} \sin u_k(t_{k+1} - \xi)
\]

\[
c_{22}(\xi) = e^{-\alpha_k(t_{k+1} - \xi)} \left[ \cos u_k(t_{k+1} - \xi) - \frac{\alpha_k}{u_k} \sin u_k(t_{k+1} - \xi) \right]
\]

\[a_r = \text{radius of circular target orbit.}\]

From linearity considerations it is clear that

\[
E(T, t_k) = E(T, t_{m-1}) + E(t_{m-1}, t_{m-2}) + \cdots + E(t_{k+1}, t_k)
\]
and

\[ g(T, t_k) = E(T, t_k) \cdot s_r(t_k) \]

The \( g \) vector, computed according to the above process, is then used in the Kalman filter equations and the extrapolation process.
6.0 SOME COMPUTATIONAL RESULTS

6.1 TARGET AND INTERCEPTOR VEHICLE ORBITS AND PARAMETER VALUES

To test the effect of linearizing assumptions on the optimal control process, system performance must be checked against a true optimal solution of the problem. As pointed out in Section 1.2, this is possible for the intercept problem if the duration of the terminal guidance process is short enough to permit the effects of natural perturbations to be neglected. We shall make this assumption and define a two-body orbit representing a true optimal trajectory. This two-body orbit is optimal in the sense that an interceptor, which has the proper set of initial values of its state variables, will travel this orbit and, with no expenditure of fuel, collide with the target vehicle at a specified time. Obviously, the performance index defined in Section 2.3 will have zero value for such an orbit.

The target orbit used to obtain all of the computational results is a 300-mi altitude circular orbit inclined at 65° to the earth's equatorial plane. The chosen intercept point is where the target is at 30° latitude above the equator after passing its ascending node. The optimal two-body orbit is chosen to lie in the target's orbital plane with its apogee at the chosen intercept point. Its semi-major axis is 1.05046334 earth-radii and its eccentricity is 0.024. These same orbits are used for all the intercept and rendezvous cases, except where the effect of eccentricity is investigated.

For all computations 40 estimation intervals, 3 observation intervals per estimation interval, and 10 extrapolation intervals per observation interval are used (see Fig. 2).
6.2 EFFECTS OF LINEARIZING THE CONTROL PROCESS

The effects of linearizing assumptions on the optimal control process are tested by assigning the necessary set of initial values to the interceptor state variables to cause it to follow the optimal two-body orbit to collision with the target vehicle in the absence of controlling thrust. The terminal guidance system, optimized according to linearizing assumptions and the performance index specified in Section 2.3, is allowed to control the interceptor during the terminal guidance process. The terminal errors and the fuel mass ratio for the process are computed as measures of the deviation of system performance from true optimality.

The mass ratio is computed by the following formula:

\[
\frac{m_f}{m_0} = \frac{\text{Mass of fuel consumed}}{\text{Initial mass of vehicle plus fuel}}
\]

\[
= 1 - \exp \left[ -\frac{1}{c} \int_0^\tau |a(t)| \, dt \right]
\]

where \( \tau \) is the duration of the process and \( c \) the rocket exhaust velocity, assumed to be constant.

The computational results for five different values of the process duration \( \tau \) are presented in Table 1 for the intercept case and in Table 2 for the rendezvous case. Perfect observations of the state variables are assumed in computing the optimal control vector. By this is meant that the values of the relative state variables obtained from the integration of the nonlinear differential equations of motion are used as
Table 1
EFFECT OF $\tau$ ON THE INTERCEPT CASE WITH PERFECT
ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Duration (min)</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>.1030</td>
<td>.00063</td>
</tr>
<tr>
<td>2.0</td>
<td>.1302</td>
<td>.00134</td>
</tr>
<tr>
<td>3.0</td>
<td>.1921</td>
<td>.00230</td>
</tr>
<tr>
<td>4.0</td>
<td>.2428</td>
<td>.00374</td>
</tr>
<tr>
<td>5.0</td>
<td>.1825</td>
<td>.00580</td>
</tr>
</tbody>
</table>

Table 2
EFFECT OF $\tau$ ON THE RENDEZVOUS CASE WITH PERFECT
ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Duration (min)</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.046</td>
<td>.000044</td>
<td>.1554</td>
</tr>
<tr>
<td>2.0</td>
<td>.260</td>
<td>.000056</td>
<td>.1434</td>
</tr>
<tr>
<td>3.0</td>
<td>.380</td>
<td>.000059</td>
<td>.1435</td>
</tr>
<tr>
<td>4.0</td>
<td>.328</td>
<td>.000004</td>
<td>.1435</td>
</tr>
<tr>
<td>5.0</td>
<td>.065</td>
<td>.000179</td>
<td>.1433</td>
</tr>
</tbody>
</table>
input variables to the control process (Fig. 4). The value of $\lambda$ is taken to be $10^{-3}$ in the performance index for these tables and in all numerical data to follow. This value represents a compromise between terminal errors and computational round-off errors.

The miss distance for all values of $\tau$ is of the order of several tenths of a foot, i.e., zero for all practical purposes. The mass ratio increases with the value of $\tau$, as would be expected, but for all values of $\tau$ it is very small. It would appear that the deviation from the optimal trajectory caused by the linearized guidance system uses a slight amount of fuel, but has no practical effect on the terminal errors.

There is of course no such easily defined true optimal trajectory for the rendezvous case, since some velocity adjustment will always be required. Further, it is not clear that the two-body orbit defined above is the best approximation to such a true optimal trajectory. However, it will be used for convenience. The results in Table 2 again indicate negligible terminal errors. The fuel mass ratios vary only slightly. The fuel mass ratio for the velocity difference between the target orbit and the optimal two-body orbit at the rendezvous point has a value of 0.031. This can be used as a check point to compare with the data in Table 2.

The computational results for five different values of eccentricity for the optimal two-body orbit are presented in Table 3 for the intercept case and in Table 4 for the rendezvous case. Five different optimal two-body orbits were selected, lying in the target plane,
with eccentricities of 0.1, 0.2, 0.3, 0.4 and 0.5. Then semi-major axes were adjusted to make the apogee points coincide with the desired rendezvous point. The initial values of the state variables were chosen to correspond to the point on each two-body orbit at 2.5 min before apogee. Perfect observations of the state variables are assumed in computing the optimal control vector, as for the data in Tables 1 and 2.

The terminal error data and the fuel mass ratio data increase with eccentricity, as would be expected. This is due to the linearized equations being a poorer approximation for higher eccentricities. The increase in mass ratio for the rendezvous case is explained in part by the larger velocity difference between the circular and elliptic orbits at the apogee point for higher eccentricities.

6.3 EFFECTS OF LINEARIZING THE ESTIMATION PROCESS

The recurrence equations for computing the Kalman filter process derived in Sections 3.3 and 5.7 are given by

\[
P_{k+1} = P_k - P_k \ddot{g}^T(t_{k+1}, nT) \left[ \dot{g}(t_{k+1}, nT) P_k \dot{g}^T(t_{k+1}, nT) \right]^{-1} \dot{g}(t_{k+1}, nT) P_k
\]

\[
[\Delta \dot{S}(nT)]_{k+1} = [\Delta \dot{S}(nT)]_k + [\delta \dot{S}(nT)]_k
\]

\[
[\delta \dot{S}(nT)]_k = P_{k+1} \ddot{g}^T(t_{k+1}, nT) R_s^{-1}(t_{k+1}) \left[ \Delta \ddot{S}(t_{k+1}) \right. - \dot{g}(t_{k+1}, nT) \left[ [\Delta \dot{S}(nT)]_k - \dot{s}(nT, t_{k+1}) \right]
\]
Table 3
EFFECT OF ECCENTRICITY ON THE INTERCEPT CASE WITH PERFECT ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.493</td>
<td>.0074</td>
</tr>
<tr>
<td>0.2</td>
<td>0.796</td>
<td>.0152</td>
</tr>
<tr>
<td>0.3</td>
<td>1.177</td>
<td>.0233</td>
</tr>
<tr>
<td>0.4</td>
<td>1.418</td>
<td>.0319</td>
</tr>
<tr>
<td>0.5</td>
<td>1.719</td>
<td>.0410</td>
</tr>
</tbody>
</table>

Table 4
EFFECT OF ECCENTRICITY ON THE RENDEZVOUS CASE WITH PERFECT ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>.496</td>
<td>.293 x 10^-3</td>
<td>.482</td>
</tr>
<tr>
<td>0.2</td>
<td>.621</td>
<td>.382 x 10^-3</td>
<td>.742</td>
</tr>
<tr>
<td>0.3</td>
<td>.947</td>
<td>.237 x 10^-3</td>
<td>.877</td>
</tr>
<tr>
<td>0.4</td>
<td>1.369</td>
<td>.471 x 10^-3</td>
<td>.944</td>
</tr>
<tr>
<td>0.5</td>
<td>1.774</td>
<td>.522 x 10^-3</td>
<td>.977</td>
</tr>
</tbody>
</table>
\[ g(nT, t_{k+1}) = \int_{t_k+1}^{nT} \xi(nT, \xi) G(\xi) s_x(\xi) \, d\xi \]

\[ \approx E(nT, t_{k+1}) s_x(t_k) \]

In line with the desire to make the computation process as simple as possible, the following assumptions are made initially:

\[ R_s(t_{k+1}) = I \quad \text{(a unit matrix)} \]

\[ g(nT, t_{k+1}) = 0 \]

These two assumptions, of course, greatly simplify the computation process since they eliminate a matrix inversion and an integration process.

In addition to these two assumptions, it will be assumed that no uncertainties or errors are associated with observing vector \( q(t_k) \) and computing the components of \( \Delta \xi(t_k) \).

The first tests are a rerun of the data in Tables 1 and 2, with the relative state variables used in the computation of the optimal control vector being computed by the estimation process employing the Kalman filter. These data, which show the effect of process duration on terminal errors and fuel mass ratio, are presented in Table 5 for the intercept case and in Table 6 for the rendezvous case. A comparison of Table 5 with Table 1 and Table 6 with Table 2 indicates the fuel mass ratio is not significantly affected. However, there is a
Table 5

EFFECT OF $\tau$ ON THE INTERCEPT CASE
WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Duration (min)</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.1</td>
<td>.0002</td>
</tr>
<tr>
<td>2.0</td>
<td>20.6</td>
<td>.0007</td>
</tr>
<tr>
<td>3.0</td>
<td>104.9</td>
<td>.0030</td>
</tr>
<tr>
<td>4.0</td>
<td>366.1</td>
<td>.0155</td>
</tr>
<tr>
<td>5.0</td>
<td>822.8</td>
<td>.0841</td>
</tr>
</tbody>
</table>

Table 6

EFFECT OF $\tau$ ON THE RENDEZVOUS CASE
WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Duration (min)</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.8</td>
<td>.10</td>
<td>.2549</td>
</tr>
<tr>
<td>2.0</td>
<td>42.3</td>
<td>1.85</td>
<td>.1633</td>
</tr>
<tr>
<td>3.0</td>
<td>124.9</td>
<td>5.84</td>
<td>.1554</td>
</tr>
<tr>
<td>4.0</td>
<td>366.1</td>
<td>10.99</td>
<td>.1554</td>
</tr>
<tr>
<td>5.0</td>
<td>863.6</td>
<td>15.28</td>
<td>.1581</td>
</tr>
</tbody>
</table>
significant increase in the terminal errors that is roughly proportion- 
tional to the process duration in each case.

Tables 7 and 8 show the effect of increasing eccentricity on termi- nal errors and fuel mass ratio. For these cases the process duration was fixed at 2.5 min and the appropriate optimal two-body orbits selected as for Tables 3 and 4. The data in Tables 7 and 8 can be compared with the data in Tables 3 and 4. One effect of in- 
cluding the Kalman filter is to increase the fuel mass ratios slightly. 
The effects on terminal errors are much more pronounced with the 
errors increasing more rapidly with eccentricity than in Tables 3 and 4.

The approximation for the target motion correction vector \( g(nT, t_{K+1}) \), 
described in Section 5.7, is next added to the estimation process and 
the tests in Tables 7 and 8 are rerun. The results are given in Tables 
9 and 10. A comparison of Tables 7 and 9 and Tables 8 and 10 shows 
practically no effect on fuel mass ratio, but some improvement in 
terminal errors in all cases.

The addition of the target motion correction vector improved 
terminal position error by approximately 40 ft in the intercept and 
60 ft in the rendezvous cases, and the rendezvous velocity error by 
about 3 ft/sec.

The differences between the relative positions as estimated by 
the Kalman filter and as obtained from the true relative state vector 
components were computed at the end of each estimation interval from

\[
\text{Position Error} = + \left\{ \left[ \Delta x(nT) - \Delta \hat{x}(nT) \right]^2 + \left[ \Delta y(nT) - \Delta \hat{y}(nT) \right]^2 \right. \\
+ \left[ \Delta z(nT) - \Delta \hat{z}(nT) \right]^2 \right\}^{1/2} , \quad n = 1, 2, \ldots, 40
\]
Table 7

EFFECT OF ECCENTRICITY ON THE INTERCEPT CASE
WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>209.5</td>
<td>.0059</td>
</tr>
<tr>
<td>0.2</td>
<td>419.6</td>
<td>.0119</td>
</tr>
<tr>
<td>0.3</td>
<td>630.6</td>
<td>.0182</td>
</tr>
<tr>
<td>0.4</td>
<td>842.3</td>
<td>.0247</td>
</tr>
<tr>
<td>0.5</td>
<td>1054.9</td>
<td>.0315</td>
</tr>
</tbody>
</table>

Table 8

EFFECT OF ECCENTRICITY ON THE RENDEZVOUS CASE
WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>283.0</td>
<td>.15.3</td>
<td>.5172</td>
</tr>
<tr>
<td>0.2</td>
<td>568.5</td>
<td>30.7</td>
<td>.7763</td>
</tr>
<tr>
<td>0.3</td>
<td>855.2</td>
<td>46.4</td>
<td>.9014</td>
</tr>
<tr>
<td>0.4</td>
<td>1146.2</td>
<td>62.4</td>
<td>.9591</td>
</tr>
<tr>
<td>0.5</td>
<td>1441.1</td>
<td>78.6</td>
<td>.9843</td>
</tr>
</tbody>
</table>
Table 9

EFFECT OF ECCENTRICITY ON THE INTERCEPT CASE WITH
THE KALMAN FILTER AND THE TARGET MOTION
CORRECTION VECTOR

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>167.3</td>
<td>.0056</td>
</tr>
<tr>
<td>0.2</td>
<td>377.3</td>
<td>.0116</td>
</tr>
<tr>
<td>0.3</td>
<td>588.4</td>
<td>.0179</td>
</tr>
<tr>
<td>0.4</td>
<td>800.2</td>
<td>.0243</td>
</tr>
<tr>
<td>0.5</td>
<td>1012.8</td>
<td>.0311</td>
</tr>
</tbody>
</table>

Table 10

EFFECT OF ECCENTRICITY ON THE RENDEZVOUS CASE WITH
THE KALMAN FILTER AND THE TARGET
MOTION CORRECTION VECTOR

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>219.0</td>
<td>11.9</td>
<td>.5168</td>
</tr>
<tr>
<td>0.2</td>
<td>504.5</td>
<td>27.4</td>
<td>.7761</td>
</tr>
<tr>
<td>0.3</td>
<td>791.8</td>
<td>43.1</td>
<td>.9013</td>
</tr>
<tr>
<td>0.4</td>
<td>1082.4</td>
<td>59.0</td>
<td>.9590</td>
</tr>
<tr>
<td>0.5</td>
<td>1377.4</td>
<td>75.2</td>
<td>.9842</td>
</tr>
</tbody>
</table>
Similar data were also computed for the velocity error.

Figure 5 shows a plot of these data for the intercept case with process durations of 3 and 5 min. The corresponding data for the rendezvous case are plotted in Fig. 6.

The data for the longer process duration indicate much larger errors as do the data in Tables 5 and 6. The data for the rendezvous case velocity error at the end of the 40-th estimation interval are between 2000 and 3000 ft for both values of γ and were not plotted.

There are two possible sources of these bias errors:

- Accumulation of computational round-off errors
- Linearization errors in the Kalman filter

The relative contributions of these two sources have not been investigated, but the increase of terminal errors with eccentricity indicated in Tables 7 and 8 suggests that linearization errors may be the significant contributor.

6.4 EFFECTS OF LAUNCH TIME ERROR

The purpose of the terminal guidance system is to remove state variable errors introduced by previous guidance processes, and by possible last-minute target maneuvers. An extensive performance analysis of the correction of initial errors has not been made since we are interested in the effects of linearizing assumptions on system performance. However, some data have been accumulated concerning launch time errors. These errors are equivalent to an uncertainty in knowledge of the position of the target along its orbit, which is likely to be the most significant of the initial errors.
Fig. 5—Kalman filter bias errors for intercept case
Fig. 6—Kalman filter bias errors for rendezvous case
Tables 11 and 12 show the effect of a launch time error on terminal errors and fuel mass ratio for the intercept and rendezvous cases when perfect estimation of the relative state vector is assumed. The terminal errors for both cases increase with the magnitude of the launch error, but not significantly for practical purposes. However, the fuel mass ratios do undergo significant changes. This is to be expected as launch time errors call for significant deviations from the optimal two-body orbit selected for zero launch time error.

When the Kalman filter is used for the estimation process, the fuel mass ratio is changed but little, while terminal errors increase as indicated by earlier data. These results are shown in Tables 13 and 14.

When the target motion correction vector is used, the fuel mass ratio is only slightly affected. The results for the terminal errors are improved slightly in some cases and deteriorate somewhat in others. These data are shown in Tables 15 and 16.

6.5 EFFECTS OF OBSERVATIONAL NOISE

It is assumed that each observation is corrupted by noise having a known Gaussian probability distribution and zero mean. The method of introducing the noise into the simulation is described in Section 5.3 and Fig. 4. It is necessary to specify the set of variances making up the diagonal elements of matrix \( R_q \). These are assigned the numerical values

\[
r_{11} = \sigma^2_p = 2.1k \times 10^{-12}
\]
Table 11
EFFECT OF LAUNCH TIME ERROR ON THE INTERCEPT CASE WITH PERFECT ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>10.1</td>
<td>.8467</td>
</tr>
<tr>
<td>-0.2</td>
<td>5.5</td>
<td>.6082</td>
</tr>
<tr>
<td>-0.1</td>
<td>3.2</td>
<td>.3736</td>
</tr>
<tr>
<td>0</td>
<td>.12</td>
<td>.0018</td>
</tr>
<tr>
<td>+0.1</td>
<td>2.0</td>
<td>.3756</td>
</tr>
<tr>
<td>+0.2</td>
<td>4.7</td>
<td>.6095</td>
</tr>
<tr>
<td>+0.4</td>
<td>9.5</td>
<td>.8472</td>
</tr>
</tbody>
</table>

Table 12
EFFECT OF LAUNCH TIME ERROR ON THE RENDEZVOUS CASE WITH PERFECT ESTIMATION OF THE RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>11.8</td>
<td>.0029</td>
<td>.9719</td>
</tr>
<tr>
<td>-0.2</td>
<td>6.4</td>
<td>.0015</td>
<td>.8206</td>
</tr>
<tr>
<td>-0.1</td>
<td>3.7</td>
<td>.0007</td>
<td>.5473</td>
</tr>
<tr>
<td>0</td>
<td>.2</td>
<td>.0002</td>
<td>.1435</td>
</tr>
<tr>
<td>+0.1</td>
<td>2.2</td>
<td>.0005</td>
<td>.6560</td>
</tr>
<tr>
<td>+0.2</td>
<td>5.3</td>
<td>.0013</td>
<td>.3637</td>
</tr>
<tr>
<td>+0.4</td>
<td>10.7</td>
<td>.0025</td>
<td>.9736</td>
</tr>
</tbody>
</table>
Table 13
EFFECT OF LAUNCH TIME ERROR ON THE INTERCEPT CASE WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>241.0</td>
<td>.8441</td>
</tr>
<tr>
<td>-0.2</td>
<td>139.3</td>
<td>.6051</td>
</tr>
<tr>
<td>-0.1</td>
<td>92.3</td>
<td>.3714</td>
</tr>
<tr>
<td>0</td>
<td>50.3</td>
<td>.0014</td>
</tr>
<tr>
<td>+0.1</td>
<td>8.1</td>
<td>.3722</td>
</tr>
<tr>
<td>+0.2</td>
<td>30.0</td>
<td>.6056</td>
</tr>
<tr>
<td>+0.4</td>
<td>98.4</td>
<td>.8443</td>
</tr>
</tbody>
</table>

Table 14
EFFECT OF LAUNCH TIME ERROR ON THE RENDEZVOUS CASE WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>297.8</td>
<td>9.76</td>
<td>.9809</td>
</tr>
<tr>
<td>-0.2</td>
<td>173.2</td>
<td>5.64</td>
<td>.8511</td>
</tr>
<tr>
<td>-0.1</td>
<td>119.7</td>
<td>3.90</td>
<td>.5844</td>
</tr>
<tr>
<td>0</td>
<td>68.1</td>
<td>3.65</td>
<td>.1576</td>
</tr>
<tr>
<td>+0.1</td>
<td>26.6</td>
<td>1.83</td>
<td>.6939</td>
</tr>
<tr>
<td>+0.2</td>
<td>8.5</td>
<td>.95</td>
<td>.8904</td>
</tr>
<tr>
<td>+0.4</td>
<td>69.7</td>
<td>.12</td>
<td>.9860</td>
</tr>
</tbody>
</table>
Table 15
EFFECT OF LAUNCH TIME ERROR ON THE INTERCEPT CASE WITH THE KALMAN FILTER AND THE TARGET MOTION CORRECTION VECTOR

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>198.4</td>
<td>.8440</td>
</tr>
<tr>
<td>-0.2</td>
<td>96.6</td>
<td>.6050</td>
</tr>
<tr>
<td>-0.1</td>
<td>49.8</td>
<td>.3713</td>
</tr>
<tr>
<td>0</td>
<td>8.1</td>
<td>.0012</td>
</tr>
<tr>
<td>+0.1</td>
<td>34.4</td>
<td>.3721</td>
</tr>
<tr>
<td>+0.2</td>
<td>72.6</td>
<td>.6055</td>
</tr>
<tr>
<td>+0.4</td>
<td>140.9</td>
<td>.8443</td>
</tr>
</tbody>
</table>

Table 16
EFFECT OF LAUNCH TIME ERROR ON THE RENDEZVOUS CASE WITH THE KALMAN FILTER AND THE TARGET MOTION CORRECTION CASE

<table>
<thead>
<tr>
<th>Launch Time Error (min)</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>226.8</td>
<td>6.75</td>
<td>.9809</td>
</tr>
<tr>
<td>-0.2</td>
<td>101.9</td>
<td>2.65</td>
<td>.8511</td>
</tr>
<tr>
<td>-0.1</td>
<td>47.5</td>
<td>.97</td>
<td>.5843</td>
</tr>
<tr>
<td>0</td>
<td>3.8</td>
<td>.25</td>
<td>.1570</td>
</tr>
<tr>
<td>+0.1</td>
<td>42.8</td>
<td>1.27</td>
<td>.6939</td>
</tr>
<tr>
<td>+0.2</td>
<td>78.5</td>
<td>2.12</td>
<td>.8904</td>
</tr>
<tr>
<td>+0.4</td>
<td>138.6</td>
<td>2.94</td>
<td>.9860</td>
</tr>
</tbody>
</table>
\[ r_{22} = \sigma_{\rho}^2 = 74k \times 10^{-12} \]
\[ r_{33} = \sigma_{b}^2 = 0.25k \times 10^{-6} \]
\[ r_{44} = \sigma_{b}^2 = 0.36k \times 10^{-6} \]
\[ r_{55} = \sigma_{\rho}^2 = 0.25k \times 10^{-6} \]
\[ r_{66} = \sigma_{\rho}^2 = 0.36k \times 10^{-6} \]

where \( k = 0, 1, 2, 4, 8 \)

These variances are used in generating the random values of the components of vector \( v_q(t_k) \), as well as in the evaluation of matrix \( R_q \). The constant \( k \) is referred to as the variance factor in the tables. The variance values for \( k=1 \) were arbitrarily selected and only roughly indicate the state of the art.

The data in Tables 17 and 18 show the results of adding noise to the system with the Kalman filter eliminated from the simulation. The data for the intercept case in Table 17 show an increase in terminal errors with increasing variance, as would be expected, but the fuel mass ratio is essentially unaffected. For the rendezvous case, the data in Table 18 also show terminal errors increasing with increasing variance. Here, too, the fuel mass ratio is affected only slightly until the variances increase by almost an order of magnitude.

The effect of introducing the Kalman filter is shown in Tables 19 and 20. The bias errors discussed above masked out any benefits
Table 17

EFFECT OF GAUSSIAN NOISE ON THE INTERCEPT CASE
WITH PERFECT ESTIMATION OF THE RELATIVE
STATE VECTOR

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.0</td>
<td>.3756</td>
</tr>
<tr>
<td>1</td>
<td>2.65</td>
<td>.3755</td>
</tr>
<tr>
<td>2</td>
<td>3.59</td>
<td>.3755</td>
</tr>
<tr>
<td>4</td>
<td>5.09</td>
<td>.3755</td>
</tr>
<tr>
<td>8</td>
<td>7.44</td>
<td>.3757</td>
</tr>
</tbody>
</table>

Launch time error -0.1 min

Table 18

EFFECT OF GAUSSIAN NOISE ON THE RENDEZVOUS
CASE WITH PERFECT ESTIMATION OF THE
RELATIVE STATE VECTOR

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.2</td>
<td>.0005</td>
<td>.6560</td>
</tr>
<tr>
<td>1</td>
<td>2.99</td>
<td>.661</td>
<td>.6560</td>
</tr>
<tr>
<td>2</td>
<td>4.25</td>
<td>.966</td>
<td>.6560</td>
</tr>
<tr>
<td>4</td>
<td>6.39</td>
<td>1.418</td>
<td>.6559</td>
</tr>
<tr>
<td>8</td>
<td>71.40</td>
<td>.622</td>
<td>.8026</td>
</tr>
</tbody>
</table>

Launch time error -0.1 min
Table 19

EFFECT OF GAUSSIAN NOISE ON THE INTERCEPT CASE WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.1</td>
<td>.3722</td>
</tr>
<tr>
<td>1</td>
<td>33.2</td>
<td>.3725</td>
</tr>
<tr>
<td>2</td>
<td>45.1</td>
<td>.3727</td>
</tr>
<tr>
<td>4</td>
<td>61.9</td>
<td>.3729</td>
</tr>
<tr>
<td>8</td>
<td>85.7</td>
<td>.3731</td>
</tr>
</tbody>
</table>

Launch time error - 0.1 min

Table 20

EFFECT OF GAUSSIAN NOISE ON THE RENDEZVOUS CASE WITH THE KALMAN FILTER

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>26.6</td>
<td>1.83</td>
<td>.6939</td>
</tr>
<tr>
<td>1</td>
<td>67.0</td>
<td>3.56</td>
<td>.6961</td>
</tr>
<tr>
<td>2</td>
<td>87.6</td>
<td>4.41</td>
<td>.6971</td>
</tr>
<tr>
<td>4</td>
<td>117.7</td>
<td>5.66</td>
<td>.6983</td>
</tr>
<tr>
<td>8</td>
<td>165.3</td>
<td>7.38</td>
<td>.7002</td>
</tr>
</tbody>
</table>

Launch time error - 0.1 min
derived from smoothing over all previous observations. For these cases the addition of the target motion correction vector has only a slight effect on terminal errors and fuel mass ratio, as indicated in Tables 21 and 22.

One final group of computations was made using a covariance matrix $R_s(t_k)$ computed from

$$R_s(t_k) = M(t_k)R_q M^T(t_k)$$

instead of a unit matrix. The system performance for the rendezvous case is shown in Table 23. The data differ only slightly from the data in Table 22 where $R_s(t_k)$ was taken as a unit matrix.

6.6 SOME CONCLUSIONS

From the discussion in Section 6.2 and the data presented in Tables 1 through 4, it appears that none of the linearizing assumptions had any detrimental effect on the optimal control system performance, i.e., the terminal errors are within practical tolerances. The approximation of the time-varying elements in the matrix $R(t-t)$ by simpler expressions might be possible without exceeding system performance tolerances. This would simplify the computation process still further.

The presence of the bias errors in the estimation process indicated in Figs. 5 and 6 requires further investigation. Apparently, a more sophisticated method of integrating the equations of motion is required. The terminal errors in Tables 17 and 18 raise the question as to whether the additional complication of a Kalman filter would be worth the benefit derived from it (assuming the bias errors can be eliminated). The question is also raised as to whether the use of
**Table 21**

**EFFECT OF GAUSSIAN NOISE ON THE INTERCEPT CASE WITH THE KALMAN FILTER AND THE TARGET MOTION CORRECTION VECTOR**

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>34.4</td>
<td>.3721</td>
</tr>
<tr>
<td>1</td>
<td>34.3</td>
<td>.3725</td>
</tr>
<tr>
<td>2</td>
<td>40.8</td>
<td>.3726</td>
</tr>
<tr>
<td>4</td>
<td>53.4</td>
<td>.3728</td>
</tr>
<tr>
<td>8</td>
<td>74.2</td>
<td>.3731</td>
</tr>
</tbody>
</table>

Launch time error -0.1 min

**Table 22**

**EFFECT OF GAUSSIAN NOISE ON THE RENDEZVOUS CASE WITH THE KALMAN FILTER AND THE TARGET MOTION CORRECTION VECTOR**

<table>
<thead>
<tr>
<th>Variance Factor</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>42.8</td>
<td>.127</td>
<td>.6939</td>
</tr>
<tr>
<td>1</td>
<td>48.5</td>
<td>.216</td>
<td>.6961</td>
</tr>
<tr>
<td>2</td>
<td>63.5</td>
<td>.301</td>
<td>.6970</td>
</tr>
<tr>
<td>4</td>
<td>90.7</td>
<td>.427</td>
<td>.6983</td>
</tr>
<tr>
<td>8</td>
<td>133.8</td>
<td>.604</td>
<td>.7001</td>
</tr>
</tbody>
</table>

Launch time error -0.1 min
Table 23

EFFECT OF GAUSSIAN NOISE ON THE RENDEZVOUS CASE
WITH THE KALMAN FILTER, THE TARGET MOTION CORRECTION VECTOR, AND \( R_\kappa = M \varphi_{M}^{T} \)

<table>
<thead>
<tr>
<th>Variance</th>
<th>Miss Distance (ft)</th>
<th>Velocity Error (ft/sec)</th>
<th>Mass Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>14.9</td>
<td>10.9</td>
<td>.7863</td>
</tr>
<tr>
<td>1</td>
<td>16.9</td>
<td>14.0</td>
<td>.7916</td>
</tr>
<tr>
<td>2</td>
<td>26.2</td>
<td>15.1</td>
<td>.7936</td>
</tr>
<tr>
<td>4</td>
<td>39.4</td>
<td>16.8</td>
<td>.7962</td>
</tr>
<tr>
<td>8</td>
<td>53.9</td>
<td>18.6</td>
<td>.7999</td>
</tr>
</tbody>
</table>

Launch time error -0.1 min
matrix $R_{e}(t_{e})$ in place of a unit matrix is worth the additional computational complexity.

Further investigation is needed to determine whether fewer observational variables could be used without the observations becoming too ill-conditioned to be of practical use.

From some computational data not included above, it appears possible to use an adaptive control loop to reduce terminal errors resulting from launch time errors. This adaptive loop requires an estimation of the launch time error from observational data obtained during the early part of the terminal guidance process. On the basis of this estimate, the value of the parameter $\tau$ in the control matrix $\Gamma(\tau-t)$ is then appropriately adjusted to a new value which is used during the remainder of the process. This correction to $\tau$ appears to be very nearly a linear function of the launch time error.
REFERENCES


