MEMORANDUM
RM-5198-PR
DECEMBER 1966

# A COMPARISON OF AVERAGE LIKELIHOOD AND MAXIMUM LIKELIHOOD RATIO TESTS FOR DETECTING RADAR TARGETS OF UNKNOWN DOPPLER FREQUENCY

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
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This research is supported by the United States Air Force under Project RAND—Contract No. AF 49(638)-1700—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.



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Published by The RAND Corporation

# PREFACE

This Memorandum was prepared as part of RAND's continuing study of radar detection theory. It compares the method of detection commonly employed in coherent radars (maximum likelihood ratio test) with a theoretically better, but more complex, procedure (average likelihood ratio test). The results show that the performances of the two tests are not significantly different, and, thus, that implementation of the more complex detector is not warranted in search radars for the assumed target model. The Memorandum should be useful to those concerned with radar design, including Air Force personnel and contractors.

RM-5198-PR



## SUMMARY

In a coherent search radar, the pulse-to-pulse doppler shift of a signal is generally not known a priori. Given the distribution of this parameter, the best test variable for detection is the average of the likelihood ratio with respect to target doppler frequency. Most coherent search radars employ a maximum likelihood ratio detector, that is, a bank of independent doppler filters, for detection. The average likelihood and maximum likelihood tests are compared here for a target with the Rayleigh amplitude distribution. It is shown that, over a wide range of detection and false-alarm probabilities, the performances of the two tests do not differ significantly. For this target model, the likelihood ratio has the Pareto distribution, which arises in some statistical problems in economics. The new results obtained here for the distribution of the sum of two or more Pareto-distributed variables are of considerable general interest.

# CONTENTS

# LIST OF FIGURES

## I. INTRODUCTION

In each resolution cell of a pulsed search radar, a choice is made between the null hypothesis that noise only is present ($H_0$), and the signal-plus-noise hypothesis ($H_1$) based on a set of radar returns. The usual method of deciding between $H_0$ and $H_1$ is to form a likelihood ratio,[1] or some function of the set of returns which approximates a likelihood ratio, and to choose $H_1$ if and only if this function exceeds a threshold. The likelihood ratio for a set of coherent radar returns is a function of three signal parameters which are generally not known a priori: amplitude, phase, and doppler frequency. When the distributions of these parameters are known a priori, the best test variable is the likelihood ratio averaged over the unknown parameters.[2]

In nearly all radars of practical interest, the signal phase is unknown a priori and is uniformly distributed over $(0,2\pi)$ with respect to an arbitrary reference signal. Similarly, the a priori uncertainty in doppler frequency is usually large compared to the radar pulse repetition frequency, so the pulse-to-pulse phase shift due to target doppler is also uniformly distributed over $(0,2\pi)$. Selin[3] discusses the average likelihood ratio test under these assumptions for a target of known and nonfluctuating amplitude. Because of the intractability of the expression for average likelihood ratio in this case, no quantitative results were given by Selin[3] for the performance of the average likelihood test (averaged over signal phase and doppler frequency).

For a slowly fluctuating target with Rayleigh amplitude distribution (Case I, Swerling[4]), the equation for average likelihood ratio is more tractable and a more exact evaluation of the average likelihood test is possible. This problem is analyzed below. In no case is the difference between average and maximum likelihood tests (with respect to target doppler frequency) found to be significant. It is also shown that the test based on the average of the logarithm of the likelihood ratio is less sensitive than the maximum likelihood ratio test.

In Section II, the distribution of the likelihood ratio is shown to be a Pareto distribution. In Appendix A, it is shown that the sum of N independent variables, each having the Pareto distribution, does not approach a normal distribution. Instead it approaches a distribution which is, again, for large values of the variable, a Pareto distribution. In Appendix B, examples are given which further illustrate the self-reproducing property of the Pareto distribution. Finally, in Appendix C an extension of the Chernoff bound is derived for positive random variables. This is used to relate the false-alarm probabilities of the average and maximum likelihood ratio tests.

 



## II. PROBABILITY DENSITY FUNCTION OF AVERAGE LIKELIHOOD RATIO

Let the set of N complex quantities $\{z_1, z_2, \ldots, z_N\} = Z$ denote the N pulsed radar returns from one resolution cell.

$$z_n = x_n + iy_n + ae^{i(\psi - n\varphi)} \tag{1}$$

where $x_n$ and $y_n$ are normally distributed with zero mean and unit variance, and

$a$ = normalized signal amplitude ($a = 0$, $H_0$; $a \neq 0$, $H_1$)

$\psi$ = signal initial phase

$\varphi$ = signal phase shift due to target doppler frequency

When the likelihood ratio

$$L(Z|a,\psi,\varphi) = \frac{p(Z|a,\psi,\varphi)}{p(Z|0)} \tag{2}$$

is averaged over signal phase, $\psi$, it reduces to (see, for example, Selin[3] Eq. (6))

$$L(Z|a,\varphi) = \exp\left\{-\frac{Na^2}{2}\right\} I_0\left(a\left|\sum_{n=1}^{N} z_n e^{in\varphi}\right|\right) \tag{3}$$

where $I_o$ is the modified zero-order Bessel function.

If the design signal, a, is Rayleigh distributed

$$\begin{aligned} p(a) &= \frac{a}{S} e^{-a^2/2S} && a \geq 0 \\ &= 0 && a < 0 \end{aligned} \tag{4}$$

the likelihood ratio can be averaged over a, using Weber's first exponential integral[5]

$$L(Z|\varphi) = \int_0^\infty p(a)L(Z|a,\varphi)da$$

$$= \frac{1}{1+NS} \exp \left\{ S|V(\varphi)|^2/2(1+NS) \right\} \tag{5}$$

where $S = E\left\{a^2/2\right\}$ is the design average signal-to-noise (S/N) ratio and

$$V(\varphi) = \sum_{n=1}^{N} z_n e^{in\varphi} \tag{6}$$

When noise only of unit variance is present ($H_0$), the density function for $U = |V(\varphi)|$, for any $\varphi$, is

$$p(U) = \frac{U}{N} e^{-U^2/2N} \qquad U \geq 0$$
$$= 0 \qquad U < 0 \tag{7}$$

When signal and noise are present ($H_1$), and $\varphi$ is matched to the signal doppler frequency,

$$p(U) = \frac{U}{N(1+NS)} e^{-U^2/2N(1+NS)} \qquad U \geq 0$$
$$= 0 \qquad U < 0 \tag{8}$$

From Eqs. (5) through (8), the density function for $L(Z|\varphi)$ is

$$p(x) = \frac{k-1}{x^k} \qquad x > 1$$
$$x = (1+NS)\, L(Z|\varphi) \tag{9}$$

The quantity k has the following values in the noise-only and signal-plus-noise cases:

$$k = k_0 = 2 + 1/NS; \qquad H_0$$
$$k = k_1 = 1 + 1/NS; \qquad H_1 \tag{10}$$

The probability density function of Eq. (9) is the Pareto density. This function is discussed in the statistical literature as an approximate fit to the distribution of incomes.[6] Note that x has a first moment but no second moment when $k = k_0$, and has neither a first nor second moment when $k = k_1$.

The density functions of Eqs. (9) and (10) characterize the conditional likelihood ratio $L(Z|\varphi)$. The average likelihood ratio is the average of this function over $\varphi$, that is,

$$L(Z) = \frac{1}{2\pi}\int_0^{2\pi} L(Z|\varphi)\,d\varphi \tag{11}$$

where $L(Z|\varphi)$ is given by Eq. (5). Since no exact method was found for computing the distribution of L(Z), we will restrict all ensuing analysis to the following discrete approximation to L(Z)

$$\Lambda(Z) = \underset{j}{\text{Ave}}\left\{L(Z|\varphi_j)\right\} = \frac{1}{N}\sum_{j=1}^{N} L(Z|\varphi_j) \tag{12}$$

where the $\varphi_j$ are selected as follows:

$$\varphi_j = \frac{2\pi j}{N} \tag{13}$$

The resulting

$$V_j \equiv V(\varphi_j) = \sum_{n=1}^{N} z_n e^{i \frac{2\pi jn}{N}} \qquad (14)$$

are independent. In the noise-only case, independence of the $V_j$ is proved by noting that $V_j$ are normally distributed in two dimensions and are uncorrelated, that is,

$$E\left\{V_i V_j^*\right\} = E\left\{V_i V_j\right\} = 0 \qquad i \neq j \qquad (15)$$

This result is readily shown by substituting Eq. (14) in Eq. (15) and taking expected values.

Since the $V_j$ are independent, the corresponding samples of the conditional likelihood ratio $L(Z|\varphi_j)$ are also independent. The quantity X,

$$X = \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} (1 + NS)\ L(Z|\varphi_j) \qquad (16)$$

is then proportional to the average likelihood ratio, $\Lambda(Z)$, and is the sum of N independent $x_j$ with the density functions given by Eqs. (9) and (10). This is a convenient variable to use in evaluating the average likelihood ratio tests.

In the noise-only case ($H_0$), all N of the $x_j$ have the density function of Eq. (9) with $k = k_0$. In the signal-plus-noise case, N-1 of the variables have the density function with $k = k_0$, and one has $k = k_1$. This representation of the average likelihood ratio is an approximation and assumes that, when a signal is present, it has a doppler frequency corresponding exactly to one of the $\varphi_j$ of Eq. (13). It is considered a good approximation for the purpose of comparing maximum likelihood tests with average likelihood tests. This representation is exact when the signal doppler shift is known to have one of N values, which are equally probable and known a priori.

## III. MAXIMUM LIKELIHOOD RATIO TESTS

In comparing average likelihood with maximum likelihood tests, it is convenient to use the variables $x_n$ and the distribution of Eqs. (9) and (10) for both cases. In the maximum likelihood test, the maximum of N independent variables $x_n$ is compared with a threshold $\lambda$. The probability of false alarm is then

$$P_f = 1 - \left[1 - \lambda^{(1-k_0)}\right]^N \tag{17}$$

and the detection probability, assuming one of the N variables has the distribution exponent $k_1$, is

$$P_d = 1 - \left[1 - \lambda^{(1-k_0)}\right]^{(N-1)} \left[1 - \lambda^{(1-k_1)}\right] \tag{18}$$

## IV. AVERAGE LIKELIHOOD RATIO TESTS

To evaluate the performance of the average likelihood ratio test, we require the distribution or density function for the sum of N variables, $x_n$, each having a Pareto distribution. Although this distribution has a very simple form, no exact method was found for computing the distribution of the sum for arbitrary N. In fact, the tail of the density function of X (Eq. (16)) has the asymptotic form

$$p(X) \simeq \frac{N(k-1)}{X^k} \tag{19}$$

for large X, and does not approach the normal. This is the portion of the distribution curve which is of principal interest in radar when low false-alarm probabilities are required. This asymptotic expression is derived in Appendix A.

### EXACT SOLUTION FOR N = 2

An exact solution is obtained in series form for the case of N = 2. The density function for the sum of two variables with the same exponent k is obtained by convolution

$$p(X) = \int_1^{X-1} \frac{(k-1)^2}{x_1^k (X-x_1)^k} \, dx_1 \tag{20}$$

Expanding $(1 - x_1/X)^{-k}$ in a power series and noting that the integrand is symmetric about X/2,

$$p(X) = \frac{2(k-1)^2}{X^k} \int_1^{X/2} \sum_{\nu=0}^{\infty} \frac{(k-1)_\nu}{\nu!} \left(\frac{x_1}{X}\right)^\nu \frac{dx_1}{x_1^k}$$

$$= (k-1)^2 \sum_{\nu=0}^{\infty} \frac{(k-1)_\nu}{\nu!(\nu-k+1)} \left[\frac{2^{k-\nu}}{X^{2k-1}} - \frac{2}{X^{k+\nu}}\right]; \qquad X \geq 2 \tag{21}$$

where $(k-1)_\nu = \Gamma(k+\nu)/\Gamma(k)$. The false-alarm probability is then

$$P_f = \int_\lambda^\infty p(X)\; dX$$

$$= (k_0-1)^2 \sum_{\nu=0}^{\infty} \frac{(k_0-1)_\nu}{\nu!(\nu-k_0+1)} \left[\frac{2^{k_0-\nu-1}}{(k_0-1)\lambda^{2k_0-2}} - \frac{2}{(k_0+\nu-1)^{k_0+\nu-1}}\right] \tag{22}$$

The same method is used to obtain an expression for the probability density of the sum of two variables with different values of k. The detection probability is

$$P_d = F(k_0,k_1,\lambda) + F(k_1,k_0,\lambda)$$

$$F(k_0,k_1,\lambda) = (k_0-1)(k_1-1) \sum_{\nu=0}^{\infty} \frac{(k_1-1)_\nu}{\nu!(\nu-k_0+1)} \left[\frac{2^{k_1-\nu-1}}{(k_1+k_0-2)\lambda^{k_0+k_1-2}} - \frac{1}{(k_1+\nu-1)\lambda^{k_1+\nu-1}}\right] \tag{23}$$

The performance of average and maximum likelihood ratio detectors is compared in Fig. 1 for the case of N = 2. The difference between

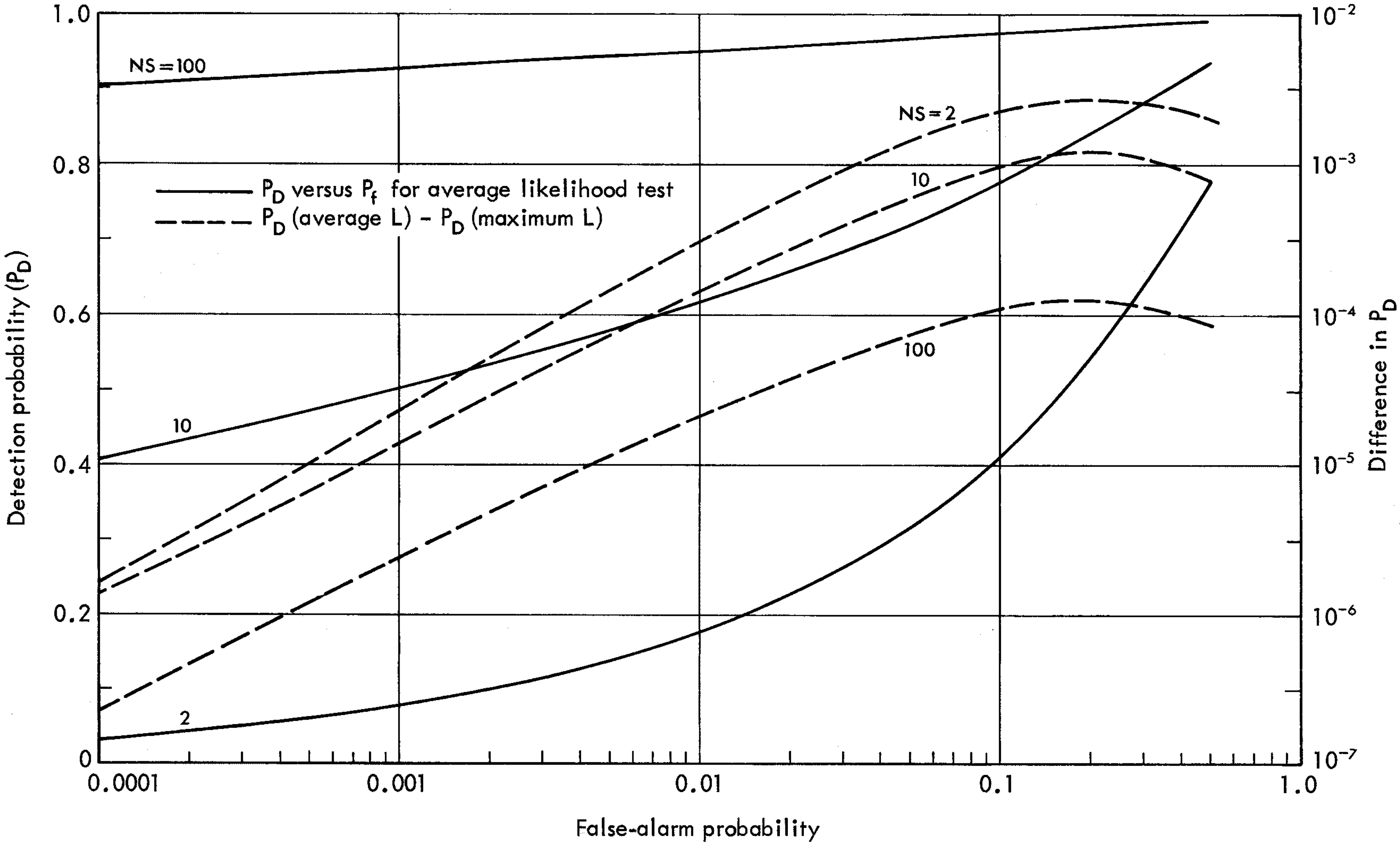


Fig. 1 — Relative performance of average and maximum likelihood tests for N = 2

detection probabilities is very small and is shown on a separate scale. In all cases the average likelihood test is slightly better, as would be expected.

## EXACT SOLUTION FOR N = 3

The probability density function of Eq. (21) can be convolved again with the density function of Eq. (9) to obtain the density for the sum of three variables. Integrating the resulting series from $\lambda$ to $\infty$ gives the following expression for detection probability:

$$P_d = \sum_{n=0}^{\infty} \frac{2(k_1-1)(k_0-1)^2(k_0-1)_n}{n!(n+1-k_0)} \left[H(k_1,k_0,n) + G(k_1,k_0,n)\right]$$

$$- (k_1-1)(k_0-1)^2 \left[U(k_1,k_0) + V(k_1,k_0)\right] \sum_{n=0}^{\infty} \frac{(k_0-1)_n}{n!2^{n-k_0}(n-k_0+1)} \qquad (24)$$

where

$$H(k_1,k_0,n) = \sum_{m=0}^{\infty} \frac{(k_1-1)_m}{m!(m+1-k_0-n)} \left[\frac{\lambda^{2-k_0-k_1-n}}{(2-k_0-k_1-n)2^{m+1-k_0-n}} - \frac{2^{m+1-k_0-n}\lambda^{1-k_1-m}}{(1-k_1-m)}\right]$$

$$G(k_1,k_0,n) = \sum_{m=0}^{\infty} \frac{(k_0+n-1)_m}{m!(m+1-k_1)} \left[\frac{\lambda^{2-k_0-k_1-n}}{(2-k_0-k_1-n)2^{m-k_1+1}} - \frac{\lambda^{1-k_0-m-n}}{(1-k_0-m-n)}\right]$$

$$U(k_1,k_0) = \sum_{n=0}^{\infty} \frac{(k_1-1)_n}{n!(n+2-2k_0)} \left[\frac{\lambda^{3-2k_0-k_1}}{(3-2k_0-k_1)2^{n+2-2k_0}} - \frac{2^{n+2-2k_0}\lambda^{1-k_1-n}}{(1-k_1-n)}\right]$$

$$V(k_1,k_0) = \sum_{n=0}^{\infty} \frac{(2k-2)_n}{n!(n-k_1+1)} \left[ \frac{\lambda^{3-k_1-2k_0}}{(3-2k_0-k_1)2^{n-k_1+1}} - \frac{\lambda^{2-2k_0-n}}{(2-2k_0-n)} \right]$$

The same equation yields false-alarm probability when $k_1$ is replaced by $k_0$.

The performance of the average and maximum likelihood tests for $N = 3$ is shown in Fig. 2. Again the average likelihood ratio test is slightly superior, but the difference between the two tests is very small.

## APPROXIMATE RESULTS FOR LARGE N

Exact solutions for the performance of the average likelihood test were not obtained for N larger than 3. For large false-alarm probabilities, results were obtained by simulation and are shown in Fig. 3 along with the exact performance of the corresponding maximum likelihood tests. A sample of 500 trials was used to obtain each false-alarm probability and each detection probability. Again the average and maximum likelihood tests are almost identical in performance.

At low false-alarm probabilities, simulation becomes impractical because of the large sample size required. An asymptotic expression is derived in Appendix A for the density function of the sum of N noise-only variables. This density function is given in Eq. (19). The corresponding false-alarm probability for a threshold $\lambda$ is

$$P_f = \frac{N}{\lambda^{k_0-1}} \tag{25}$$

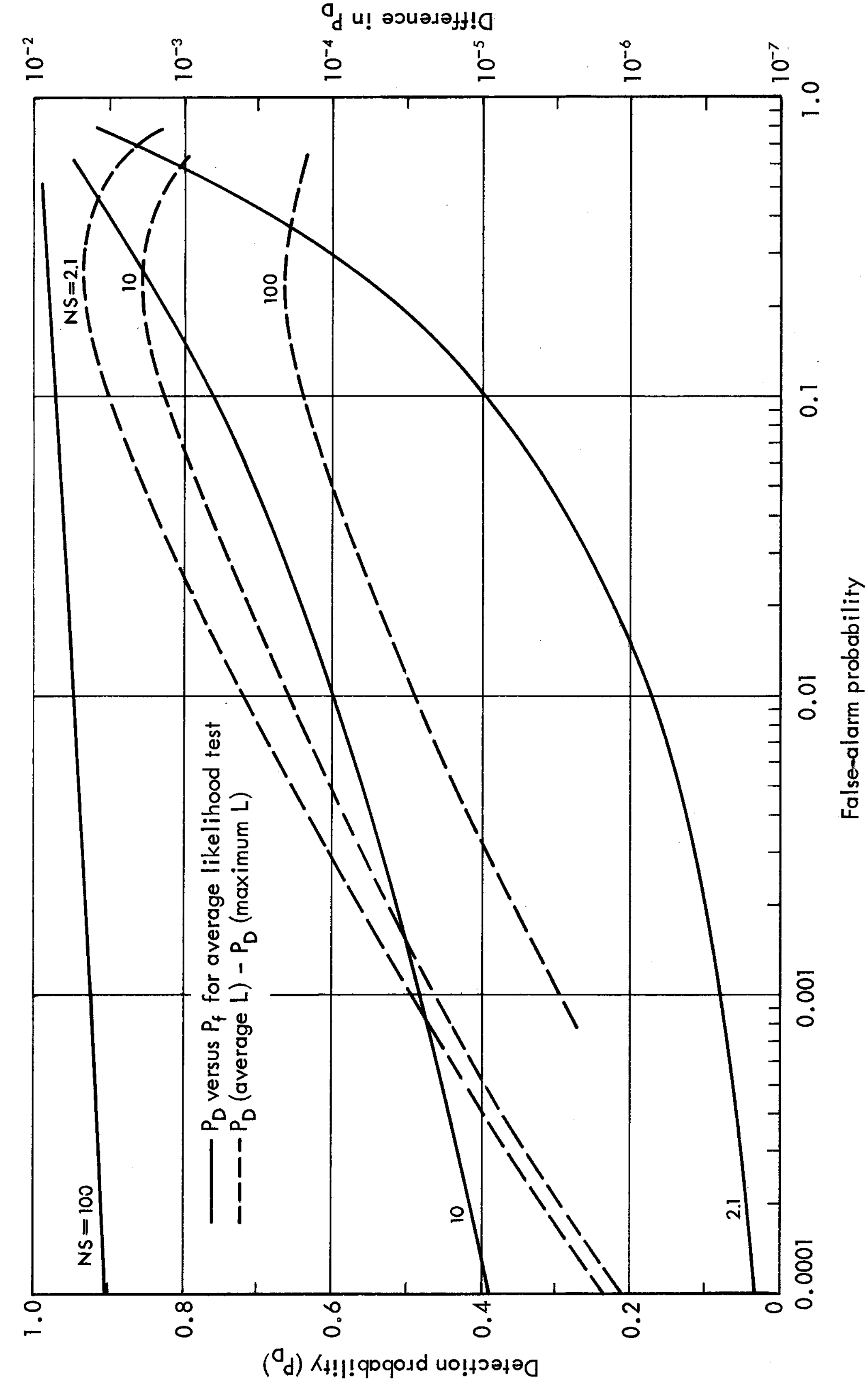


Fig. 2 — Relative performance of average and maximum likelihood tests for N = 3

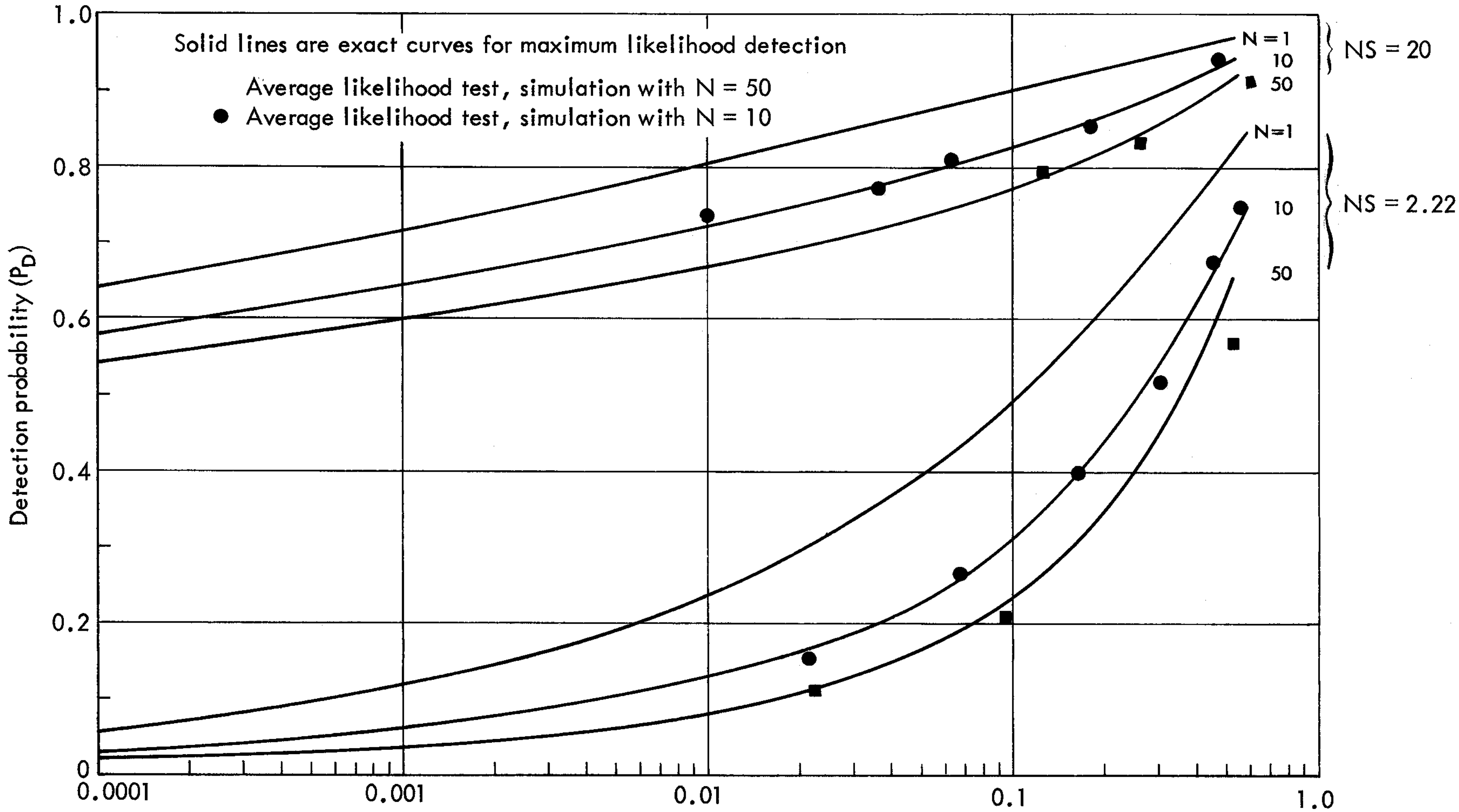


Fig. 3 — Relative performance of maximum and average likelihood tests for large N

This same approximate expression is obtained for the maximum likelihood test (from Eq. (17)) when $\lambda$ is large. Several detection probabilities were obtained by simulation for the $N \geq 2$, large $\lambda$ case, and the results again verified that the performance of the average likelihood ratio test is very close to that of the maximum likelihood test.

In Appendix B, the special case of NS = 2 (where $k_0 = 5/2$ and $k_1 = 3/2$) is considered further. Using functions which approach the Pareto distribution for large X and are more tractable, it is shown that the density function for the sum of M noise variables and N-M signal-plus-noise variables is approximately

$$p(X) = \frac{C_1(N-M)}{X^{3/2}} + \frac{C_2M + C_3M(N-M)}{X^{5/2}} \tag{26}$$

For sufficiently large X, the first term dominates and detection probability varies as $\lambda^{-1/2}$. This same dependence of $P_d$ on $\lambda$ is obtained in the maximum likelihood test (Eq. (18)) for sufficiently large $\lambda$.

An upper bound on false-alarm probability, based on a modification of the Chernoff bound, is obtained in Appendix C. Again, for large $\lambda$, this bound approaches the $P_f$ of the maximum likelihood test.

AVERAGE OF LOGARITHM OF LIKELIHOOD RATIO

Exact expressions can be obtained for the detection and false alarm probabilities when the variables $y_n = \log x_n$ are added. The density function for $y_n$ is

$$p(y_n) = (k-1)\, e^{-(k-1)y_n} \quad ; \qquad y_n \geq 0 \tag{27}$$

The density function for the sum of N of these variables can be obtained using characteristic functions:

$$p(y) = \frac{(k-1)^N y^{N-1}}{\Gamma(N)} e^{-(k-1)y} \quad ; \quad y \geq 0 \tag{28}$$

$$y = \sum_{n=1}^{N} y_n$$

The corresponding false-alarm and detection probabilities are

$$P_f = \Gamma\Big(N,\ (k_0-1)\lambda\Big)/\Gamma(N)$$

and

$$P_d = (k_0-1)^{N-1}\left[e^{-(k_1-1)\lambda} - \sum_{\nu=0}^{N-2} \frac{(k_1-1)}{\nu!k_1^{\nu+1}}\Gamma(\nu+1,\ k_1\lambda)\right] \tag{29}$$

where $\Gamma(n,a)$ is the incomplete gamma function. The maximum likelihood ratio is a better test variable than the average of the logarithm of the likelihood ratio for detecting targets of unknown doppler frequency. Two examples of the relative performance of the two tests are shown in Fig. 4.

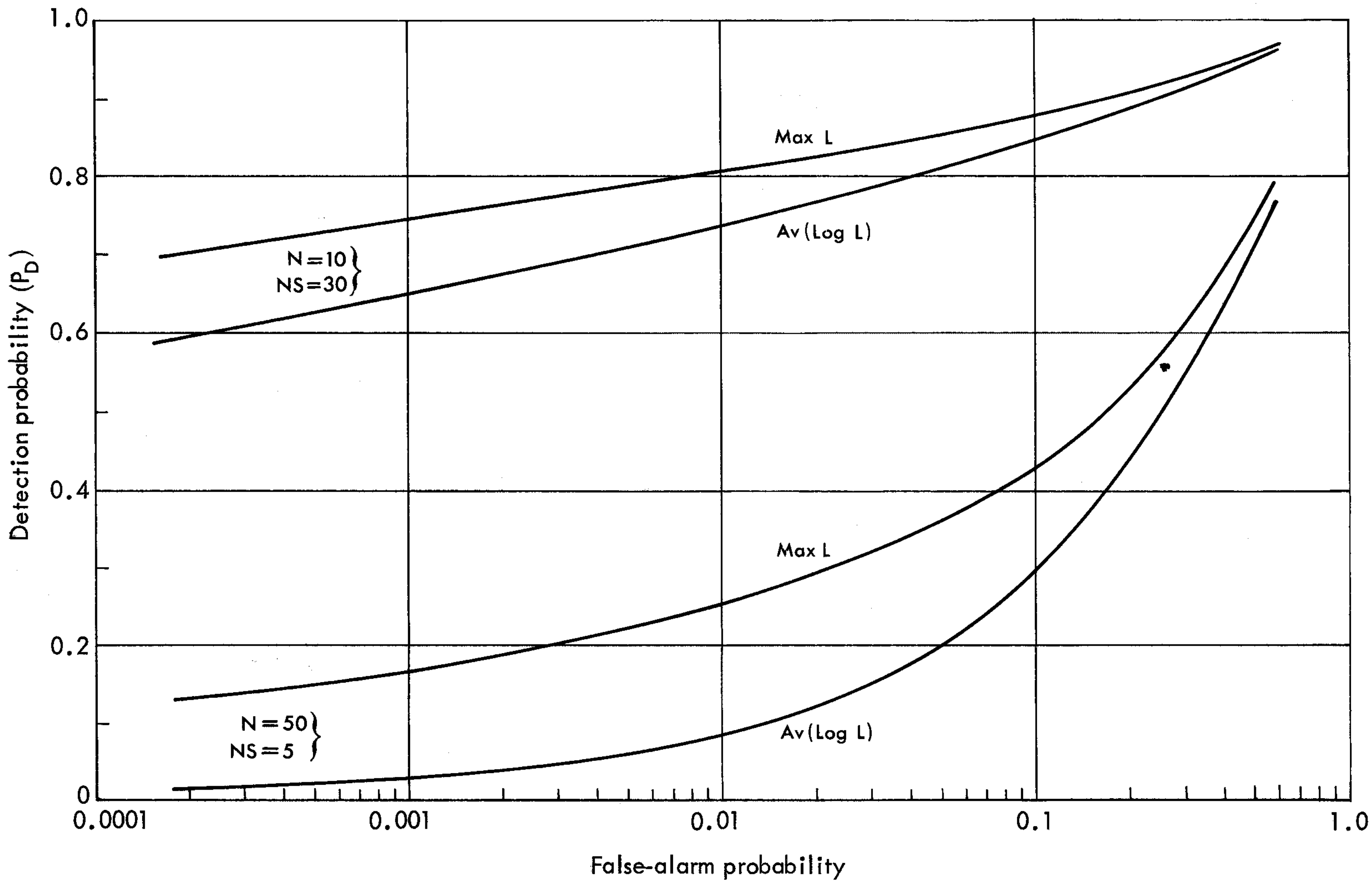


Fig. 4 — Relative performance of maximum likelihood and average of log (likelihood) tests

## V. CONCLUSIONS

When the amplitude of a radar echo is Rayleigh distributed (Case 1, Swerling), the likelihood ratio for a given target doppler frequency has the Pareto distribution. Detection criteria based on the average (over target doppler frequency) likelihood ratio and the maximum likelihood ratio were compared. An exact comparison of the two tests was made for N = 2 and 3, where N is the number of possible target doppler frequencies. For large N results were obtained by simulation and asymptotic approximations to the tail of the distribution curve. The average likelihood test is known to be superior, but it was shown that the difference between the performances of the two tests is very small.

Appendix A

ASYMPTOTIC FORM OF THE DENSITY FUNCTION

Although the exact form of the density function cannot be found for arbitrary N, it is possible to find an asymptotic expression for the tail of the function for sufficiently large X. The simple form given in Eq. (19) is the first term of an expansion in inverse powers of X; this expansion will be presented to third order in this appendix.

The probability density function $P_N(X)$ is defined as the density function for the sum of N independent samples of the variable x, where x is distributed according to the Pareto distribution.[6] This density function can be represented as the inverse Laplace transform of its own moment-generating function, so that

$$P_N(X) \equiv \frac{1}{2\pi i} \int_{c_1} e^{\sigma X} E_N(\sigma)\, d\sigma \qquad \text{(A-1)}$$

$$E_N(\sigma) = \text{expected value of } \exp(-\sigma x) \qquad \text{(A-2)}$$

where the contour $c_1$ is a straight line parallel to the imaginary axis lying to the right of all the singularities of the integrand. Since the N samples are independent and drawn from the same population, the moment-generating function $E_N(\sigma)$ is the $N^{th}$ power of the moment-generating function $E_1(\sigma)$, defined by

$$E_1(\sigma) = \int_0^\infty e^{-\sigma x}\, p(x)\,dx \tag{A-3}$$

Here p(x) is the Pareto distribution.

The integral $E_1(\sigma)$ can be evaluated in terms of confluent hypergeometric functions. Thus, inserting the Pareto distribution into the integral (A-3) and making the translation $x = 1 + t$, there results

$$E_1(\sigma) = (k-1)e^{-\sigma} \int_0^\infty e^{-\sigma t}\,(1+t)^{-k}\,dt \tag{A-4}$$

This is a standard form for the confluent hypergeometric function of the second kind[7]

$$E_1(\sigma) = (k-1)e^{-\sigma}\,U(1,2-k,\sigma) \tag{A-5}$$

This function can be expressed in terms of more familiar functions by using the connection formulas relating the confluent hypergeometric functions of the first and second kind given on pp. 504-505 of Ref. 7. After some algebraic simplifications, the result is

$$E_1(\sigma) = {}_1F_1(1-k,2-k,-\sigma) - \Gamma(2-k)\sigma^{k-1} \tag{A-6}$$

For $\sigma$ large and in the right half-plane, the asymptotic form of $E_1(\sigma)$, derived from Eq. (A-5), is $(k-1)e^{-\sigma}/\sigma$. The form (A-6) and the analyticity properties of the confluent hypergeometric function show that the only singularity of $E_1(\sigma)$ in the finite part of the $\sigma$-plane is a branch point of the origin. It follows that the contour integral for $P_N$ can be closed in the right half-plane if $X < N$, and the integral is zero. This result is also a consequence of the vanishing of the probability distribution $p(x)$ for $x < 1$, from which the sum of N samples must have a vanishing probability distribution for $X < N$.

When $X > N$, the integration path must be closed in the left half-plane. The analyticity properties of $E_1$ show that it is algebraic for $\sigma$ large in the left half plane. Because of the branch point at the origin, a cut must be placed along the negative real axis, and the contour may be closed by two quarter-circles in the left half-plane and a loop around the branch cut. The quarter-circles do not contribute to the integral, so that the result is

$$P_N(X) = \frac{1}{2\pi i} \int_{c_2} d\sigma e^{\sigma X} \left[ {}_1F_1(1-k,2-k,-\sigma) - \Gamma(2-k)\sigma^{k-1} \right]^N \qquad \text{(A-7)}$$

where the contour $c_2$ starts at $-\infty$, encircles the origin counterclockwise, and returns to $-\infty$. The phase of $\sigma$ increases from $-\pi$ to $\pi$ as the contour is traversed.

An asymptotic expansion of the integral may be obtained by using a generalized form of Watson's lemma.(8) First, the integration contour may be taken as the lines $\sigma = te^{-i\pi}$ and $\sigma = te^{i\pi}$; there is no contribution from the small circle around the origin if $k > 0$. The integral then becomes

$$P_N(X) = \frac{1}{2\pi i}\int_0^\infty dt\; e^{-tX}\left[\begin{array}{l}\left\{{}_1F_1(1-k,2-k,t)-\Gamma(2-k)\,t^{k-1}e^{-i\pi(k-1)}\right\}^N \\ -\left\{{}_1F_1(1-k,2-k,t)-\Gamma(2-k)\,t^{k-1}e^{\,i\pi(k-1)}\right\}^N\end{array}\right] \qquad \text{(A-8)}$$

The lemma of Ref. 8 may now be stated. Let the function enclosed in brackets in Eq. (A-8) be called $\varphi_N(t)$. Define a sequence of numbers $\lambda_n$ such that

$$\varphi_N(t) \sim \sum_1^M a_n\, t^{\lambda_n - 1} \qquad \text{as t approaches 0} \qquad \text{(A-9)}$$

Then $P_N(X)$ has the asymptotic representation

$$P_N(X) \sim \sum_1^M \Gamma(\lambda_n)\, a_n\, X^{-\lambda_n} \qquad \text{as X approaches infinity} \qquad \text{(A-10)}$$

The $N^{th}$ power appearing in the integral may be expanded by binomial theorem, with the result

$$\frac{1}{2\pi i}\,\varphi_N(t) = \sum_1^N \frac{N!\,(-1)^{\ell}[\Gamma(2-k)]^{\ell}}{\ell!\,(N-\ell)!}\,\frac{\sin \pi\ell(k-1)}{\pi}\, t^{\ell(k-1)} \qquad \text{(A-11)}$$
$$[{}_1F_1(1-k,2-k,t)]^{N-\ell}$$

The hypergeometric function is a series of positive integer powers, from which a power of the hypergeometric function is also a series of positive integer powers. Thus, a set of coefficients $c_{N-\ell,m}$ can be defined by

$$[{}_1F_1(1-k,2-k,t)]^{N-\ell} = \sum_0^\infty c_{N-\ell,m}\, t^m \tag{A-12}$$

Then the asymptotic expansion of $P_N(X)$ for large X is given by

$$P_N(X) \sim \sum_1^N \frac{N!\ [\Gamma(2-k)]^\ell}{\ell!\,(N-\ell)!}\ \frac{\sin \pi\ \ell(k-1)(-1)^\ell}{\pi}$$

$$\sum_0^\infty c_{N-\ell,m}\, \Gamma[\ell(k-1)+m+1]X^{-[\ell(k-1)+m+1]} \tag{A-13}$$

The sequence of exponents in this expansion requires investigation. For $2 < k < 3$, the first five terms are always k, k+1, 2k-1, k+2, 2k. The sixth and seventh terms are 3k-2 and k+3, in the indicated order for $2 < k < 5/2$ and in reverse order for $5/2 < k < 3$. The eighth term is 2k+1, and then the next four are 3k-1, 4k-3, k+4, 2k+2 for $2 < k < 7/3$; 3k-1, k+4, 4k-3, 2k+2 for $7/3 < k < 5/2$; k+4, 3k-1, 2k+2, 4k-3 for $5/2 < k < 3$. If the series is truncated with the sixth term, the error term will always be of order $X^{-5}$ or higher. Therefore, after some transformations, to terms of order less than $X^{-5}$, $P_N(X)$ is given by the asymptotic expansion

$$P_N(X) \sim \frac{N(k-1)}{X^k}\left[1+\frac{k\, c_{N-1,1}}{X}+\frac{k(k+1)c_{N-1,2}}{X^2}\right]$$

$$+\frac{N(N-1)}{2}\frac{[\Gamma(2-k)]^2}{\Gamma(2-2k)X^{2k-1}}\left[1+\frac{(2k-1)c_{N-2,1}}{X}\right] \tag{A-14}$$

$$-\frac{N(N-1)\ (N-2)}{6}\frac{[\Gamma(2-k)]^3}{\Gamma(3-3k)X^{3k-2}}+0(X^{-5})$$

The coefficients appearing here may be found immediately from Eq. (A-12). The factors in brackets may be regarded as expansions of inverse powers of appropriate arguments, and the entire expression may be brought into the form

$$P_N(X) \sim N(k-1)\left[X - \frac{(N-1)\ (k-1)}{k-2} + \frac{(N-1)\ (k^2-1)}{2(2-k)^2(3-k)X}\right]^{-k}$$
$$+ \frac{N(N-1)}{2}\ \frac{[\Gamma(2-k)]^2}{\Gamma(2-2k)}\left[X - \frac{(N-2)\ (k-1)}{k-2}\right]^{-(2k-1)} \qquad \text{(A-15)}$$
$$- \frac{N(N-1)\ (N-1)}{6}\ \frac{[\Gamma(2-k)]^3}{\Gamma(3-3k)}\ X^{-(3k-2)} + O(X^{-5})$$

The first term in the expansion, $N(k-1)X^{-k}$, shows characteristic behavior in that the tail of the distribution of $P_N$ is N times the tail of the distribution of p. The terms subtracted from X correspond to moving the tail of the distribution out to well beyond the average value of $P_N$, the average value being $N(k-1)/(k-2)$. It may be shown directly that Eq. (A-15) reduces to the correct expression for N = 2 and 3 as given in the text.

For $1 < k < 2$, the situation is not so simple. There will now be clusters of terms in the vicinity of each integer. With $k = 1+1/NS$ from Eq. (8), there will be N terms with exponents between $m+1+1/NS$ and $m+1+1/S$. While the lowest order term is still $Np(x)$, the error in the representation is very difficult to evaluate when S is at all large, and it has not been considered further.

The principal result of this appendix, Eq. (A-15), shows how the tail of the distribution behaves for sufficiently large N when p(x) is a Pareto distribution. In Appendix B, some distributions will be investigated which behave like $x^{-3/2}$ and $x^{-5/2}$, but which are such that $P_N(X)$ can be evaluated in closed form. These distributions display behavior quite similar to Eq. (A-15).

Appendix B

SOME EXPLICITLY REDUCIBLE PROBABILITY DISTRIBUTIONS

While the theory in the text of this Memorandum has dealt with the Pareto distribution, the general intractability of the analysis has led to a search for distributions which will have the general properties of the Pareto distribution, but will provide simpler results. Several such distributions have been found, and they will be considered in this appendix.

The first distributions are based on the Laplace transform*

$$\int_0^\infty dx\; e^{-px}\, I_\nu(ax)x^{-1} = \nu^{-1}\left[a/\left\{p+\sqrt{p^2-a^2}\right\}\right]^\nu \qquad \mathrm{Re}\; p > a \qquad \text{(B-1)}$$

where $I_\nu$ is the modified Bessel function. This leads to consideration of the probability distribution

$$p_1(x) = \nu\, I_\nu(ax)e^{-ax}/x \qquad X > 0 \qquad \text{(B-2)}$$

This is normalized, and for $\nu \geq 1$ it is finite for all positive x. For x large compared to $\nu^2/a$, the distribution has the form

$$p_1(x) \rightarrow \frac{\nu}{\sqrt{2\pi a}}\, x^{-3/2}\left[1-\frac{4\nu^2-1}{8\;ax}\right] \qquad x \gg \nu^2/a \qquad \text{(B-3)}$$

This distribution therefore behaves like a Pareto distribution with k = 3/2. It possesses neither mean nor variance. For $\nu > 1$, the distribution has a single maximum, which for large $\nu$ occurs in the vicinity of $x = \nu/a$.

*Reference 9, p. 195, pair 4.

The moment-generating function $E_1(\sigma)$ is given by writing $p = \sigma + a$ in Eq. (B-1) and becomes

$$E_1(\sigma) = \left[a/\left\{\sigma + a + \sqrt{\sigma^2 + 2\sigma a}\right\}\right]^{\nu} \tag{B-4}$$

The distribution function $P_N(X)$ is then given from Appendix A as

$$\begin{aligned} P_N(X) &= \frac{1}{2\pi i}\int d\sigma\; e^{\sigma X}\left[a/\left\{\sigma + a + \sqrt{\sigma^2 + 2\sigma a}\right\}\right]^{N\nu} \\ &= \frac{1}{2\pi i}\, e^{-aX}\int dp\; e^{pX}\left[a/\left\{p + \sqrt{p^2 - a^2}\right\}\right]^{N\nu} \end{aligned} \tag{B-5}$$

This is equivalent to the inverse of Eq. (B-1) with $\nu$ replaced by $N\nu$, and yields

$$P_N(X) = N\nu I_{N\nu}(aX)e^{-aX}/X \tag{B-6}$$

This is a distribution of the same type as Eq. (B-2). It also has a single maximum which occurs near $X = N\nu/a$. For X large compared to $N^2\nu^2/a$, the distribution has the form

$$P_N(X) \rightarrow \frac{N\nu}{\sqrt{2\pi a}}\, X^{-3/2}\left[1 - \frac{4N^2\nu^2 - 1}{8\,a\,X}\right] \qquad X \gg N^2\nu^2/a \tag{B-7}$$

The standard relation $P_N(X) \rightarrow N\, p_1(x)$ obeyed by this distribution, and Eq. (B-6), show that the position of the peak has moved out by a factor of N. The self-reproducing property of the distribution $p_1(x)$ is indeed much simpler than that of the Pareto distribution.

To obtain a distribution which behaves like $x^{-5/2}$ at infinity, replace p by q in Eq. (B-1), and integrate over q from p to infinity.

Changing the order of integration on the left and performing the q integration provides another factor $x^{-1}$ on the left side of Eq. (B-1). The right side may be integrated by the substitution q = a cosh z. This results in the transform pair

$$\int_0^\infty dx\, e^{-px}\, I_\nu(ax)x^{-2} = \left[\frac{a}{p+\sqrt{p^2-a^2}}\right]^\nu \cdot \frac{p+\nu\sqrt{p^2-a^2}}{\nu(\nu^2-1)} \qquad \text{(B-8)}$$

Thus, consider the probability distribution

$$p_2(x) = \frac{\nu(\nu^2-1)}{a}\, I_\nu(ax)e^{-ax}\, x^{-2} \qquad \text{(B-9)}$$

This is normalized, and has the mean value $\bar{x} = (\nu^2-1)/a$. When x is large compared to $\nu^2/a$, this distribution has the asymptotic form

$$p_2(x) \rightarrow \frac{\nu(\nu^2-1)}{\sqrt{2\pi}\, a^{3/2}}\, x^{-5/2}\left[1-\frac{4\nu^2-1}{8\ a\ x}\right] \qquad \text{(B-10)}$$

Thus, $p_2(x)$ has a mean, but no variance.

Using $p_2(x)$ as the input probability distribution, the distribution of the sum $P_N(X)$ is given by

$$P_N(X) = \frac{1}{a^N}\, e^{-aX}\, \frac{1}{2\pi i}\int dp e^{pX}\left[\frac{a}{p+\sqrt{p^2-a^2}}\right]^{N\nu}\left[p+\nu\sqrt{p^2-a^2}\right]^N \qquad \text{(B-11)}$$

This may be reduced by the algebraic relation

$$p+\nu\sqrt{p^2-a^2} = \frac{1}{2}\left[(\nu+1)\ (p+\sqrt{p^2-a^2}) - (\nu-1)\ \frac{a^2}{p+\sqrt{p^2-a^2}}\right] \qquad \text{(B-12)}$$

Expansion of the $N^{th}$ power of the right side of Eq. (B-12) yields

$$P_N(X) = \frac{e^{-ax}}{(2a)^N} \sum_{0}^{N} \frac{N!(-1)^{\ell}}{\ell!(N-\ell)!} (\nu-1)^{\ell} a^{2\ell}(\nu+1)^{N-\ell} \frac{1}{2\pi i}$$

$$\int dp e^{pX} \frac{a^{N\nu}}{\left[p + \sqrt{p^2-a^2}\right]} N(\nu-1) + 2\ell \qquad \text{(B-13)}$$

Using Eq. (B-1) to evaluate the integral yields

$$P_N(X) = \frac{e^{-aX}}{2^N} \sum_{0}^{N} \frac{N!(-1)^{\ell}}{\ell!(N-\ell)!} (\nu-1)^{\ell} (\nu+1)^{N-\ell} [N(\nu-1) + 2\ell] \frac{I_{N(\nu-1) + 2\ell}(aX)}{X} \qquad \text{(B-14)}$$

This is not a closed form, but it is a finite sum. The order of the Bessel functions within the sum lies between $N(\nu-1)$ and $N(\nu+1)$. When aX is large compared to $N^2(\nu+1)^2$, the Bessel functions may be expanded in an asymptotic representation. The terms of order $X^{-3/2}$ cancel on summation over $\ell$. The terms of order $X^{-5/2}$ and $X^{-7/2}$ may be summed after lengthy algebraic transformations, with the result

$$P_N(X) \rightarrow \frac{N\nu(\nu^2-1)}{\sqrt{2\pi}\, a^{3/2}\, X^{5/2}} \left[1 + \frac{20N(\nu^2-1)-(24\nu^2-21)}{8aX}\right] \; ; \; X \gg N^2\nu^2/a \qquad \text{(B-15)}$$

Again the characteristic behavior $P_N(X) \rightarrow Np(x)$ occurs for sufficiently large X, and the peak of the distribution moves out by a factor of N. However, there is an interesting contrast in the behavior of the second terms in the asymptotic representations Eqs. (B-7) and (B-15). In Eq. (B-7) the second term is negative, so the distribution lies below the asymptotic form Np(x). In Eq. (B-15) the second term is positive for N > 1, $\nu > 1$, so the distribution lies above the asymptotic form.

Another form of probability distribution which yields closed-form results is based on the pair*

$$\int_0^\infty dx\ e^{-px-\frac{a}{2x}}\ x^{-\frac{(n+1)}{2}}\ He_n\left(\left(\frac{a}{x}\right)^{1/2}\right) = \sqrt{\pi}\ 2^{\frac{n}{2}}\ p^{\frac{n-1}{2}}\ e^{-(2ap)^{1/2}} \qquad \text{(B-16)}$$

Here $He_n$ denotes the $n^{th}$-order Hermite polynomial, defined by

$$He_n(z) = (-1)^n\ e^{z^2/2}\left(\frac{d}{dz}\right)^n e^{-z^2/2} \qquad \text{(B-17)}$$

The special case n = 1 of Eq. (D-16) yields

$$\int_0^\infty dx\ e^{-px-\frac{a}{2x}}\ \frac{a^{1/2}}{\sqrt{2\pi}\ x^{3/2}} = e^{-(2ap)^{1/2}} \qquad \text{(B-18)}$$

and the derivative of this with respect to a yields

$$\int_0^\infty dx\ e^{-px-\frac{a}{2x}}\ \frac{a^{3/2}}{\sqrt{2\pi}\ x^{5/2}} = e^{-(2ap)^{1/2}}\left[1 + (2ap)^{1/2}\right] \qquad \text{(B-19)}$$

These distributions are appropriate for considering the sum of M noise samples and N-M signal samples, where the signals are distributed according to Eq. (B-18), and the noise according to Eq. (B-19), where the parameter a may be different for the two distributions. Note that the distribution corresponding to the coefficient of $e^{-px}$ in Eq. (B-18)

*Reference 9, p. 173, pair 18.

is normalized, has no mean, and has a maximum at $x = a/3$. The distribution of Eq. (B-19) is normalized, has a mean value a, no variance, and a maximum at $x = a/5$. The sum of M noise samples, distributed according to Eq. (B-19) with parameter a, and N-M signal samples, distributed according to Eq. (B-18) with parameter b, will have the distribution

$$P_{NM}(X) = \frac{1}{2\pi i} \int dp \; e^{pX-(2p)^{1/2}\left[Ma^{1/2}+(N-M)b^{1/2}\right]} \left[1 + (2ap)^{1/2}\right]^M \qquad \text{(B-20)}$$

Let the coefficient of $(2p)^{1/2}$ in the exponent be called $c^{1/2}$. The transformation

$$1 + (2ap)^{1/2} = (2aq)^{1/2} \qquad \text{(B-21)}$$

followed by simplification of the exponent yields

$$P_{NM}(X) = e^{\frac{X}{2a} + \left(\frac{c}{a}\right)^{1/2}} \frac{1}{2\pi i} \int dq \left(1 - \frac{1}{\sqrt{2aq}}\right)(2aq)^{M/2} \; e^{qX-(2q)^{1/2}\left(\frac{X}{\sqrt{a}} + c^{1/2}\right)} \qquad \text{(B-22)}$$

This is a combination of two expressions of the form Eq. (B-16). After further manipulation, $P_{NM}$ takes the form

$$P_{NM}(X) = \sqrt{\frac{\pi}{2}}\,\frac{1}{X}\, e^{-\frac{c}{2X}} \left(\frac{a}{X}\right)^{M/2} \left[He_{M+1}\left(\sqrt{\frac{X}{a}} + \sqrt{\frac{c}{X}}\right) - \sqrt{\frac{X}{a}}\, He_M\left(\sqrt{\frac{X}{a}} + \sqrt{\frac{c}{X}}\right)\right] \qquad \text{(B-23)}$$

This is a closed-form expression. For $X \gg \sqrt{ac},\ aM^2$, it may be expanded in inverse powers of X yielding

$$P_{NM}(X) \rightarrow \sqrt{\frac{\pi}{2}} \left[ \frac{(N-M)b^{1/2}}{X^{3/2}} e^{-(N-M)^2 \frac{b}{2X}} + \frac{Ma^{3/2}}{X^{5/2}} + \frac{3}{2} \frac{M(N-M)ab^{1/2}}{X^{5/2}} \right] \quad \text{(B-24)}$$

In this order of approximation, the first term in brackets is the probability distribution of the N-M signal samples, the second term is the distribution of the M noise samples, and the third term is an interaction term. Thus, for sufficiently large values of X, the signal term will dominate.

Appendix C

## A LOWER BOUND ON FALSE-ALARM PROBABILITY FOR THE AVERAGE LIKELIHOOD TEST

One of many methods attempted for approximating the false-alarm probability was a modification of the Chernoff bound. Although the tightest possible bound did not turn out to be useful for the Pareto distribution, it did yield the overall bound given by Eq. (C-5) below.

For any $\sigma \geq 0$ and all $z_n \geq 0$ $(n = 1,2,\ldots N)$,

$$\int \cdots \int_{\sum_{n=1}^{N} z_n \leq \lambda} p(z_1 \ldots z_n) \exp \left\{ - \sigma \sum_{n=1}^{N} z_n \right\} dz_1 \ldots dz_n$$

$$\geq e^{-\sigma\lambda} \int \cdots \int_{\sum_{n=1}^{N} z_n \leq \lambda} p(z_1 \ldots z_n) \, dz_1 \ldots dz_n \qquad \text{(C-1)}$$

When the $z_n$ are independent this reduces to

$$\left[ \int_1^{\lambda} p(z_1) \, e^{-\sigma z_1} \, dz_1 \right]^N \geq e^{-\sigma\lambda} \, \text{prob} \left\{ \sum_{n=1}^{N} z_n \leq \lambda \right\} \qquad \text{(C-2)}$$

Note that this bound is tighter than the Chernoff bound. For the Chernoff bound $\infty$ replaces $\lambda$ on the left side of this equation. In the case of interest here, the $z_n$ have the Pareto density of Eq. (7) and the integral on the left of Eq. (C-2) can be expressed in terms of the incomplete gamma function

$$g(\sigma) = \int_1^\lambda \frac{k-1}{z_1^k} e^{-\sigma z_1} dz_1 = (k-1)\,\sigma^{k-1}\,[\Gamma(1-k, \sigma) - \Gamma(1-k, \sigma\lambda)] \qquad \text{(C-3)}$$

Then, for any $\sigma \geq 0$

$$\text{Prob}\left\{\sum_{n=1}^{N} z_n \leq \lambda\right\} \leq \underset{\sigma\geq 0}{\text{Min}}\left[e^{\sigma\lambda}\, g^N(\sigma)\right] \qquad \text{(C-4)}$$

when $\sigma = 0$, this equation yields the following overall bound for $z_n \geq 0$

$$\text{Prob}\left\{\sum_{n=1}^{N} z_n \leq \lambda\right\} \leq \text{Prob}\left\{\underset{n}{\text{Max}}\; z_n \leq \lambda\right\} \qquad \text{(C-5)}$$

The right side of Eq. (C-4) was evaluated as a function of $\sigma$, and the minimum was found for a few representative cases. The bound was not found to be significantly closer than the bound of Eq. (C-5). It can be shown that for $\lambda > N\bar{z}_n$ the minimum of Eq. (C-4) occurs at $\sigma = 0$.

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