A CLASS OF GAMES WITH UNIQUE SOLUTIONS

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SUMMARY  
In a game with payoff $M(x,y) = \phi(xy) + \rho(x) + \tau(y)$ played over the unit square (such that $\rho, \tau$ are continuous and $\phi$ is analytic and with sufficiently many non-vanishing coefficients in its power series expansion about zero) if either player has a non-step function\(^1\)optimal strategy, the opposing player has a unique optimal strategy. Examples are included which illustrate the fact that games with well-behaved payoffs can have unique solutions\(^2\)which are more or less pathological.  

\(\S \)1. For any distribution $f$ (which we may consider as a measure) we may define the spectrum of $f$, $\sigma(f)$, as the complement of all open sets of $f$-measure zero. The set $\sigma(f)$ is a closed Borel set, since we may obtain $\sigma(f)$ by deleting the intervals of $f$-measure zero which have rational end points. If one is given a constant, $v$, strategies $f$ and $g$, and functions $\phi$, $H$, $K$, such that  

1. $\phi$ is continuous on the unit square  
2. $H(x) \leq v$, and $H(x) = v$ on $\sigma(f)$, $H$ continuous  
3. $K(y) \geq v$, and $K(y) = v$ on $\sigma(g)$, $K$ continuous  

then by setting  

\(4\) $M(x,y) = \phi(x,y) - \int \phi df(x) - \int \phi dg(y) + \int \int \phi df(x)dg(y) - v + H(x) + K(y)$,  

one obtains the payoff $M$ of a game which has value $v$, and $(f,g)$ as a solution.  

\(^1\) By a step function we mean a distribution based on a finite set of points.  
\(^2\) By a solution we mean a pair $(f,g)$ consisting of an optimal strategy $f$ for player I, $g$ for player II.
For
\[ \int \mathcal{M}df(x) = \int \phi df(x) - \int \phi df(x) - \int \phi d\mathcal{D}g + \int \phi d\mathcal{F}dg \]
\[ - v + v + K(y) = K(y) \geq v , \]
since \( H(x) = v \) on \( \sigma(f) \), and similarly
\[ \int \mathcal{M}dg(y) = H(x) \leq v. \]

The representation (4) of the payoff holds trivially in the case of any payoff \( \mathcal{M} \), if we select for \( (f,g) \) any solution of the game with payoff \( \mathcal{M} \), since we may then set \( v \) equal to the value and

\[ \phi = \mathcal{M} , \quad H(x) = \int \mathcal{M}dg(y) , \quad K(y) = \int \mathcal{M}df(x) , \]

and obtain (4) as a trivial identity. (4) has, however, some non-trivial consequences if we replace \( \phi(x,y) \) by a function of the product \( xy \).

**Theorem 1:** Let \( f \) and \( g \) be non-step functions, and let \( k, K, \) and \( v \) satisfy (2) and (3) (above). Let \( \phi \) be an analytic function such that
\[ \phi(t) = \sum_{j=0}^{\infty} a_j t^n j \quad \text{for} \quad |t| \leq r , \quad r > 1 , \]
\[ a_j + 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty . \]

Then the game with payoff \( \mathcal{M} \) defined by
\[ \mathcal{M}(x,y) = \phi(xy) - \int \phi(xy)df(x) - \int \phi(xy)dg(y) + \int \phi(xy)df(x)dg(y) \]
\[ - v + H(x) + K(y) \]
has \( (f,g) \) as its unique solution.

Proof: Because of the uniform convergence we have assumed for \( \phi \) we have
\[ \phi(xy) - \int \phi(xy) df(x) - \int \phi(xy) dg(y) + \iint \phi(xy) df(x) dg(y) \]

\[ = \sum a_j (x^n_j y^n_j - f_{n_j} x^n_j y^n_j - x^n_j g_{n_j} + f_{n_j} g_{n_j}) \]

\[ = \sum a_j (x^n_j - f_{n_j}) (y^n_j - g_{n_j}) \]

where \( f_n \) is the \( n \)-th moment of \( f \). Hence we may write

\[ M(x, y) = \sum a_j (x^n_j - f_{n_j}) (y^n_j - g_{n_j}) - v + H(x) + K(y). \]

Of course \((f, g)\) is a solution of the game, and we only have to show uniqueness. Let \( f' \) be an optimal strategy for player I. Then

\[ \int M f'(x) = \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) - v + \int H(x) df'(x) + K(y) \]

\[ = \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) - v + v + K(y) \geq v. \]

But \( K(y) = v \) on \( \sigma(g) \) so that

\[ \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) \geq 0 \quad \text{for } y \in \sigma(g). \]

Actually we must have equality on \( \sigma(g) \), since otherwise there exists a \( y_0 \in \sigma(g) \) such that

\[ \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) > 0, \]

and hence an interval containing \( y_0 \) in which this is true;

however since \( \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) \) is non-negative on \( \sigma(g) \), from

\[ \int \sum a_j (f'_n - f_{n_j}) (y^n_j - g_{n_j}) dg(y) = \sum a_j (f'_n - f_{n_j}) (g_{n_j} - g_{n_j}) = 0 \]
we conclude that this interval is of $g$-measure zero, hence that
$y_0 \notin \sigma(g)$ which is a contradiction. Thus

$$\sum a_j (f'_n_{nj} - f_n_{nj})(y_n^{nj} - g_n^{nj}) = 0 \text{ on } \sigma(g)$$

and since $\sigma(g)$ is not a finite set of points the analytic function
on the left is identically zero, whence

$$f'_n_{nj} = f_n_{nj} \quad j = 1, 2, \ldots$$

However, this implies $f' = f$, as is shown in [1] say, since

$$\sum \frac{1}{n_j} = \infty.$$  A similar argument suffices to show $g$ is unique.

As is evident from the above proof Theorem 1 may be stated in the
following one-sided form:

Corollary 1: Let $M, \phi, H, K, v$, satisfy the requirements of
Theorem 1. Then if either player has a non-step function optimal
strategy, his opponent has a unique optimal strategy.

§2. Theorem 1 may be simplified to

Theorem 2: Let

$$M(x, y) = \phi(xy) + \rho(x) + \tau(y)$$

(where $\rho$ and $\tau$ are continuous on $[0, 1]$ and

$$\phi(t) = \sum_{j=0}^{\infty} a_j t^{n_j} \quad \text{for } |t| \leq r, r > 1$$

and $a_j \neq 0$, $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$) be the payoff of a game

in which each player has a non-step function optimal strategy. Then
the optimal strategies are unique.

Proof: Let $f$ and $g$ be the non-step function strategies for
I and II. Then
\[ K(y) = \int Mdf = \int \phi(xy) df(x) + \int \rho(x) dt(x) + \tau(y) \geq v \]
\[ H(x) = \int Mdg = \int \phi(xy) dg(y) + \rho(x) + \int \tau(y) dg(y) \leq v \]
\[ v = \int H(x) df(x) = \int \phi(xy) df + \int \rho(x) df + \int \sigma(y) dg \]

and \( H, K \) and \( v \) obviously satisfy (2) and (3). Moreover writing
\[ M(x, y) = (M(x, y) - H(x) - K(y) + v) - v + H(x) + K(y) \]
and replacing the terms in the parentheses we obtain
\[ M(x, y) = \phi(xy) + \rho(x) + \tau(y) - \int \phi(xy) df(x) - \int \rho(x) df(x) - \tau(y) \]
\[ - \int \phi(xy) dg(y) - \rho(x) - \int \tau(y) dg(y) \]
\[ - \int \int \phi(xy) df(x) dg(y) - \int \rho(x) df(x) - \int \tau(y) dg(y) \]
\[ - v + H(x) + K(y) \]
or
\[ M(x, y) = \phi(xy) - \int \phi(xy) df(x) - \int \phi(xy) dg(y) \]
\[ + \int \int \phi(xy) df(x) dg(y) - v + H(x) + K(y) \]
so that Theorem 1 immediately applies.

Theorem 2 may also be put in a one-sided form

**Corollary 2:** Let \( M(x, y) = \phi(xy) + \rho(x) + \sigma(y) \), \( \rho, \sigma \)

continuous, \( \phi(t) = \sum_{j=1}^{\infty} a_j t^j \), for \( |t| \leq r, r > 1 \) and \( a_j \neq 0 \),
\[ \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty \]. If either player in the game with payoff \( M \) has a non-step function optimal strategy then his opponent has a unique optimal strategy.

3. Examples

The first example is a game with a rational payoff function with a unique solution consisting of distributions with countable spectra. Set
\[ \phi(xy) = \frac{2}{x-xy} - \frac{2}{4-xy} \]

and

\[ f(x) = g(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, I_{2^{-n}}(x) \]

Then

\[ \int \phi(xy) \, df(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left( \frac{2}{2-2^{-n}y} - \frac{2}{4-2^{-n}y} \right) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1} - y} - \frac{1}{2^{n+2} - y} \right) = \frac{1}{2-y} \]

and by symmetry

\[ \int \phi(xy) \, dg(y) = \frac{1}{2-x} . \]

Setting \( H(x) \equiv v \equiv K(y) \), and forming the function \( M \) given by (4) (omitting constants) we obtain

\[ M(x,y) = \frac{2}{x-xy} - \frac{2}{4-xy} - \frac{1}{2-y} - \frac{1}{2-x} \]

as the payoff of a game having \((f,g)\) as a solution (the strategy \( f = g \) is not a step function in our terminology!), and since \( M \) is of the form required by Theorem 2, the solution is indeed unique.

Our second example is

\[ M(x,y) = e^{xy} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, e^{x} \cos^{2n} - \sum_{n=0}^{\infty} \frac{1}{n!} \, e^{y} \sin^{2n-1} \]

which is formed from

\[ \phi(xy) = e^{xy} \]

\[ H(x) \equiv v \equiv K(y) \]

\[ f(x) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{1}{n!} \, I_{\sin^{2n}}(x) \]

\[ g(y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, I_{\cos^{2n}}(y) \]
(again omitting constants). The strategies $f$ and $g$ have jumps at a dense set of points in $[0,1]$, and are the unique strategies by Theorem 2. We note that the payoff in this example is the sum of two payoffs which have saddle points

$$e^{xy} \text{ and } -\frac{1}{e} \sum \frac{1}{n!} e^y \sin^2 n - \sum \frac{1}{2^{n+1}} e^x \cos^2 n$$

Reference

[1]. I. Glicksberg and O. Gross, A Class of Games with Unique Density Function Solutions, RM-501