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RESEARCH MEMORANDUM

ON THE OPTIMAL USE OF GUIDED MISSILES—II:

DUMMY MISSILES

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SUMMARY

This is the second of a series of papers devoted to mathematical problems arising in the use of guided missiles. As in the first paper of the series, RM-1741, we study the optimal allocation of missiles against a given target system, now assuming that dummy missiles are allowable.

CONTENTS

ON THE OPTIMAL USE OF GUIDED MISSILES—II: DUMMY MISSILES

1. INTRODUCTION

In the first paper of this series [1], we showed that the problem of allocating missiles to attack a given target system can be treated by the methods of dynamic programming, under certain assumptions concerning the nature of the model.

In the present paper we wish to consider the same problem, first under the assumption that dummy missiles are to be used,* and secondly under the assumption that the amount of fissile material available is limited.

Our purpose is to present a computational technique that can treat in a routine fashion target systems with hundreds of targets, some independent and some in target clusters, under realistic assumptions concerning probability of survival of the individual missile and the type of defense employed.

2. GENERAL DESCRIPTION OF THE MODEL

Let us begin with a set of N independent target systems, T_1, T_2, ..., T_N, which may themselves be single or multiple targets. Our basic assumption is that we know, to some degree of accuracy, the probability $P_i(y, z)$ that the i^{th} target will be destroyed by a combination of y live missiles and z dummy

*A discussion of the value of dummy missiles was given in [2]. At that time, since we were not able to treat large-scale systems, we illustrated the idea by means of some small-scale calculations.

missiles, when an optimal salvo policy is employed. The problem of determining the optimal salvo policy will be considered below.

We shall begin with the case where there is no constraint upon the quantity of fissile material available.

Assume that we have a quantity x of money, which is to be divided into N parts, $x = \sum_{i=1}^{N} x_i$, with the quantity x_i to be used for the construction of live and dummy missiles directed against the i^{th} target. In more general situations some of this money may be required for the construction of bases and launching platforms, for upkeep, and so on.

The problem is to determine the allocation of resources that maximizes the expected damage done to the target system, given the values v_i of the individual targets. The method we employ may also be utilized to treat the same process where the criterion is that of maximizing the probability of obtaining a damage of at least D_0.

3. ANALYTIC FORMULATION

Let us write

$$(1) \qquad x = \sum_{i=1}^{N} x_i ,$$

where x_i is the amount of money allocated to the destruction of the i^{th} target. In turn, x_i is divided into two quantities, y_i and z_i, where y_i is the amount of money used to construct live missiles, and z_i is the amount of money used to construct dummy missiles.

These quantities, y_i and z_i, are to be determined so as to maximize the probability of destroying the i^{th} target, employing an optimal salvo policy.

Let a be the cost of a live missile, and b the cost of a dummy missile. Then the probability of destroying the i^{th} target is

$$(2) \qquad P_i = P_i(y_i/a,\ z_i/b).$$

Define

$$(3) \qquad q_i(x_i) = \max_{\substack{y_i+z_i=x_i \\ y_i,\ z_i \geq 0}} P_i .$$

The total expected damage is then

$$(4) \qquad D_N(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} v_i q_i(x_i).$$

We now wish to maximize D_N subject to the restrictions $x_i \geq 0$, $x = \sum_{i=1}^{N} x_i$.

4. FUNCTIONAL EQUATIONS

Let us define

$$(1) \qquad f_N(x) = \max_{\{x_i\}} D_N(x_1, x_2, \ldots, x_N),$$

where the x_i are subject to the above restrictions.

Then

$$2) \qquad f_1(x) = v_1 q_1(x),$$

and

(3) $$f_{k+1}(x) = \max_{0 \leq x_{k+1} \leq x} \left[v_{k+1} q_{k+1}(x_{k+1}) + f_k(x - x_{k+1}) \right],$$

for $k = 1, 2, \ldots, N - 1$.

5. DISCUSSION

Let us now consider the computational aspects of a solution of the above form.

Consider a target system of one hundred distinct targets, with an allocation of one billion dollars. Take the unit of dollar allocation to be one million dollars, a "megabuck." Then x ranges between 0 and 1000 in units of 1, for each $f_k(x)$. We thus have to compute one hundred functions of x, the sequence $f_k(x)$, using (4.3), over the range 0–1000. This computation requires very little time on a modern digital computer, and the machine automatically determines the maximizing x_{k+1} at the same time.

As we have pointed out before, an advantage of this approach is that we automatically perform a sensitivity analysis over the range of the number of targets and the quantity of money allocated.

The problem has now been reduced to determining the functions $q_i(x_i)$.*

*Once a table of $q_i(x_i)$ has been determined, experience indicates that the above problem could be calculated at the rate of about one target every 2 minutes. The continuity of the policy x_{k+1} as a function of x makes the maximization particularly simple.

6. SALVO POLICY

Let us now consider the problem of determining optimal firing policy, given an initial stockpile of y live and z dummy missiles. We assume that we know

(1) $Q(y_1, z_1)$ = the probability of destroying the target on a salvo of y_1 live and z_1 dummy missiles

We shall discuss this point again below.

The functional equation determining P(y, z) is

$$(2) \quad P(y, z) = \max_{\substack{0 \le y_1 \le y \\ 0 \le z_1 \le z}} \left[Q(y_1, z_1) + (1 - Q(y_1, z_1))P(y - y_1, z - z_1) \right].$$

If, as is sometimes done, we take the probability of survival of any one of $y_1 + z_1$ missiles to be given by

$$(3) \quad P = e^{-a_i/(y_1+z_1)},$$

where a_i is a constant characteristic of the i^{th} target, we see that

$$(4) \quad Q(y_1, z_1) = 1 - (1 - e^{-a_i/(y_1+z_1)})^{y_1} .$$

In this case, it is not difficult to show that the optimal policy is to fire all the missiles at once; cf. [1].

In general, considering a defense consisting of many zones, with different types of detection and interception, it may not be true that optimal salvo policies possess this same property. In any case, given the nature of the defense, optimal policy may be determined from (2).

7. RESTRICTIONS ON FISSILE MATERIAL

Let us now consider the same allocation process under the assumption that there is a restriction on the quantity of fissile material available. Let w be the total amount of fissile material available, and let w_i be the quantity allotted to the destruction of the i^{th} target. The restriction is now

(1) $$\sum_{i=1}^{N} w_i \leq w, \qquad w_i \geq 0,$$

since it is not clear, a priori, that all the available fissile material will be employed.

Assume that each missile carries a quantity c of fissile material.* Then w_i/c live missiles will be required to utilize this quantity of material. Consequently, the initial stockpile will consist of w_i/c live missiles and $(x_i - aw_i/c)/b$ dummy missiles. It follows that a first constraint we must impose is

(2) $$x_i \geq aw_i/c, \qquad i = 1, 2, \ldots, N.$$

The probability of destroying the target is now

(3) $$P_i(w_i/c, (x_i - aw_i/c)/b),$$

and the total expected damage is

(4) $$D_N(x_i, w_i) = \sum_{i=1}^{N} v_i P_i(w_i/c, (x_i - aw_i/c)/b).$$

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*This suggests the important problem of determining the optimal size, or sizes, of payload. We shall not consider this here, although it may be treated by the methods sketched above.

We wish to maximize this function subject to the inequalities

$$(5)\qquad \text{(a)}\quad \sum_{i=1}^{N} x_i = x, \qquad x_i \geq 0,$$

$$\text{(b)}\quad \sum_{i=1}^{N} w_i \leq w, \qquad w_i \geq 0,$$

$$\text{(c)}\quad a w_i \leq c x_i, \qquad i = 1, 2, \ldots, N.$$

Set

$$(6)\qquad f_1(x, w) = v_1 \max_{0 \leq w_1 \leq \min [w, cx/a]} P_1(w_1/c, (x_3 - aw_1/c)/b),$$

and define, generally,

$$(7)\qquad f_N(x, w) = \max D_N(x_i, w_i),$$

over the region described by (5). Then, for $N \geq 1$,

$$(8)\qquad f_{N+1}(x, w) = \max_R \Big[v_{N+1} P_{N+1}\Big(\frac{w_{N+1}}{c}, \frac{x_{N+1}}{b} - \frac{aw_{N+1}}{bc} \Big) + f_N(x - x_{N+1}, w - w_{N+1}) \Big],$$

where the maximization is over the region R described by

$$(9)\qquad \text{(a)}\quad 0 \leq x_{N+1} \leq x,$$

$$\text{(b)}\quad 0 < w_{N+1} < \min \Big(w, \frac{cx_{N+1}}{a}\Big).$$

In performing the calculation certain properties of the function can be exploited. Once the policy (x_N, w_N) that maximizes $f_N(x, w)$ is determined for a particular x and w, it follows that the maximizing policy for neighboring points will neighbor the policy (x_N, w_N). Consequently only a small portion of the 2-dimensional policy region need be searched. The ranges of the variables x and w are necessarily limited by the need for a grid in 2 dimensions, so a coarser mesh should be constructed. With these simplifications the calculation using a 1000-point grid should take about 10 minutes per target on a high-speed digital computer.

Although this computation is more complicated than the one described in Sec. 4, it can readily be performed on existing computers.

REFERENCES

1. Bellman, R., Notes on the Theory of Guided Missiles—I, Allocation of Missiles, The RAND Corporation, Research Memorandum RM-1741, July 2, 1956. [redacted]L.

2. Bellman, R., On the Optimal Use of Guided Missiles, The RAND Corporation, Research Memorandum RM-812, April 16, 1952. C[redacted].