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RESEARCH MEMORANDUM

ON THE COMPUTATIONAL SOLUTION OF DYNAMIC-PROGRAMMING PROCESSES--XIV: MISSILE-ALLOCATION PROBLEMS

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SUMMARY

The purpose of this research memorandum is to show the applicability of the techniques of dynamic programming to the treatment of problems connected with the allocation of attack to the destruction of an enemy target system. It is shown by means of some calculations that large-scale problems involving thousands of attackers and hundreds of targets can be resolved computationally in a reasonable period of time with the use of such computers as the RAND Johnniac or the IBM 704.

In case there are two types of attackers--either manned aircraft and decoys, or missiles of different capabilities, say--it is shown that an application of the Lagrange-multiplier technique reduces the computational solution to one involving sequences of functions of one variable. Examples of this type of calculation are given. The method can be extended to cover more general situations.

By means of these calculations, it is shown that certain analytic approximations for optimal policies furnish excellent results.

In the Appendix, a result is presented that greatly speeds the computational solution.

CONTENTS

ON THE COMPUTATIONAL SOLUTION OF DYNAMIC-PROGRAMMING PROCESSES--XIV: MISSILE-ALLOCATION PROBLEMS

1. INTRODUCTION

In this research memorandum we consider some problems involving the optimal allocation of attack against a target system and the optimal allocation of defense against this attack. As is usual, certain simplifying assumptions must be introduced. We shall see, however, that dynamic-programming techniques can produce numerical solutions to problems that five years ago were of intolerable size and complexity. These results are then used to complement mathematical analysis; and this analysis is of use, in turn, in explaining the general analytic structure of the numerical solution.

In the Appendix we present a result that plays an essential role in reducing the computing time required to derive the numerical solution.

2. THE PROBLEM

Consider a target complex consisting of N independently defended targets. In many cases, targets jointly defended can be lumped together and considered to be one target for the sake of initial study. For each target, we assume known a curve giving expected residual value as a function of the attack strength. A knowledge of this curve implies estimation of the following factors:

1. Value of the target.
2. Defense level.
3. "Hardness" of defense.
4. Defense strategy.

[redacted]

For the present study we have assumed that E_M, the residual value of the M-th target after attack, is given by a function of the form

$$(2.1) \qquad E_M(y_M) = V_M(1 - \alpha e^{-x_M/y_M})^{y_M}$$

where

V_M is the value of the M-th target,

x_M is its defense level,

y_M is the attack strength,

and

α is the probability that an attacker surviving the defense destroys the target.

See Fig. 1.

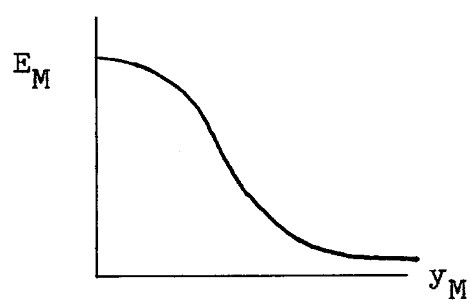

Fig. 1. The residual value of the M-th target as a function of the attack strength y_M.

An attack may be made either by vehicles of a single type, say manned aircraft or ICBM, or by vehicles of several types simultaneously. An attack by manned aircraft and decoys, for example, can be treated by taking E_M to be a function of two variables, the number of attacking aircraft and the number of decoys.

If several kinds of attacking vehicles are involved, E_M will be a function of several variables. The only such model that we shall investigate here involves an attack on the M-th target by y_M attackers and z_M decoys. For previous discussion of these concepts, see [1] and [2]. We shall assume in this case that

$$(2.2) \qquad E_M(y_M, z_M) = V_M\left[1 - \alpha e^{-x_M/(y_M+z_M)}\right]^{y_M}.$$

We shall consider the problem of allocating a quantity X of attacking vehicles against the N targets in such a way as to maximize expected damage, or, equivalently, to minimize the sum of the expected residual values. This will lead us to the problem of optimal allocation of money to various kinds of attackers that, in turn, are allocated to targets. Finally, we shall consider the problem of assigning defense against an expected optimal attack.

It should be stressed that this study was undertaken primarily to illustrate techniques, and secondly to determine the structure of policies. The numerical results obtained here have little actual meaning because of the many hypothetical inputs of the model, in the form of survival functions, parameters, and so on. However, for given functions of any validity, the methods we develop here can be used to determine classes of good policies.

3. MATHEMATICAL FORMULATION

The conventional formulation of this problem involves the minimization of a function of N variables subject to restrictions on the sums of the variables. Since we are dealing with missiles or airplanes, we shall restrict our attention to integral values of the variables.

In mathematical form, we wish to determine the quantity

$$\min_{y_i} \sum_{i=1}^{N} E_i(y_i), \tag{3.1}$$

subject to the constraints

$$\sum_{i=1}^{N} y_i = Y, \tag{3.2}$$

where the y_i are integers, $i = 1,2,\ldots,N$.

If attackers and decoys or two kinds of attackers are involved, we wish to evaluate

$$\min_{\{y_i, z_i\}} \sum_{i=1}^{N} E_i(y_i, z_i), \tag{3.3}$$

subject to the constraints

$$\sum_{i=1}^{N} y_i = Y, \tag{3.4}$$

$$\sum_{i=1}^{N} z_i = Z,$$

where the y_i, z_i are integers, $i = 1,2,\ldots,N$. Thus we have a function of $2N$ variables subject to 2 constraints.

4. DYNAMIC-PROGRAMMING FORMULATION

Since conventional mathematical techniques and current computers seem to be inadequate for the solution of such a problem, we adopt the dynamic-programming methodology. By this means, we reduce the N-dimensional problem to a sequence of one-dimensional problems. Observe that the total residual value of the targets after an optimal attack depends on only one quantity, the number of attackers in the offense strike. Define

$$f_M(y) = \text{residual value of first } M \text{ targets after an optimal attack by } y \text{ attackers,}$$

where the targets have been numbered in any convenient order. By the "principle of optimality" (cf. [3]), we have

$$(4.1) \qquad f_M(y) = \min_{0 \le y_M \le y} \left[E_M(y_M) + f_{M-1}(y - y_M) \right], \qquad M = 2,3,\ldots,N,$$

and

$$(4.2) \qquad f_1(y) = \min_{0 \le y_1 \le y} E_1(y_1) = E_1(y)$$

if $E_1(y)$ is a monotone decreasing function.

To solve the problem, we compute $f_1(y)$ using (4.2) for all values of y_1 from 0 to y, then use (4.1) to compute recurrently $f_2(y),\ldots,f_N(y)$.

An entirely analogous argument leads to the following recurrence relationship for the attacker-decoy problem:

$$(4.3) \qquad f_M(y,z) = \min_{\substack{0\leq y_M\leq y \\ 0\leq z_M\leq z}} \left[E_M(y_M,z_M) + f_{M-1}(y - y_M, z - z_M)\right].$$

In this case we can solve a 2N-dimensional problem by means of a sequence of N functions of two variables. However, even functions of only two variables present certain difficulties from the computational viewpoint. There is a severe restriction on the range of the variables because of space limitations within the computer. If $0 \leq y \leq 200$ and $0 \leq z \leq 600$ (reasonable sizes for attacking air forces), a table of 120,000 numbers would be necessary to represent $f_M(y,z)$ and much time would be required to compute 120,000 values at each iteration. In the next section, we discuss a technique for further reducing the dimension of the problem.

5. THE LAGRANGE-MULTIPLIER TECHNIQUE

The formulation given in the preceding section is simple and straightforward; however, as we have mentioned above, computer memory facilities are easily exhausted by functions of two variables, and, in addition, a great deal of time is required to carry out the calculations. Consequently, it is of importance to develop alternative techniques that permit a reduction in dimension.

A combination of the classical Lagrange-multiplier technique with the functional-equation technique of dynamic programming permits us to reduce the problem we are considering to a sequence of problems involving functions of <u>one</u> variable.

In place of the problem considered above, where there is a constraint on the number of attackers of the second type that are available, let us consider the situation in which there is no constraint on the quantity available. However, to compensate for this, we assume that there is a cost attached to the use of an attacker of the second type, a quantity λ.

It is reasonable to suppose that if this unit cost is chosen correctly, the number of attackers of the second type that are used in an optimal strategic campaign will equal the quantity originally available; see [4].

We then formulate the problem of minimizing the sum

$$\sum_{i=1}^{N} \left[E_i(y_i, z_i) + \lambda z_i \right], \tag{5.1}$$

subject to the constraints

$$\text{(5.2)(a)} \qquad \sum_{i=1}^{N} y_i = Y,$$

$$\text{(b)} \qquad z_i \geq 0,$$

where the y_i and z_i are integers, $i = 1, 2, \ldots, N$.

For fixed λ, the minimum is a function of y, $f_M(y)$, determined recursively by the relationships

$$f_M(y) = \min_{\substack{0 \leq y_M \leq y \\ 0 \leq z_M}} \left[E_M(y_M, z_M) + \lambda z_M + f_{M-1}(y - y_M) \right]. \tag{5.3}$$

The original problem is solved by fixing λ, solving a one-dimensional problem, examining the resulting $\Sigma\, y_i$, adjusting λ to make $\Sigma\, y_i$ approximately equal to Y, and re-solving the problem for the new λ. We repeat this cycle until the λ yielding $\Sigma\, y_i = Y$ is found. Usually three or four iterations suffice, depending on the effort expended in determining the new λ at each iteration. There is generally an optimal procedure for this λ search, but a one-dimensional dynamic-programming calculation is sufficiently fast to make such sophistication unnecessary. By the use of several Lagrange multipliers, problems of even higher dimension can be solved. The λ search will then, as is to be expected, consume more time. However, the Lagrange-multiplier technique does serve to reduce presently unsolvable problems of high dimension to the range of computability.

One 2-dimensional calculation would yield the returns and optimal policies for all combinations of y and z less than or equal to their upper bounds. In the course of a solution obtained in this way we calculate the values of a function of two variables over a <u>region</u> of the (y,z) plane. Using the Lagrange-mulitplier approach, we calculate coordinates for points of a space curve giving the returns and policies over a curve in the (y,z) plane for each particular λ. Several values of λ result in several such curves in space; from this, one can deduce the general form of the complete function of two variables. This technique will be illustrated in the following section.

6. COMPUTATIONAL ASPECTS

Consider first the process for an attack by vehicles of a single type, as discussed in Sec. 2. This is the simplest type of dynamic-programming process. The coding of such a problem, for a high-speed digital computer, can be accomplished in a couple of days. Figure 2 shows the flow chart for the solution of this problem. To allocate 100 attackers to 20 targets in an optimal fashion would require about 10 minutes of Johnniac computing time.

We now consider, in greater detail, the two-dimensional model, as reduced to one dimension by use of a Lagrange multiplier. For fixed λ, the functional equation is

$$f_M(y) = \min_{\substack{0 \leq y_M \leq y \\ 0 \leq z_M}} \left[E_M(y_M, z_M) + \lambda z_M + f_{M-1}(y - y_M) \right]. \tag{6.1}$$

Note that $f_{M-1}(y - y_M)$ does not depend on z_M, so that we can minimize $E_M(y_M, z_M) + \lambda z_M$ over all z_M before performing the minimization over the y's. Let $\phi_M(y_M)$ denote

$$\min_{0 \leq z_M} \left[E_M(y_M, z_M) + \lambda z_M \right]. \tag{6.2}$$

We now have a purely one-dimensional process,

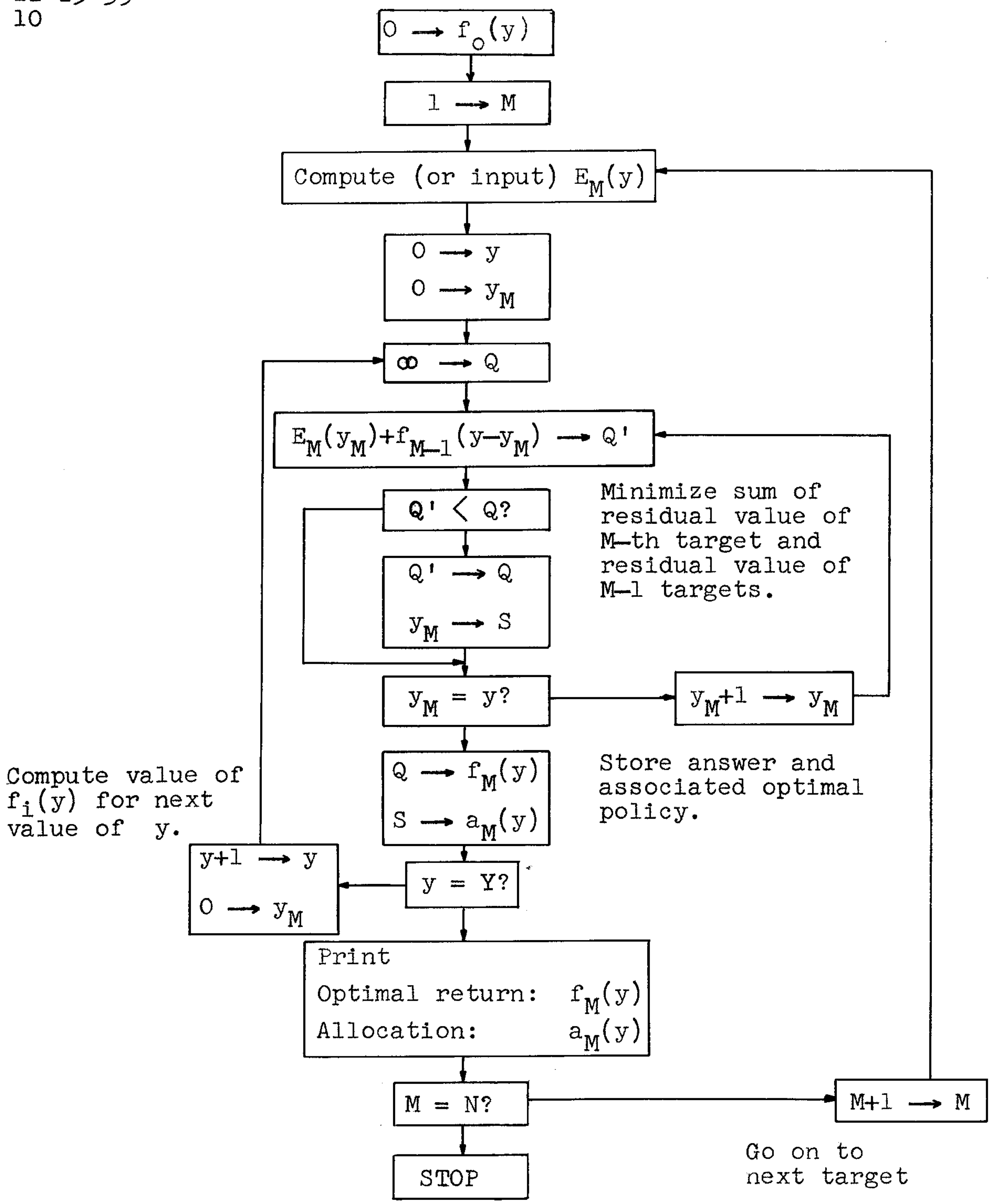

Fig. 2. Flow chart for solution of problem involving an attack by vehicles of a single type.

$$(6.3) \qquad f_M(y) = \min_{0 \leq y_M \leq y} \left[\phi_M(y_M) + f_{M-1}(y - y_M) \right],$$

and we use the technique diagrammed in Fig. 2 to solve.

We further note that if $\bar{z}_M > 0$ minimizes $E_M + \lambda z_M$ for $y = \bar{y}_M$, then, for $y > \bar{y}_M$, the minimizing z_M will be less than $\bar{z}_M$. An elaborate investigation of this essential point is found in the Appendix; the device greatly reduces the search necessary to determine the values of $\phi_M(y_M)$.

7. ANALYSIS OF RESULTS

For the one-dimensional model, the numerical solution consists of a set of N tables, each one showing the optimal initial decision for a process involving M targets ($M = 1,2,\ldots,N$) for all possible initial resources $(0 \leq y \leq Y)$. To interpret these results we first examine the last output, $f_N(y)$, and determine y_N, the optimal allocation to the N-th target for initial supply Y. We then examine the next-to-last output to find our optimal allocation to target $N - 1$ for initial supply $Y - y_N$, etc. If the tables are punched, instead of printed, this can be done automatically by the computer. Since the tables must be re-examined in the opposite order from that in which they are produced, a technique has been developed whereby the cards are read back into the computer upside down. If the tables are stored on tape, then this device is unnecessary.

Table 1 shows the optimal allocation of attackers to 20 targets numbered 1 through 20, where target values and defense levels are assumed equal to the target number.

Table 1

OPTIMAL ALLOCATION OF 100
WEAPONS AGAINST 20 TARGETS

Target value and defense	1 thru 9	10	11	12	13	14	15	16	17	18	19	20
Weapon allocation	0	7	7	8	8	9	9	10	10	10	11	11

In Fig. 3 we see that highly nonlinear individual return curves still lead to a near-linear optimal-damage curve.

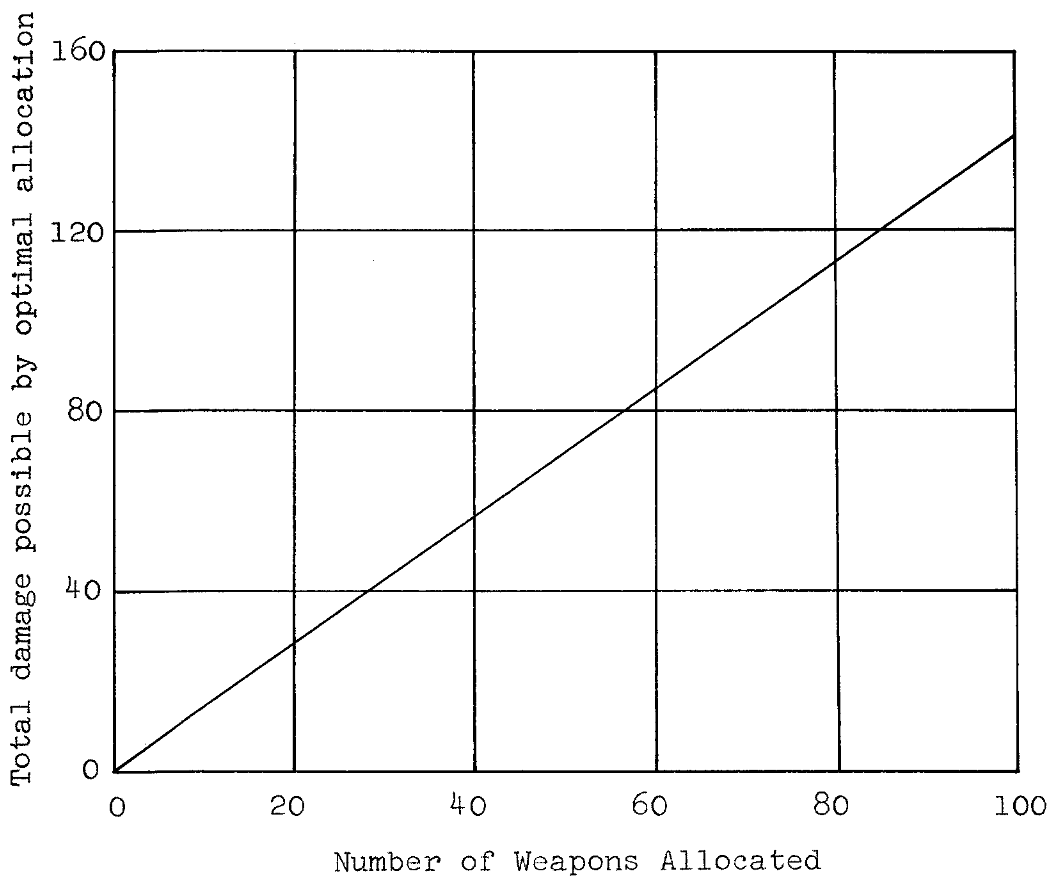

Fig. 3. Total damage as function of weapons allocated under optimal policy.

It is important to note that, no matter how pathological the individual functions may be, the optimal policy has the effect of taking the best from each curve and yielding a smooth return curve. This remark is of value if the sequence of functions f_M is to be approximated by a curve instead of tabulated as in this example.

Now let us consider the solution of the attacker-decoy model by means of a Lagrange multiplier. For a fixed value of λ we compute recursively a function giving, for each y between 0 and Y, a number of decoys and an expected damage. The number z is the optimal number of decoys to be purchased at price λ. Consequently, for each fixed λ we get a curve, usually discontinuous and of the type shown in Fig. 4, in the (y,z) plane, and a space curve $f(y,z)$ giving expected damage for arguments along this curve in the (y,z) plane.

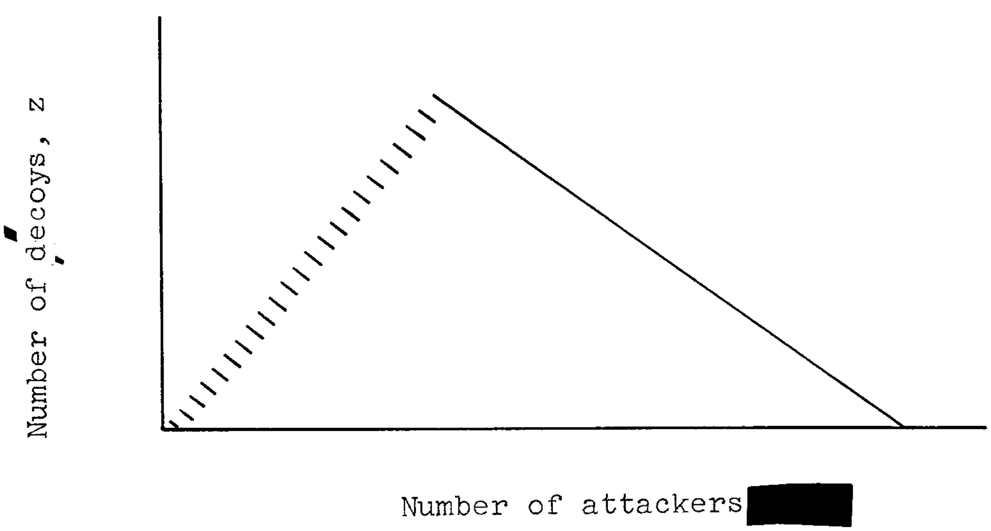

Fig. 4. Number of decoys if optimal policy is used, for fixed λ.

After obtaining several such space curves, using various values of λ, we can draw the contours of the two-dimensional return function (see Fig. 5).

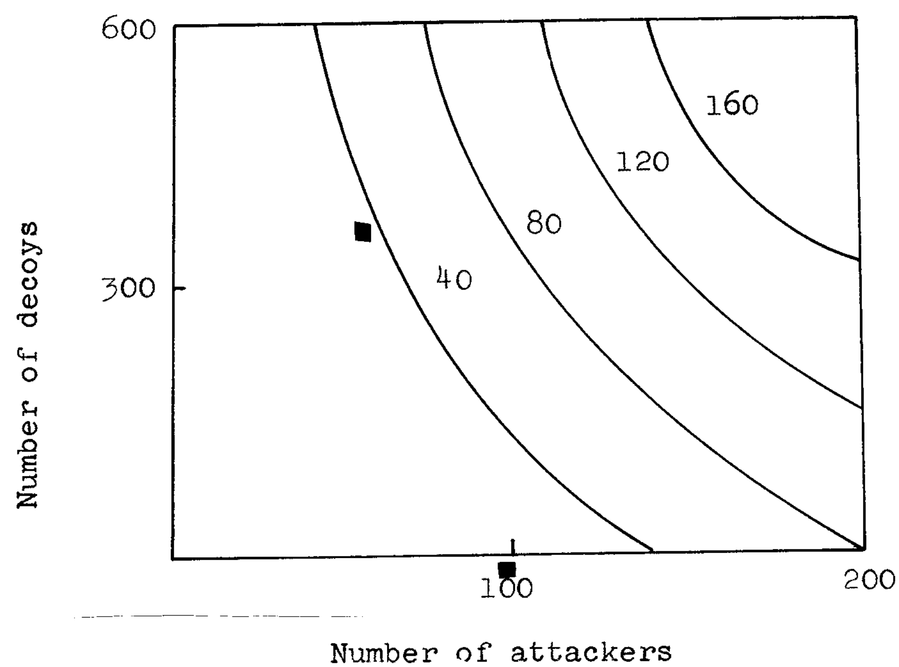

Fig. 5. Total damage obtained by optimal allocation of various combinations of attackers and decoys.

We have analyzed a two-dimensional process. By the straightforward two-dimensional dynamic-programming approach, such an analysis would involve function tables of $200 \times 600 = 120{,}000$ values and hundreds of hours of computing time. By the Lagrange-multiplier approach, equivalent results were obtained, using 1000 memory cells, in about three hours.

When we have solved this problem, we also have the answer to the more basic problem of allocation of funds to various types of weapons. Let attackers cost \$a each, and decoys \$b each. If \$c are to be allocated to the attack, then we examine Fig. 5 along the line $ay + bz = c$. The point at which the expected damage function is maximum yields the correct combination of attackers and decoys, and the strategy that produces the value is the optimal assignment of the weapon.

A logical continuation of this study leads one to a consideration of defense assignment. Assuming defense is additive and $\sum_{i=1}^{N} x_i = D$, can the x's be chosen in a better manner than proportional to the target value? This question was investigated computationally by reassigning defense after the attack to be the average of (a) a defense proportional to the attack and (b) the previous defense. In this way an iterative scheme was determined, which it was hoped would converge to the optimal defense. It was found that for fixed, and known, attack strength a defense could be found that was better than the proportional defense. In the following section we analyze this problem from a mathematical viewpoint, using our computational experience as a guide.

8. MATHEMATICAL ANALYSIS OF RELATED PROBLEMS

We shall now discuss what can be done analytically toward solving several types of allocation problems. These are usually difficult when the solutions are on the boundary of the region over which the maximization is carried out, and also when the allocation is restricted to partitions into integers.

In some cases in which the optimal allocation is an interior solution, and can be found, this analysis will serve as a comparison and may help to further the understanding of the numerical solutions of the more difficult cases.

In all these problems we assume the expected residual value of a target as a function of the attack strength y to be of the sort indicated in Fig. 6; cf. Fig. 1. That is, we assume that $dE(y)/dy \leq 0$, $dE/dy = 0$ at $y = 0$, and $dE/dy \to 0$ as $y \to \infty$. There is one inflection point.

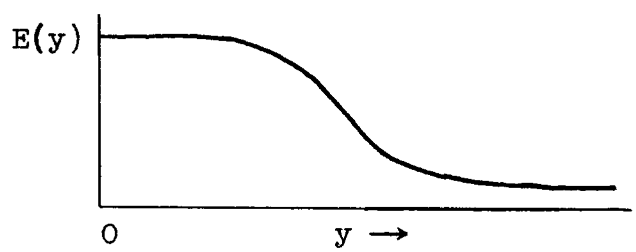

Fig. 6. Expected residual value $E(y)$ of target as function of attack strength y.

Problem 1. Let

V_i = value of i-th target T_i for $i = 1,\ldots,N$,

x_i = defense allocation for T_i,

$D = \Sigma x_i$,

y_i = attack allocation for T_i,

$A = \Sigma y_i$.

Choose (y_i) to minimize $\sum_i E_i(x_i, y_i)$.

In case the second weapon assigned to each target does less marginal damage than the first assigned to that target, the solution is easy: Simply assign the weapons by sequentially maximizing the marginal expected damage at each stage. This solution follows immediately from the fact that the function $E(y)$ has one inflection point, so that the marginal damage on that target would be a decreasing function of y_i. In particular, the abscissa of the inflection point lies to the left of $y = 2$.

If this condition does not hold, then the problem can be solved numerically on a computing machine by means of the dynamic-programming techniques discussed earlier in this paper.

Problem 2. Consider the two-move game in which the defender must deploy his forces first. Both sides know each other's total strength and the attacker will optimize knowing the defense deployment.

The optimal defense deployment X is given by

$$(8.1) \qquad \max_X \min_Y \sum_{i=1}^{N} E_i(x_i, y_i),$$

subject to the conditions $\Sigma x_i = D$ and $\Sigma y_i = A$.

In particular, let

$$(8.2) \qquad E_i(x_i, y_i) = V_i(1 - \alpha e^{-x_i/y_i})^{y_i},$$

where α = the probability that a weapon surviving the defense will destroy the target.

This problem can be solved analytically under certain conditions whereby the attack is sufficiently strong to make it optimal to hit all targets when opposed by the particular defense that will be defined below. In this way we avoid the difficulty of boundary solutions. Precisely, let

$$E(X,Y) = \sum_i E_i(x_i, y_i), \tag{8.3}$$

and let

$$M(X,Y) = E(X,Y) + \lambda(\Sigma x_i - D) + \mu(\Sigma y_i - A), \tag{8.4}$$

where λ and μ are Lagrange multipliers.

Setting $\partial M/\partial x_i = 0$ and $\partial M/\partial y_i = 0$ for each i gives $\hat{y}_i$, $\hat{x}_i$ as follows. Let

$$\begin{aligned} L &= \alpha e^{-D/A}, \\ V &= \sqrt[N]{V_1 \; V_2 \; \cdots \; V_N}, \\ \hat{y}_i &= \frac{A}{N} - \frac{\ln (V_i/V)}{\ln(1 - L)}, \\ \hat{x}_i &= \frac{D}{A}\, \hat{y}_i . \end{aligned} \tag{8.5}$$

<u>Theorem</u> 1. <u>If</u> $\hat{y}_i \geq 1$ <u>for all</u> i, <u>and if</u> $\hat{Y} = (\hat{y}_i)$ <u>is the optimal attack against</u> $\hat{X} = (\hat{x}_i)$, <u>then</u> $\hat{X}$ <u>is the optimal defense deployment of</u> D <u>against</u> A; i.e.,

$$\max_X \min_Y E(X,Y) \leq E(\hat{X},\hat{Y}), \tag{8.6}$$

<u>with equality holding only for</u> $X = \hat{X}$. (See [5].)

Proof. Observe first that

$$(8.7)\qquad E_i(\hat{x}_i,\hat{y}_i) = V(1 - L)^{A/N},$$

which is constant for all i. For any defense x_i, let $y_i(x_i)$ be the attack level giving the same residual target value as for $\hat{x}_i,\hat{y}_i$. Thus $y_i(x_i)$ is defined by

$$(8.8)\qquad E_i(x_i,y_i(x_i)) = E_i(\hat{x}_i,\hat{y}_i),$$

or

$$(1 - \alpha e^{-x_i/y_i(x_i)})^{y_i(x_i)} = \frac{K}{V_i}.$$

It can be shown that

$$(8.9)\qquad \frac{dy_i(x_i)}{dx_i} = \frac{1}{x_i/y_i - [(1 - U)\log(1 - U_i)]/U_i} > 0,$$

where

$$U_i = \alpha e^{-x_i/y_i}.$$

Thus, it requires a bigger attack to reach the same residual target level against a bigger defense, as is to be expected. Also, for this attrition function, we have

$$(8.10)\qquad \frac{d^2y_i(x_i)}{dx_i^{\,2}} < 0,$$

so that $y_i(x_i)$ is a strictly concave function of x_i. From this it follows easily that for $\sum_i x_i = D$ and α a Lagrange multiplier, the function

(8.11) $$\phi(X) = \sum_i y_i(x_i) + \alpha(D - \sum_i^N x_i)$$

is a strictly concave function of $X = (x_1, x_2, \ldots, x_N)$. Moreover,

(8.12) $$\frac{\partial\phi}{\partial x_i}(X) = 0$$

for every i, since from (8.9) we have

(8.13) $$\left.\frac{dy_i(x_i)}{dx_i}\right|_{x_i=\hat{x}_i} = \frac{1}{D/A - [(1 - L)\log(1 - L)]/L},$$

a constant independent of i. Thus the unique maximum of $\phi(X)$ occurs at $\hat{X}$ since $\Sigma x_i = D$; accordingly, against any other defense, the attacker could achieve the same damage with a smaller force. This implies in turn that for any $X \neq \hat{X}$,

(8.14) $$\min_Y M(X,Y) < M(\hat{X},\hat{Y}).$$

Now if $\hat{Y}$ is optimal against $\hat{X}$, then

(8.15) $$\min_Y M(\hat{X},Y) = M(\hat{X},\hat{Y}),$$

so that by (8.14) and (8.15) we have

(8.16) $$\min_Y M(X,Y) \leq \min_Y (\hat{X},\hat{Y}),$$

with equality holding only for $X = \hat{X}$. Thus $\hat{X}$ is the unique optimal defense, as was to be proved.

We have reduced the max min problem to a simpler minimization problem (which in general is still too complicated to handle analytically), that of verifying whether or not $\hat{Y}$ is optimal against $\hat{X}$.

It should be recalled that

$$\frac{\partial M}{\partial x_i}(\hat{X},\hat{Y}) = 0 \quad \text{and} \quad \frac{\partial M}{\partial y_i}(\hat{X},\hat{Y}) = 0$$

for each i; in fact, these equations defined $\hat{X}$, $\hat{Y}$. Also, working out the value of $\partial^2 M(\hat{X},\hat{Y})/\partial y_i{}^2$ analytically tells us whether or not $\hat{Y}$ is locally optimal against $\hat{X}$. But even when this necessary condition for an over-all optimum is satisfied, there are cases when $\hat{Y}$ is not the over-all optimal strategy against $\hat{X}$. Thus we are left with machine computations to verify whether or not $\hat{Y}$ is optimal against $\hat{X}$.

Now it is clear that in general the $\hat{y}_i$'s will not be integers; but in order to apply the dynamic-programming techniques of verification, we propose to adjust the target values to assure integral values of $\hat{y}_i$, and treat this new problem as a sufficiently close approximation to the original to bring out the general behavior of the optimal strategies. To do this, let

$$(8.17) \qquad V_i = (1 - \alpha e^{-D/A})^{k_i},$$

where the k_i's are appropriate integers,

$$(8.18) \qquad \sum_{i=1}^{N} k_i = 0,$$

and $A/N - k_i$ is a positive integer for each i.

Then

$$(8.19) \qquad \hat{y}_i = \frac{A}{N} - k_i, \qquad \hat{x}_i = \frac{D}{A}\hat{y}_i.$$

But now since the $\hat{y}_i$ are integers, we can use the dynamic-programming techniques to verify numerically whether or not $\hat{Y}$ is optimal against $\hat{X}$. If it is optimal then our problem has the solution $\hat{X}$, $\hat{Y}$. This problem was carried out in several large examples that otherwise would have been very difficult to verify.

The solution of a problem involving 25 targets is presented in Table 2, with the optimal defense and attack deployments listed for the case $\alpha = 0.5$. The reader is cautioned not to give undue practical interpretation to this example, for its main purpose is to illustrate the mathematical techniques involved in a case of extremely wide range of target values. In order that $\hat{X}$, $\hat{Y}$ be the solution to this problem, it is easy to see that A must be larger than D. For problems involving target values more comparable in size, the $\hat{X}$, $\hat{Y}$ deployment is sometimes optimal for cases where the defense can save a larger part of his target system.

If the attack level is not quite strong enough to satisfy the conditions of the theorem, it appears, on the basis of numerical examples, that even in this case the bigger targets are attacked in a manner very close to $\hat{y}_i$ (making these residual values nearly constant) while the rest of the target system is attacked by sending individual attackers against the most valuable of these targets. On the other hand, for radically different situations the optimal strategies will not resemble $\hat{X}$, $\hat{Y}$.

Table 2

OPTIMAL DEFENSE AND ATTACK DEPLOYMENT

Target	k_i	Value	Defense	Attack
1	−9	58.66	6.5	13
2	−6	15.10	5.0	10
3	−3	3.89	3.5	7
4	−3	3.89	3.5	7
5	−2	2.47	3.0	6
6	−2	2.47	3.0	6
7	−2	2.47	3.0	6
8	−1	1.57	2.5	5
9	−1	1.57	2.5	5
10	−1	1.57	2.5	5
11	−1	1.57	2.5	5
12	0	1.00	2.0	4
13	0	1.00	2.0	4
14	1	0.64	1.5	3
15	2	0.41	1.0	2
16	2	0.41	1.0	2
17	2	0.41	1.0	2
18	3	0.26	0.5	1
19	3	0.26	0.5	1
20	3	0.26	0.5	1
21	3	0.26	0.5	1
22	3	0.26	0.5	1
23	3	0.26	0.5	1
24	3	0.26	0.5	1
25	3	0.26	0.5	1
Total	0	101.18	50.0	100

Generalization. A result similar to Theorem 1 holds for more general attrition functions $\phi_i(x_i/y_i)^{y_i}$ satisfying conditions of type (8.9) and (8.10).

Problem 3. Consider the following variation of Problem 2, where the attack consists of both aircraft and decoys. As before, we adjust target values slightly to assure integral values y_i and z_i defined below.

Theorem 2. *Let*

$$\begin{aligned}
\Sigma x_i &= D, && \text{the total defense kill potential;} \\
\Sigma y_i &= F, && \text{the total number of attackers;} \\
\Sigma z_i &= G, && \text{the total number of decoys;} \\
A &= F + G; && \\
V_i &= (1 - e^{-D/A})^{k_i} && = \text{value of } i\text{-th target, where } k_i, \\
& && (F/N) - k_i,\ (G/N) - (G/F)k_i \\
& && \text{are positive integers;} \\
V &= \sqrt[N]{V_1 V_2 \cdots V_N} = 1; && \\
\Sigma k_i &= 0; && \\
\hat{y}_i &= \frac{F}{N} - k_i; && \\
\hat{z}_i &= \frac{G}{N} - \frac{G}{F} k_i; && \\
\hat{x}_i &= \frac{D}{F} \hat{y}_i . &&
\end{aligned}$$

If $(\hat{Y}, \hat{Z})$ *is optimal against* $\hat{X}$, *then*

$$\max_X \min_{Y,Z} E(X;Y,Z) \le E(\hat{X};\hat{Y},\hat{Z}), \tag{8.20}$$

with equality holding only for $X = \hat{X}$.

The proof is similar to that of Theorem 1 and will be omitted.

Now the problem has been reduced to a two-dimensional dynamic-programming minimization problem of a type that would be extremely laborious to handle in any other way known to the authors.

A ten-target attacker-and-decoy problem of this type was verified to have the optimal defense $\hat{X}$, since $(\hat{X}, \hat{Z})$ was optimal against $\hat{X}$.

Problem 4. A three-move game. Here we suppose the attack has a single missile budget with which to buy both attackers and decoys.

Let

D = total defense kill potential,

B = total attack budget measured in units of the cost of one decoy,

F = total number of attackers,

G = total number of decoys,

λ = cost ratio of attacker to decoy,

$B = G + \lambda F$,

$A = G + F = B - (\lambda - 1)F$ = total attack.

Both sides know amounts B, D, λ. The sequence of decisions follows:

1. Attack chooses optimal budget split F and G.

2. Defense chooses optimal defense deployment knowing this split F and G.

3. Attack chooses optimal target allocation of attackers and decoys knowing this defense deployment.

Let us assume as before that the attack level is strong enough to assure an interior optimal solution $\hat{Y}$, $\hat{Z}$ against the defense $\hat{X}$. Next, suppose some G and F are chosen. Then we are faced with Problem 3 again.

Thus if $(\hat{Y},\hat{Z})$ are optimal for $\hat{X}$ as defined in Problem 3 then $\hat{X}$ is the optimal defense and we have

$$(8.21) \qquad E(\hat{X};\hat{Y},\hat{Z}) = \Sigma\, V_i(1 - \alpha e^{-D/A})^{y_i} = NV(1 - \alpha e^{-D/A})^{F/N}.$$

To minimize $E(\hat{X};\hat{Y},\hat{Z})$ subject to $B = G + \lambda F$, etc., we minimize

$$(8.22) \qquad S(A) = (B - A) \log (1 - \alpha e^{-D/A}) = (B - A) \log (1 - L),$$

since $(\lambda - 1)F = B - A$ and $L = \alpha e^{-D/A}$. Solving $dS/dA = 0$ gives the relationship

$$(8.23) \qquad \phi\left(\frac{A}{D}\right) = 1 - \frac{A}{D}\left(\frac{1 - L}{L}\right) \log (1 - L) = \frac{B}{A}$$

for the optimal total attack A. This in turn gives the optimal allocation of attackers and decoys.

This relationship can be conveniently solved by the nomogram shown in Fig. 7.

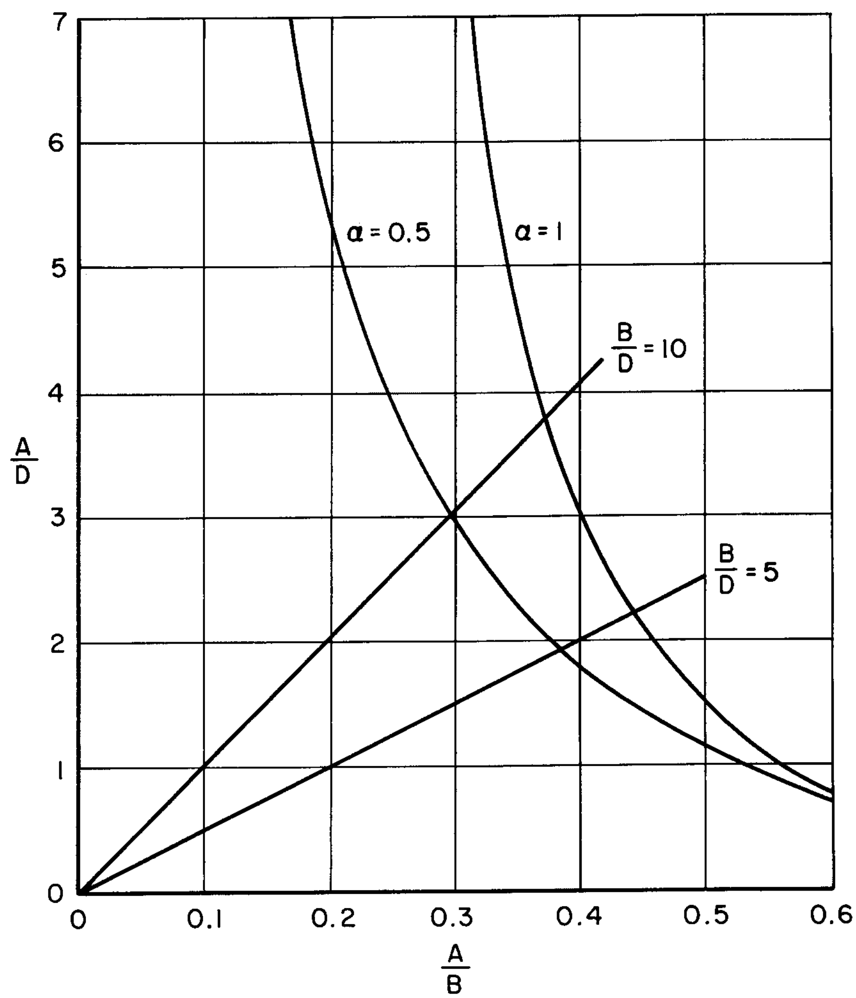

Fig. 7. Nomogram for optimal allocation of attackers and decoys.

Plot

$$\frac{A}{B} = \frac{1}{\phi(A/D)}$$

for fixed α against A/D. Draw a straight line through the origin with slope B/D. Its intersection with the proper curve $1/\phi(A/D)$ will give the optimal A/B ratio. The optimal split $F = (B - A)/(\lambda - 1)$ follows.

For example, if $B/D = 10$, $\lambda = 10$, $\alpha = 1$, then $A/B = 0.37$ and $F = 0.7B$.

These results are meant only to serve as a rough guide to more realistic situations.

Appendix

ON A TRANSCENDENTAL CURVE

The following discussion is concerned with the inflexion points of a family of curves defined by the relationship

$$(1) \qquad f(z) = \left[1 - e^{-x/(y+z)}\right]^{y} + \lambda z, \quad z \geq 0,$$

with the parameters satisfying $y \geq 1$, $x > 0$, $\lambda > 0$.

1. THE NUMBER OF INFLEXION POINTS

We assert that each member of the family has at most one point of inflexion. To show this, it is enough to verify that the extended curve on the range $z > -y$ has exactly one. We proceed to do this.

Note that, for the foregoing result, the presence of λ in (1) is inessential. Also, since nonsingular linear transformations preserve the number of points of inflexion, we can set $z = xw - y$. The condition $z > -y$ is then equivalent to $w > 0$ and the problem thus reduces to determining the number of points of inflexion of

$$(2) \qquad \phi(w) = (1 - e^{-1/w})^{y}, \quad w > 0.$$

From (2) we get

$$(3) \qquad \frac{-\phi'(w)}{y} = (1 - e^{-1/w})^{y-1} e^{-1/w} \frac{1}{w^2}.$$

Now, the inflexion points of (2) correspond to the relative minima and maxima of the function (3). But since the transformation $w = 1/u$ is a homeomorphic mapping of $(0,\infty)$ with $dw/du < 0$, we see that the inflexion points of (2) correspond to the relative interior minima and maxima of

$$(4) \qquad X(u) = (1 - e^{-u})^{y-1}e^{-u}u^2, \quad u > 0.$$

These, in turn, are included in the set of positive roots of $X'(u) = 0$. Now, since $e^{-u}u^2$ has exactly one interior local maximum $(u = 2)$ and no interior local minima, we can assume that $y > 1$. Thus, from (4) we get

$$(5) \qquad X'(u) = 2u(1 - e^{-u})^{y-1}e^{-u} - (1 - e^{-u})^{y-1}e^{-u}u^2$$
$$+ (y - 1)(1 - e^{-u})^{y-2}e^{-2u}u^2 = 0.$$

Since

$$u(1 - e^{-u})^{y-2}e^{-u} \neq 0,$$

from (5) we obtain

$$2(1 - e^{-u}) - u(1 - e^{-u}) + (y - 1)ue^{-u} = 0,$$

or

$$2 - u + e^{-u}(uy - 2) = 0.$$

Now since $y > 1$, $u = 2$ is not a root of the above equation, and we obtain

$$(6) \qquad e^u = \frac{uy - 2}{u - 2}, \quad 2 \neq u > 0.$$

Since $e^u > 1$ if $u > 0$, and $(uy - 2)/(u - 2) < 1$ if

$0 < u < 2$, we see that there is no root in $(0,2)$. On the other hand, e^u strictly increases to infinity while, for $u > 2$, $(uy - 2)/(u - 2)$ strictly decreases from ∞ to a finite value. Thus, we see that there is exactly one positive root of the above equation. Since the order of contact is clearly zero, it follows that this root corresponds to a unique inflexion point of (2) and our assertion is proved.

2. A NECESSARY AND SUFFICIENT CONDITION FOR CONVEXITY

We observe from (1) that the term not involving λ is positive and tends to zero as $z \to \infty$. The presence of the unique inflexion point on the extended curve then guarantees that f is eventually convex. It follows that a necessary and sufficient condition that f be convex on the range $z > 0$ is that the inflexion point, z, on the extended curve satisfy $z \leq 0$, i.e., in accordance with the order of the transformations of Sec. 1, $xw - y \leq 0$ or $x/u - y \leq 0$, where, from (6),

$$e^u = \frac{uy - 2}{u - 2}.$$

Solving this last equation for y in terms of u, we obtain

$$y = e^u + \frac{2}{u}(1 - e^u) = F(u). \tag{7}$$

Now, noting that the right-hand side of (6) is a strictly increasing function of y for $u > 2$ and that e^u is strictly increasing, we see that the root u is a strictly increasing function of y, and thus we see that a necessary and sufficient condition that $x/u - y \leq 0$, i.e., that $u \geq x/y$, is $y \geq F(x/y)$. In other words,

$$y \geq e^{x/y} + \frac{2y}{x}(1 - e^{x/y}).$$

The boundary curve $y = F(x/y)$ of course can be parametrized as follows:

(8) $$x = tF(t),$$

$$y = F(t), \quad t \geq 2,$$

where

$$F(t) = e^t + \frac{2}{t}(1 - e^t).$$

We thus obtain the region in the (x,y) plane in which f is convex, as shown in Fig. 8.

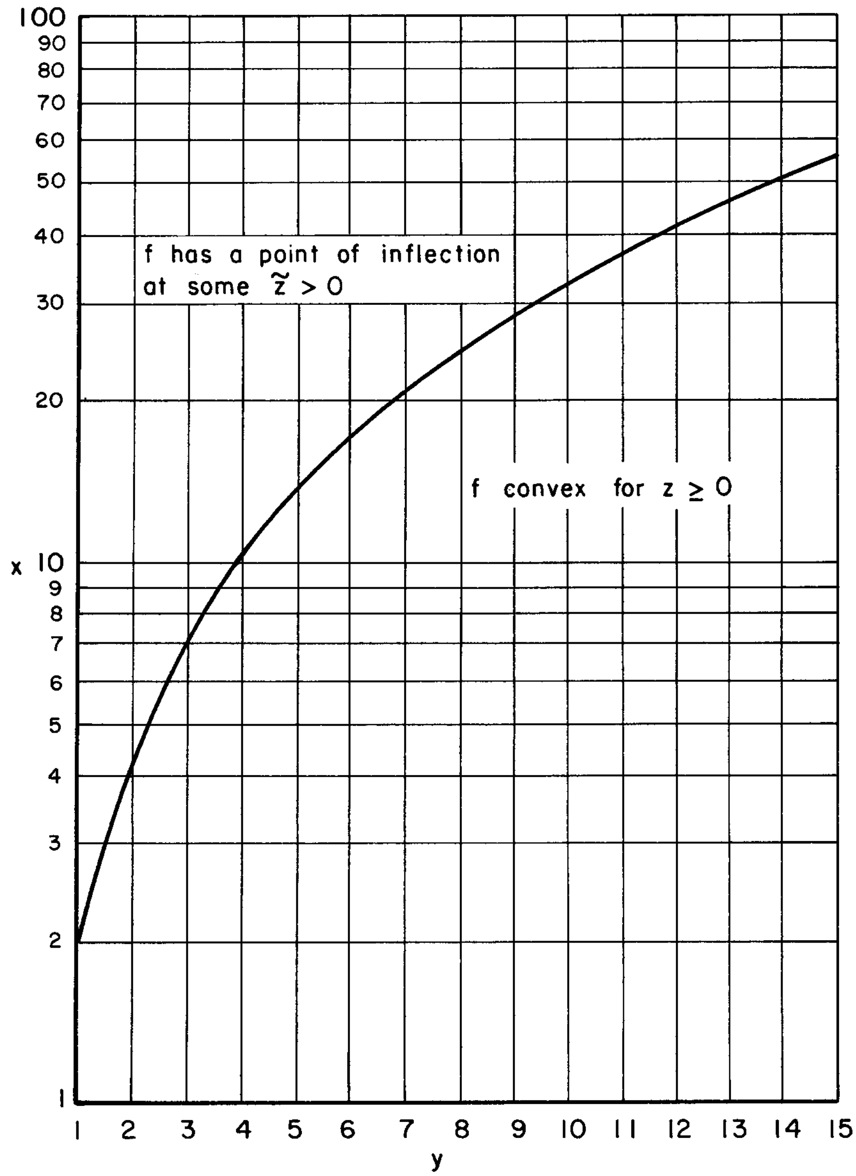

Fig. 8. Region of convexity of the function $f(z) = \left[1 - e^{-x(y+z)}\right]^y + \lambda z$, $\lambda > 0$, for $z \geq 0$ with parameters satisfying $y \geq 1$, $x > 0$.

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