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RESEARCH MEMORANDUM

Notes on Linear Programming: Part II

DUALITY THEOREMS

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SUMMARY

Since the simplex procedure itself yields as a natural by-product proofs of several important theorems concerned with "Duality" in the field of linear inequalities, we demonstrate them here.

DUALITY THEOREMS

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We shall consider one of many forms of the duality theorem — this one due to von Neumann [1] (for references to other equivalent forms see [2], [3], [4]). Let A be an m x n matrix, b an m-component column vector, and c an n-component row vector.

Theorem 1: If column vectors $X = \{x_1, x_2, \cdots, x_n\}$ exist which satisfy (1) and if the corresponding values of x_o in (1.1) have a finite upper bound, then row vectors $Y = (y_1, y_2, \cdots, y_m)$ exist which satisfy (2) and the corresponding values of y_o in (2.1) have a finite lower bound:

(1)	$x_i \geq 0 \quad (i = 1, \cdots, m)$	(2)	$y_j \geq 0 \quad (j = 1, \cdots, n)$
	$AX \leq b$		$c \leq YA$
(1.1)	$cX = x_o$;	(2.1)	$y_o = Yb$;

moreover, there exist two vectors $X = X^*$ and $Y = Y^*$ such that the corresponding values $x_o = x_o^*$ and $y_o = y_o^*$ satisfy

$$\text{Max } x_o = x_o^* = y_o^* = \text{Min } y_o \; . \qquad (3)$$

Assuming there exist a solution Y to (2) and a solution X to (1), we may multiply $AX \leq b$ by Y on the left without affecting the inequality (because Y has nonnegative components) and similarly multiply $c \leq YA$ on the right by X to obtain

$$(4) \qquad x_o = bX \leq YAX \leq bY = y_o \; .$$

This shows that y_o forms an upper bound for the values of x_o, and x_o a lower bound for values of y_o; hence if we exhibit a pair of solutions X and Y with the property $x_o = y_o$, this must be a maximizing solution for X and a minimizing solution for Y, and the duality theorem is established.

We shall now show that we can obtain such a pair as an immediate corollary of the properties of an optimum basic solution of the simplex method [5]. For this purpose we transform (1) into the equivalent system of linear equations in nonnegative variables by introducing nonnegative variables $W = \{x_{n+1}, \cdots, x_{n+m}\}$

$$(5) \qquad \begin{aligned} x_o - cX &= 0 \qquad x_j \geq 0 \qquad (j = 1,2,\cdots,n+m) \\ AX + I_m W &= b \end{aligned}$$

where I_m is the m x m identity matrix. One of the main results of the generalized simplex method[1] is that when a system such as (5) has solutions and a finite upper bound exists for values of x_o, there exists a solution $x_j = x_j^*$ satisfying (5) and a row vector β^* (Theorem VI in [5]) with the properties

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[1] The linear equation system in [5] (second section) contains a redundant equation to which the stated result applies more directly — however, it is a simple matter to show that it may be omitted as here.

$$(6) \qquad \beta^* P_o = 1, \quad \beta^* \begin{bmatrix} o \\ b \end{bmatrix} = x_o^*, \quad \beta^* P_j \geq 0 \qquad (j = 1,2,\cdots,n),$$

where P_j is the column vector of coefficients associated with the variable x_j in (5). It is easy to verify from $\beta^* P_o = 1$ that the first component of β^* is unity. Accordingly we define

$$(7) \qquad \beta^* = [1, y_1^*, y_2^*, \cdots, y_n^*] = [1, Y^*]$$

and set

$$(8) \qquad y_o^* = \beta^* \begin{bmatrix} o \\ b \end{bmatrix} = Y^* b \ .$$

It is also easy to verify that the other properties of β^* in (6) are precisely the same as (2) and (3), establishing the Minmax Theorem.

Theorem 2: If either system has a solution but the associated linear form is unbounded, then the dual system has no solution.[1]

Proof: If (on the contrary) the dual system also has a solution Y, then the linear forms denoted by x_o and y_o satisfy, by (4), $x_o \leq y_o$; whence y_o is an upper bound for x_o, contradicting our hypothesis.

Theorem 3: Whenever inequality occurs in the k-th relation of either system for an optimizing solution, then the k-th variable of an optimizing solution of the dual system

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[1] Note: Both a system and its dual may have no solution; for example, $x_1 - x_2 \leq 1$, $-x_1 + x_2 \leq -2$, $2x_1 - x_2 = \max$, and $y_1 - y_2 \geq 2$, $-y_1 + y_2 \geq 1$, $y_1 - 2y_2 = \min$, where $x_i \geq 0$, $y_j \geq 0$.

vanishes. Conversely, if the k-th variable is positive of the dual system, then k-th relation of the original system is an equality.

Proof: Let $AX^* + IW^* = b$ where X^* is an optimizing solution to (1). Multiply this expression by Y^*, an optimizing solution to (2); then

$$Y^*AX^* + Y^*W^* = Y^*b \; ; \tag{9}$$

from (3) and (4) follows $Y^*AX^* = Y^*b$ or

$$Y^*W^* = \sum_1^m y_i^* x_{n+i}^* = 0 \qquad (x_{n+i}^* \geq 0,\; y_i^* \geq 0) \; . \tag{10}$$

Since $x_{n+i}^* > 0$ means an inequality in the k-th relation of the first problem, it follows $y_i^* = 0$; similarly if $y_i^* > 0$, then $x_{n+i}^* = 0$, proving the theorem.

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