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GAMES WITH INFORMATION LAG

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SUMMARY

We consider the problem of obtaining a set of conditions for the existence of a value and optimal strategies for a 2-person game in which each player has a denumerable sequence of choices to make. It is shown that as far as the existence theorem is concerned, the information pattern may be neglected.

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In this note we consider two-person games in which each player has a sequence of choices indexed by a denumerable time parameter, with a payoff which is a function of the pair of sequences chosen. If no information is exchanged, the sequences themselves are the pure strategies. If there is information, then the strategy spaces are much more complicated than the sequence spaces. What we shall show here is that general conditions for the existence of a value and optimal strategies, stated in terms of a payoff defined over the product of the pure strategy spaces, may be translated into similar conditions stated in terms of the payoff defined on the spaces of sequences. As a special case, we show that the discrete evasion game considered by Isaacs [5], Isaacs and Karlin [6], and Dubins [1] has a value for arbitrary time lag, and an optimal strategy for the evader. Also, stochastic games [8] with imperfect information will possess values and optimal strategies for both players, provided that the total payments are bounded.

We assume the possible <u>actions</u> a_t, b_t of the players at each time to be chosen from finite sets. Let $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$ denote action sequences, or <u>activities</u>, and let A, B denote the spaces of all possible sequences, which will then be compact in the ordinary product topology. A metric ρ may be introduced on each of these spaces. Pure strategies for the first [second] player will be transformations from B to A [A to B], but with certain conditions imposed.

The first player is supposed to know b only up to b_{n-k} when he chooses a_n, and the second player is supposed to know a only up to $a_{n-\ell}$ when choosing b_n. We assume $k + \ell > 0$, and permit k or $\ell = +\infty$. (When k and ℓ are finite only their sum is significant.) The quantity $d = k + \ell - 1$ may be called the <u>delay</u> of the game: $d = 0$ is equivalent to perfect information; $d = 1$ is the case of simultaneous moves with otherwise perfect information; etc. The restriction on strategies T of the first player is the the following:

(1) If $a = Tb$, $a' = Tb'$, and if $b_j = b_j'$ for $j = 1,2,\ldots,n$, then $a_i = a_i'$ for $i = 1,2,\ldots,n+k$;

and similarly for the second player.

This property (a sort of Lipschitz condition) ensures that the strategies are continuous functions, and in fact that the sets of strategies are equicontinuous families. Moreover, they are closed in the topology induced by the metric:

$$\rho(T,T') = \sup_B \rho(Tb,T'b)$$

It follows (Ascoli's theorem) that they are compact spaces.

Property (1) obviously guarantees that a strategy pair (T,U) uniquely determines a sequence pair (a,b), (In fact, a and b are unique fixed points of TU and UT.) The mapping $(T,U) \rightarrow (a,b)$ is easily seen to be continuous. Therefore, if the sequence payoff function P(a,b) is continuous (in both variables together), then the strategy payoff M(T,U) is likewise. Again, if P is upper [lower] semicontinuous, then so is M, by virtue of the representation of P by a monotonic decreasing [increasing] sequence of continuous functions.

In the continuous case, the existence of a value to the game and optimal mixed strategies for both players is assured by a well-known theorem. In the semi-continuous cases, a value exists, and an optimal strategy for the second [first] player, by a theorem of Glicksberg [2].

We shall now give several examples of infinite games which are included in the conditions of our results. Let us first consider the "discrete evasion" game [1,5,6]. In this game the minimizing player, player II, at each time instant, moves either one unit to the left or one unit to the right, so that his activity space may be represented by a denumerable sequence of $\pm$ 1's. I has complete information about the past of player II, but his actions consist of a pair of decisions, the first of which is a time point, the second a guess of player II's position k moves later. (In the case which has been solved, k was equal to 2.) His activity may then be represented by a sequence of integers a_t: $a_t = 0,1,\ldots,k+1$, a zero denoting no guess. Only the first non-zero integer (and its time index) is significant in the payoff. The payoff is 1 if player I guesses correctly, 0 if he guesses incorrectly or does not guess at all. The "delay" in this game is essentially equal to k.

The payoff is easily seen to be continuous except in the neighborhood of points (a,b) where a = (0,0...); and hence is certainly lower semi-continuous. Therefore, the game has a value, player II has an optimal strategy, and player I has at least ε-optimal strategies.

The cause of the semi-continuity is the fact that one player is attempting to achieve a specific event, which occurs at a definite time if at all, and the other player to avoid the occurence of this event. In any game of this type, we may expect the player attempting to avoid the specific event to have an optimal strategy, and the other to have ε-optimal strategies.

Another example of this general dictum may be found in a modified version of the iterated matrix game discussed by Hausner [3,4] and Peisakoff [7]. In the original version of this game, the two players initially have a certain sum of money at their disposal, and they repeat a finite matrix game until one or the other is ruined. The ruined player pays 1 unit to the other player, and in the case where the play does not terminate, the final payoff is zero. It is easy to see that in this case the game is neither continuous nor semi-continuous, and the game is not covered by our discussion. On the other hand, if we modify the game so that player I attempts to ruin player II, and player II attempts to survive, then the payoff function is semi-continuous and player II has an optimal strategy.

The iterated matrix game may be modified in another way, so as to produce optimal strategies for both players. Let a be a number between zero and one, and let the payoff function be defined in the following way: if one player ruins the other on the n^{th} move, then he receives a^n. Then the payoff function is continuous and each player has an optimal strategy.

The latter is substantially a stochastic game in the sense of [8] (cf. remark No. 2 there—the only difference is the possibility here of an infinite number of states in case the matrix game has incommensurable entries). Our previous discussion can be modified (to allow for the chance moves) to show that an information lag can be introduced in these games without upsetting the validity of the existence theorems.

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