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ARTIFICIAL SATELLITES OF THE MOON

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SUMMARY

In this paper the requirements for establishing artificial satellites of the moon are discussed. The orbit parameters of lunar satellites are considered and limits are derived for avoiding the fatal extremes of impact on the moon and recapture by the earth. Lunar satellite parameters are related to the initial conditions at the earth end of the transit trajectory to the moon. Sample transit trajectories and moon satellite orbits are presented. Computations of motion are based on an earth-moon model that represents the earth and moon as spheres rotating uniformly about their common center of mass. The magnitudes of effects due to factors neglected in this model are estimated. The problems of attitude stabilization during the earth-moon transit and of velocity adjustment near the moon are considered. Lunar satellite size and mass are related to visual magnitude, and approximate system weight requirements are estimated.

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List of Symbols

$\bar{R}$	Mean radius of the moon
H	Altitude above the mean radius of the moon
M	Mass of the moon
D	Distance between centers of earth and moon
M_o	Total mass of the earth-moon system
G	Gravitational constant
K	GM_o
μ	M/M_o
r_1	Distance from center of the earth
r_2	Distance from the center of the moon
ω	Angular velocity of the earth-moon system about its center of mass
C	Integration constant in Jacobi's Integral.
v	Magnitude of velocity in (x,y,z) space
V_i	Magnitude of initial velocity of a lunar satellite in (x,y,z) space
V_{im}	Magnitude of initial velocity of a lunar satellite in (x_m,y_m,z_m)space
V_{ip}	Maximum allowable value of V_i
V_{iq}	Maximum allowable value of V_{im}
V_{ib}	Minimum allowable value of V_{im}
V_m	Mean of V_{ib} and V_{iq}; design objective for initial satellite velocity in (x_m,y_m,z_m)
V_B	Magnitude of velocity reduction required to achieve $V_{im} = V_m$
V_e	Velocity in (x,y,z) at the initial point in the transit trajectory
γ	Projection angle, above the local earth horizontal, at the initial point in the transit trajectory

ARTIFICIAL SATELLITES OF THE MOON

Robert W. Buchheim

INTRODUCTION

Now that plans are being implemented to establish artificial satellites of the earth during the International Geophysical Year, it is interesting to consider in some detail the requirements for establishing an artificial satellite of our moon. It is the lunar satellite that we shall discuss here. We shall consider such things as general orbit properties, limiting orbit parameters, the problem of getting the satellite to the neighborhood of the moon, attitude control during transit, and visibility of the satellite on orbit.

PARAMETERS OF THE EARTH-MOON SYSTEM

First of all, let us set down the principal dimensions of the earth-moon system. The earth and moon are separated by a distance of about one-quarter million miles. The diameter of the earth is about 8000 miles and that of the moon about 2000 miles. The mass of the earth is about 80 times that of the moon, and the acceleration of gravity at the earth's surface is about 6 times that at the surface of the moon. The center of mass of the earth-moon system is inside the earth, about 3000 miles from the center toward the moon. Both earth and moon revolve about this common center of mass at a rate of about one-quarter radian per day.

In the analysis to follow it will be assumed that the earth and moon are spheres, isolated in space, separated by a fixed distance, revolving around their common center of mass at a constant rate. The exact parameters used in computation are shown in Fig. 1. The sources of these numerical

values are discussed in Appendix A. Factors that have been neglected in the earth-moon model are discussed in Appendix B.

CIRCULAR ORBITS ABOUT AN ISOLATED MOON

The first things we would like to know about a lunar satellite are its height above the moon's surface, its velocity in orbit and its orbit period. Considering, for a moment, only the gravitational field of the moon, the period P_c in a circular orbit is given by

$$P_c = \frac{2\pi (H + \bar{R})^{3/2}}{(GM)^{\frac{1}{2}}} \qquad (1)$$

with

P_c = unperturbed period in circular orbit

$\bar{R}$ = mean radius of moon

H = altitude of orbit above moon's mean surface

G = gravitational constant

M = mass of moon

For the values of the constants adopted here, this is

$$P_c = (0.183365)\ (H + 1079.93)^{3/2} \text{ seconds} \qquad (2)$$

with H in statute miles.

The velocity V_c in orbit is

$$V_c = \left[\frac{GM}{H + \bar{R}}\right]^{\frac{1}{2}} \qquad (3)$$

or, numerically,

$$V_c = \frac{1.80924 \times 10^5}{(H + 1079.93)^{\frac{1}{2}}} \text{ ft/sec} \qquad (4)$$

where, again, H is in statute miles.

Plots of P_c and V_c against H are shown in Fig. 2.

It is of some interest to note that a lunar satellite in an 8-hour orbit is moving relative to the moon at about the same speed as the moon is moving relative to the earth, about 3360 ft/sec. Thus a moon satellite with a period longer than 8 hours will always move relative to the earth in the same direction as the natural moon; a moon satellite with a period shorter than 8 hours would sometimes move relative to the earth in the same direction as the moon and sometimes in the opposite direction. This is illustrated in Fig. 3, showing the motion, relative to the earth, of lunar satellites in circular orbits with periods of about 5, 8 and 13 hours.

It is also of interest to note that a lunar satellite in a circular orbit at an altitude of about 5000 miles would have an unperturbed period of 24 hours.

MOTION IN EARTH-MOON SPACE

We cannot proceed very far in an examination of lunar satellites on the basis of the assumption of an isolated moon - we must consider the motion of the satellite under the combined influences of earth and moon.

The coordinate systems to be used in the investigations are shown in Fig. 4.

The (x_o,y_o,z_o) set is an inertial reference system with origin at the center of mass of earth and moon. The (x,y,z) set, with its origin at the earth-moon center of mass, is rotating at the same rate as the earth and moon from an initial position so chosen that the earth and moon always lie on the x axis. The (x_m,y_m,z_m) set is a non-rotating system with origin fixed at the center of mass of the moon; (x_m,y_m,z_m) is translating relative

to (x_o,y_o,z_o) with the moon's velocity and is accelerating in the $(-x)$ direction relative to (x_o,y_o,z_o) due to the moon's motion. The earth's center is at

$$x_1 = -D\mu$$

and the moon's center is at

$$x_2 = D(1 - \mu)$$

where D is the distance between centers, and μ is the fraction of the total system mass that is in the moon.

Given positions and velocities relative to the (x,y,z) system, quantities in the (x_o,y_o,z_o) and (x_m,y_m,z_m) systems are obtained through the following transformations:

$$x_o = x \cos \omega t - y \sin \omega t \tag{5a}$$

$$y_o = x \sin \omega t + y \cos \omega t \tag{5b}$$

$$\dot{x}_o = (\dot{x} - \omega y) \cos \omega t - (\omega x + \dot{y}) \sin \omega t \tag{5c}$$

$$\dot{y}_o = (\dot{x} - \omega y) \sin \omega t + (\omega x + \dot{y}) \cos \omega t \tag{5d}$$

$$x_m = \left[x - D(1 - \mu)\right] \cos \omega t - y \sin \omega t \tag{6a}$$

$$y_m = \left[x - D(1 - \mu)\right] \sin \omega t + y \cos \omega t \tag{6b}$$

$$\dot{x}_m = (\dot{x} - \omega y) \cos \omega t - \left[\dot{y} + \omega x - \omega D(1 - \mu)\right] \sin \omega t \tag{6c}$$

$$\dot{y}_m = (\dot{x} - \omega y) \sin \omega t + \left[\dot{y} + \omega x - \omega D(1 - \mu)\right] \cos \omega t \tag{6d}$$

The equations of motion of a small body in earth-moon space can be set up most directly in the (x_o,y_o,z_o) system by writing the x_o, y_o, and z_o components of the gravitational attraction of earth and moon. These equations are

$$\ddot{x}_o = \frac{-K(1-\mu)(x_o-x_{o1})}{r_1^3} - \frac{K\mu(x_o-x_{o2})}{r_2^3}$$

$$\ddot{y}_o = \frac{-K(1-\mu)(y_o-y_{o1})}{r_1^3} - \frac{K\mu(y_o-y_{o2})}{r_2^3}$$

$$\ddot{z}_o = \frac{-K(1-\mu)z_o}{r_1^3} - \frac{K\mu z_o}{r_2^3}$$

where K is the product of G times the total mass of earth and moon.

Transformed to (x,y,z) coordinates, these are

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x - \frac{K(1-\mu)(x-x_1)}{r_1^3} - \frac{K\mu(x-x_2)}{r_2^3} \qquad (7a)$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \frac{K(1-\mu)y}{r_1^3} - \frac{K\mu y}{r_2^3} \qquad (7b)$$

$$\ddot{z} = \frac{-K(1-\mu)z}{r_1^3} - \frac{K\mu z}{r_2^3} \qquad (7c)$$

The terms $2\omega\dot{y}$ and $2\omega\dot{x}$ are the Coriolis accelerations and the $\omega^2 x$ and $\omega^2 y$ are centrifugal terms.

SMALL MOTIONS NORMAL TO THE MOON'S ORBITAL PLANE

In the work to follow we will limit our consideration to lunar satellite orbits in the (x,y) plane, the plane of the moon's orbit. As a first application of equations (7), however, we will investigate the first-order character of the orbit under the influence of an initial displacement or

velocity in the z direction.

To obtain an approximation to small motions in the z direction, we consider equation (7c) on the assumption that the motion of the satellite projected on the x,y plane is nominally circular. Then $r_2 = R_c$, the radius of this circular orbit, and r_1 is approximately equal to the distance D from earth to moon. Thus the equation of motion in z is approximately

$$\frac{d^2z}{dt^2} + \left[\frac{K(1-\mu)}{D^3} + \frac{K\mu}{R_c^3}\right] z = 0$$

and the solution is

$$z = \left[\frac{\dot{z}(0)}{\Omega}\right] \sin \Omega t + z(0)\cos \Omega t$$

where $z(0)$ and $\dot{z}(0)$ are the initial displacement and velocity in z, and Ω is

$$\Omega = \left[\frac{K(1-\mu)}{D^3} + \frac{K\mu}{R_c^3}\right]^{\frac{1}{2}}$$

So the motion in z is oscillatory, with a period

$$P_z = \frac{2\pi}{\Omega}$$

Since the period P_c of circular motion is approximately as given by equation (1), with $(H + \bar{R}) = R_c$ and $GM = K\mu$; and since the period P_m of the moon's orbital motion is given by

$$P_m^2 = \frac{(2\pi)^2 D^3}{K}$$

the expression for P_z can be written

$$P_z = \left[\frac{1-\mu}{P_m^2} + \frac{1}{P_c^2}\right]^{-\frac{1}{2}} \qquad (8)$$

From equation (8) we find that the period of orbit motion in z is only slightly less than the period of circular motion; P_z is within 0.04 per cent of P_c for orbit altitudes less than 5000 miles, and within 0.5 per cent for altitudes less than 10,000 miles.

It can be seen that slight errors in the z direction tilt the plane of the circular orbit slightly with respect to the (x,y) plane. This orbit plane precesses slowly in such a fashion that the time between alternate passages of the satellite through the (x,y) plane is slightly less than the time between successive closures of the orbit in the orbital plane.

REGIONS OF POSSIBLE MOTION IN THE MOON'S ORBITAL PLANE

Since no complete solution to the set of equations (7) is known, we cannot analytically describe a lunar satellite's motion in detail. There is, however, one known integral, due to Jacobi, that furnishes much of the essential information. Jacobi's integral is[1]

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 = v^2 = \omega^2(x^2 + y^2) + \frac{2K(1-\mu)}{r_1} + \frac{2K\mu}{r_2} - C \qquad (9)$$

with v the scalar velocity in (x,y,z) space and C a constant of integration.*

* It might be noted here that free motion of a body in earth-moon space is not conservative - neither the body's total energy nor its angular momentum are constant over a given trajectory. In fact, Jacobi's integral shows that the quantity conserved over a trajectory is $[U_o - H_z]$ where U_o is total energy and H_z is angular momentum about the z (or z_o) axis.

Following the reasoning first employed by Hill in the development of his Lunar Theory[2] we recognize that v^2 can take on only positive values in real motion. Thus, for a given value of the constant C, if substitution of the coordinates of a given location in (9) results in a positive v^2, that location lies in a region of possible motion; if the resulting v^2 is negative, the given location lies in a region in which motion of the body is not possible. The boundaries between regions of possible motion and excluded regions are contours on which $v = 0$, called "curves of zero relative velocity." A real particle cannot cross one of these curves. Several such sets of contours in the (x,y) plane are shown in Fig. 5.* For initial conditions corresponding to $C = C_1$, the body can move either in a closed region about the earth or in a closed region about the moon -- it is forever restricted to the neighborhood of its parent and could not travel from earth to moon. The International Geophysical Year satellite will be a body of this kind, restricted to the neighborhood of the earth; and our satellite of the moon must also be such a body, restricted to the neighborhood of the moon. For a set of initial conditions corresponding to $C = C_3$ the body is restricted to a closed contour around the earth-moon system but motion from one body to another is now possible. For a case like $C = C_4$, the contour delimiting the region of possible motion is open behind the moon, indicating that the body can escape entirely from the earth-moon system. Clearly $C = C_2$ is the limiting case separating those in which motion between earth and moon is, and is not, possible. The C values are in order $C_1 > C_2 > C_3 > C_4$.

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* The contours of Fig. 5 are only sketches showing the form of the regions; they are not drawn to scale.

MAXIMUM ALLOWABLE SATELLITE VELOCITY

If we wish to send a body from the earth to the vicinity of the moon we must place it on a trajectory with initial conditions such that $C < C_2$ so it can move in a region that is open between earth and moon. However, once it is near the moon, some force must be applied to the satellite to alter its trajectory to one with $C > C_2$ to preclude the possibility of recapture by the earth and thus insure that it will remain permanently in the immediate neighborhood of the moon. The value of C that is of critical importance in study of the lunar satellite is, then, C_2.

In order to determine C_2 we first determine the location of the double point in the C_2 contour.* This double point can be shown[1] to occur at one of the three real roots of

$$\omega^2 x - \frac{K(1-\mu)(x-x_1)}{r_1^3} - \frac{K\mu(x-x_2)}{r_2^3} = 0$$

This expression is identical with the right-hand member of equation (7a). Solution of this algebraic quintic shows that the double point occurs at $x_c = 0.83702354D$. We now substitute $x = x_c$, $y = 0$, $z = 0$ into Jacobi's integral, equation (9), with $v^2 = 0$, and solve for C to find $C_2 = 3.5990604 \times 10^7 \ (\text{ft/sec})^2$.

We can now determine the maximum allowable value V_{ip} of the initial velocity V_i in the moon satellite orbit. With $C = C_2$ in (9) the value of v^2 computed for any set of coordinates near the moon is V_{ip}^2, so

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* This critical point on the limiting C_2 contour lies at one of the five "centers of libration" of the system. This point is not at the so-called "zero-gravity" point of equality of earth and moon gravitational acceleration. Because of the effects of the orbital motions of earth and moon, the zero-gravity point has no direct significance.

$$V_{ip} = \left[\omega^2(x^2 + y^2) + \frac{2K(1-\mu)}{r_1} + \frac{2K\mu}{r_2} - C_2\right]^{\frac{1}{2}} \qquad (10)$$

This maximum allowable velocity V_{ip} is in the rotating (x,y,z) space. For comparison with a minimum allowable velocity to be computed below, we wish to convert V_{ip} to an equivalent upper limit V_{iq} in the moon's non-rotating system (x_m,y_m,z_m).

Using (10) and the transformations (6c) and (6d) we find that the initial velocity V_{im} in (x_m,y_m,z_m) is related to the initial velocity V_i in (x,y,z) by

$$V_{im}^2 = V_i^2 + \omega^2(x^2 + y^2) + 2\omega(x\dot{y} - y\dot{x})$$

$$- 2\omega D(1-\mu)(\dot{y} + \omega x) + \omega^2 D^2(1-\mu)^2$$

In terms of the polar coordinate system (R,θ) with origin at the center of the moon, as shown in Fig. 4, this relation is

$$V_{im}^2 = V_i^2 + \omega^2 R^2 + \begin{bmatrix}(+)\ \text{ccw}\\(-)\ \text{cw}\end{bmatrix} (2\omega R)V_i \cos\psi \qquad (11a)$$

where V_i in terms of R and θ is

$$V_i = \left\{ \omega^2\left[D^2(1-\mu)^2 + R^2 + 2DR(1-\mu)\cos\theta\right] + \frac{2K(1-\mu)}{(D^2 + R^2 + 2DR\cos\theta)^{\frac{1}{2}}} + \frac{2K\mu}{R} - C \right\}^{\frac{1}{2}} \qquad (11b)$$

ψ = the acute angle between the V_i vector and the local horizontal and where

$\begin{bmatrix}(+)\ \text{ccw} \\ (-)\ \text{cw}\end{bmatrix}$ indicates that the sign of the term is (+) for a satellite revolving about the moon in a counterclockwise direction, and (-) for a clockwise satellite.

With $C = C_2$ in (11b), $V_i = V_{ip}$. Letting $V_i = V_{ip}$ in (11a) we obtain $V_{im} = V_{iq}$, and V_{iq} is the maximum allowable initial velocity in the moon's non-rotating system (x_m, y_m, z_m).

In considering the dependence of the conversion from V_{ip} to V_{iq} on the direction of rotation, we note that a counterclockwise satellite orbit is direct; that is, its rotation is in the same direction as that of the earth-moon system; while a clockwise orbit is retrograde, having a rotation opposite in direction to that of the earth-moon system.

V_{ip} is plotted against H_i in Fig. 6. The dependence on θ is too slight to be distinguished in such a plot. The change in V_{ip} over a full circle of θ values is less than 0.1 per cent for $H_i < 5000$ miles.

The angle ψ will be nearly zero if a nominally circular orbit is to be established, so we will assume $\cos \psi \approx 1$.

The V_{iq} for a clockwise orbit is plotted against H_i in Fig. 7. The V_{iq} for a counterclockwise orbit is 30 to 172 ft/sec higher than these values for $0 < H_i < 5000$ miles. Any combination of initial velocity and altitude below the applicable (V_{iq}, H_i) curve will insure against the moon satellite's being recaptured by the earth.

MINIMUM ALLOWABLE INITIAL VELOCITY

Besides the danger of recapture by the earth, the moon satellite can also be lost through impact on the moon. In order to prevent such a

catastrophe it is necessary to establish the satellite in an orbit that is always at least one moon radius from the moon's center.

To establish limiting conditions for prevention of destructive impact on the moon due to a velocity deficiency, let us suppose the moon to be isolated in space. Then the satellite is acted upon only by the moon's field, and (x_m, y_m, z_m) can be treated exactly as an inertial system. The assumption of an isolated moon is, of course, only completely compatible with reality at the center of mass of the moon where the gravitational attraction of the earth and the effect of the acceleration of (x_m, y_m, z_m) exactly cancel. It is, however, adequate for the establishment of a useful lower bound on allowable initial velocity. Quantitative estimates of the limitations on these results are given in Appendix C, where it is shown that the error in the minimum velocities computed here is less than 10 ft/sec for initial altitudes less than 5000 miles.

The basic relations on orbit in inertial space around an isolated moon are

$$V^2 = GM \left[\frac{2}{R} - \frac{1}{a}\right] \tag{13}$$

$$h = VR \cos \alpha \tag{14}$$

where a is the semi-major axis of the orbit conic, α is the angle between the velocity vector and the local horizontal, and h is the constant angular momentum of a unit mass in the orbit. We are interested only in elliptical orbits. At the point of closest approach $\alpha = 0$ so (14) becomes

$$h = V_n r_n$$

with V_n and r_n the velocity and distance from the moon's center at

closest approach. In terms of V_n and r_n, (13) is

$$h^2 = GM \left[2r_n - \frac{r_n^2}{a} \right] \tag{15}$$

But substitution of the initial velocity V_{im}, initial radius R_i and initial trajectory angle α_i in (14) gives

$$h = V_{im} R_i \cos \alpha_i$$

and (15) can be then written

$$V_{im}^2 = \frac{2GM}{R_i} \left[\frac{r_n}{R_i} \right] \left[\frac{1 - (r_n/R_i)}{\cos^2 \alpha_i - (r_n/R_i)^2} \right] \tag{16}$$

In order for the satellite to avoid collision with the moon, the pericenter radius r_n must be

$$r_n \geq 1079.93 \text{ statute miles}$$

and the minimum allowable value of V_{im} is then

$$V_{ib} = \left[\frac{6.0621751(10^7)\rho}{(1+\rho)\left[(1+\rho)^2 \cos^2 \alpha_i - 1\right]} \right]^{\frac{1}{2}} \text{ft/sec} \tag{17}$$

with

$$\rho = \frac{H_i}{1079.93}$$

and H_i in statute miles. This minimum allowable velocity is plotted against H_i for $\alpha_i = 1^\circ$ and 20° in Fig. 8. In Fig. 7, V_{ib} is plotted against H_i for a particular case to be discussed below.

Taking r_n = 1079.93 miles is consistent with the assumption that the moon is exactly spherical. Perhaps this value ought to be increased by a few miles to account for mountains and the non-spherical lunar figure; however, the resulting change in V_{ib} would be very slight, less than 10 ft/sec for $0 < H_i < 5000$ miles.

PERMISSIBLE RANGE OF INITIAL VELOCITY AND ALTITUDE

Curves of V_{ib} and V_{iq} against H_i, like those in Fig. 7, set off a domain of combinations of initial altitude and velocity that will preclude recapture of the satellite by the earth, which would result from excessive initial velocity, and destructive impact on the moon as a result of insufficient initial velocity. Thus we can assert that it is necessary for the initial conditions to lie in the region bounded by the curves of V_{ib} and V_{iq} against H_i.

Unfortunately, however, we cannot assert that this region establishes sufficient conditions for initiation of a successful satellite orbit. The specific difficulty lies in the fact that the conditions we have established for preclusion of impact on the moon are not complete. A set of initial conditions below the V_{ib} curve will result in impact; a set of conditions above the V_{ib} curve may result in impact. Analytical treatment of this latter possibility requires the complete solution of the set of equations (7a) and (7b) and this is not available. Therefore we must rely on numerical experimentation to discover any special regions between the V_{iq} and V_{ib} curves that should be excluded.

TRIAL ORBITS

To make an exploration of the permissible region of initial conditions, specific sets of values of V_i and H_i were selected in the arbitrarily

chosen range $500 \leq H_i \leq 2500$ miles, and the corresponding trajectories were computed by integration of (7a) and (7b) on a digital computer, the JOHNNIAC, at The RAND Corporation, Santa Monica, California. The solutions were all run through at least 10 days of satellite operation (unless interrupted by impact on the moon) and a few were extended to 20 days. In all cases the initial point was assumed to lie either at $\theta = 90^\circ$ or at $\theta = -90^\circ$, and the trajectory angle α_i with respect to the local horizontal was taken to be $\alpha_i = \psi = 0$. Some samples of the resulting orbits are shown in Figs. 9 through 22. Most of these are plotted in (x,y,z) space because the rotation of the coordinate system prevents the occurrence of a great many orbit segments in the same part of a plot. On the orbit plots the initial velocity is given as V_i in (x,y,z) space as related to V_{ip}, the maximum allowable, in that space; and is also given as V_{im} in (x_m,y_m,z_m) space as related to V_{iq}, the maximum allowable, in that space. In all trajectory plots a scale of dimensions is used such that one unit corresponds to the distance between the earth and moon, $D = 239{,}074.3$ statute miles. The time marks shown indicate time in days from the initial point.

Exploration of the region near and between the V_{iq} and V_{ib} curves was directed principally at the case of retrograde orbits because we will find somewhat less restrictive accuracy requirements associated with retrograde orbits, suggesting that they would be the more likely choice for any early attempt at a moon satellite. Examples of retrograde orbits are given in Figs. 9 through 17. Figure 9 illustrates the result of a velocity deficiency -- prompt collision with the moon. Figure 10 is an example of an orbit resulting from conditions near the V_{iq} limit but not exceeding it. No difficulties of any sort were encountered in the region between V_{ib} and

V_{iq} for the retrograde case. In fact, as illustrated in Figs. 11 through 15, it was found that V_{iq} can be exceeded slightly without introducing any apparent undesirable results. This additional freedom on V_{im} above V_{iq} is, however, not extensive. When V_{im} exceeds V_{iq} by more than about 2-3 per cent the satellite is quickly recaptured by the field of the earth, as is illustrated by Figs. 16 and 17.

From the results of exploration by automatic computer we can conclude that, for the retrograde orbit case, the region between the V_{ib} and V_{iq} curves is a valid description of the range of initial conditions that will result in establishment of a successful moon satellite.

As a qualification of these results it must be pointed out that the persistence of a well-behaved orbit for 10 or 20 days does not represent a proof of its long-term persistence in this form. In preparation for establishment of a specific moon satellite it would be necessary to more closely explore the stability of the proposed orbits.

The case of direct orbits was also explored, but not so thoroughly as the retrograde case. An example of a direct orbit is shown in Fig. 18. A significant aspect of the direct case is the fact that difficulties are encountered at initial velocities somewhat below V_{iq}. In these orbits, illustrated by Figs. 19, 20 and 21, the satellite collides with the moon after a few revolutions. Thus, for the direct case, the effective upper limit on V_{im} is somewhat below V_{iq} whereas in the retrograde case it is found to be a bit above V_{iq}.

A case of a large, nearly-circular orbit is shown in Fig. 22.

As an interesting experiment, the results of high-speed vertical firings near the moon were investigated. The results are shown in Figs. 23

and 24. These results indicate the possibility of establishing a long-duration "semi-satellite" orbit from vertical projection as well as horizontal. It is pertinent to note that vertical firing leads to retrograde trajectories.

TRANSIT FROM EARTH TO MOON

In order to establish a lunar satellite on orbit, it is obviously necessary first to transport it from the earth to the vicinity of the moon.

For the transit trajectory from earth to moon we shall use roughly the same conditions assumed in a discussion of a moon rocket by Mr. George Clement at a Franklin Institute Symposium on April 18, 1956;[8] and we shall consider the establishment of retrograde orbits. Integration of equations (7a) and (7b) on the digital computer shows that a satisfactory transit trajectory can be established if the rocket booster sets the vehicle on its way with a speed of 35,100 ft/sec (in x,y,z space) from the position shown in Fig. 25, and in the direction indicated. The resulting trajectory is shown in rotating (x,y) space in Fig. 26, in the (x_o,y_o) inertial system in Fig. 27, and in the (x_m,y_m) system in Fig. 28. In this trajectory, the vehicle passes the moon in a direction substantially aligned with the local horizontal at an altitude of 1000 miles, at $\theta = 39.6^\circ$, 2.12 days after it left the earth. At this point its speed is about 7736 ft/sec in (x,y,z) space, and about 7707 ft/sec in (x_m,y_m,z_m).

At an initial altitude of 1000 miles above the surface of the moon the maximum allowable satellite velocity in (x_m,y_m,z_m) is $V_{iq} = 5338$ ft/sec. Arbitrarily assuming a direction error $\alpha_i = 5^\circ$ due to incorrect vehicle attitude or other causes, the minimum allowable satellite velocity at $H_i = 1000$ miles is $V_{ib} = 3280$ ft/sec. The median value between V_{iq} and

V_{ib} is V_m = 4309 ft/sec. This value will be taken as the design objective. The orbit resulting from initial conditions at H_i = 1000 miles, V_{im} = 4309 ft/sec and $\alpha_i = 0$ is shown in Figs. 29 and 30.

To establish the lunar satellite on orbit, then, it is necessary to reduce the vehicle velocity from the arrival value of 7707 ft/sec to V_m = 4309 ft/sec, a reduction of 3398 ft/sec, by firing a rocket along the vehicle's line of travel. What we wish to do, of course, is to fire at the instant when the vehicle is at the desired altitude above the moon and traveling along a local horizontal. For the present purpose we will assume that the firing is signaled by radio command from a tracking station on the earth or by a vehicle-borne device that indicates arrival in the proper vicinity of the moon.

If the initial conditions at the earth end of the transit trajectory are varied (by guidance and control errors) there will be resulting changes in the arrival velocity V_a, altitude of nearest approach H_a, and azimuthal position of nearest approach θ_a, at the moon end. If the velocity is reduced by a fixed amount V_B = 3398 ft/sec by firing a preset rocket at nearest approach (so $H_i = H_a$ and $\theta_i = \theta_a$), the resulting initial satellite velocity will be

$$V_i = \left[(V_a - V_B \cos \bar{\mu})^2 + (V_B \sin \bar{\mu})^2\right]^{\frac{1}{2}} \qquad (18)$$

and the direction angle relative to the local horizontal will be

$$\alpha_i = \arctan \left[\frac{V_B \sin \bar{\mu}}{V_a - V_B \cos \bar{\mu}}\right] \qquad (19)$$

where $\bar{\mu}$ is $\left|\theta_i - 39.6^\circ\right|$, the difference between the actual θ at nearest approach and the value 39.6° for the design objective trajectory. In these equations it is assumed that the roll axis of the vehicle, and, therefore, the thrust line of the decelerating rocket, are aligned exactly along the local horizontal at $\theta = 39.6^\circ$. We can allow for an angular misalignment β of the roll axis by replacing $\bar{\mu}$ above with $\mu' = \bar{\mu} + \beta$. The initial velocity V_{im} in (x_m, y_m, z_m) corresponding to V_i can be computed from equation (11a).

The initial conditions at the earth end were varied by changing the magnitude of the initial velocity V_e by an amount δV_e and the direction angle γ by $\delta\gamma$. Assuming that the decelerating rocket is fired at such time that $H_i = H_a$, that $V_B = 3382$ ft/sec and that $\beta = 5^\circ$, the resulting arrival velocities V_a were used in equation (18) to calculate V_i; and the corresponding velocities V_{im} in (x_m, y_m, z_m) were calculated from equation (11a). The resulting (V_{im}, H_i) loci are shown in Fig. 7. The values of $\bar{\mu}$ and δT, the variation in total transit time from the earth, as well as the variations δV_e and $\delta\gamma$ are plotted against the resulting H_i in Fig. 31. The V_{ib} curve in Fig. 7 was calculated for the larger α_i, computed from equation (19), at each H_i in Fig. 31.

Both (V_{im}, H_i) loci terminate at the left in Fig. 7 because of incipient impact with the moon during either the transit trajectory or the subsequent satellite orbit. The allowable errors δV_e and $\delta\gamma$ at these terminal points are about

$$(\delta V_e)_L = -33 \text{ ft/sec}$$

$$(\delta\gamma)_L = +0.135^\circ$$

The (V_{im},H_i) locus for δV_e variations crosses the V_{iq} curve at about H_i = 2200 miles, thus defining the other required limit on δV_e as

$$(\delta V_e)_R = +77 \text{ ft/sec}$$

The (V_{im},H_i) locus for $(\delta\gamma)$ variations trends to the right in about the same way as the limit curves, implying a limit $(\delta\gamma)_R$ on this side of the design point that is a good deal larger in magnitude than $(\delta\gamma)_L$. With the information at hand, we may state the following allowable tolerances:

$$-33 \leq V_e \leq +77 \text{ ft/sec}$$

$$-\approx 0.3^\circ < \delta\gamma \leq +0.135^\circ$$

An error of a few per cent in the velocity reduction V_B, due to uncertainties in the performance of the decelerating rocket, will narrow the limits between V_{ib} and V_{iq} by a corresponding amount but will not significantly affect these tolerances on δV_e and $\delta\gamma$. If a solid rocket is used for deceleration, the skin coating of the vehicle will have to be designed with due regard for the equilibrium temperature range that can be tolerated by the propellant. The exact nature of the surface required would depend on the time of month of launching because it is strongly influenced by the aspect of the vehicle relative to incoming rays of sunlight.

It is significant to note that a variation of initial transit trajectory conditions in the ranges determined above will result in a variation of transit time from the earth to H_i of about 4 hours. During 4 hours the vehicle would move about 7 moon diameters. This large uncertainty in arrival time makes it impractical to initiate the decelerating rocket by

a preset clock.

The effects of small velocities in the z direction at the beginning of the transit trajectory can be investigated by solving equation (7c) on the assumption that r_1 and r_2 are functions of x and y only. This assumption is valid so long as

$$z^2 \ll (x-x_1)^2 + y^2$$

$$z^2 \ll (x-x_2)^2 + y^2$$

For the transit trajectory of Fig. 26, numerical integration of (7c) shows that the value z_t of z at the moon is roughly

$$z_t = \left[2 \frac{\text{miles}}{\text{ft/sec}}\right] \dot{z}(0)$$

So, for a nominal specification of z_t = 250 miles, the specification on $\dot{z}(0)$ is about

$$\dot{z}(0) \leq 125 \text{ ft/sec.}$$

It is interesting to compare the accuracy requirements computed above with the performance implied in information released about the orbit of the United States satellite for the International Geophysical Year. Reports have indicated an initial orbit altitude H_{ie} = 300 miles, a perigee altitude of 200 miles and an apogee altitude as great as 1400 miles. If we use these data in (16), with the constants applicable for the earth, we can compute V_{ie} and α_{ie}, finding them to be

$$\alpha_{ie} = 4.1 \text{ degrees}$$

$$V_{ie} = 1.0512\, V_{ec}$$

where $V_{ec} \approx 25{,}000$ ft/sec, the circular velocity at an altitude of 300 miles.

A velocity higher than V_{ec} will not drive the perigee below 200 miles, but a velocity only slightly below V_{ec} will cause the perigee to drop sharply; therefore, we might infer that the expected range of V_{ie} is

$$V_{ec} \leq V_{ie} \leq 1.0512\, V_{ec}$$

so the expected error in initial velocity is roughly

$$\Delta V_{ie} = \tfrac{1}{2}(1.0512 - 1)\, V_{ec} = 640 \text{ ft/sec}$$

Assuming that the error in velocity magnitude is proportional to velocity, this earth satellite data would indicate the feasibility of an error δV_t for the earth-moon trajectory of

$$\delta V_t = \Delta V_{ie} \left[\frac{35{,}000}{25{,}000}\right] = 896 \text{ ft/sec}$$

which is about 20 times the allowable value of roughly 50 ft/sec.

If velocity error normal to the trajectory is also proportional to nominal velocity magnitude, the value of $\delta\gamma$ would be the same for an earth-moon trajectory as for the IGY earth-satellite case, i.e., 4.1 degrees. This is about 20 times the allowable value of roughly 0.2 degrees.

The tolerances derived above on velocity error δV_e and projection angle error $\delta\gamma$ at the earth end of the transit trajectory apply only to the case of retrograde lunar satellite orbits. For comparison, a preliminary exploration was conducted of the problem of establishing a direct lunar satellite orbit.

The basic qualitative difference between the two cases is the fact that the transit trajectory for the retrograde satellite passes above the

moon in a plot like Fig. 26, while that for a direct orbit passes below the moon. A negative δV_e reduces the altitude of closest approach to the moon in the first case, and a positive δV_e reduces this altitude in the second case.

Results of investigation indicate that the allowable tolerance on δV_e for the direct case is about 0.45 of that for the retrograde case, and that the allowable tolerances on $\delta\gamma$ are roughly the same in the two cases. These results contribute to the general conclusion that the establishment of a direct lunar satellite requires significantly more refined guidance and control accuracy than does the establishment of a retrograde orbit.

ATTITUDE CONTROL OF THE VEHICLE DURING EARTH-MOON TRANSIT

In the discussion of the use of the decelerating rocket, it was assumed that the orientation of the vehicle is under control. Since attitude error is one of the factors making up the trajectory direction uncertainty α, it is desirable to consider the feasibility of this concept in a reasonably simple vehicle. For conventional vehicle shapes the thrust line of the retarding rocket will lie along the roll axis, so for proper decelerating action the roll axis of the vehicle should be aligned along the velocity vector of the vehicle at the time of rocket firing. This direction is well defined by the planned trajectory at the time of launching from the earth, so the satellite vehicle's roll axis can be oriented for proper alignment of the decelerating thrust immediately at the end of powered flight from the earth. The attitude control problem, then, is one of holding this fixed alignment during the several days of transit to the moon. The principal disturbances acting to destroy the initial alignment are initial pitch and yaw rates, and the gravitational gradient of the earth.

These are discussed briefly in Appendix D, where it is shown that roll axis alignment can be held within about one degree of any fixed position by spinning the vehicle in roll at a moderate rate, one or two revolutions per second.

VISIBILITY OF A MOON SATELLITE

Suppose that we have established a body on a permanent orbit around the moon -- how do we observe it? Certainly we could put in a radio transmitter to make it "visible," but its effective life would be rather severely circumscribed by power-supply capacity. It would be highly desirable to make it optically observable as a passive body -- its useful life could then be essentially endless.

One way to enhance the visibility of the satellite is to arrange the design so that the vehicle, once on orbit, is made into a white reflecting thin-walled metal sphere. It is estimated that the visual magnitude of such a sphere, in full sunlight, as seen from the earth, would be about

$$M_v = 20.6 - 5 \log_{10} D_s$$

where D_s is its diameter in feet. The required diameter is plotted against visual magnitude in Fig. 32. This figure also shows the weight of the satellite skin, computed on the assumption that the skin is made up of aluminum foil 1/10,000 inch thick. The weight plotted is the weight that would be measured at earth sea level.

The satellite would have to be observed in the immediate vicinity of the moon, which is of visual magnitude (-12.55) when full. To prepare an estimate of required satellite diameter we need data on the visual magnitude that must obtain for an object very near the moon in order to make it

reliably observable. Unfortunately, exact data on this point are not now at hand, although the necessary observation experiments are under way. On the basis of the general information available, such as an examination of stars used in occultation programs and conversations with experienced observers, it appears that a satellite of magnitude 6 could be seen on the bright limb of the moon with a 12-inch telescope, and that one of magnitude of about 10 could be observed at about a moon diameter from the bright limb.

As seen from Fig. 32, satellites of visual magnitudes 10 to 6 would be 132 to 832 feet in diameter and would have a skin weight of 79 to 3140 pounds.

PROJECTION ROCKET SYSTEM

Clement has shown that a rocket system with an over-all weight of about 1,000,000 pounds could project a 500-pound payload from the earth onto a trajectory similar to that in Fig. 26.(8)

If such a rocket were indeed used to establish a lunar satellite, about one-half the total payload would have to be devoted to the decelerating rocket at the moon. Suppose, arbitrarily, that one-half of the remaining 250 pounds of payload weight could be devoted to the satellite skin and the other half were assigned to erecting mechanisms, ribs, and other associated equipment. We find from Fig. 32 that skin weight of 125 pounds corresponds to a satellite diameter of 166 feet. Such a satellite, when in full phase, would have a visual magnitude of 9.5, which falls within the range estimated to be required for reliable observation at about one diameter from the moon's bright limb.

CONCLUSIONS

The above analysis indicates that a permanent, optically observable satellite of our moon could be established in a retrograde orbit through the use of a booster rocket system weighing something more than 1,000,000 pounds, if the guidance and control functions were performed within errors about 1/20 of those that appear to be expected in the case of the United States satellite for the International Geophysical year. A similar satellite could be established in a direct orbit if guidance and control errors were further reduced by a factor of about 2.

Appendix A

PARAMETERS OF THE EARTH-MOON SYSTEM

The parameters that appear in the coefficients of the equations of motion (7a), (7b), (7c) are:

1. ω, the angular velocity of earth and moon about their common center of mass.
2. M_o, the total mass of the system, i.e., the mass M_e of the earth plus the mass M of the moon.
3. G, the gravitational constant.
4. μ, the fraction of M_o in the moon, i.e., $\mu = M/M_o$.
5. x_1, the x-coordinate of the center of the earth. The magnitude of x_1 is the distance from the center of the earth to the center of mass of the system. Adopting the convention that the earth will lie on the negative x-axis, we have $x_1 = -D\mu$ where D is the distance between centers of earth and moon.
6. x_m, the x-coordinate of the center of the moon. With the convention described above $x_2 = D(1-\mu)$.

The other two quantities r_1 and r_2 in the equations of motion are simply the distances from the vehicle to the center of the earth and the moon, respectively.

Noting that G and M_o appear only in the form of their product GM_o and that x_1 and x_2 are completely defined by D and μ, we see that the parameters that must be numerically specified are:

$$\omega, (GM_o), \mu, D$$

These parameters appear in the following relations which must be satisfied:

$$\omega^2 = \frac{GM_o}{D^3} \tag{1A}$$

$$g = \frac{GM_e}{R_e^2} = \frac{GM_o(1-\mu)}{R_e^2} \tag{2A}$$

g is the gravitational acceleration at the surface of the assumed spherical earth, after removal of the centrifugal components. R_e is the radius of the spherical earth.

It should be noted that R_e is of direct importance although it does not appear in the differential equations of motion since it enters in the specification of initial conditions. Positions are measured relative to points on the surface of the earth and then reduced to coordinates relative to the earth's center for use in the equations of motion, and this reduction necessarily involves R_e.

Combining (1A) and (2A) we find that the following relation must hold among D, g, R_e, and ω:

$$D^3 = \frac{R_e^2 g}{(1-\mu)\omega^2} \tag{3A}$$

To arrive at a consistent set of parameters from available information we will adopt values for g, R_e, μ and ω from observation and calculate a D from these.

The value for ω is quite accurately known to be:[3]

$$\omega = 2.6616995 \times 10^{-6} \text{ rad/sec.}$$

A nominal value of $\mu = 1/82.45$ will be adopted on the authority of

the American Ephemeris.[3]

A value for R_e can be calculated by computing the radius of the sphere having the same volume as that enclosed by the International Ellipsoid of Reference. This ellipsoid has an equatorial radius:[3]

$a = 6.378388 \times 10^8$ centimeters, and a polar radius

$b = 6.356912 \times 10^8$ centimeters

For equal volumes, R_e is given by

$$R_e = (a^2 b)^{1/3}$$

so

$$R_e = 6.371221 \times 10^8 \text{ centimeters.}$$

To obtain a g value for the assumed spherical earth, we recognize that a homogeneous ellipsoid attracts with a gravitational intensity, at geographic latitude 35° 15' 52", that is equal to the intensity on a sphere of equal mass and volume.[1] On the assumption that the earth's attraction conforms substantially to this rule, we can find g by calculating the earth's gravitational acceleration at latitude 35° 15' 52" and removing the centrifugal acceleration.

$$g_j = 978.032 \left[1 + 0.0052895 \sin^2\phi - 0.0000059 \sin^2 2\phi\right] \text{ cm/sec}^2$$

where ϕ is geographic latitude.

At $\phi = 35^\circ\ 15'\ 52''$, g_j is

$$\bar{g}_j = 979.75131 \text{ cm/sec}^2$$

The radius of the International Ellipsoid at geographic latitude $\phi = 35^\circ\ 15'\ 52''$ is

$$\bar{r} = 6.3712695 \times 10^8 \text{ cm.}$$

The perpendicular distance to the polar axis from a point on the International Ellipsoid is

$$\rho' = r \cos \psi'$$

where ψ' is the geocentric latitude at the point. At geographic latitude $\phi = 35^\circ\ 15'\ 52''$, the geocentric latitude is

$$\bar{\psi}' = 35^\circ\ 4'\ 56''$$

and at this point ρ' is

$$\bar{\rho}' = 5.2137888 \times 10^8 \text{ cm.}$$

The centrifugal acceleration at a point on the ellipsoid is

$$g_c = \Omega_e^2 \rho'$$

where Ω_e is the angular velocity of the earth about its polar axis. This is[3]

$$\Omega_e = 7.29211585 \times 10^{-5} \text{ rad/sec}$$

When $\rho' = \bar{\rho}'$, g_c is

$$\bar{g}_c = 2.7724298 \text{ cm/sec}^2$$

The net value of g then is

$$g = \left\{\left[\bar{\bar{g}}_j + \bar{g}_c \cos\ (35^\circ 15' 52'')\right]^2 + \left[\bar{g}_c \sin(35^\circ 15' 52'')\right]^2\right\}^{\frac{1}{2}}$$

$$g = 982.016 \text{ cm/sec}^2$$

Using the above parameter values in (3A) to compute D we have in summary:

$$\omega = 2.6616995 \times 10^{-6} \text{ rad/sec}$$

$$\mu = 1/82.45$$

$$R_e = 6.371221 \times 10^8 \text{ centimeters}$$

R_e = 7917.786 statute miles

g = 982.016 cm/sec^2

g = 32.21837 ft/sec^2

D = 3.847528 x 10^{10} centimeters

D = 239,074.3 statute miles

No information on the value of gravity on the moon's surface is available to relate GM and $\bar{R}$ the moon's mean radius. The value[5]

$\bar{R}$ = 1079.93 statute miles

will be used for the present purpose.

If the actual values of g, R_e and μ depart from the nominal values g_o, R_o and μ_o by amounts Δg, ΔR and $\Delta\mu$ the value of D will depart from the nominal value D_o by ΔD, where

$$\frac{\Delta D}{D_o} \approx \frac{1}{3}\left[\frac{2\Delta R}{R_o} + \frac{\Delta g}{g_o} + \frac{\Delta \mu}{1-\mu_o}\right]$$

Discussions of likely errors in the dimensions of the earth indicate that $\Delta R \approx 1000$ feet is a fairly conservative estimate. The component of ΔD due to a ΔR = 1000 feet is given by

$$\left[\frac{\Delta D}{D_o}\right]_{\Delta R} = \frac{1}{31,400}$$

The largest correction to the gravity formula suggested in recent times is the 17 milligals by which the value of Jeffreys[4] differs from the International Gravity Formula.[3] On the assumption that Δg = 25 milligals is a reasonable outside estimate of error in g we have

$$\left[\frac{\Delta D}{D_o}\right]_{\Delta g} = \frac{1}{118,000}$$

Review of the various determinations of the moon's mass indicates that $\Delta\mu = 0.25/(82.5)^2$ would probably cover the current uncertainty in μ. Thus

$$\left[\frac{\Delta D}{D_o}\right]_{\Delta\mu} = \frac{1}{26,800}$$

For the nominal

$$D_o = 239,074.3 \text{ statute miles.}$$

These uncertainties amount to

$$(\Delta D)_{\Delta R} = 7.61 \text{ miles}$$

$$(\Delta D)_{\Delta g} = 2.02 \text{ miles}$$

$$(\Delta D)_{\Delta\mu} = 8.93 \text{ miles}$$

The distance between earth and moon computed here differs by 217 miles from the mean value of 238,857 miles determined by observation.[5] The principle reason for this difference is the action of the sun on the moon, which has been neglected in the present study.

Appendix B

APPROXIMATE EFFECTS OF NEGLECTED FACTORS

Trajectory computations have been based on the assumption that the vehicle moves in a space devoid of all other bodies except the earth and moon. It has further been assumed that the earth and moon are spherical bodies, gravitating like point masses, revolving in circles about their common center of mass.

Factors neglected in these calculations include:

1. The gravitational field of the sun
2. Pressure of solar radiation
3. Oblateness of the earth
4. Eccentricity of the moon's orbit
5. Inclination of the moon's orbit to the earth's equatorial plane

We can prepare rough quantitative estimates of the importance of these factors.

1. Gravitational field of the sun

An observer at the center of mass of the earth-moon system would not be aware of the presence of any force field due to the sun. He would, however, be able to determine a gravitational gradient due to the sun, and would encounter a gravitational acceleration due to this body as he moved away from the center of mass toward or away from the sun.

Relative to an inertial system (x_o, y_o, z_o) fixed at the earth-moon center of mass, a body at a point removed from the center of mass by a distance r_o will experience a maximum gravitational acceleration g_{sm} due to the sun, where

$$g_{sm} = 2 \Omega_o r_o$$

with Ω_o the angular velocity of the earth in its orbit about the sun. The mean value of Ω_o is about

$$\Omega_o = 2 \times 10^{-7} \text{ radians/second}$$

so for $r_o = D$, the distance from earth to moon, the value of g_{sm} is

$$g_{sm} = 10^{-4} \text{ ft/sec}^2$$

As a comparison, the value g_e of the earth's field at the moon is about

$$g_e = 0.885 \times 10^{-2} \text{ ft/sec}^2$$

or about 100 times g_{sm}.

The mean value of g_{sm} over an earth-moon rocket trajectory is about 5×10^{-5} ft/sec, and integration over a 2.5 day transit would indicate need for an alteration of about 10 ft/sec in the required initial velocity in order to account for the sun's gravitational influence. This would be an adjustment of about 0.03 per cent in the nominal initial velocity of 35,000 ft/sec.

These figures indicate that the effect of the sun's field is small but of sufficient magnitude to warrant inclusion in trajectory calculations for any actual flight attempt.

2. Pressure of solar radiation

The pressure on a perfect absorber due to solar radiation at about the earth's distance from the sun is

$$p = 4.49 \times 10^{-5} \text{ dynes/cm}^2$$

On a nearly perfect reflector this pressure is about twice p, so the total force due to radiation pressure that can act on a bright vehicle with a projected area $\bar{A}$ is $2p\bar{A}$, and the acceleration a_r produced in a vehicle with an earth-weight of W pounds is

$$a_r = 6.04(10^{-6})\ (\bar{A}/W)\ \text{ft/sec}^2$$

with $\bar{A}$ in square feet. If we represent the vehicle, for rough estimating purposes, as a body with the projected area and volume of an equivalent sphere of radius R_v and mean density ρ_v, we can write a_r as

$$a_r = 1.81\ (10^{-5})\ (\rho_v R_v)^{-1}\ \text{ft/sec}^2$$

We will assume that the vehicle radius lies in the range

$$1 < R_v < 1000\ \text{ft}$$

and that its mean density lies in the range

$$10^{-3} < \rho_v < 10^{2}\ \text{lb/ft}^3$$

Then a_r will lie in the range

$$2(10^{-10}) < a_r < 2(10^{-2})\ \text{ft/sec}^2$$

from which we see that a_r is very small over a wide range of values for R and ρ_v.

3. Oblateness of the earth

In the trajectory calculations, it has been assumed that the gravitational field of the earth varies with distance r from the center as

$$g_a = \frac{-GM_e}{r_1^2}$$

This relation does not hold exactly for a non-spherical earth. To evaluate the effect of oblateness, consider the field of an oblate spheroid along an axis x' in the equatorial plane with origin at the center of the spheroid. The approximate expression for gravitational acceleration g' in this case is[1]

$$g' = g_a \left[1 + \frac{3}{10} y^2\right]$$

with

$$y = \frac{ae}{x'}$$

and

a = equatorial radius of the earth

e = eccentricity of spheroidal earth

ae = 324.96227 statute miles

Values of $\delta g = g' - g_a$ are tabulated below.

x'(Miles)	δg(ft/sec^2)
4,000	6.32×10^{-2}
6,000	1.25×10^{-2}
10,000	1.61×10^{-3}
20,000	1.01×10^{-4}
50,000	2.56×10^{-6}
100,000	1.61×10^{-9}
239,000	4.95×10^{-11}

One effect of the difference between g' and g_a is its influence on the distance D between earth and moon. Assuming the moon's orbit is substantially coincident with the earth's equatorial plane we can use the expression

$$\omega^2 = \frac{G(M_e' + M)}{D^3}$$

to approximately relate ω, the moon's angular velocity, and the orbital radius D, provided the mass M_e' is taken to be the "apparent" mass of the earth, where

$$M_e' = M_e(g'/g_a)$$

with (g'/g_a) evaluated at D = 239,000 miles, the approximate distance of the moon from the earth. Thus we have

$$\frac{(g'g_a)}{D_o^3(1 + \Delta D/D_o)^3} = 1$$

with D_o the distance computed on the assumption of a spherical earth. From this we find

$$\frac{\Delta D}{D_o} = \frac{1}{5.37 \times 10^6}$$

or

$$\Delta D = 0.0445 \text{ miles}$$

If we numerically integrate $\delta g(x)$ over the transit trajectory shown in Fig. 26 we find an approximate velocity accumulation of about 20 ft/sec.

This figure indicates that the effects of oblateness, like those due to the sun, while small, are large enough to require inclusion in trajectory calculations for an actual flight.

4. Eccentricity of the moon's orbit

As a result principally of the mean eccentricity of its orbit, the moon's distance from the earth actually varies between 221,463 miles and 252,710 miles, about a time average of 238,857 miles. We can obtain a first-order estimate of the effect of this variation by finding the change

in C_2 that it produces.

We will assume that over the space of a few days the moon's distance and angular velocity are essentially constant with values D and ω that are equal to the mean of the actual values over the interval to be considered. Then Jacobi's integral (9) is a valid solution to the equations of motion in earth-moon space, and the critical C value, C_2, is

$$C_2 = \omega^2 x_c^2 + \frac{2K(1-\mu)}{r_{1c}} + \frac{2K\mu}{r_{2c}} \qquad (1B)$$

where

$$r_{1c} = x_c + D\mu$$
$$r_{2c} = D(1-\mu) - x_c$$
$$x_c = k_2 D$$

with x_c the coordinate of the double point in the C_2 contour, so $x_c = 0.83702354\ D$, and $k_2 = 0.83702354$.

The instantaneous ω and D are related by

$$D^2\omega = h$$

with h the constant angular momentum per unit mass of the moon about the center of the earth. We will let

$$\omega = \omega_o + \delta\omega$$
$$D = D_o + \delta D$$

with ω_o and D_o the mean values used in the trajectory computations in the body of this paper. Then C_2 is

$$C_2 = C_{2o} + \delta C$$

with $C_{2o} = 3.5990604(10^7)(\text{ft/sec})^2$, the value computed for $\omega = \omega_o$ and $D = D_o$.

Use of these relations in the equation (1B) we find, approximately,

$$\delta C = -(\omega_o^2 k_2^2 D_o^2 + C_{2o})(\delta D/D_o) \qquad (2B)$$

Now the velocity v_i required at an initial position x_i, y_i, z_i to establish a trajectory with $C = C_2$ is given by

$$v_i^2 = (v_o + \delta v)^2 = \omega^2 \left[D\mu - r_1\right]^2 + \frac{2K(1-\mu)}{r_1} + \frac{2K\mu}{D-r_1} - (C_2 + \delta C)$$

where $x_i = D\mu - r_1$; r_1 = distance from center of earth at initial point and $y_i = z_i = 0$ has been assumed for purposes of this estimate.

Use of (2B) and the expressions above for ω and D, we obtain, approximately

$$\delta v = \left\{ 2\omega_o^2 (D_o\mu - r_1)\left[\mu - \frac{4(D_o\mu - r_1)}{D_o}\right] - \frac{2K\mu}{(D_o - r_1)^2} + \frac{C_2 + \omega_o^2 k_2^2 D_o^2}{D_o} \right\} \frac{\delta D}{2v_o} \qquad (3B)$$

This δv can be assumed to apply for trajectories like those of Fig. 26 because the initial V_e there is very nearly v_o. For the cases of interest, $v_o \approx 35{,}000$ ft/sec, $r_1 \approx 4300$ miles; and the maximum departure δD from $D_o = 238{,}857$ miles is 17,388 miles. Thus δv is about

$$\delta v = 45 \text{ ft/sec}$$

indicating that the effect of the eccentricity of the moon's orbit is large enough to warrant inclusion in trajectory programming of any actual flight but not large enough to noticeably affect the general nature of the motion.

5. Inclination of moon's orbit to the equator

The effects of an inclination of the moon's orbit away from the

earth's equatorial plane are associated only with the oblateness of the earth and the magnitudes of these effects, therefore, should be no more serious than those indicated in item (3) above.

Appendix C

ACCURACY OF V_{ib} CALCULATIONS

To estimate the effect of the earth's field on the minimum allowable velocity V_{ib}, we recognize that the maximum effective gravitational acceleration produced by the earth near the moon, in the (x_m, y_m, z_m) system is

$$g_{em} = 2\,\omega^2 R$$

where R is the distance from the center of the moon to the point being considered. To obtain an upper bound on the effect of the earth we will assume that the effective field remains constant at g_{ei} from the initial point at distance R_i to the point of nearest approach to the moon. This point of nearest approach is to be at $R = \bar{R}$, the radius of the moon; but it will be disturbed by an amount d_e due to g_{ei}, with

$$d_e = \tfrac{1}{2} g_{ei} T^2$$

where T is the time of travel of the satellite from the initial point to the point of closest approach. For an initial direction angle $\alpha_i \approx 0$, we have

$$T \approx \tfrac{1}{2} P$$

with P the period in an elliptical orbit with semi-major axis

$$a = \tfrac{1}{2}(R_i + \bar{R})$$

P can be found from Fig. 2 by using $(H_i/2)$ for orbit altitude H.

Using these relations and $\omega = 2.66 \times 10^{-6}$ rad/sec, we can calculate d_e as a function of initial altitude H_i. The calculation results are shown in Fig. 33. It can be seen that d_e is very small compared to R

for H_i below about 5000 miles.

From (16) we find that a disturbance d_e in the distance of nearest approach requires that the lowest allowable velocity V_{ib} be changed by an amount δV in order to prevent impact on the moon, where

$$\delta V \approx \frac{V_{ib} d_e}{2R\left[1 - \left(\frac{\bar{R}}{H_i + \bar{R}}\right)^2\right]}$$

Using V_{ib} from Fig. 10, we find δV as a function of H_i as plotted in Fig. 33.

Appendix D

ATTITUDE DISTURBANCES

The significant sources of torque tending to disturb the roll attitude of a space vehicle on an earth-moon trajectory are:

a. Initial angular rates in pitch and yaw

b. Gravitational gradients

From Euler's Equations,[6] we find that the maximum magnitude of the roll attitude error due to an initial angular rate ν in pitch or yaw is

$$\beta_1 = \frac{2\,\nu\,(A/C)}{\Omega_r}$$

where (A/C) is the ratio of the moment of inertia in pitch and yaw to that in roll, and Ω_r is the vehicle spin rate about the roll axis. For a conventional vehicle shape, and roughly uniform mass distribution, (A/C) would fall in the approximate range

$$2 < A/C < 10$$

Therefore, for a nominal specification of allowable attitude error of $\beta = 1^\circ$, the ratio (Ω_r / ν) should lie in the range

$$230 < \frac{\Omega_r}{\nu} < 1500$$

Experience with sounding rockets like the Viking indicates that a free-flight pitch/yaw residual rate limitation of about $\nu = 0.01$ radian/second is readily achieved. So Ω_r would probably not need to exceed a value in the approximate range

$$0.4 < \Omega_r < 2 \text{ RPS}$$

The torque due to the gradient of the earth's field does not exceed[7]

$$L_g = \frac{3}{2} \frac{GM_e}{r_1^3} \left| C - A \right|$$

with r_1 the distance of the vehicle from the center of the earth. Allowing this torque to act on the vehicle for an earth-moon transit of 2.5 days, we find a maximum roll attitude error β_g due to this source of about

$$\beta_g = \left| \frac{1 - (A/C)}{4\ \Omega_r} \right| \text{ degrees}$$

For (A/C) and Ω_r in the ranges indicated above, β_g would not exceed

$$\beta_g = 0.45 \text{ degrees}$$

The attitude error due to the moon's field would be about $(\beta_g/80)$ due to its smaller mass.

The attitude stabilization requirements for the earth-moon transit of a moon satellite are easily satisfied by imparting a roll spin rate of a few revolutions per second to the vehicle at the start of the transit trajectory.

There would also be an appreciable torque due to solar radiation pressure if the radiation properties of the surface were not so arranged that the "center of pressure" substantially coincided with the center of mass.

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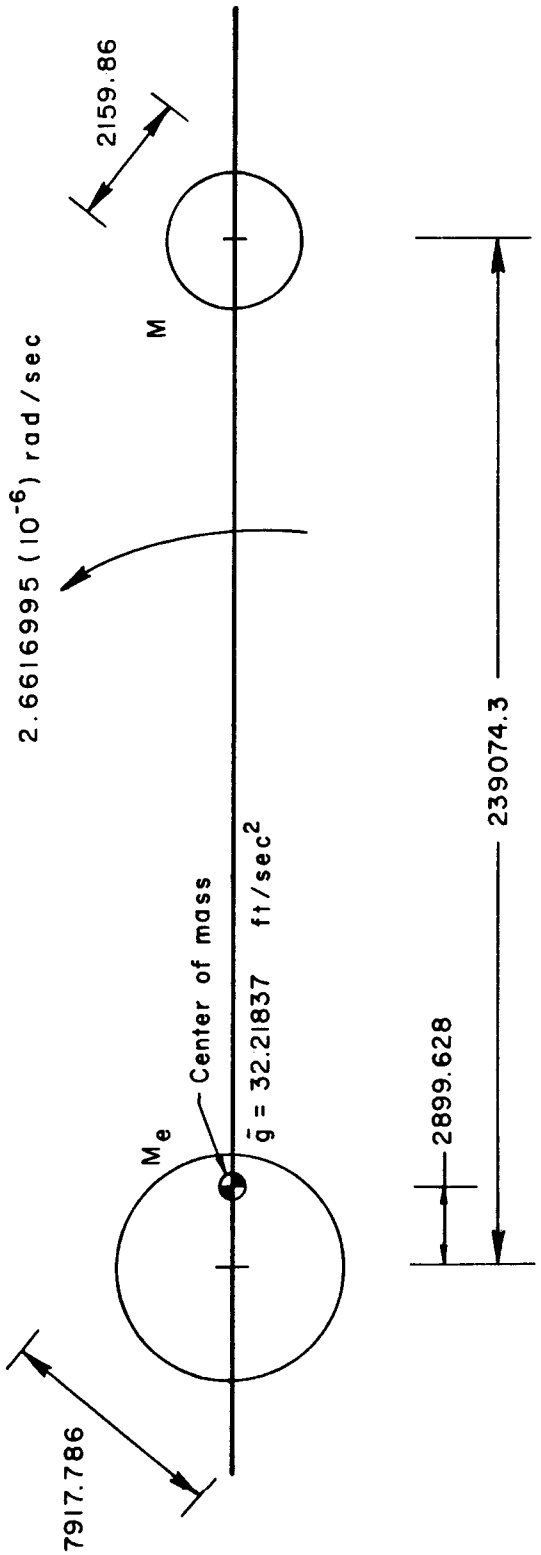

Fig. 1 - Parameters of earth - moon model

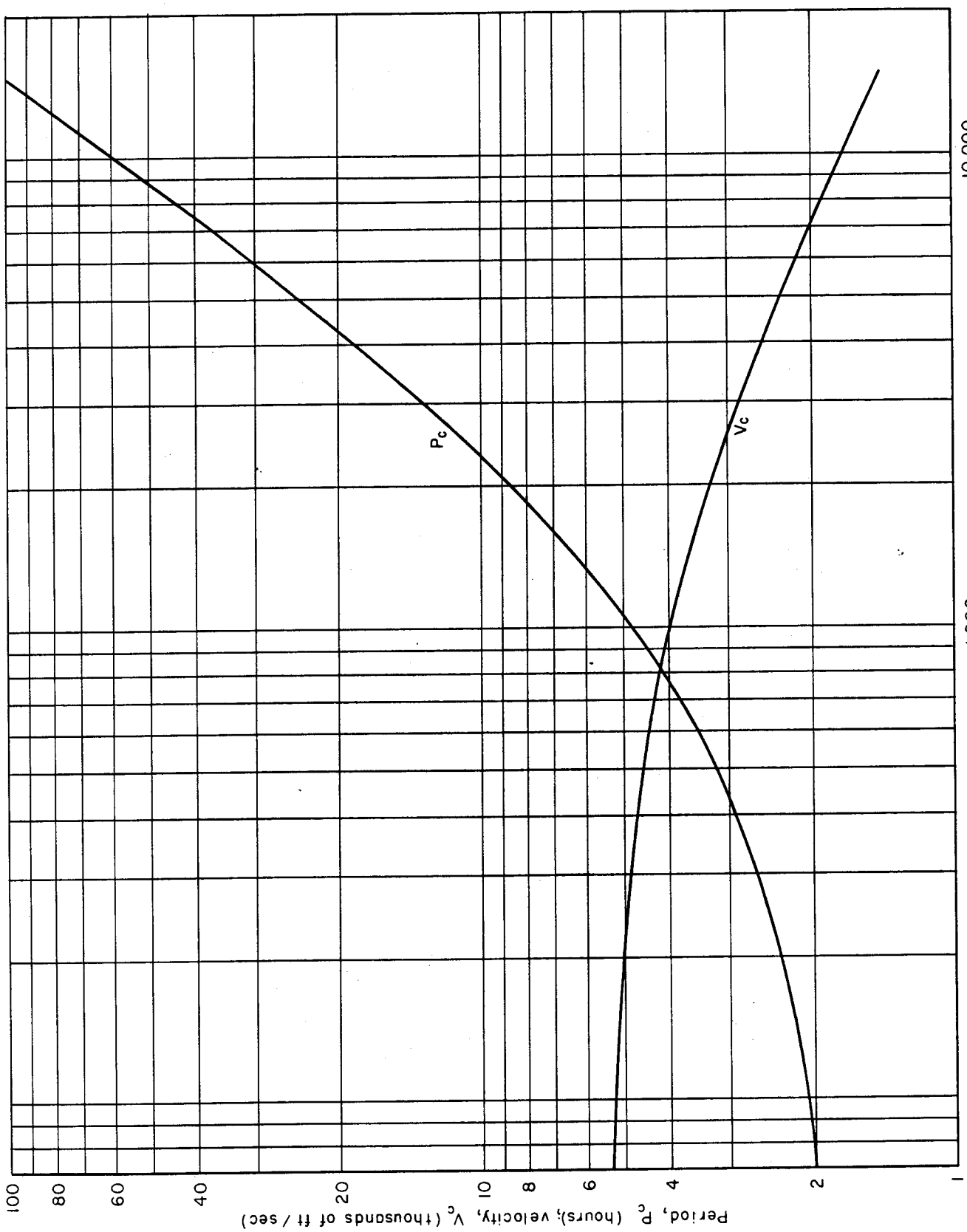

Fig. 2 - Period and velocity in circular orbit

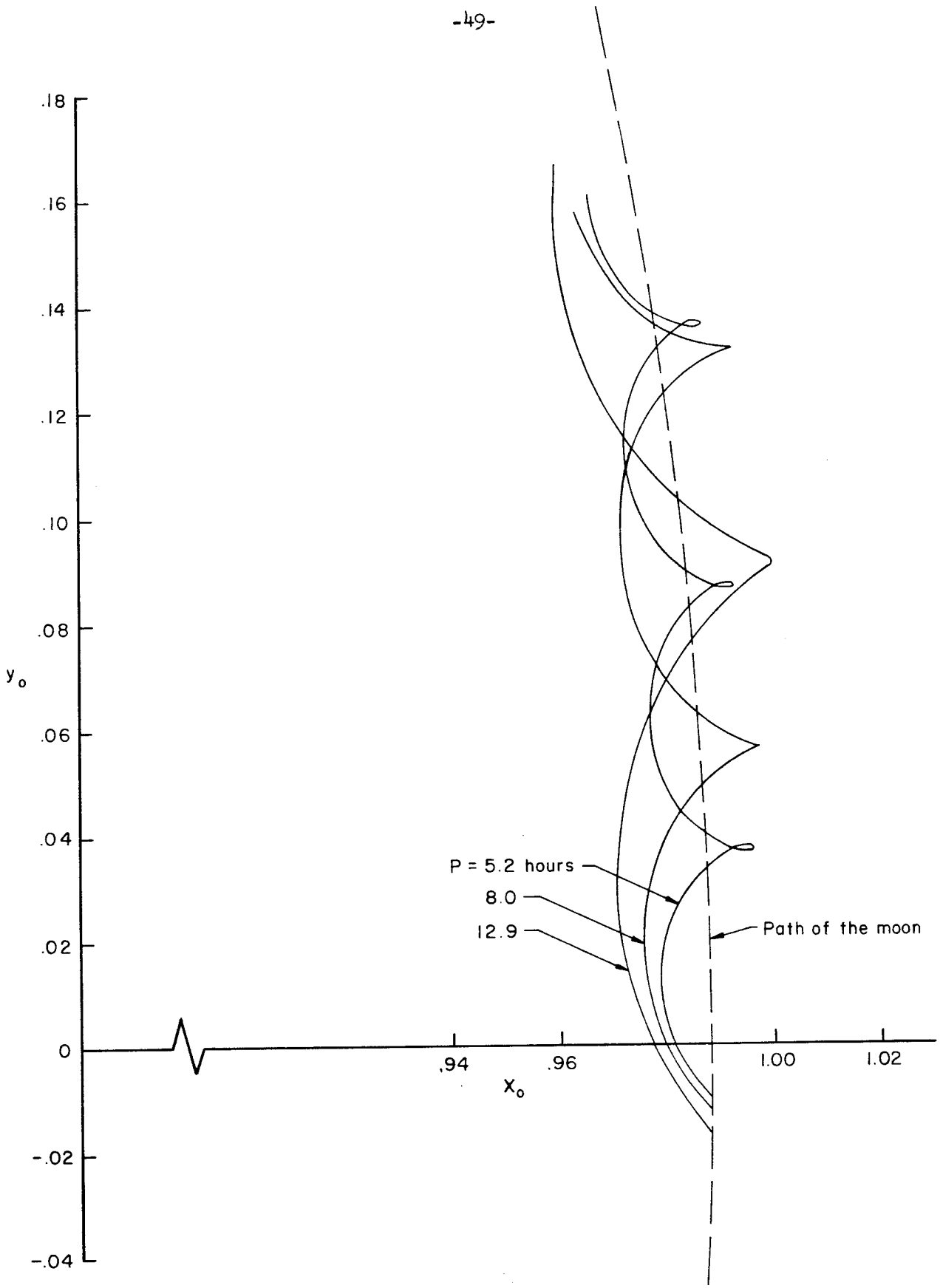

Fig. 3 - Motion of lunar satellites in circular orbits as seen by an observer near the earth

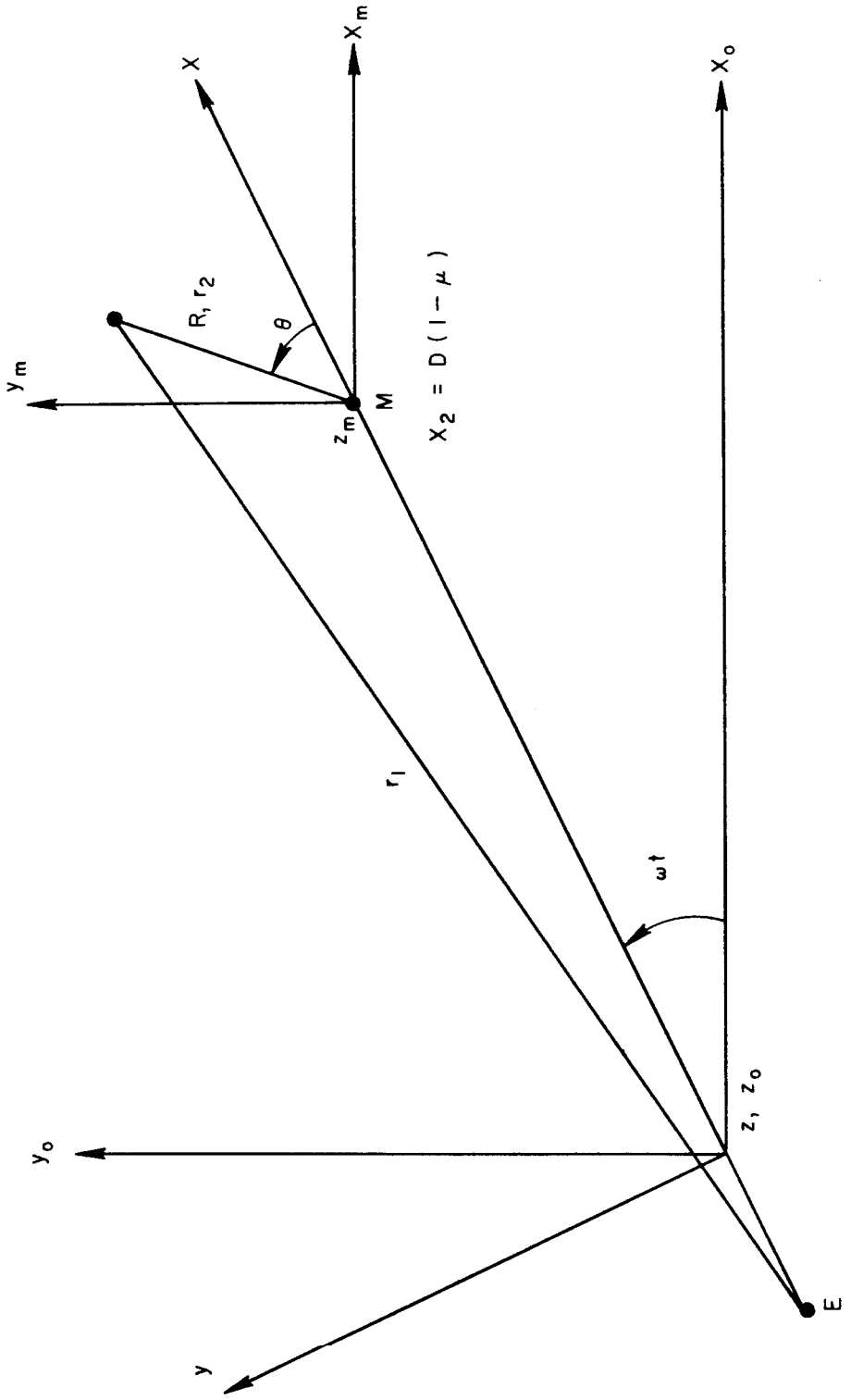

Fig. 4 - Coordinate systems

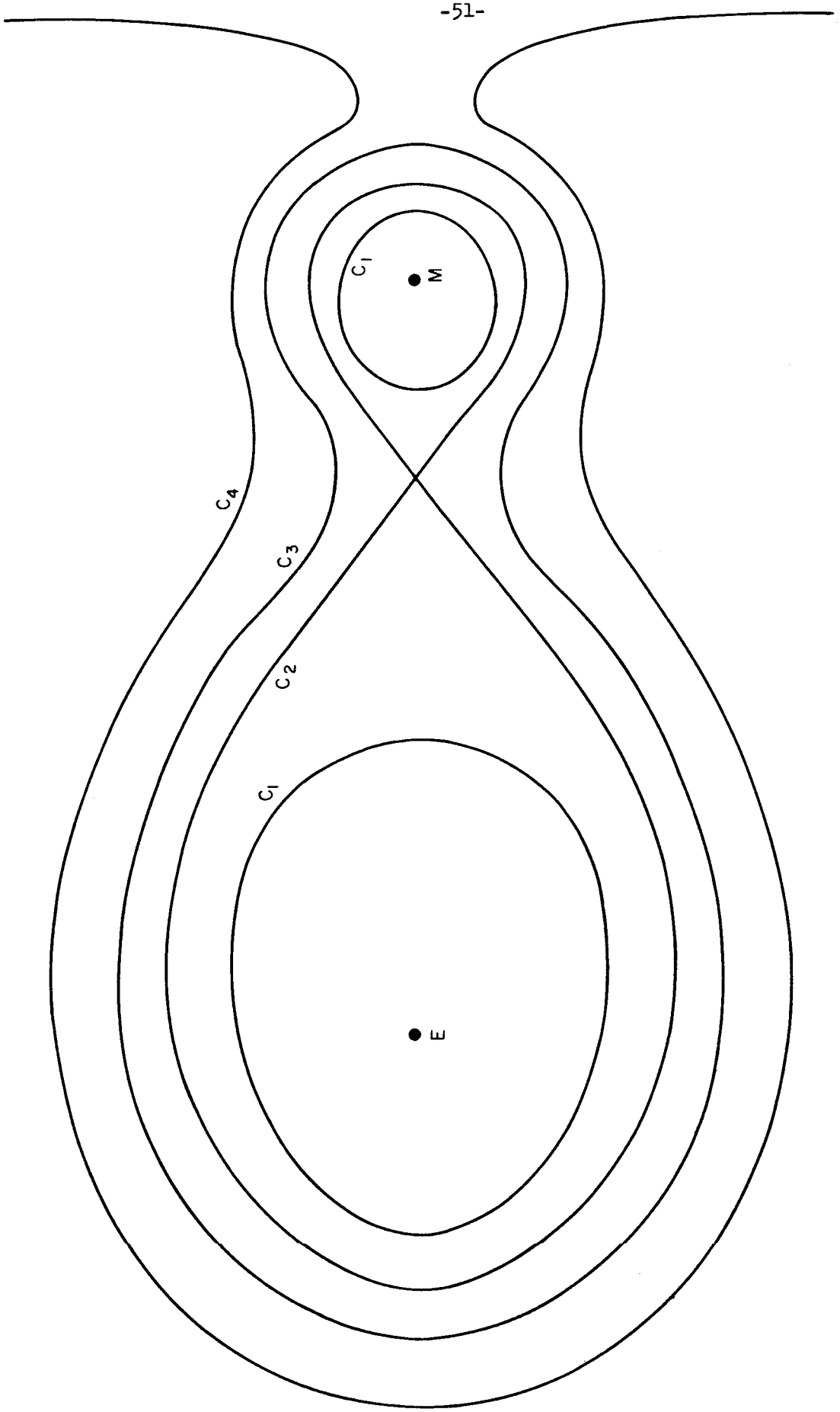

Fig. 5 - Contours of zero relative velocity
(not to scale)

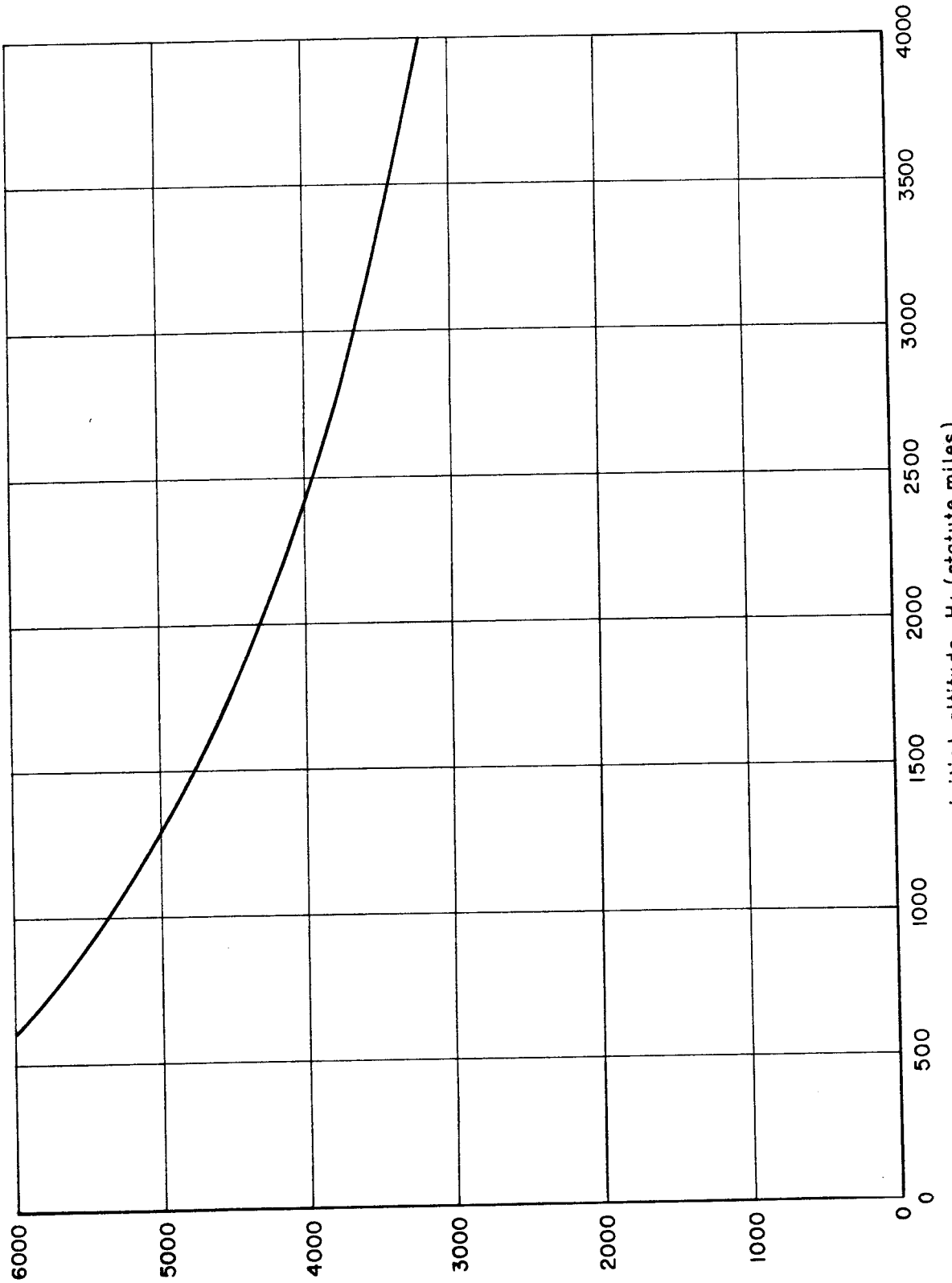

Fig. 6 - Maximum allowable initial velocity

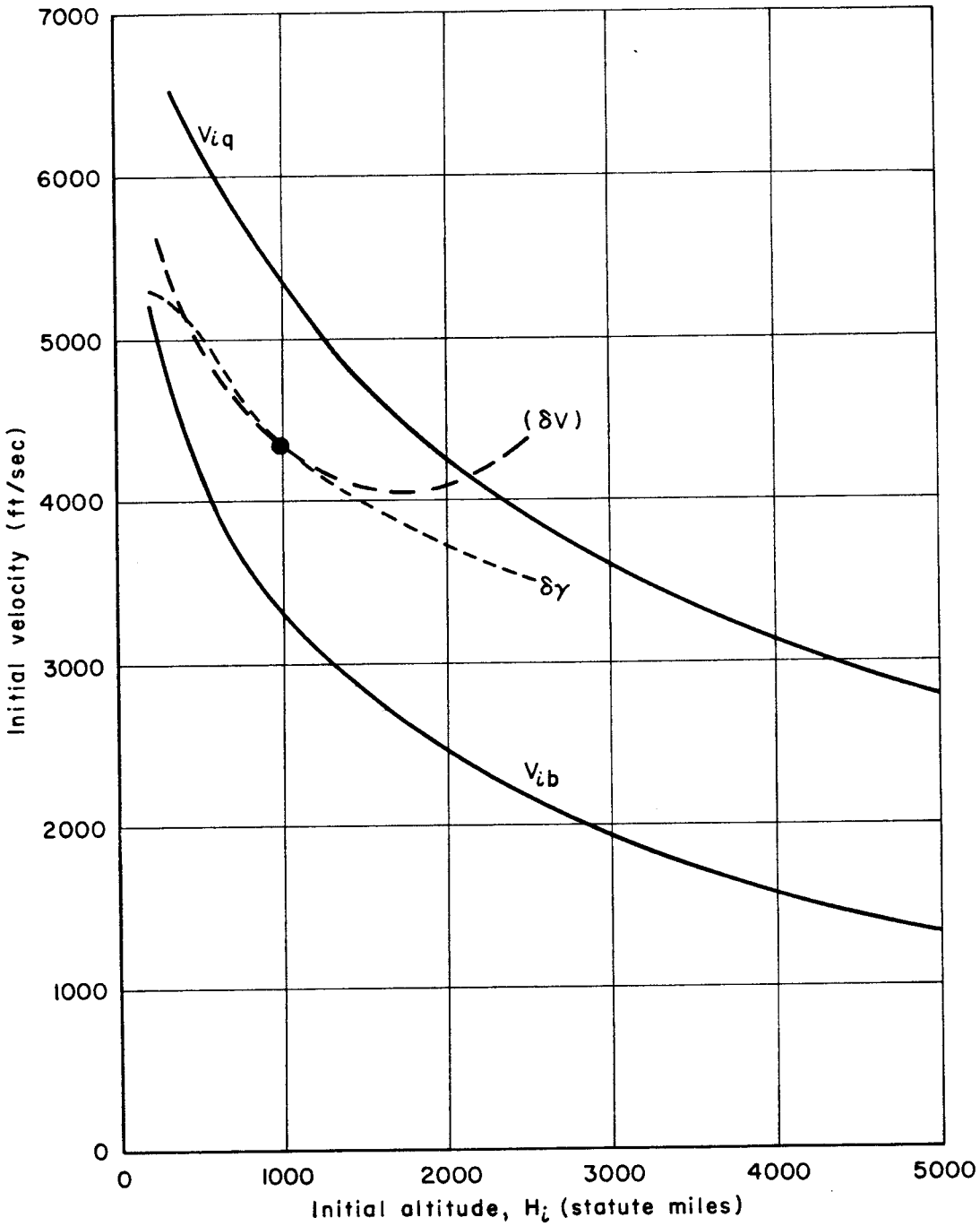

Fig. 7 - Satellite initial conditions

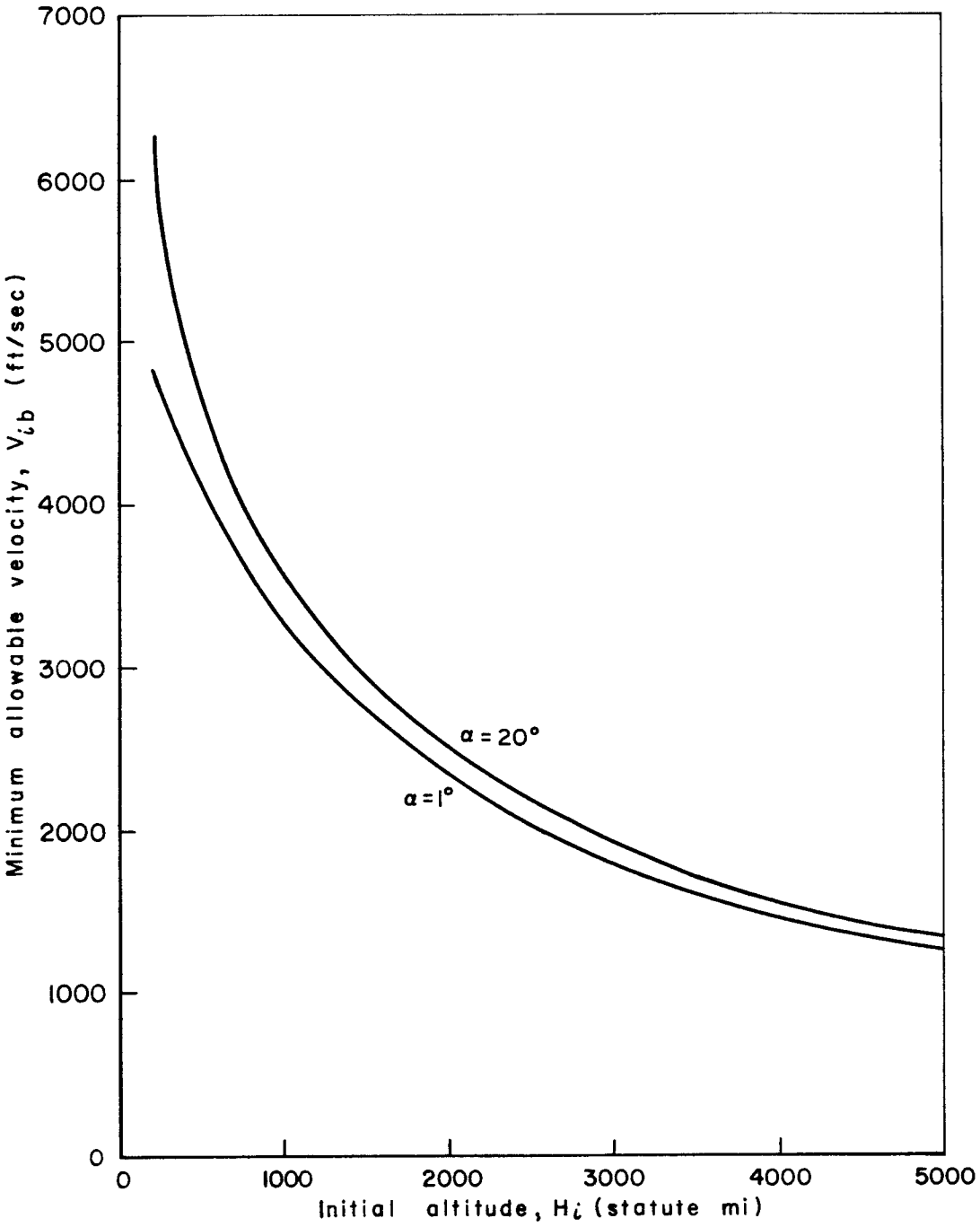

Fig. 8 - Minimum allowable initial velocity

Fig. 9 - Collision due to insufficient initial velocity

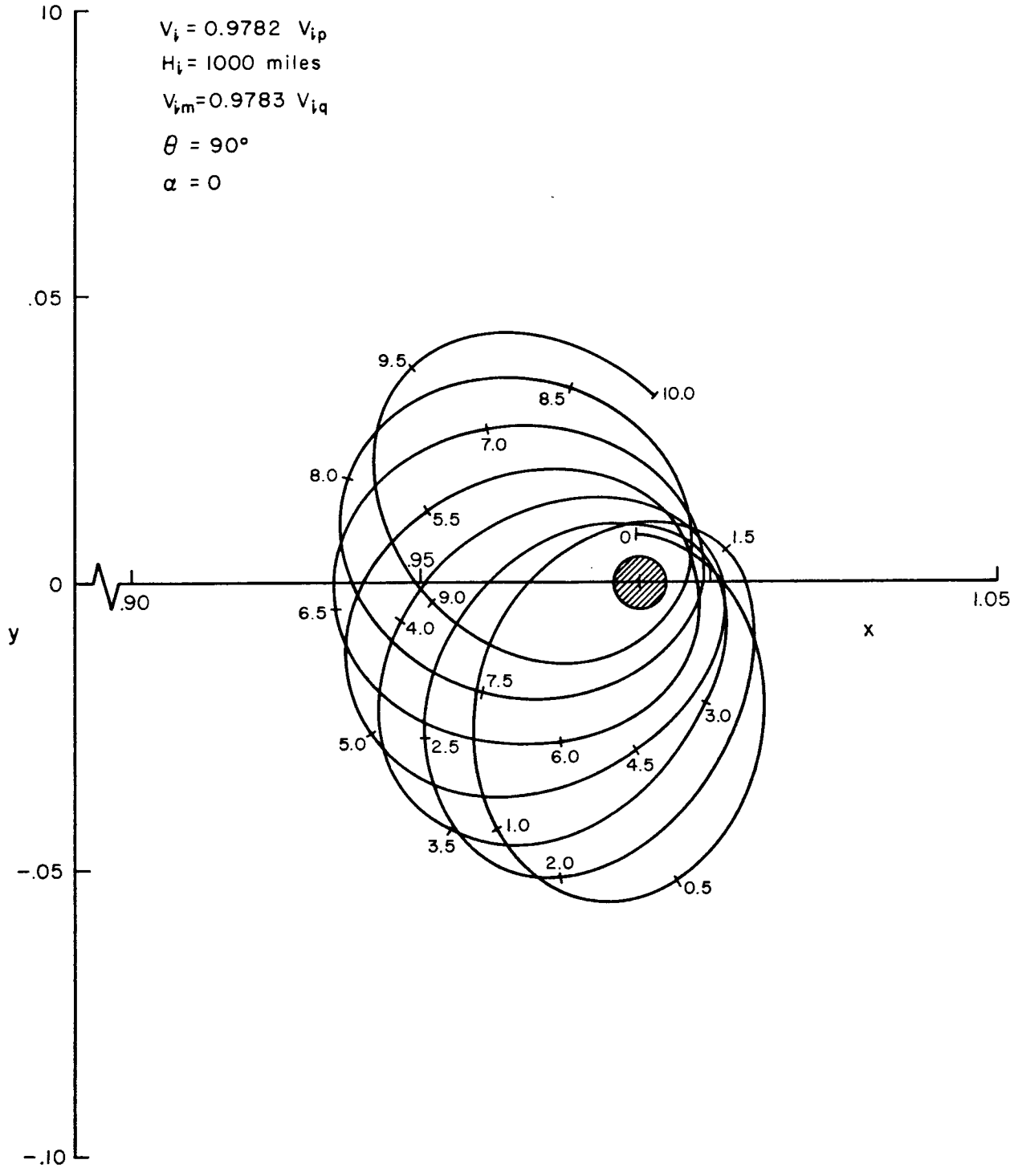

Fig. 10 - Retrograde orbit

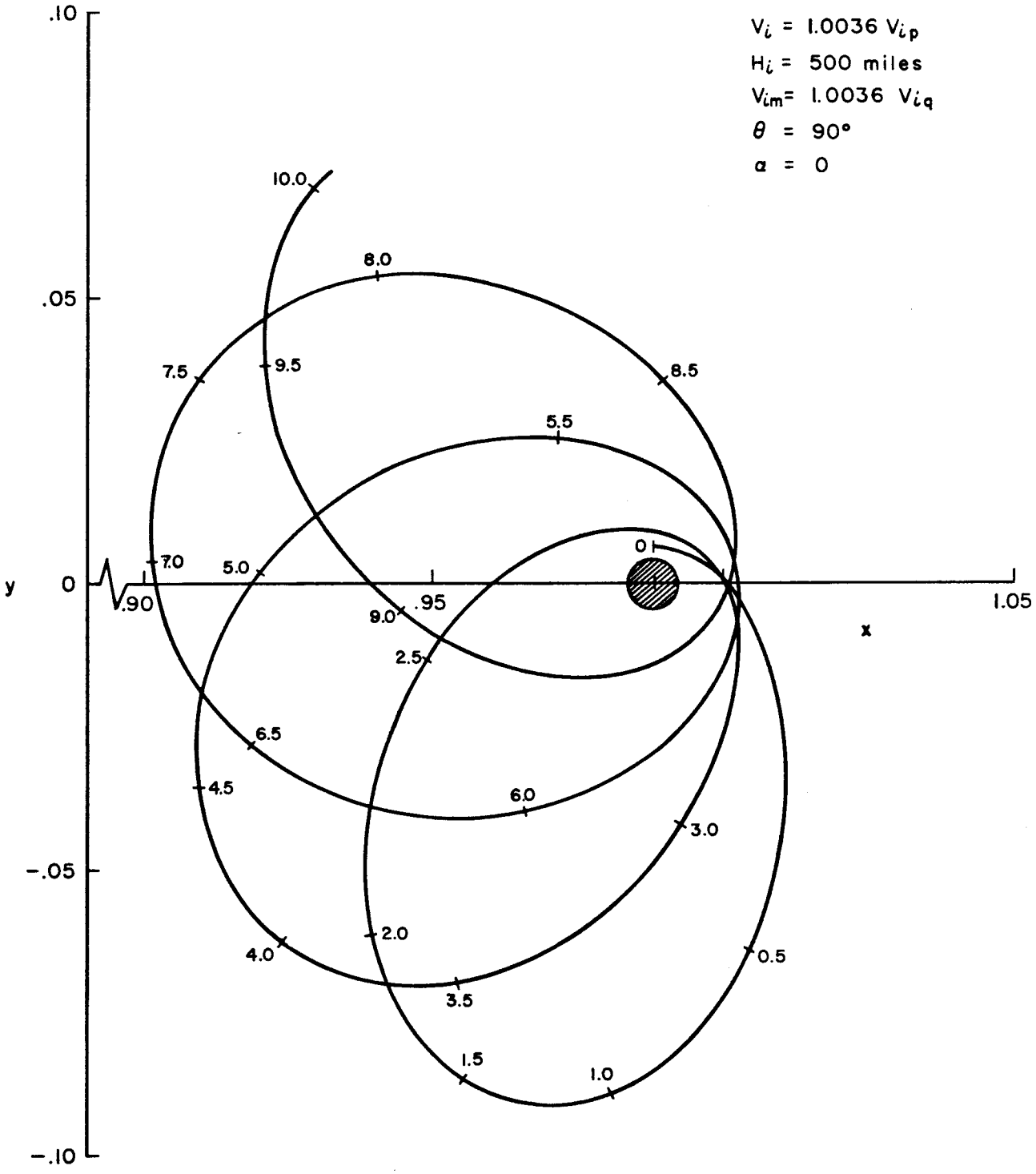

Fig. 11 - Retrograde orbit

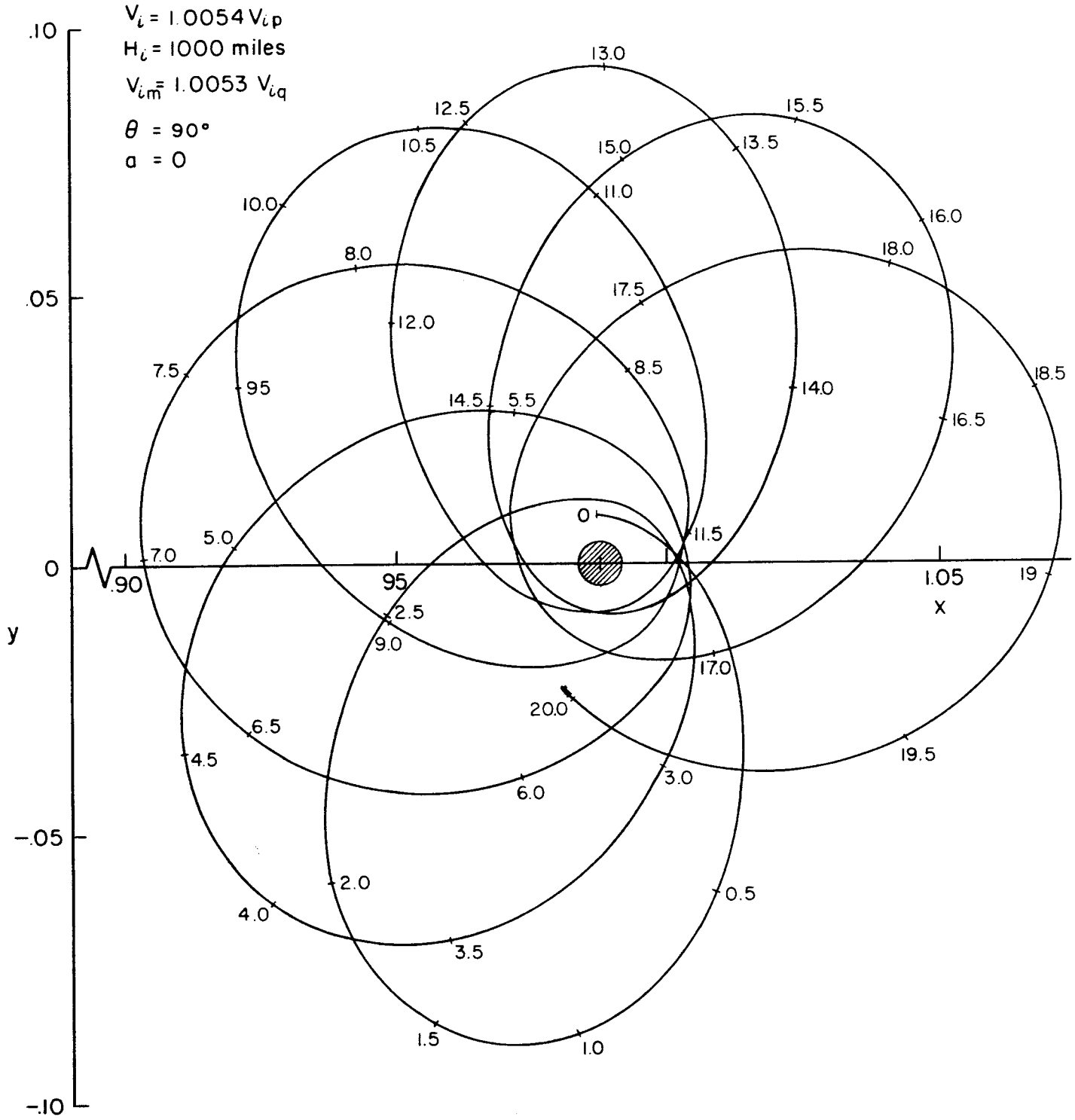

Fig. 12 - Retrograde orbit

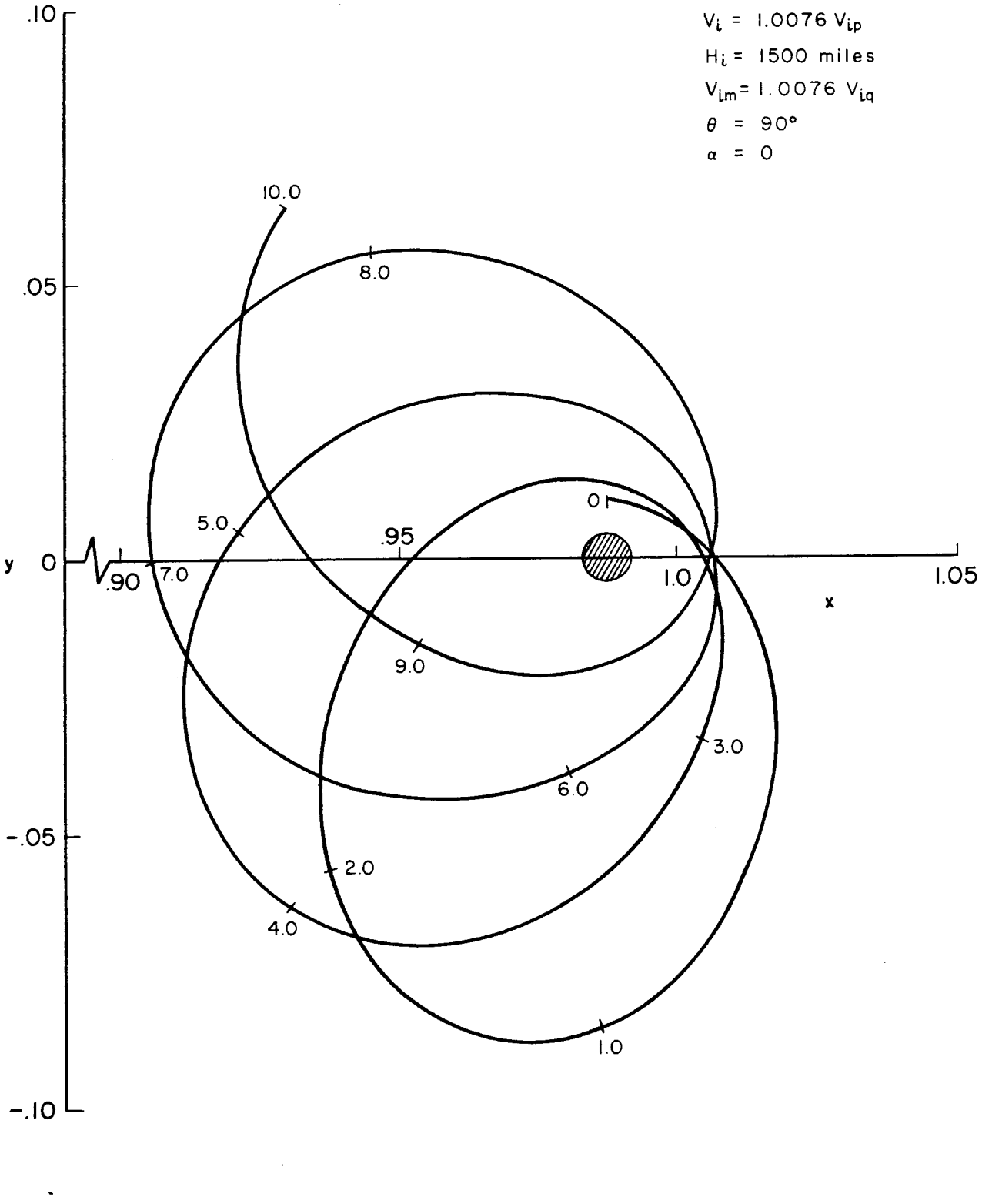

Fig. 13 - Retrograde orbit

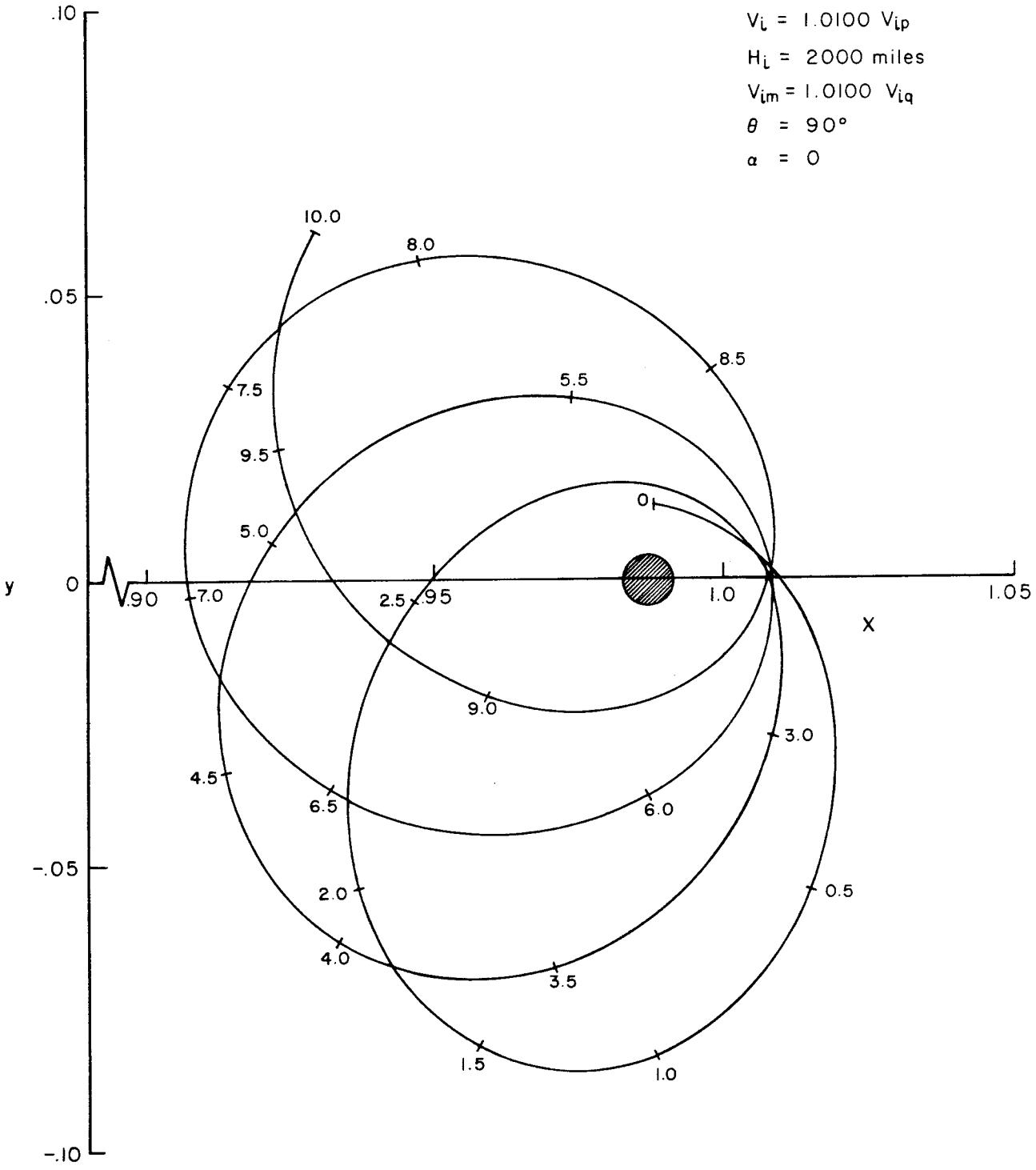

Fig. 14 - Retrograde orbit

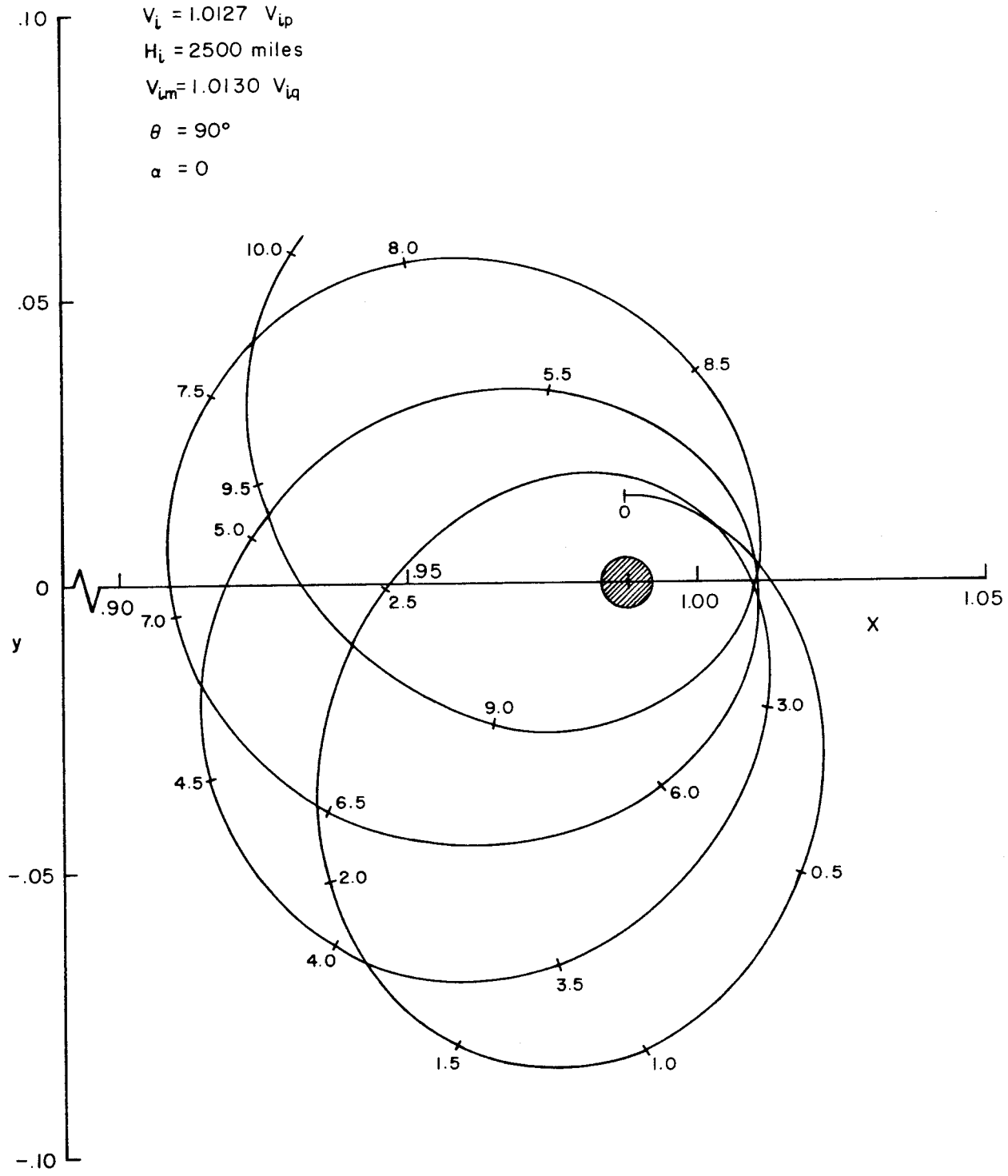

Fig. 15 - Retrograde orbit

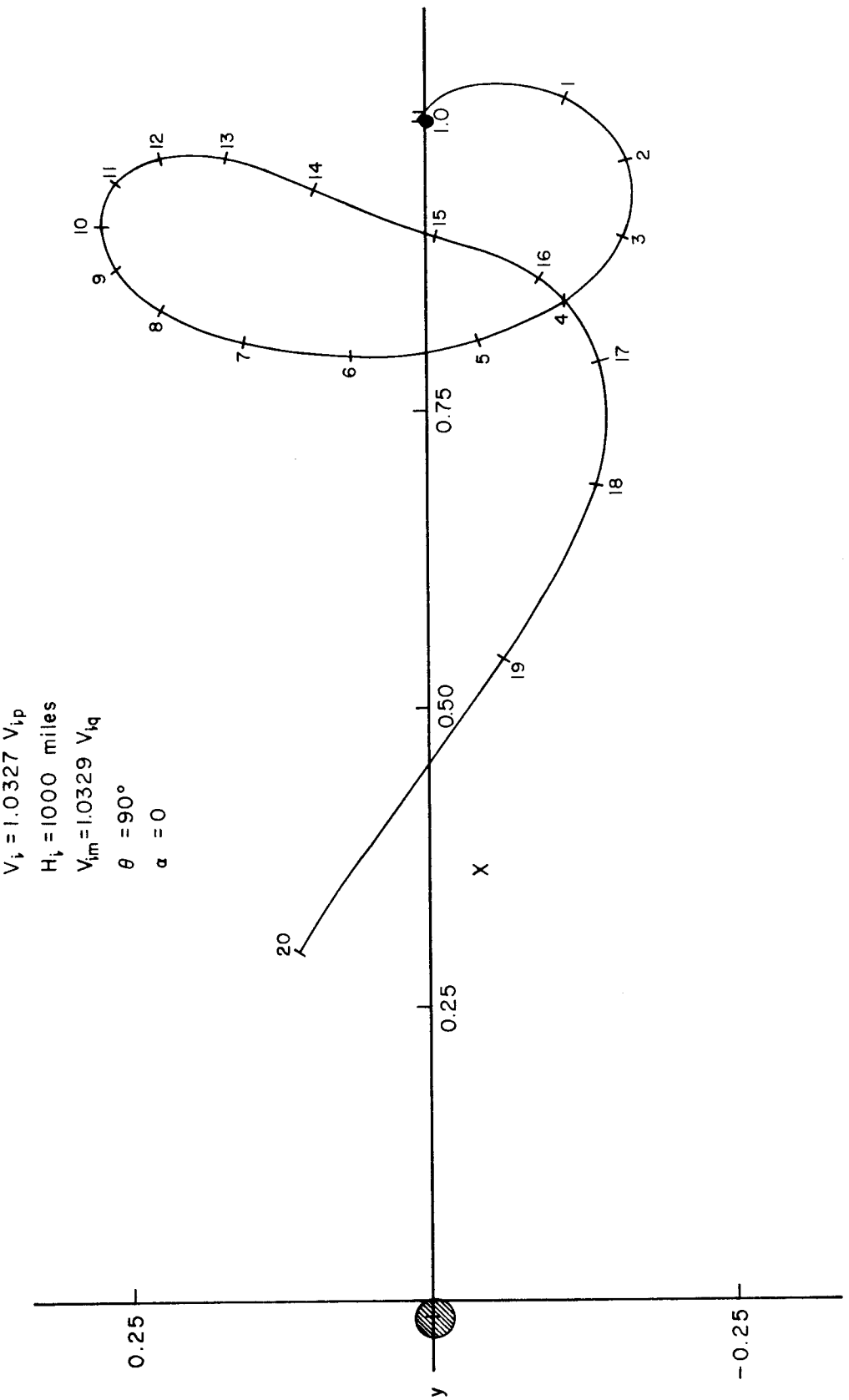

Fig. 16 - Recapture of lunar satellite by the earth

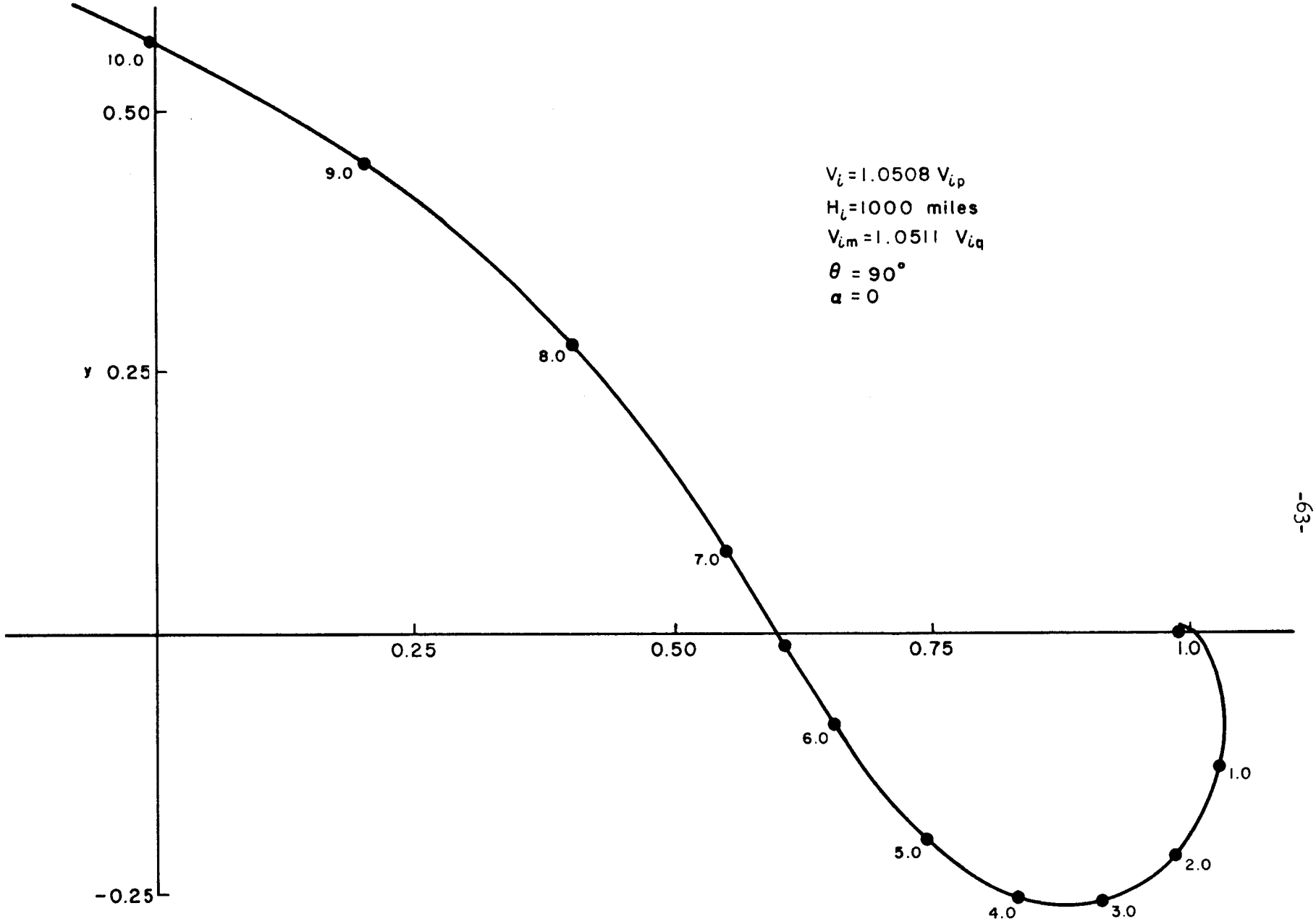

Fig. 17 - Recapture of lunar satellite by the earth

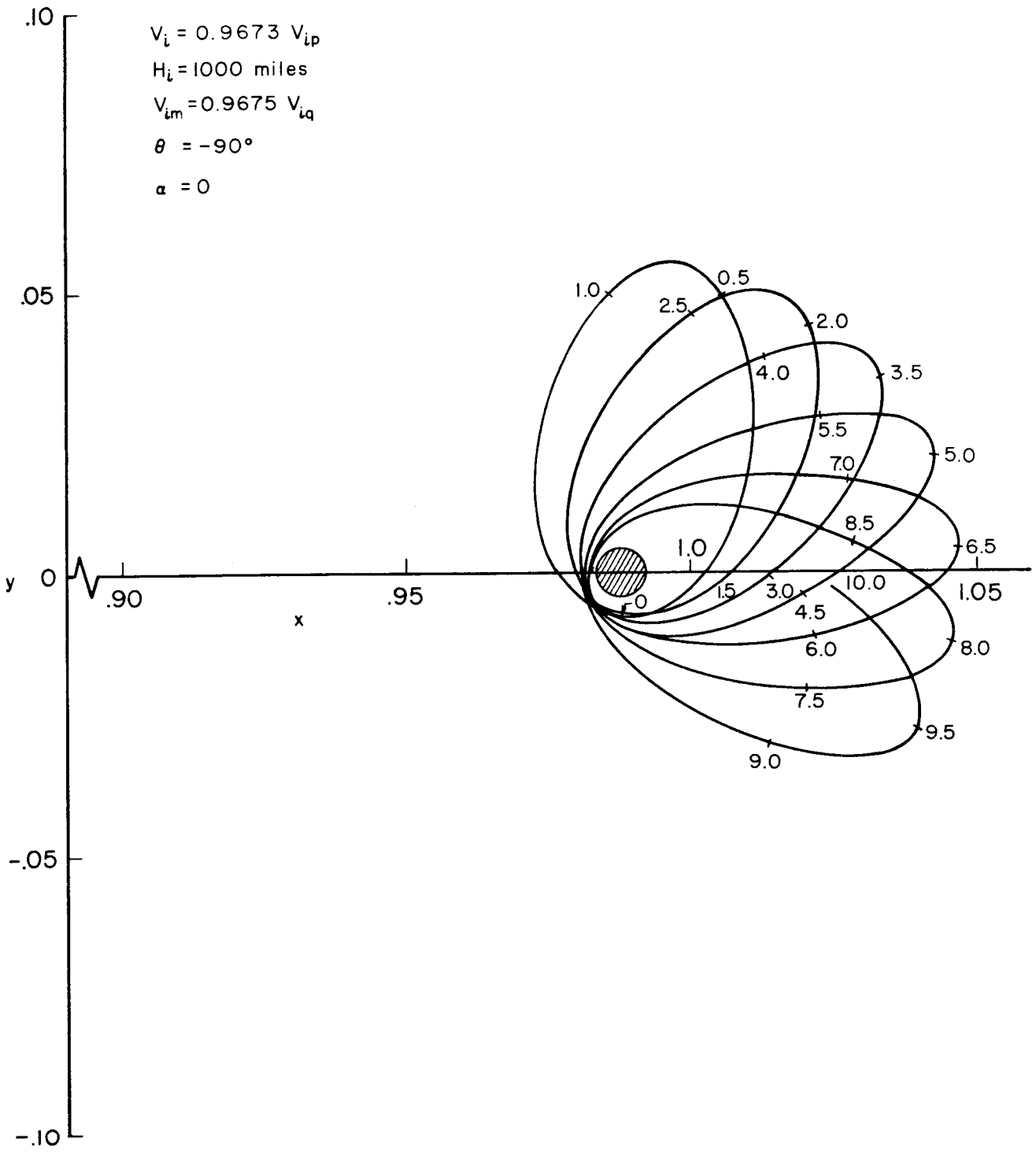

Fig. 18 - Direct orbit

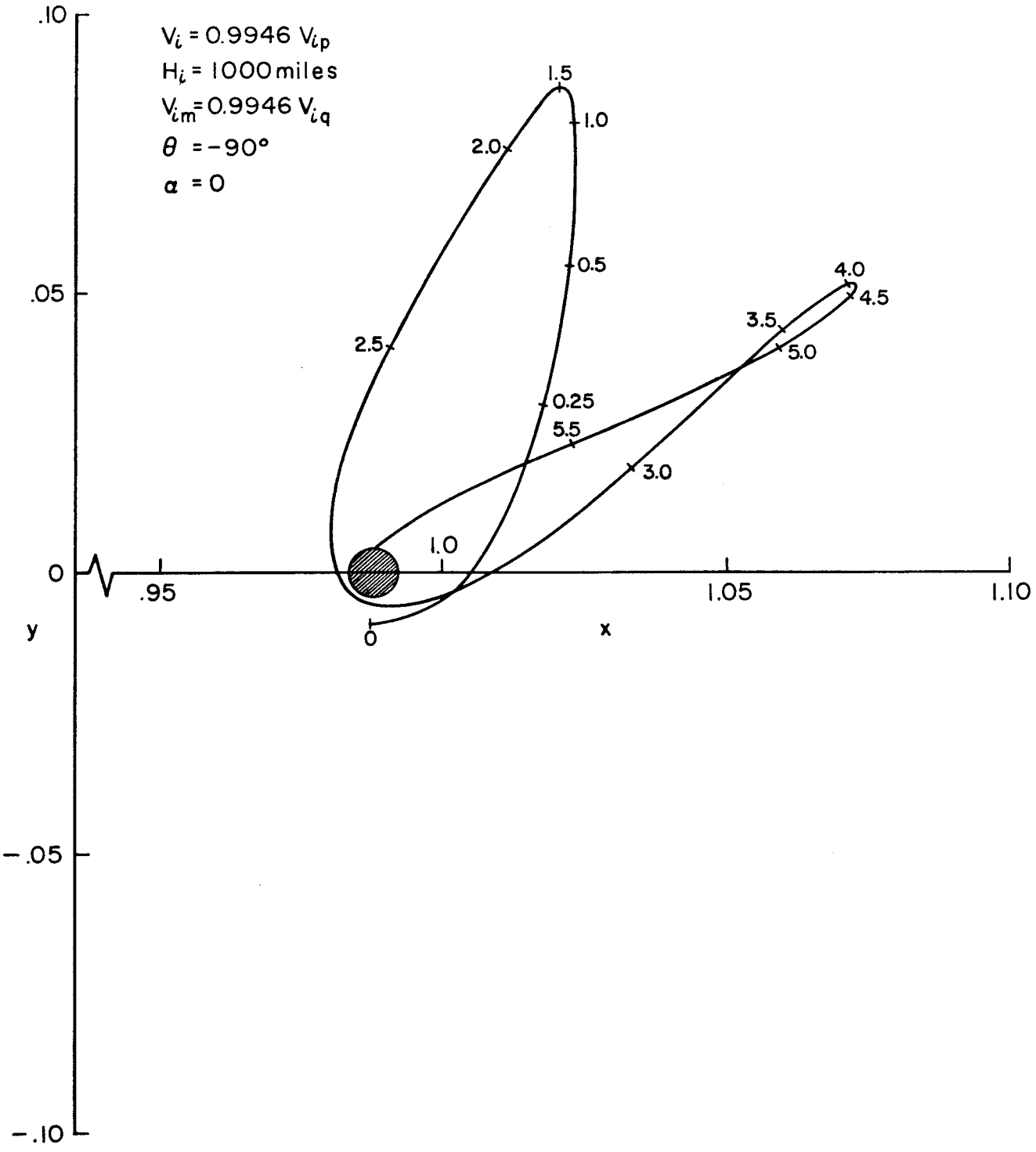

Fig. 19 - Direct orbit

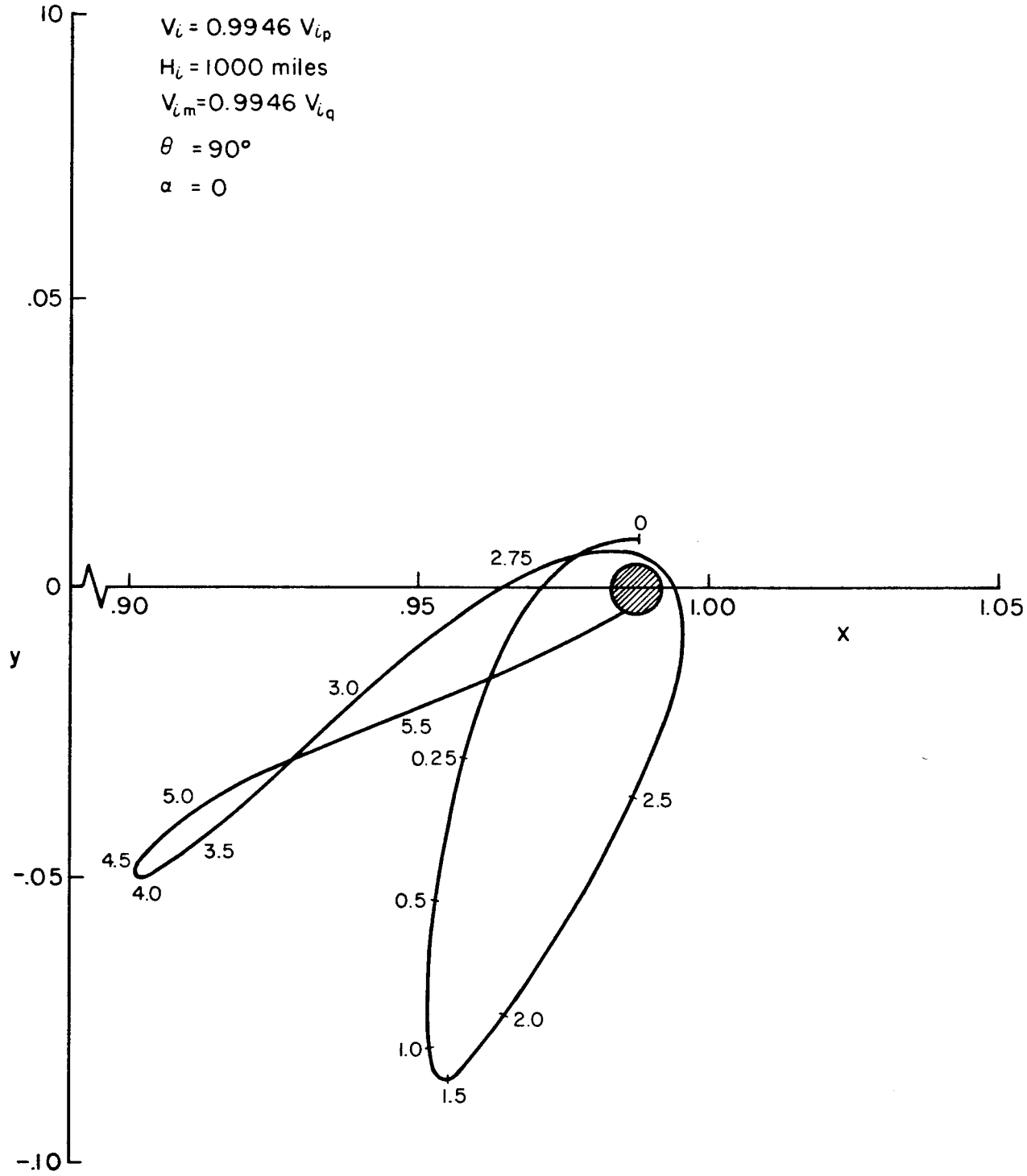

Fig. 20 - Direct orbit

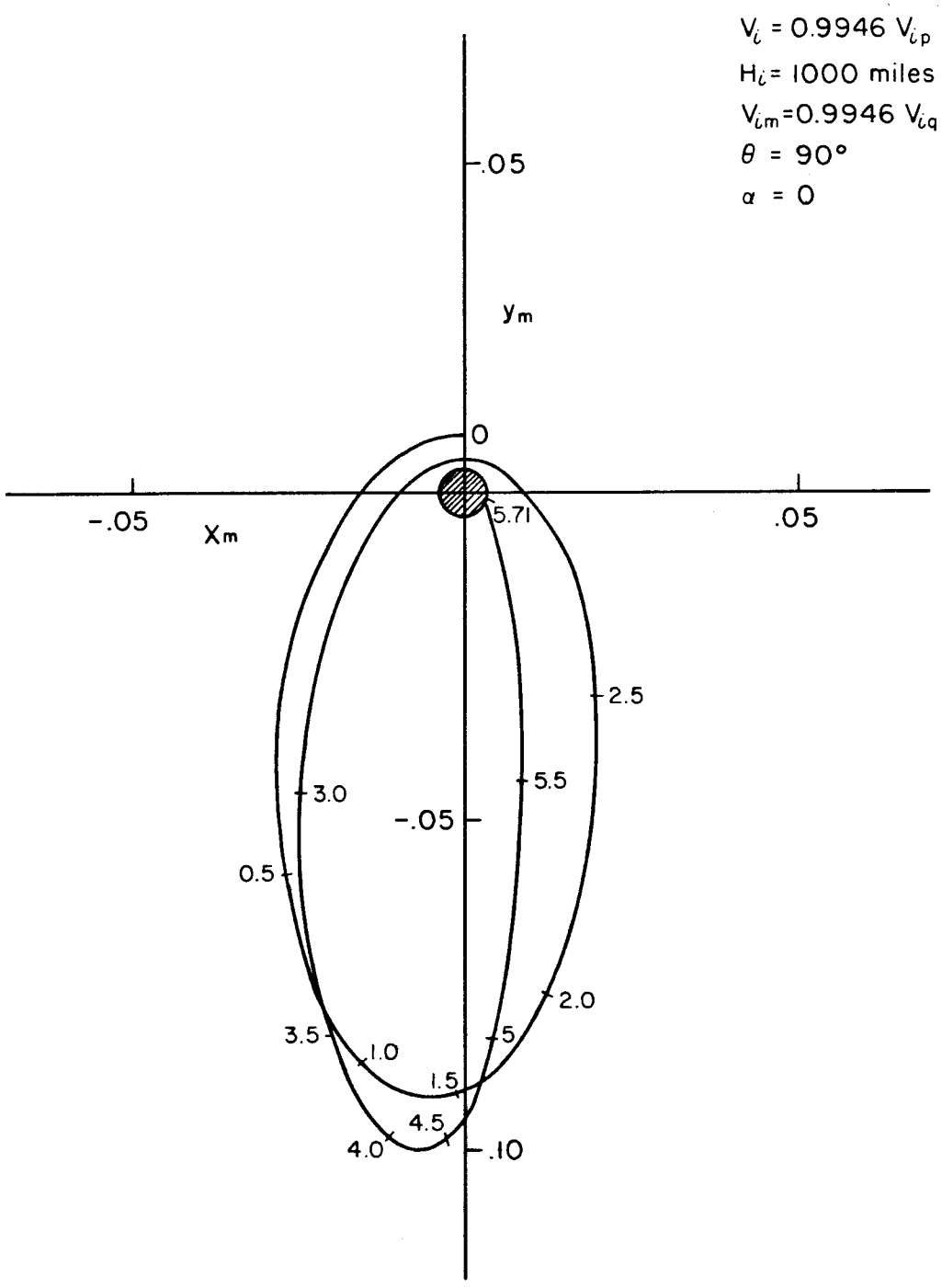

Fig. 21 - Direct orbit

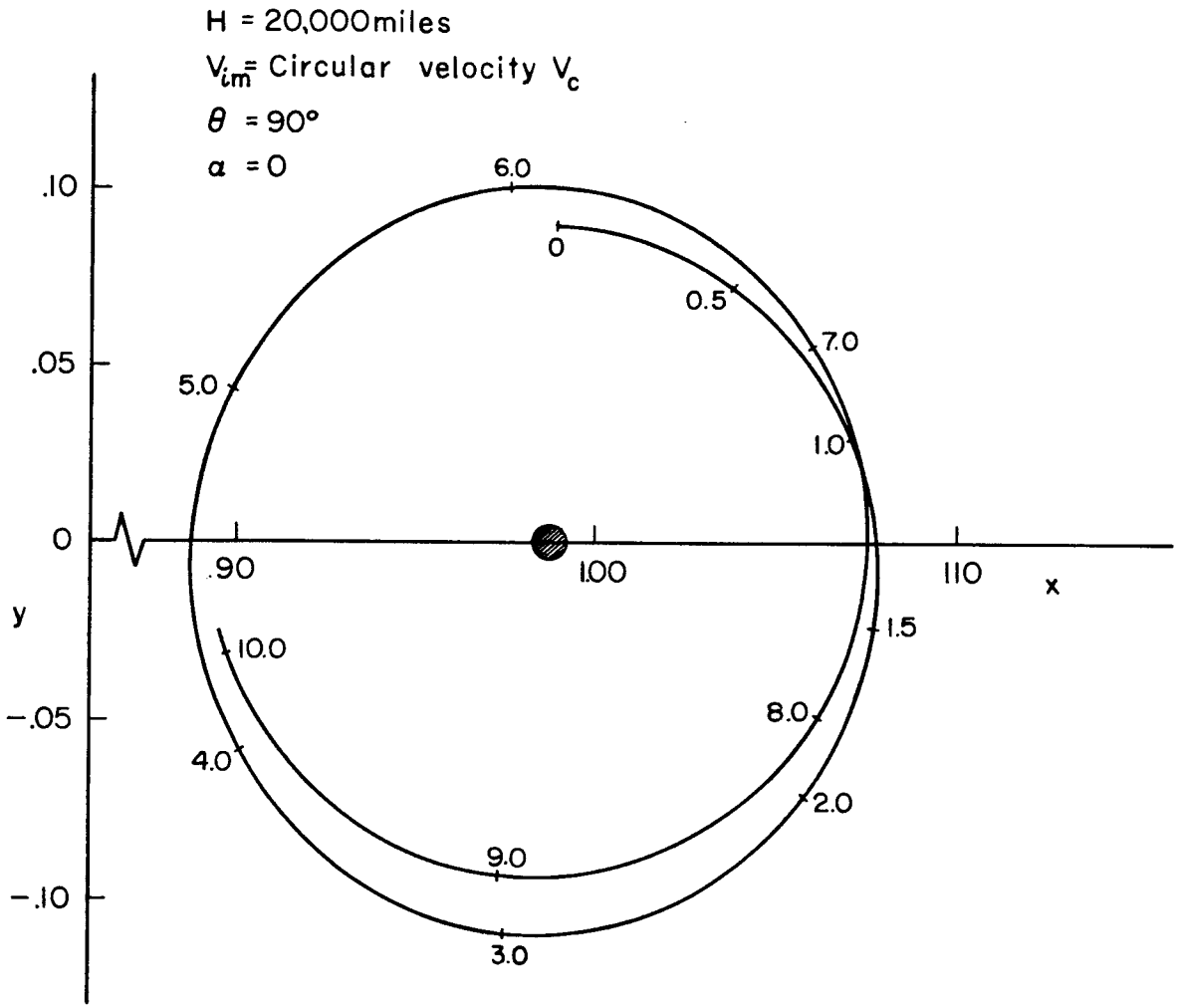

Fig. 22 - Nearly circular orbit

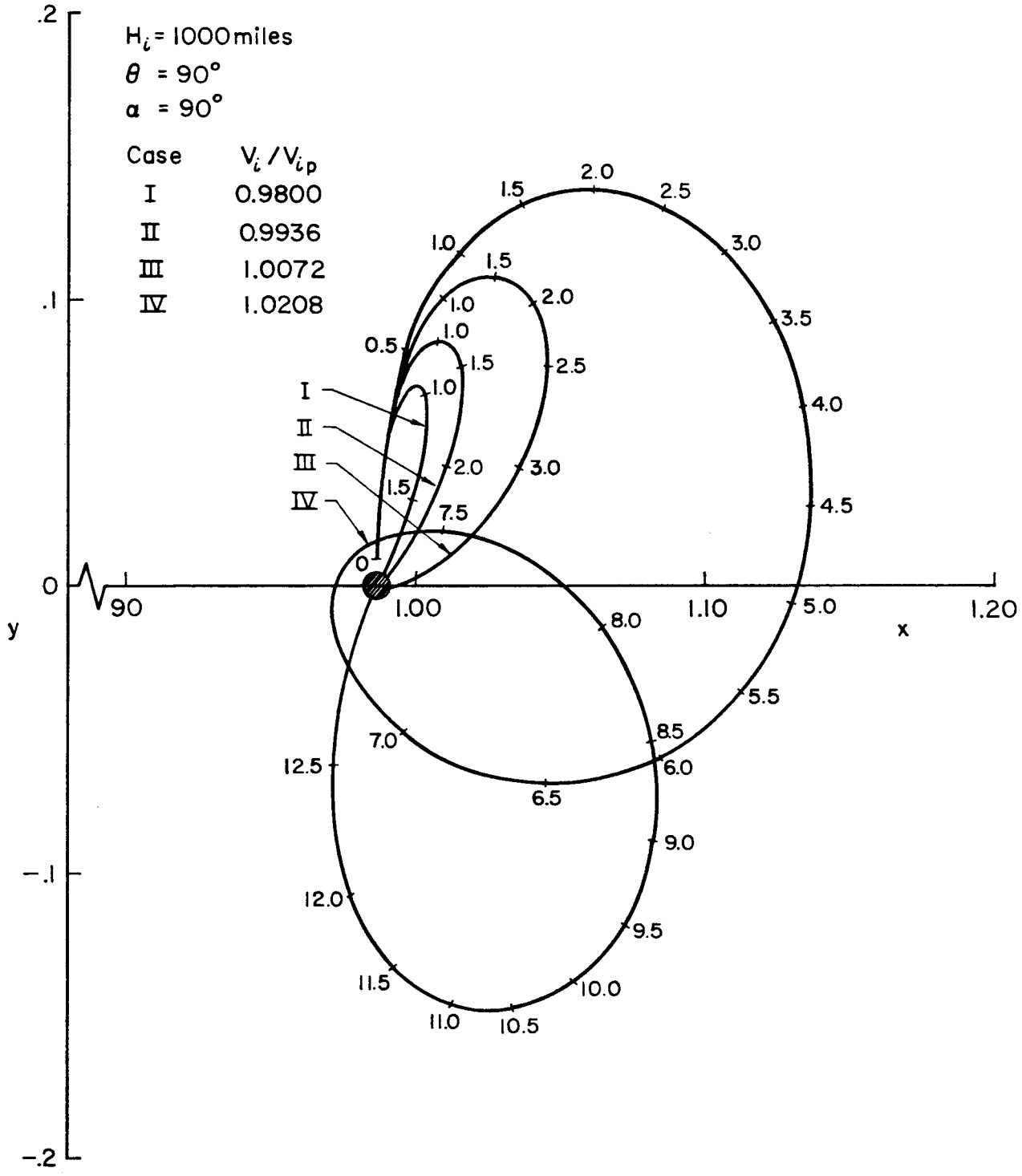

Fig. 23 - Vertical projection from the moon

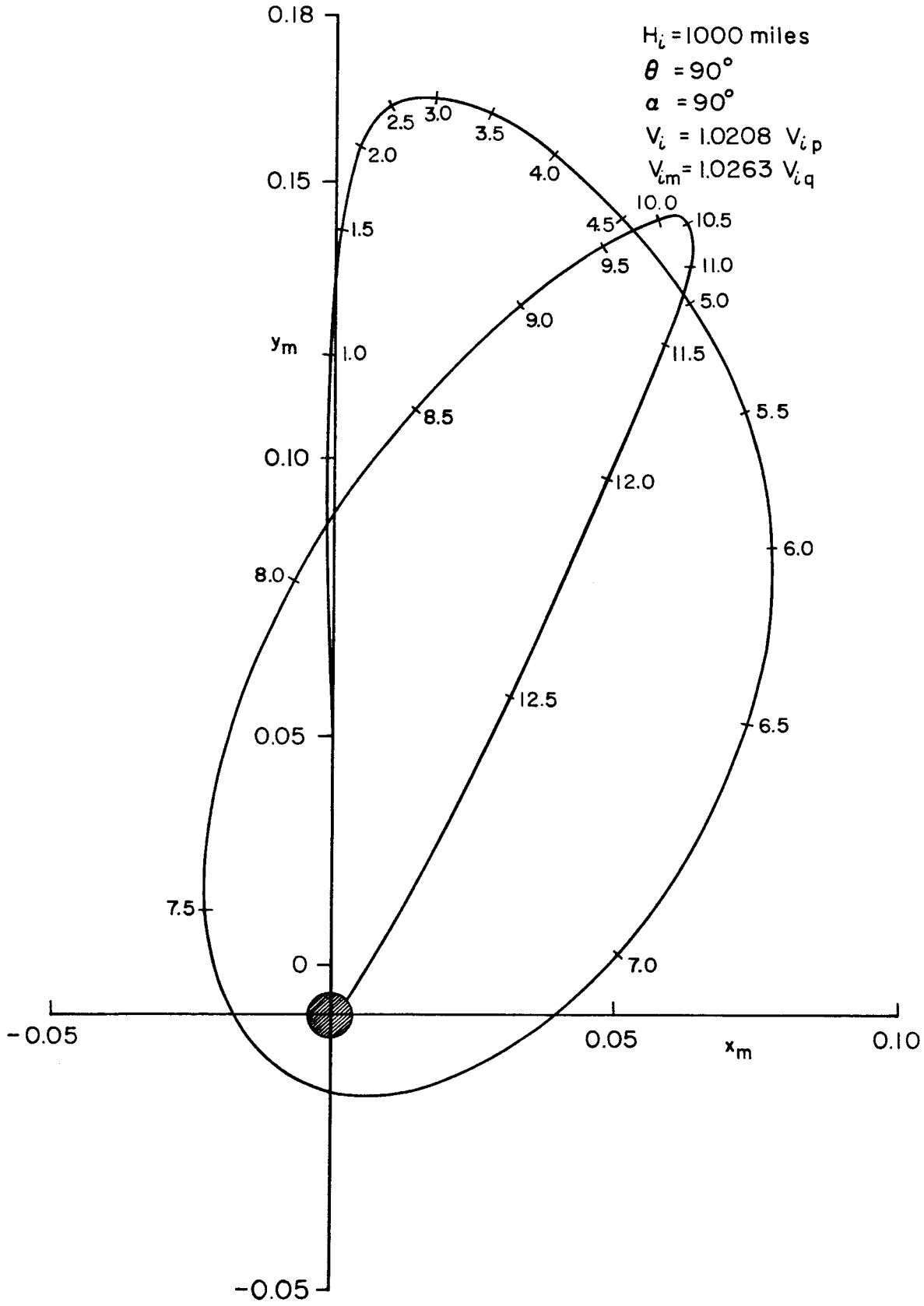

Fig. 24 - Vertical projection from the moon

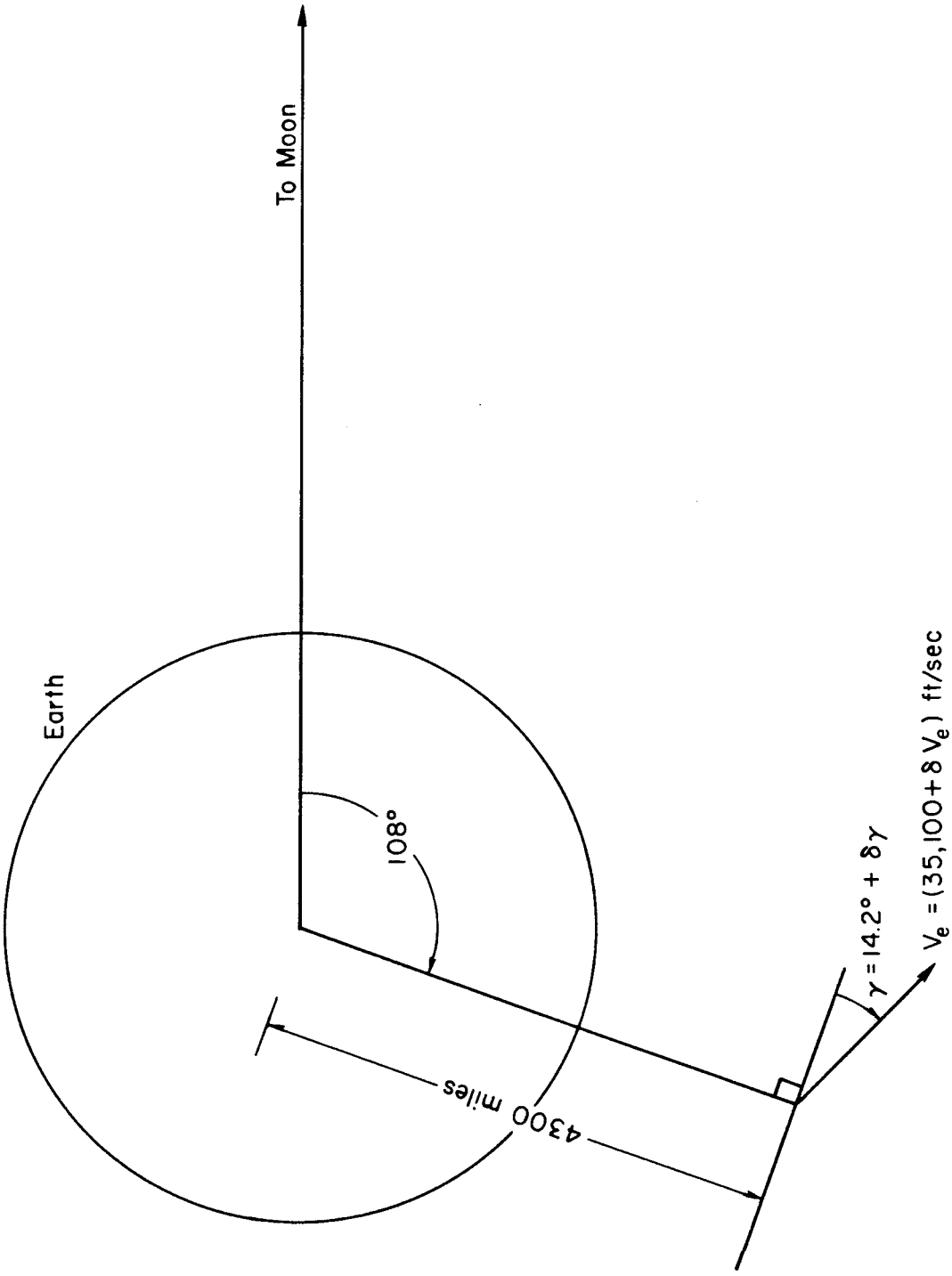

Fig. 25 - Initial conditions for Earth-Moon transit

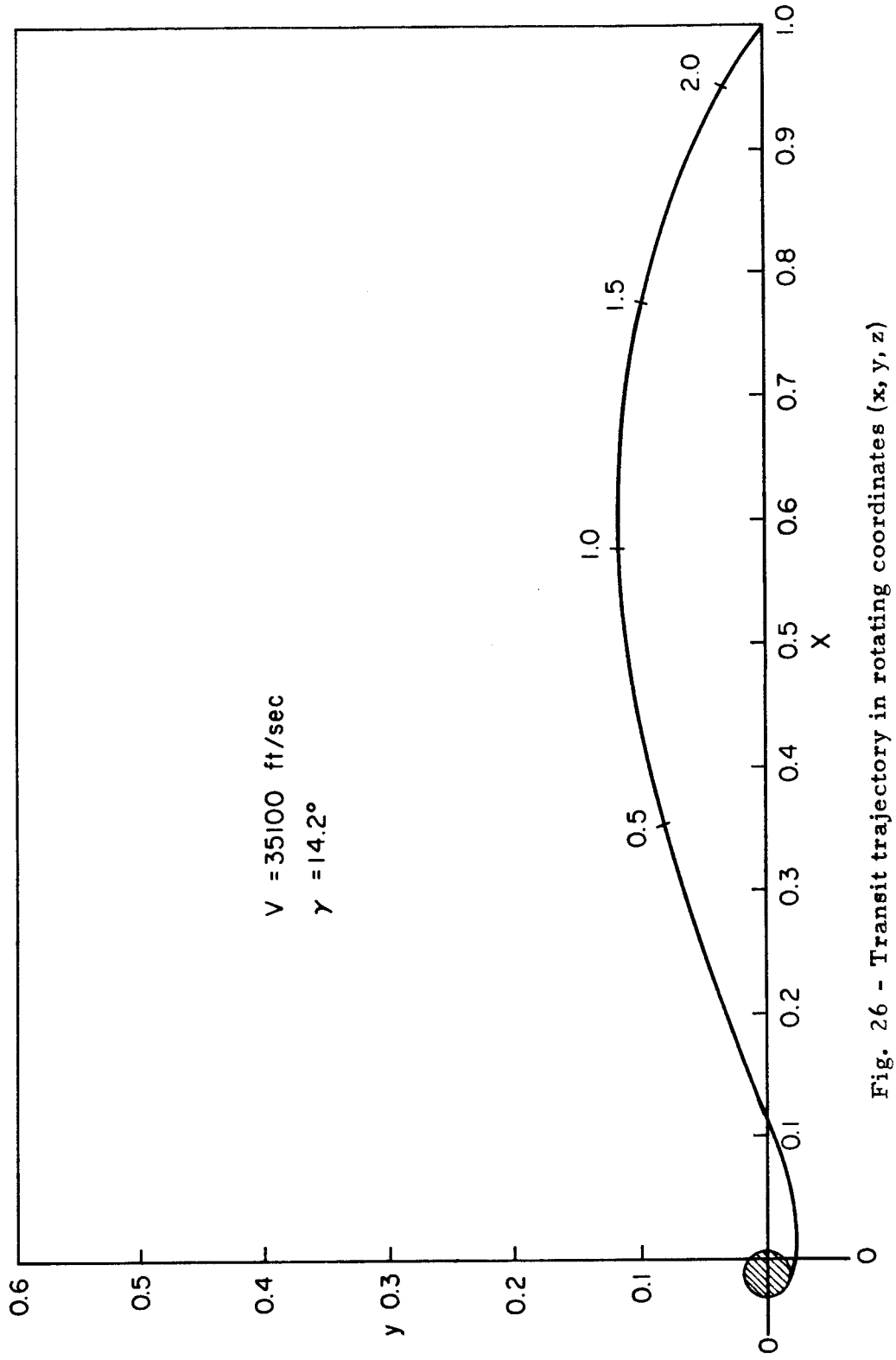

Fig. 26 - Transit trajectory in rotating coordinates (x, y, z)

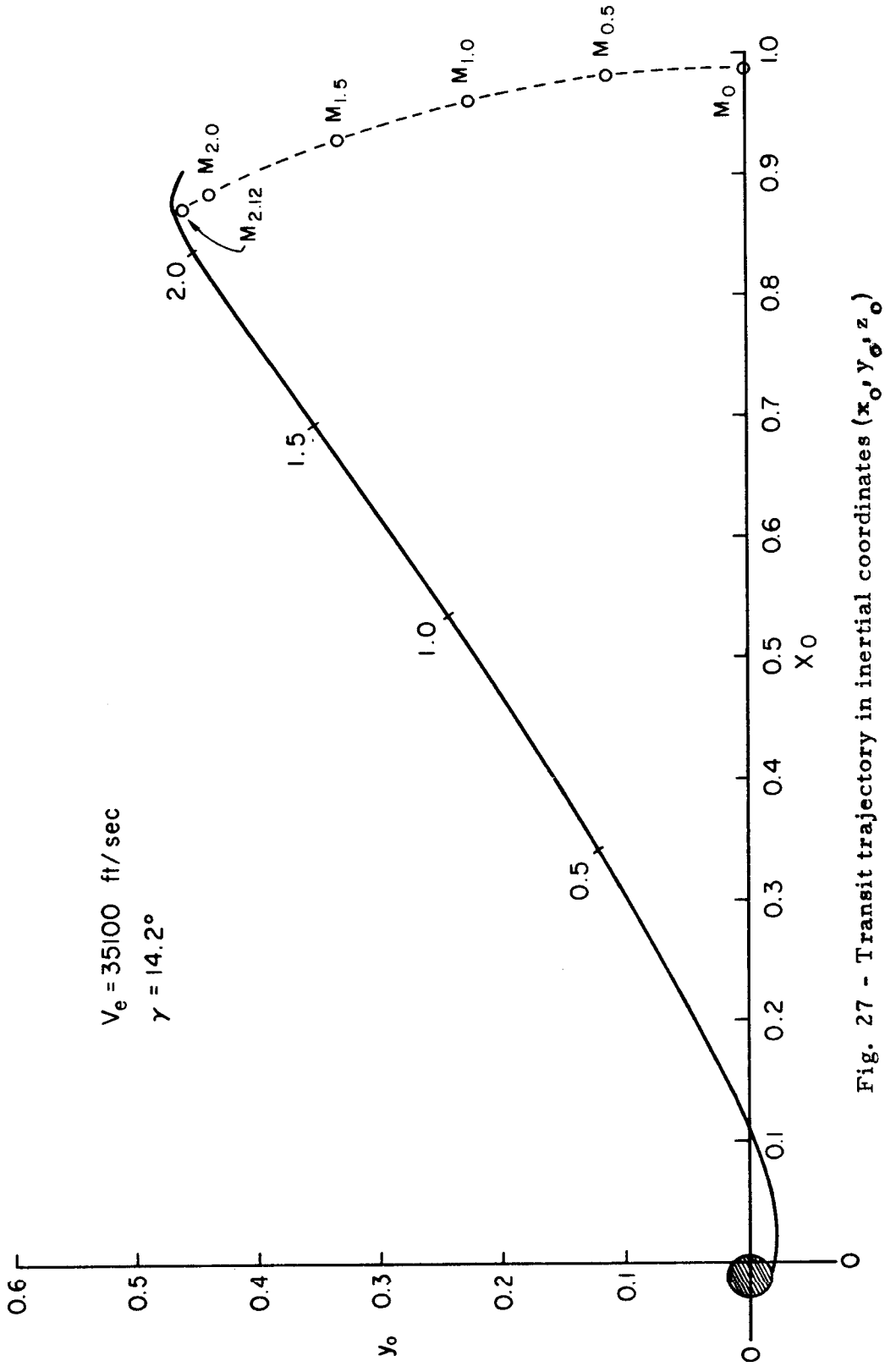

Fig. 27 - Transit trajectory in inertial coordinates (x_o, y_o, z_o)

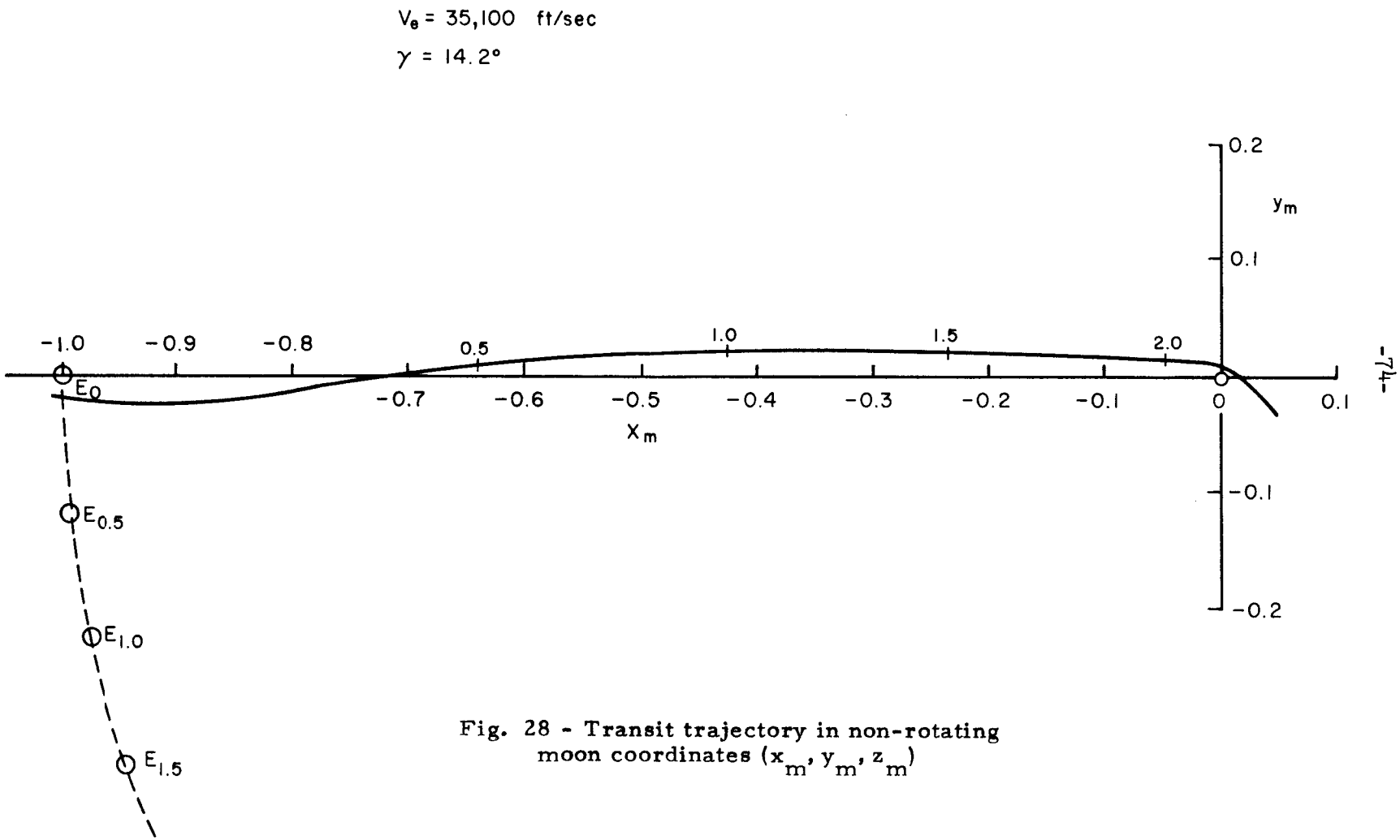

Fig. 28 - Transit trajectory in non-rotating moon coordinates (x_m, y_m, z_m)

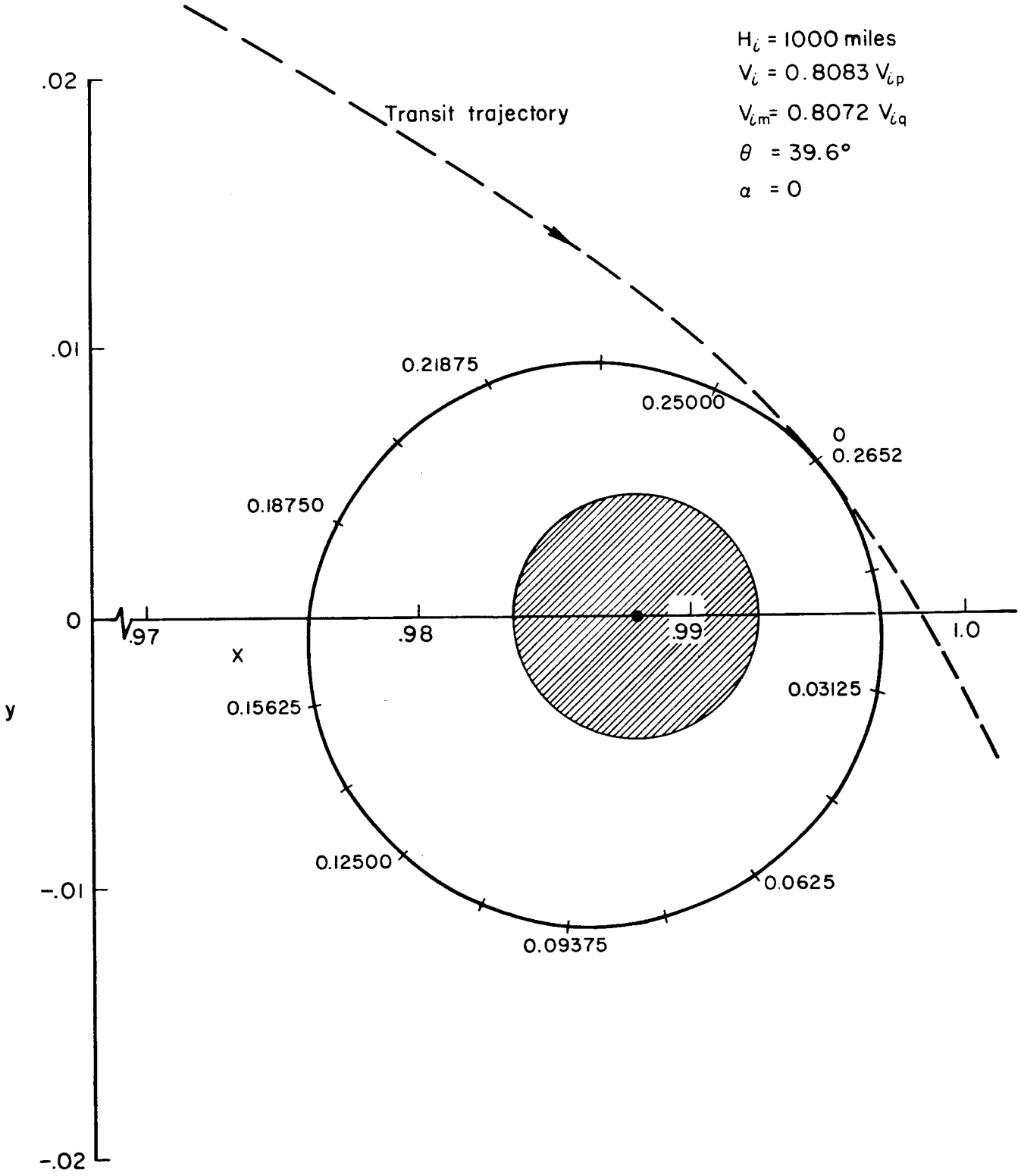

Fig. 29 - Moon satellite orbit resulting from design objective initial conditions

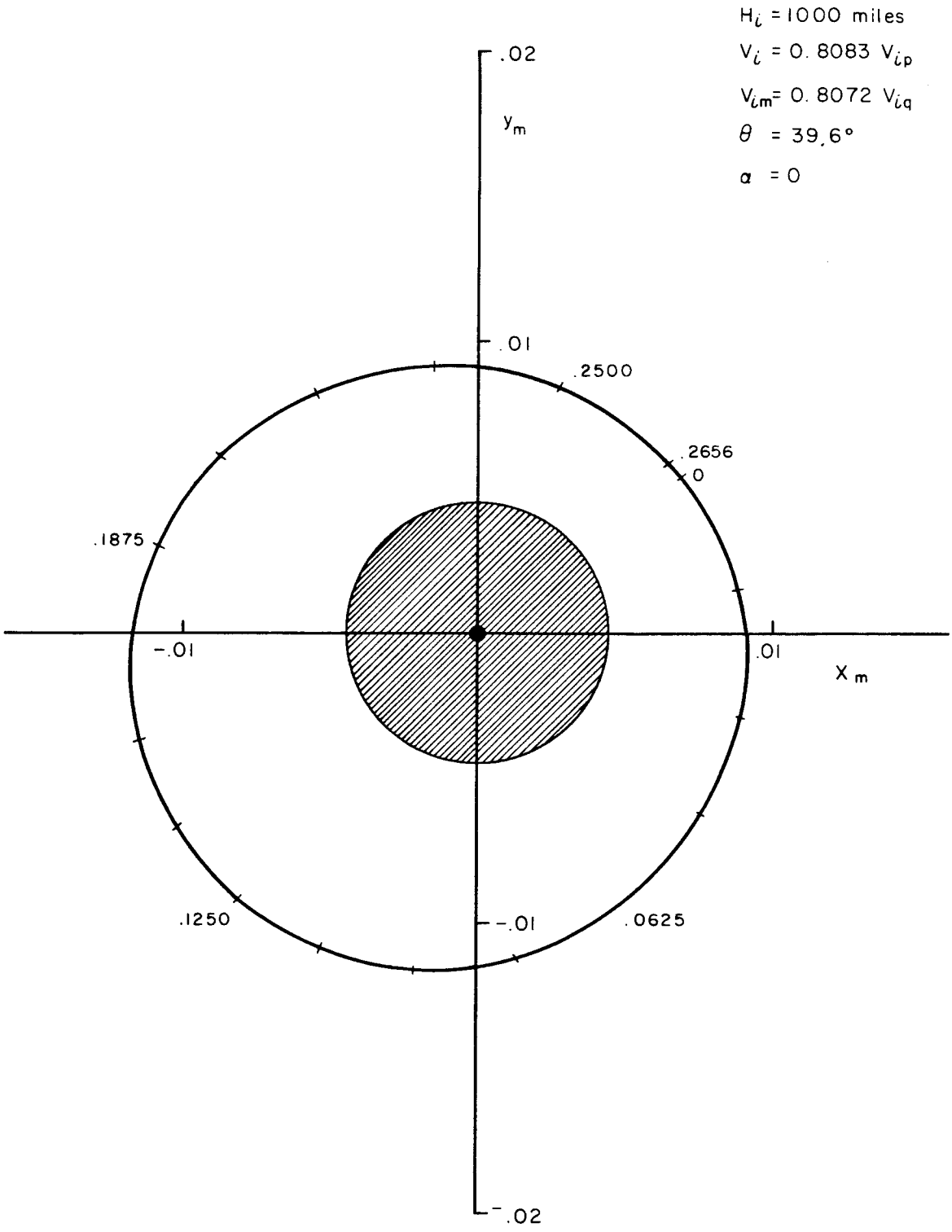

Fig. 30 - Moon satellite orbit resulting from design objective initial conditions

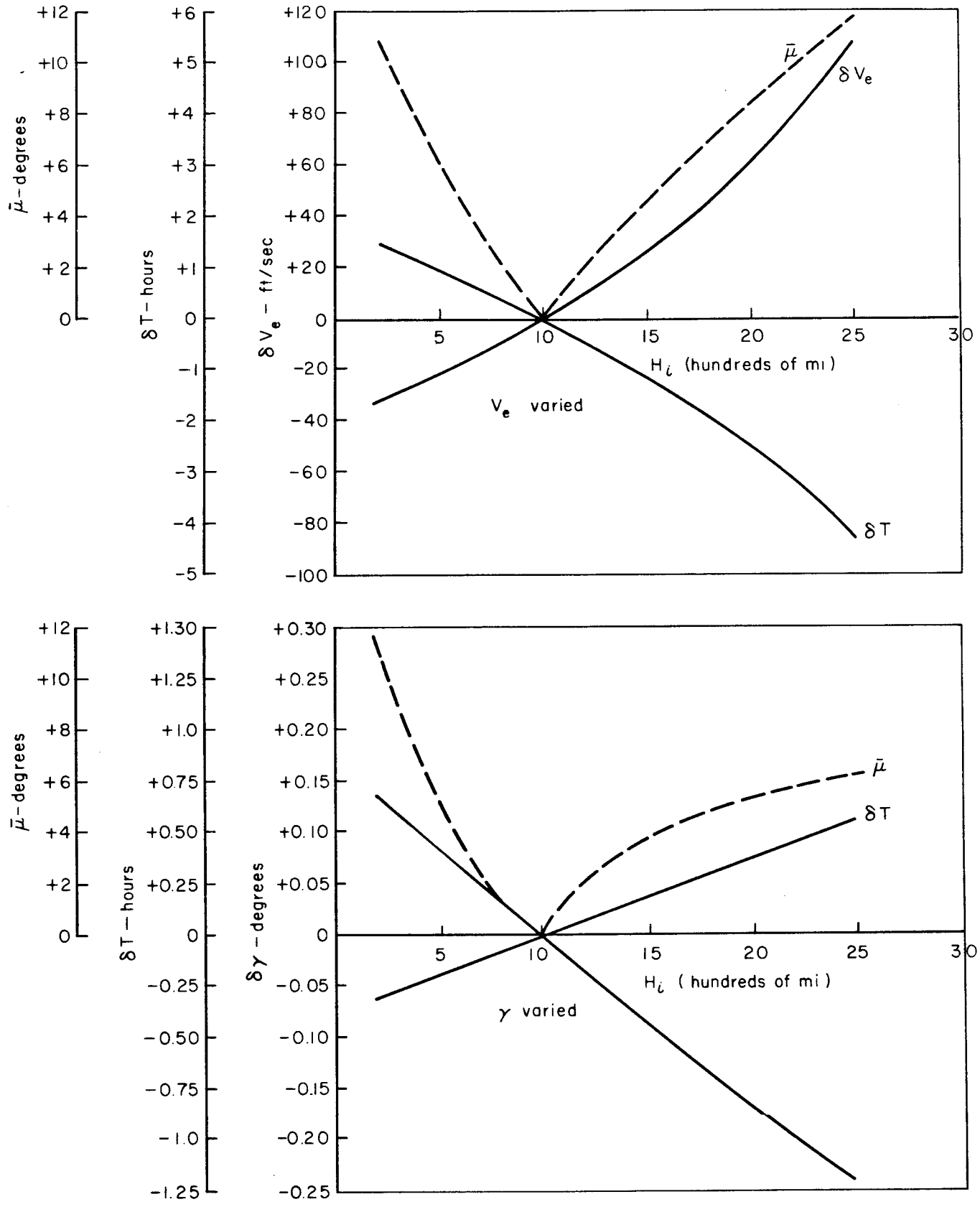

Fig. 31 - Transit trajectory parameters

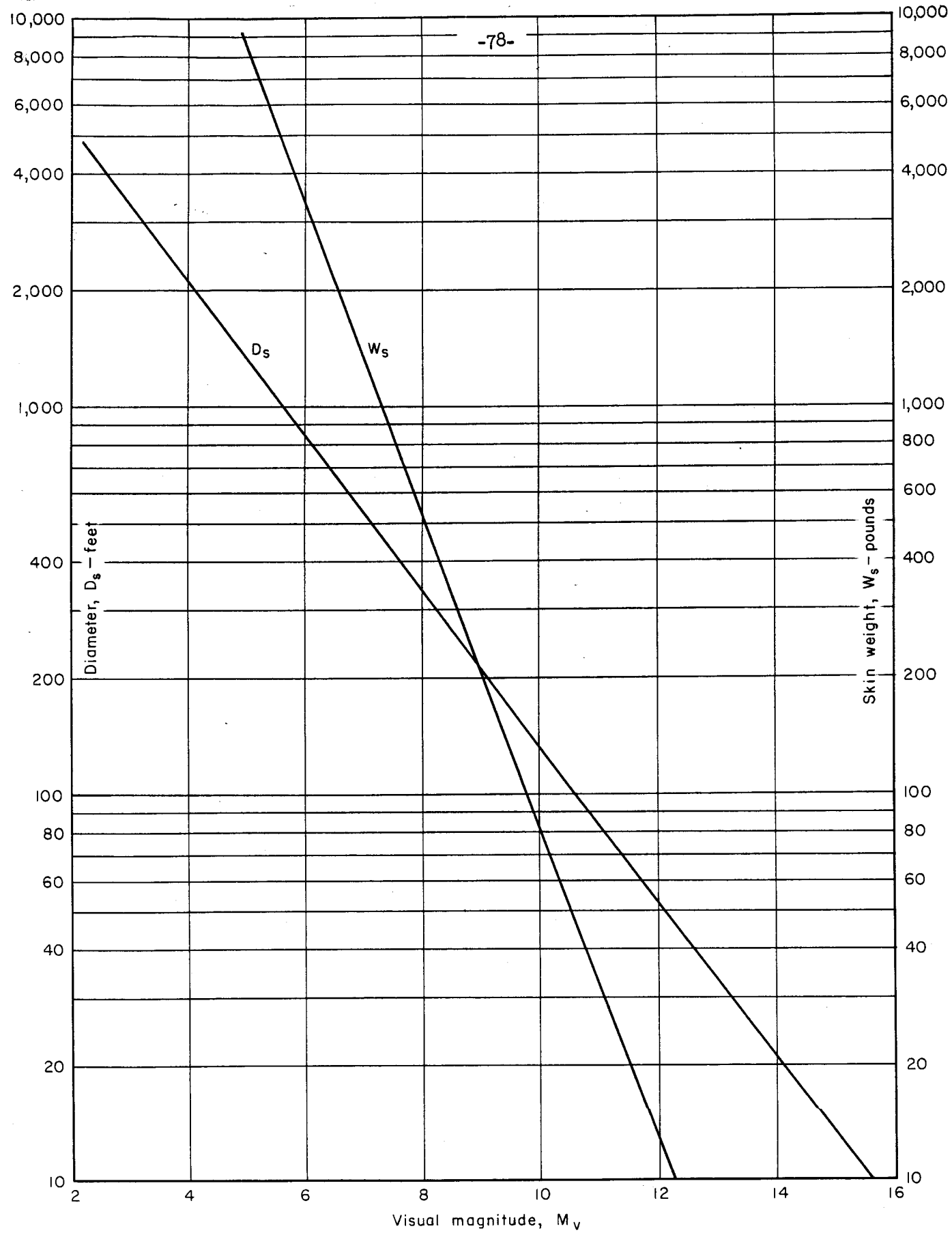

Fig. 32 - Diameter and skin weight of satellite vs full-phase visual magnitude

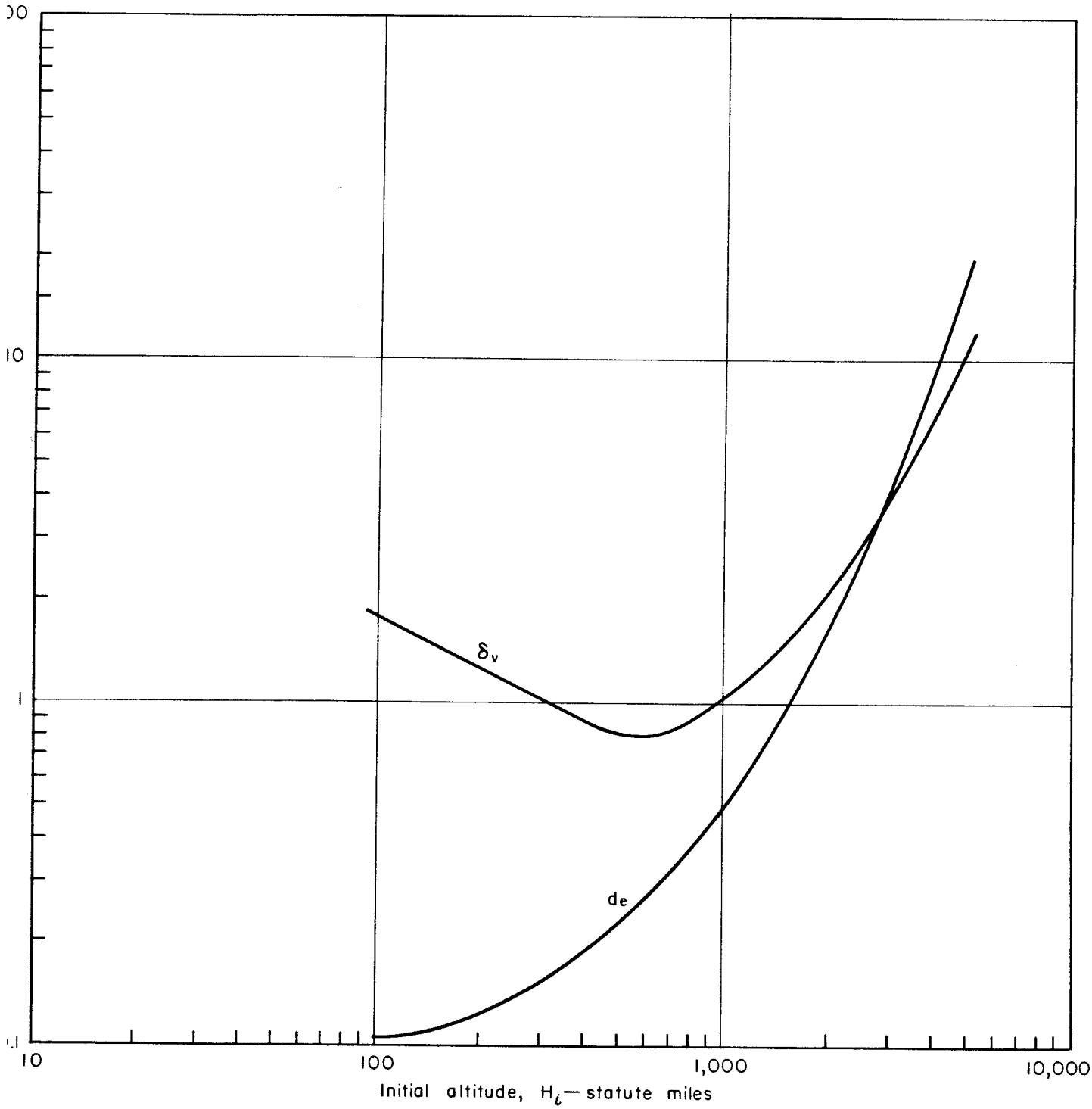

Fig. 33 - Errors in estimating V_{ib}