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RESEARCH MEMORANDUM

A METHOD FOR DETERMINING APPROXIMATE PROPULSION CUTOFF CONDITIONS FOR BALLISTIC INTERPLANETARY TRAJECTORIES

L. N. Rowell

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SUMMARY

This memorandum presents a technique for determining approximate propulsion cutoff conditions for ballistic-type interplanetary trajectories. Development of the technique, which is based on a repeated solution of the two-body problem with a desk calculator, uses elliptical and noncoplanar planetary orbits.

The total trajectory, extending from the cutoff point to the destination planet, is assumed to consist of two elliptical orbits. The first of these orbits is geocentric (earth-centered), the second heliocentric (sun-centered). Planetary perturbations are ignored.

A heliocentric elliptical orbit, assumed to extend from the center of the departure planet to the center of the destination planet, is determined for a given transfer angle. This determination is subject to the assumption that the planets are massless and that the perihelion point of the heliocentric orbit coincides with the center of the departure planet.

Since transition from the geocentric to the heliocentric orbit supposedly will occur at a point remote from the departure planet, the geocentric hyperbolic orbit is computed by using the heliocentric orbital elements.

Finally, the approximate cutoff conditions are obtained analytically by using a simple "orbit-patching" technique. This assumes that the velocity vector of the vehicle on the geocentric orbit is equal to the velocity vector of a reference vehicle on the heliocentric orbit, at a point remote from the earth.

By solving the two-body problem repeatedly with a desk calculator, it is possible to obtain a close approximation of the correct cutoff conditions

for an interplanetary trajectory. Since these cutoff conditions will be satisfactory for most preliminary design studies, it will not be necessary to compute the entire trajectory on a large-scale computer.

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SYMBOLS

a	semi-major axis of the conic section
a_h	semi-major axis of the geocentric hyperbolic orbit
E_{di}	position of the earth at the departure date
E_s	heliocentric, eccentric anomaly of the vehicle
e	eccentricity of the conic section
e_h	eccentricity of the geocentric hyperbolic orbit
F	the eccentric anomaly of the vehicle in the geocentric hyperbolic orbit
G	the square root of the sum of the squares of the residual velocities
i	the inclination of the earth's equatorial plane to the ecliptic plane
i_s	inclination angle of the vehicle's heliocentric orbit plane to the ecliptic plane
k, k_1, k_2	differences between the x, y, and z components respectively, of the velocities of the reference vehicle and the earth's center
M_{ai}	position of Mars at arrival date
n_s	the mean angular rate of the vehicle in the heliocentric orbit
$n_\oplus$, n_σ	mean angular rates of the Earth and Mars, respectively
$\bar{r}$	radius vector from the focus (center of attraction) to the body
r_e	distance of the earth from the sun
r_p	distance between the vehicle and the earth's center
r_s	distance of the vehicle from the sun
r_{co}	distance of the vehicle from the center of earth at propulsion cutoff
r_{ed}	distance between earth and sun at departure date
r_{ma}	distance between Mars and sun at arrival date
$\dot{r}_{ed}$	time rate of change of the distance between the earth and sun at the departure date

S_{ai}	position of the vehicle at arrival date
S_{di}	position of the vehicle at departure date
Δt	time required for the vehicle to move from the propulsion cutoff point a distance r_p from the earth's center
t_s	vehicle transfer time in the heliocentric orbit
Δt_d	change in the initial departure date
t_{ai}	dates of arrival at the destination planet
t_{co}	time of propulsion cutoff
t_{di}	dates of departure on heliocentric orbits
$\bar{V}_e$	velocity vector of the earth's center
V_p	velocity of the vehicle at a point on the geocentric hyperbolic orbit
$\bar{V}_p$	velocity vector of the vehicle at a point on the geocentric hyperbolic orbit
$\bar{V}_s$	velocity vector of the vehicle
V_∞	hyperbolic excess velocity relative to the earth
ΔV_d	increment of velocity to be added to the earth's velocity
V_{co}	velocity at propulsion cutoff
V_{esc}	velocity of escape from the earth
v	true anomaly
v_s	heliocentric transfer angle
v_∞	true anomaly of the vehicle in the geocentric hyperbolic orbit at an infinite distance from the earth
Δv	incremental change in the vehicle's true anomaly in the geocentric hyperbolic orbit
Δv_s	incremental change of the vehicle's true anomaly in the heliocentric orbit
v_{co}	true anomaly of the vehicle in the geocentric hyperbolic orbit at the propulsion cutoff point
x, y, z	geocentric equatorial coordinate system with origin at the earth's center, the x axis in the direction of the vernal equinox, and the equatorial plane defined by the x, y plane

x_h, y_h, z_h	geocentric hyperbolic plane coordinate system with origin at earth's center and the orbit plane defined by x_h, z_h plane
$\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$	residual component velocities
l_x, l_y, l_z	unit vectors along the x, y, z axes, respectively
l_{x_h}, l_{y_h}, l_{z_h}	unit vectors along the x_h, y_h, and z_h axes, respectively
γ_p	elevation angle of the vehicle's velocity vector on the geocentric hyperbolic orbit
γ_s	elevation of the vehicle's velocity vector in the heliocentric orbit
γ_{co}	elevation angle of the vehicle's velocity vector at propulsion cutoff
γ_{ed}	elevation angle of the velocity vector of the earth's center
δ	declination (earth latitude) of the propulsion cutoff point
θ	angle between the meridian planes containing the propulsion cutoff point and the vernal equinox direction
$\Delta\Lambda$	incremental change in the heliocentric longitude of the earth
Λ_{ma}	heliocentric longitude of Mars at arrival date
Λ_{edi}	heliocentric longitude of the earth at the departure date
Λ_{sdi}	heliocentric longitude of the vehicle at the departure date
λ_{ma}	heliocentric latitude of Mars at arrival date
ψ	azimuth direction from 0^o true north, of the cutoff velocity vector

I. INTRODUCTION

This memorandum presents and discusses a technique for determining the approximate cutoff conditions for ballistic-type interplanetary trajectories.

The total trajectory from propulsion cutoff near earth to close proximity with the destination planet is broken in two parts, as in some current interplanetary trajectory studies.(1) First, the elements of a heliocentric transfer orbit between massless points which coincide with the centers of the departure and destination planets are calculated. Second, the elements of a geocentric, hyperbolic orbit are determined. This orbit extends from the cutoff point to some specified point on the reference heliocentric orbit remote from the earth.

Criteria used in establishing heliocentric transfer orbits depend largely on the mission to be accomplished. When the object is to determine all possible trajectories having the same cutoff date and cutoff velocity, these same quantities are held fixed and quantities such as transfer angle, transfer time, and angular orientation of the cutoff velocity vector are varied in order to obtain the suitable trajectories. However, when time of flight or guidance and control problems are foremost, certain other parameters would be permitted to vary. In practice a trajectory will be established only after all the problems and limitations concerning guidance and control, available thrust, desired time of transfer, etc., have been accounted for.

In the technique discussed here the approximate date of arrival at the destination planet is chosen as the most important consideration. The transfer angle is fixed, and the initial conditions for the heliocentric

orbit are varied in order to obtain a trajectory which begins at the center of the earth and ends at the center of the destination planet. By approaching the problem in this way, it is possible to schedule arrival when the destination planet is at some favorable position in its orbit; when, for example, it is near the ecliptic plane or favorably located relative to a certain background of celestial bodies. A transfer trajectory nearly coincident with the ecliptic plane will reduce the thrust required for a given payload, and a pre-planned background of celestial bodies may simplify the terminal guidance phase.

After the heliocentric transfer orbit has been established, the vehicle, initially on a geocentric orbit, must be placed on the reference heliocentric orbit at some point in space, at the proper time, and with the proper velocity vector. We assume that propulsion cutoff occurs directly above the earth-based launch site and that transition from the geocentric to the heliocentric orbit occurs at a point of about 10^6 st mi from the earth. (At this distance the perturbing acceleration due to the sun about equals the acceleration due to the earth.)[2] Actually, this transition is a gradual one. However, by proceeding under the assumption of point transition the approximate cutoff conditions can be obtained.

The method developed and discussed in this memorandum permits the determination of orbital elements and initial conditions for trajectories between planetary orbits, i.e., between elliptical and noncoplanar orbits. The positions and velocities of the planets may be computed from either the orbital elements or an ephemeris of planets.

The problem of transition from the heliocentric transfer orbit to an orbit around the destination planet is not discussed since it is similar (although approximately in reverse) to the problem of the initial transition

from the geocentric to the heliocentric orbit. Then again, the action to be taken in the vicinity of the destination depends primarily on the mission.

II. SIMPLIFYING ASSUMPTIONS

An interplanetary trajectory can be thought of as consisting of two parts, the first of which is a geocentric orbit with the earth considered as the only attracting body, and the second a heliocentric ellipse with the sun considered as the only attracting body.

Transition from a geocentric orbit to a heliocentric orbit does not occur at a sharply defined point or region. It is therefore necessary to make certain simplifying assumptions if an approximate set of initial conditions is to be obtained by using relatively simple analytical expressions combined with simple iterative processes. Thus we assume that transition from a geocentric to a heliocentric orbit takes place at a point about one million miles from the earth.(2)

For the heliocentric transfer orbit computations we assume that all the planets are massless and that the transfer orbit extends from the center of the departure planet to the center of the destination planet. We also assume that a complete set of data concerning departure date, transit angle, transit time, initial orientation of the departure and destination planets, etc., is available from the solution of the simplified problem of transfer between circular, coplanar orbits. These data are nonessential for the method presented here, but serve to reduce the amount of preliminary computation by permitting a reasonable first guess at the departure date, the transfer time, and the arrival date.

For the geocentric orbit we assume that only the earth exerts attraction and that the geocentric orbit starts at 200 n mi above the launch site. The assumption that the orbit starts directly above the launch point simplifies the computations. It is acceptable since the powered ascent part

of the trajectory can be programmed so that the actual cutoff conditions will be correct (taking into account the geocentric and heliocentric orbits) for some point downrange from the launch point.

III. THE GEOMETRY OF THE HELIOCENTRIC TRANSFER ORBIT

For the purpose of illustration and discussion, the relative positions of the sun, the departure planet (earth), the destination planet (Mars), and the vehicle, along with the orientation of their orbital planes relative to the ecliptic plane are projected onto a unit sphere (see Fig. 1). The initial positions of the bodies approximate the positions they will have near the end of 1960. The transfer angle is the angle, measured at the sun, between a radius to the departure planet at the time of departure and a radius to the destination planet at the time of arrival.

In order to establish a particular transfer orbit (i.e., determine the orbital elements) it is necessary to obtain a departure date which will permit arrival at a position on the destination planet's orbit at the same time that the planet occupies that position. In the method presented here, this intersection must occur when the vehicle has traversed the specified transfer angle.

At this point a general explanation is given of the manner in which the proper departure date is obtained. The detailed computation procedure for determining this quantity is presented in Section V.

In Fig. 1 three sets of positions of the earth and vehicle at departure dates t_{d1}, t_{d2}, and t_d and for Mars and the vehicle at the arrival dates t_{a1}, t_{a2}, and t_a are shown. The final set of positions for the earth and vehicle at departure and arrival and the proper times of departure and arrival are defined E_d, S_d, M_a, S_a, and t_d, t_a, respectively. A transfer orbit is established when S_{di} coincides with E_{di}, at t_{di} and S_{ai} coincides with M_{ai} at t_{ai}.

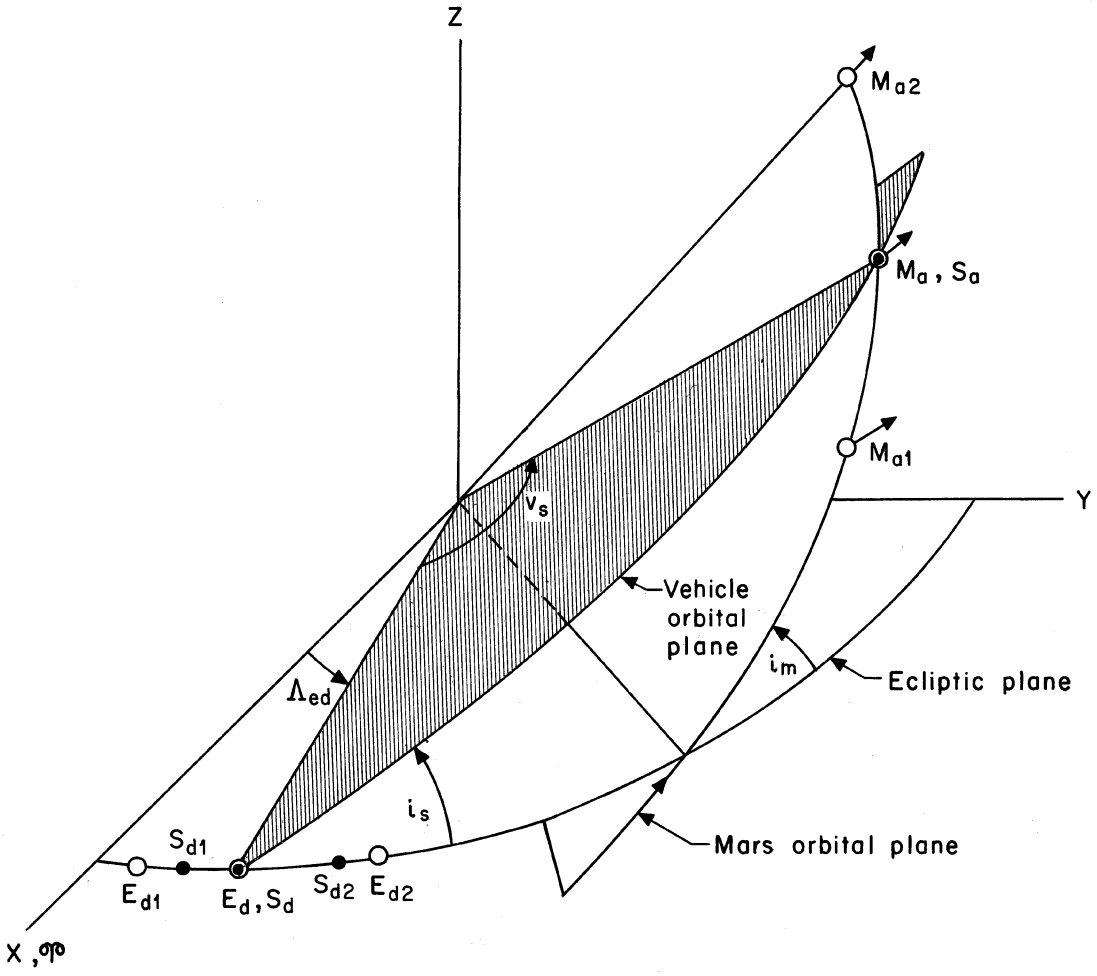


Fig. 1 — A unit sphere representation of the geometry for a typical Earth to Mars orbital transfer

The geometry of the problem and the motion of the bodies are as follows: Position M_{ai} is the position of Mars at the first selected arrival date. Positions E_{di} and S_{di} are the corresponding positions of the earth and the required position of the vehicle at the date t_{di}. The required heliocentric longitude of the vehicle at t_{di} is determined using the latitude of M_{ai} and the specified angle ν_s.

The heliocentric longitude of the earth, E_{di}, is obtained from an ephemeris for the time, t_{di}, computed by subtracting the approximate transfer time from the approximate arrival time, t_{ai}. As noted previously, the transfer time for a given transfer angle is initially based on the two-dimensional simplified solution.

If the heliocentric longitude of E_{di} is less than that of S_{di}, then a delay in the estimated departure time is required since the mean angular rate of the earth about the sun exceeds that of point S_{di}, which has approximately the same mean angular rate as Mars. In general, a good estimate of the required amount of change in the initial departure time, Δt, can be obtained from

$$\Delta t = \frac{\Lambda_{sdi} - \Lambda_{edi}}{n_{\oplus} - n_{\sigma'}} \qquad (1)$$

Here a positive value of Δt means delay, the Λ's are the heliocentric longitudes of the earth and the vehicle, and the n's are the mean angular rates of the earth and Mars about the sun.

The reference coordinate system has its origin at the sun; the positive X axis is directed toward the vernal equinox, and the Y axis is in the plane of the ecliptic.

The inclination of the vehicle's orbit plane to the ecliptic, i_s, is a function of the transfer angle, ν_s, and the heliocentric latitude of Mars at the arrival date. It is clear from the figure that as ν_s approaches 180 deg, i_s approaches 90 deg if the latitude of Mars at arrival is not zero and that for transfer orbits starting at the earth and having a single plane, a transfer angle of 180 deg can be realized only if Mars is in the ecliptic plane at the arrival date.

IV. EQUATIONS FOR THE DETERMINATION OF THE HELIOCENTRIC TRANSFER ORBIT DATA

In order to obtain the orbital data for both the heliocentric and geocentric orbits, the two-body problem was solved first for the sun as the central body and then for the earth as the central body. The vector differential equation of orbital motion of a point mass, which is negligibly small compared to the central body and moves only under the influence of an inverse square force field, is[3]

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\mu\bar{r}}{r^3} \tag{2}$$

where μ is a combination of the gravitational constant and the mass of the central body. The solution of this equation yields

$$r = \frac{a(1-e^2)}{1+e\cos(v-v_o)} \tag{3}$$

and

$$r^2\dot{v} = \sqrt{\mu a(1-e^2)} = \text{constant}$$

where r is the distance of the vehicle from the center of force, a is the semi-major axis, and e is the eccentricity of the orbit. In accordance with this differential equation of motion the standard two-body formulas are used to determine the orbital elements and related useful data.

The elements of the heliocentric transfer orbit and other heliocentric orbit data useful in the determination of the geocentric orbit and the conditions at thrust cutoff may be computed by using the equations below. Some of these equations are derived in the appendices.

In the following equations the subscript s indicates that the quantity refers to the reference vehicle.

The eccentricity of the transfer orbit is given by

$$e_s = \frac{r_{ma} - r_{ed}}{r_{ed} - r_{ma} \cos v_s} \tag{4}$$

where r_{ma} is the distance of Mars' center from the sun at the date of arrival, r_{ed} is the distance of the earth's center from the sun at the date of departure, and since $v_o = 0$, v_s is the true anomaly of the vehicle in its orbit when at Mars. Since the vehicle starts at perihelion of the transfer orbit the elevation angle of the departure velocity vector is 0 deg.

The semi-major axis can be obtained directly using r_{ma}, r_{ed}, and v_s, or by assuming e_s is obtained first and then using the equation

$$a_s = \frac{r_{ed} (1 + e_s \cos v_s)}{1 - e_s^2} \tag{5}$$

The inclination of the transfer orbit plane to the ecliptic is

$$i_s = \sin^{-1} \left(\frac{\sin \lambda_{ma}}{\sin v_s}\right) \tag{6}$$

where λ_{ma} is the heliocentric latitude of Mars at the arrival date.

The heliocentric longitude of the reference vehicle at departure (or in this case the nodal line of the transfer orbit) is

$$\Lambda_{sd} = \Lambda_{ma} - \cos^{-1} \left(\frac{\cos v_s}{\cos \lambda_{ma}}\right) \tag{7}$$

It is possible at this point to calculate the cutoff velocity required for transfer from near the earth's surface to the center of Mars. The cutoff velocity is given by

$$V_{co} = \sqrt{V_{esc}^2 + V_\infty^2} \tag{8}$$

where V_{esc} is the velocity of escape from a point near the earth and V_∞ is

the velocity remaining at an infinite distance from the earth.

Since the reference heliocentric transfer orbit is based on the assumption of a massless earth and a massless destination planet, V_∞ is the increment of velocity which should be added to the velocity of the earth's center in order to obtain the velocity required at perihelion. The velocity at infinity is

$$V_\infty = \sqrt{V_{sd}^2 + V_{ed}^2 - 2\, V_{sd}\, V_{ed} \cos \gamma_{ed} \cos i_s} \qquad (9)$$

where

$$V_{sd} = \sqrt{\mu\left(\frac{2}{r_{ed}} - \frac{1}{a_s}\right)}$$

$$V_{ed} = \sqrt{\mu\left(\frac{2}{r_{ed}} - \frac{1}{a_e}\right)}$$

$$\gamma_{ed} = \pm \cos^{-1} \sqrt{\frac{a_e(1-e_e^2)}{r_{ed}\left(2 - \frac{r_{ed}}{a_e}\right)}}$$

The + sign applies if the earth's distance from the sun is increasing.

V. PROCEDURE FOR COMPUTING HELIOCENTRIC TRANSFER ORBIT ELEMENTS AND DEPARTURE AND ARRIVAL DATES FOR SPECIFIED TRANSFER ANGLES

The following is a detailed, step-by-step procedure for computing the pertinent heliocentric orbit data. The equations used are those presented in Section IV. The initial steps utilize information obtained from solutions of the simplified problem which considers heliocentric transfer between circular and coplanar planetary orbits.

1. Obtain the approximate arrival date at Mars, t_a, for a specified transfer angle v_s, from the solutions of the simplified problem.

2. For the date t_a, the heliocentric longitude Λ_{ma}, the latitude λ_{ma} of Mars and the distance of Mars from the sun r_{ma}, may be obtained from an ephemeris of planets or computed using the orbital elements of Mars at the epoch.

3. The approximate departure date (time at the perihelion point of the heliocentric transfer orbit) of the reference vehicle is determined by

$$t_d = t_a - t_s \tag{10}$$

where t_s is the transfer time corresponding to transfer angle v_s based on the solutions of the simplified problem.

4. At t_d the distance of the earth from the sun, r_{ed}, is obtained from an ephemeris or computed. In general, the position of the earth's center will not, at this stage of the computation, coincide with the perihelion point of the transfer orbit at the date of departure. Even so, r_{ed} will probably be closer to its final value than to its mean value and it is thus better to use the approximate r_{ed}.

5. Using the known quantities r_{ma}, r_{ed}, and v_s, the eccentricity and semi-major axis are

$$e_s = \frac{r_{ma} - r_{ed}}{r_{ed} - r_{ma} \cos v_s} \tag{11}$$

and

$$a_s = \frac{r_{ed}\,(r_{ed} - r_{ma} \cos v_s)}{2r_{ed} - r_{ma}\,(1 + \cos v_s)} \tag{12}$$

6. The eccentric anomaly, E_s, used in Kepler's equation for the determination of the transfer time, is

$$E_s = \sin^{-1} \left(\frac{\sqrt{1-e_s^2} \sin v_s}{1+e_s \cos v_s} \right) \tag{13}$$

7. The transfer time for the angle v_s is

$$t_s = \frac{a_s^{3/2}}{\sqrt{\mu}} (E_s - e_s \sin E_s) \tag{14}$$

8. The t_s of step 7 is closer to the true value than the approximate value used in step 3; thus t_d is again computed by

$$t_d = t_a - t_s = \text{computed value} \tag{15}$$

If $\left|t_d \text{ (computed)} - t_d \text{ (approximate)}\right|$ is greater than a prescribed value, then steps 4 through 8 are repeated until

$$\left|(t_d)_i - (t_d)_{i-1}\right| \leq \text{prescribed constant}$$

9. A new value for r_{ed} and the heliocentric longitude of the earth, Λ_{ed}, are obtained for the time $(t_d)_i$.

10. The required longitude of departure of the reference vehicle is

$$\Lambda_{sd} = \Lambda_{ma} - \cos^{-1} \left(\frac{\cos v_s}{\cos \lambda_{ma}}\right) \tag{16}$$

11. If $\left| \Lambda_{ed} \text{ (step 9)} - \Lambda_{sd} \text{ (step 10)} \right|$ is greater than a prescribed value, then a new t_a is obtained from

$$t_a = t_a \text{ (step 1)} + \frac{\Lambda_{sdi} - \Lambda_{edi}}{n_\oplus - n_\sigma}$$

Now steps (2) through (10) are repeated. Otherwise, the orbit is established. That is, the last computed elements and related quantities obtained in the above steps are those of the proper heliocentric transfer orbit and may be used in the subsequent computations to obtain other useful quantities.

12. The inclination of the transfer orbital plane to the ecliptic plane, i_s, is

$$i_s = \sin^{-1} \left(\frac{\sin \lambda_{ma}}{\sin \nu_s} \right) \tag{17}$$

13. The required velocity at the perihelion point (the departure point) and the velocity of the earth are

$$V_{sd} = \sqrt{\mu \left(\frac{2}{r_{ed}} - \frac{1}{a_s} \right)} \quad \text{and} \quad V_{ed} = \sqrt{\mu \left(\frac{2}{r_{ed}} - \frac{1}{a_e} \right)} \tag{18}$$

14. The elevation angle of the earth's velocity vector (the angle between the velocity vector and the line which is normal to a line to the sun) at departure is

$$\gamma_{ed} = \pm \cos^{-1} \sqrt{\frac{a_e(1-e_e^2)}{r_{ed}\left(2 - \frac{r_{ed}}{a_e} \right)}} \tag{19}$$

where the plus sign applies if $\dot{r}_{ed}$ is positive.

15. The magnitude of the velocity vector to be added to the earth's velocity is

$$\Delta V_d = \sqrt{V_{sd}^2 + V_{ed}^2 - 2V_{sd} V_{ed} \cos i_s \cos \gamma_{ed}} \tag{20}$$

where $\Delta V_d = V_\infty$, since the earth is assumed to have no mass.

16. The approximate cutoff velocity magnitude is

$$V_{co} = \sqrt{V_{esc}^2 + V_\infty^2} \quad \text{where } V_{esc}^2 = \frac{2\mu_e}{r_{co}} \tag{21}$$

Since the earth has mass and the perihelion point of the transfer orbit is never infinitely far from the earth, V_{co} is an approximate value and the vehicle will not exactly traverse the reference transfer orbit obtained.

VI. GEOCENTRIC ORBIT GEOMETRY

From the cutoff point to a point remote from the earth (about 10^6 mi) the vehicle is assumed to be on an earth-centered hyperbolic orbit. Actually the transition from a geocentric to a heliocentric orbit occurs gradually as the vehicle recedes from the earth. Thus, if the problem of obtaining a trajectory from cutoff to the center of Mars is solved by a repeated solution of the two-body problem, it is necessary to assume that transition from the geocentric to the heliocentric orbit occurs at some point in time and at a point sufficiently remote from the earth.

We assume that the vehicle travels on a geocentric orbit from cutoff to some point on the reference heliocentric orbit and that the vehicle arrives at this point at the proper time and with the correct velocity vector, i.e., at the same time as if the vehicle had moved from the perihelion point of the heliocentric orbit to the chosen point. We also assume that the sum of the vehicle's velocity vector relative to the earth's center and the earth's velocity vector must equal the velocity vector of a vehicle traveling on the heliocentric orbit at the remote point. Since the heliocentric orbit is established under the assumption that the earth is massless, the velocity vector of the vehicle relative to the earth at any point on the reference heliocentric orbit is assumed to be the velocity vector occurring at an infinite distance from the earth. In other words, after the vehicle is at the selected point the earth's mass is assumed to be zero.

The fact that the earth has mass and does exert a significant force on the vehicle during much of the trip to Mars means that the actual vehicle should have a velocity at the remote point given by

$$V_p^2 = \mu_e\left(\frac{2}{r_p} - \frac{1}{a}\right) \tag{22}$$

where

$$a = \frac{-\mu_e}{V_\infty^2} \quad \text{and} \quad V_p > V_\infty \tag{23}$$

Because the magnitude of $\bar{V}_p$ is larger than the magnitude of $\bar{V}_\infty$ but has the same direction, the actual resultant velocity vector obtained will be slightly larger in magnitude and differ slightly in direction from the velocity vector required at the point. However, the force of attraction of the earth on the vehicle after the vehicle is at the point acts to gradually decrease the magnitude of the vehicle's velocity vector and thereby causes the velocity vector to rotate toward the reference transfer orbit plane. The result is that the vehicle will not travel in the plane of the reference orbit but will travel in a nearby plane.

Because of the above simplifying assumptions the actual vehicle will arrive at the Mars orbital plane at the wrong time and with the wrong position and velocity.

Several reference heliocentric orbits, beginning at perihelion, for transfer angles varying from 90 deg to 160 deg were obtained and the elements of each were used to determine a suitable geocentric orbit to match each of the reference heliocentric orbits. The initial point of geocentric orbits was fixed at a distance of 200 n mi above the launch point. The magnitude and direction of the velocity vector at cutoff and the date and time of day of cutoff were obtained as indicated in Section VII.

VII. DETERMINATION OF THE GEOCENTRIC ORBIT INITIAL CONDITIONS

The reference heliocentric orbit is assumed to start at a massless point at the earth's center. Since this cannot occur we will match the geocentric orbit to the reference heliocentric orbit at a point remote from the earth. It is assumed that about one million mi from the earth the vehicle enters a heliocentric orbit. In order to establish this distance, the true anomaly of the vehicle on the reference orbit is varied and the corresponding elapsed times are computed using Kepler's equation. The position of the earth's center is obtained from an ephemeris for these computed times. These positions, which occur after the departure date, given by Eq. (15), are used to compute the distance between the vehicle and the earth's center, r_p. The equation for r_p derived in Appendix C is

$$r_p = \sqrt{r_s^2 + r_e^2 - 2r_s\, r_e(\cos \Delta v_s \cos \Delta\Lambda + \sin \Delta v_s \sin \Delta\Lambda \cos i_s)} \qquad (24)$$

where r_t and r_e are the distances of the reference vehicle and the earth's center from the sun. The angle Δv_s is the change in the true anomaly of the vehicle and $\Delta\Lambda$ is the heliocentric longitude of the earth's center after the departure date. The inclination angle of the reference heliocentric orbit plane to the ecliptic plane, i_s, is obtained from Eq. (6). When r_p is the required value the velocity vectors of the reference vehicle and the earth's center are derived as indicated in Appendix A.

The velocity vectors of the reference vehicle and the earth's center are derived and expressed in several different coordinate systems in Appendix A. The choice of which coordinate system to use depends to some extent on the type of computer available and whether the positions of the earth are computed or are taken from an ephemeris.

It is convenient to express the velocity vectors of the reference vehicle, the earth's center, and the actual vehicle in a coordinate system having its origin at the earth's center and its reference plane as the geocentric orbital plane. The simultaneous velocity vectors of the reference vehicle, the earth's center, and the actual vehicle are represented as follows:

$$\overline{V}_s = (V_s)_{x_h} \, l_{x_h} + (V_s)_{y_h} \, l_{y_h} + (V_s)_{z_h} \, l_{z_h} \tag{25}$$

where

$$(V_s)_{x_h} = V_s \, f_1 \, (\Delta v_s, \gamma_s, \Lambda_{ed}, \delta, \Theta, \psi, i_s, i)$$

$$(V_s)_{y_h} = V_s \, f_2 \, (\Delta v_s, \gamma_s, \Lambda_{ed}, \delta, \Theta, \psi, i_s, i)$$

$$(V_s)_{z_h} = V_s \, f_3 \, (\Delta v_s, \gamma_s, \Lambda_{ed}, \delta, \Theta, \psi, i_s, i)$$

and $$\overline{V}_e = (V_e)_{x_h} \, l_{x_h} + (V_e)_{y_h} \, I_{y_h} + (V_e)_{z_h} \, l_{z_h} \tag{26}$$

where

$$(V_e)_{x_h} = V_e \, g_1 \, (\Lambda_e, \gamma_e, \delta, \Theta, i)$$

$$(V_e)_{y_h} = V_e \, g_2 \, (\Lambda_e, \gamma_e, \delta, \Theta, i, \psi)$$

$$(V_e)_{z_h} = V_e \, g_3 \, (\Lambda_e, \gamma_e, \delta, \Theta, i, \psi)$$

and $$\overline{V}_p = (V_p)_{x_h} \, l_{x_h} + (V_p)_{y_h} \, l_{y_h} + (V_p)_{z_h} \, l_{z_h} \tag{27}$$

where

$$(V_p)_{x_h} = V_p \, h_1 \, (\gamma_p, \Delta v)$$

$$(V_p)_{y_h} = 0$$

$$(V_p)_{z_h} = V_p\, h_3\, (\gamma_p,\ \Delta v)$$

At an infinite distance from the earth (neglecting perturbations and assuming two-body motion) the elevation angle of the velocity vector of the actual vehicle, γ_p, is 90 deg. Since a massless earth was used to establish the reference heliocentric orbit, it suffices to assume that at the point remote from the earth

$$\bar{V}_p = \bar{V}_\infty \text{ and } \gamma_p = 90 \text{ deg} \tag{28}$$

By substituting for h_1 and h_3 in Eq. (27), (see Appendix C for the h's in terms of the parameters) and simplifying according to Eq. (28), the components of $\bar{V}_p$ are:

$$(V_p)_{x_h} = V_\infty \cos \Delta v$$

$$(V_p)_{y_h} = 0$$

$$(V_p)_{z_h} = V_\infty \sin \Delta v$$

The magnitudes V_s, V_e, and $V_p = V_\infty$ of the three velocity vectors obtained in Appendix A are respectively:

$$V_s = \sqrt{\mu\left(\frac{2}{r_s} - \frac{1}{a_s}\right)}$$

$$V_e = \sqrt{\mu\left(\frac{2}{r_e} - \frac{1}{a_e}\right)}$$

and

$$V_p^2 = V_\infty^2 = V_s^2 + V_e^2 - 2V_s\, V_e \left[\sin(\Delta v_s - \gamma_s) \sin(\Delta v_e - \gamma_e) + \cos(\Delta v_s - \gamma_s) \cos(\Delta v_e - \gamma_e) \cos i_s\right]$$

At the remote point on the reference heliocentric orbit we require that

$$\bar{V}_p = \bar{V}_s - \bar{V}_e = \bar{V}_\infty \tag{29}$$

By writing the vectors in Eq. (29) in component form and then equating like components, the following three simultaneous equations result:

$$V_\infty \cos \Delta v = (V_s)_{x_h} f_1 - (V_e)_{x_h} g_1 \tag{30a}$$

$$0 = (V_s)_{y_h} f_2 - (V_e)_{y_h} g_2 \tag{30b}$$

$$V_\infty \sin \Delta v = (V_s)_{z_h} f_3 - (V_e)_{z_h} g_3 \tag{30c}$$

For each reference heliocentric orbit the components of the velocity vectors of the reference vehicle and the earth's center can be expressed in the xyz coordinate system. Once these components of velocity are obtained at the time the vehicle is at the remote point they remain constant until a new reference heliocentric orbit is considered. Utilizing this fact and expressing the components of $\bar{V}_{st}$ and $\bar{V}_{et}$ in the xyz coordinate system, Eq. (30) is rewritten as:

$$V_\infty \cos \Delta v = k \cos \delta \cos \Theta + k_1 \sin \Theta \cos \delta + k_2 \sin \delta \tag{31a}$$

$$0 = (-k \sin \Theta + k_1 \cos \Theta) \cos \psi + (k \sin \delta \cos \Theta + k_1 \sin \delta \sin \Theta - k_2 \cos \delta) \sin \psi \tag{31b}$$

$$V_\infty \sin \Delta v = (-k \sin \Theta + k_1 \cos \Theta) \sin \psi + (-k \sin \delta \cos \Theta - k_1 \sin \delta \sin \Theta + k_2 \cos \delta) \cos \psi \tag{31c}$$

where

$$k = (V_s)_x - (V_e)_x \tag{32a}$$

$$k_1 = (V_s)_y - (V_e)_y \tag{32b}$$

$$k_2 = (V_s)_z - (V_e)_z \tag{32c}$$

According to Eq. (31) we now have three equations and the four unknowns Δv, δ, Θ, and ψ. By assigning a value or values to one of the unknowns it is possible to solve the three equations for the three unknowns. Because the vehicle will probably require a particular firing range and downrange tracking facilities and because launchings over population centers is avoided, it is reasonable to assume that the permissible azimuth launch directions will be known well in advance of the departure date. Therefore, the azimuth direction of the geocentric hyperbolic orbit plane is specified.

With ψ given, Eq. (31b) can be solved for Θ for a known cutoff point (i.e., for a known earth latitude, δ, and longitude). The angle Θ is measured from the vernal equinox direction, positive counter-clockwise about the celestial pole (same as earth's polar axis), to the meridian plane containing the cutoff point. The angle Θ is proportional to sidereal time and thus determines the time of day of cutoff.

If we write Eq. (29) in component form and in the xyz system we have

$$(V_\infty)_x = (V_s)_x - (V_e)_x \tag{33a}$$

$$(V_\infty)_y = (V_s)_y - (V_e)_y \tag{33b}$$

$$(V_\infty)_z = (V_s)_z - (V_e)_z \tag{33c}$$

By using matrices (M11), (M10), (M8) and (M7) of Appendix C and Eq. (28) the velocity vector of the actual vehicle at the remote point is

$$\bar{V}_\infty = V_\infty \Big[(\cos \Delta v \cos \Theta \cos \delta - \sin \Delta v \sin \Theta \sin \psi - \sin \Delta v \cos \Theta \sin \delta \cos \psi)\, l_x + (\cos \Delta v \sin \Theta \cos \delta + \sin \Delta v \cos \Theta \sin \psi - \sin \Delta v \sin \Theta \sin \delta \cos \psi)\, l_y + (\cos \Delta v \sin \delta + \sin \Delta v \cos \delta \cos \psi)\, l_z \Big] \tag{34}$$

By combining Eqs. (32), (33), and (34) we get

$$\Delta\dot{x} = k + V_\infty (- \cos \Delta v \cos \Theta \cos \delta + \sin \Delta v \sin \Theta \sin \psi + \sin \Delta v \cos \Theta \sin \delta \cos \psi) \tag{35a}$$

$$\Delta\dot{y} = k_1 + V_\infty (- \cos \Delta v \sin \Theta \cos \delta - \sin \Delta v \cos \Theta \sin \psi + \sin \Delta v \sin \Theta \sin \delta \cos \psi) \tag{35b}$$

$$\Delta\dot{z} = k_2 + V_\infty (- \cos \Delta v \sin \delta - \sin \Delta v \cos \delta \cos \psi) \tag{35c}$$

where $\Delta\dot{x}$, $\Delta\dot{y}$, and $\Delta\dot{z}$ are residual velocities, each of which must equal zero in order to satisfy Eq. (29). To insure this, different values of Δv are used in Eq. (35) until the quantity G, which is obtained from

$$G = \sqrt{(\Delta\dot{x})^2 + (\Delta\dot{y})^2 + (\Delta\dot{z})^2} \tag{36}$$

is small enough. For this study, a $G \leq 2$ ft per sec is assumed to be sufficiently small.

The angle Δv is the difference in the true anomalies of the vehicle at cutoff and at infinity on the geocentric hyperbolic orbit. Since it is not one of the cutoff conditions it is not varied directly but is computed as a function of the cutoff conditions. The equations which relate ΔV and the cutoff conditions are:

$$\Delta v = v_\infty - v_{co} \tag{37}$$

where

$$v_\infty = \cos^{-1}\left(-\frac{1}{e_h}\right)$$

$$v_{co} = \cos^{-1}\left(\left[\frac{a_h(1-e_h^2) - r_{co}}{r_{co}}\right]\frac{1}{e_h}\right)$$

The quantities a_h and e_h are given by

$$a_h = -\frac{\mu_e}{V_\infty}$$

and

$$e_h = \sqrt{1 - \left(2 - \frac{V_{co}^2 r_{co}}{\mu_e}\right)\frac{V_{co}^2 r_{co}}{\mu_e}\cos^2\gamma_{co}}$$

where V_{co} and V_∞ are obtained from Appendix A. The cutoff height, r_{co}, is specified. The elevation angle of the vehicle's velocity vector at cutoff, γ_{co}, is varied through a range of values and the corresponding Δv's are computed. These Δv's are used to obtain the velocity residuals of Eq. (35) which are then substituted into Eq. (36). The values of Θ, ψ, V_{co}, r_{co}, and γ_{co} resulting in a value of G which is small enough are the set of cutoff conditions giving a geocentric orbit that satisfies Eq. (29) if the sidereal angle Θ is consistent with the time of cutoff, which is

$$t_{co} = t_D + \Delta t_s - \Delta t$$

The true t_D is the departure date on the reference heliocentric orbit and is given by Eq. (15).

The time increment Δt_s, given by Eq. (38), is the time required for the vehicle, moving on the reference heliocentric orbit, to traverse the angle Δv_s.

The time required for the actual vehicle to reach the distance r_p, given by Eq. (24), from the earth is Δt. The equation for Δt is given in Appendix D.

At this point is is necessary to note that t_{co}, computed in this way, neglects the orientation of the meridian plane, which contains the cutoff point relative to the vernal equinox direction. The sidereal angle Θ, corresponding to a particular time of day for a given meridian of longitude, is one quantity determining the orientation of the asymptote of the geocentric hyperbolic orbit in the xyz coordinate system. Thus, t_{co} must be modified so that the corresponding Θ will be the same as the Θ obtained from the geometry (see Step 8, Sec. VIII). It will usually be convenient to express t_{co} in Julian days since planetary tapes and ephemerides give the planetary positions as a function of Julian date.

The details of the determination of the cutoff date are given in Appendix D.

VIII. PROCEDURE FOR COMPUTING INITIAL CONDITIONS OF THE GEOCENTRIC ORBIT

The step-by-step procedure below uses some of the results obtained in Section V, the equations given in Section VII, and the appendices. As indicated in Section VII, some freedom is allowed in choosing a coordinate system. The method presented here uses both the equatorial plane coordinate system and the geocentric orbital plane system.

The heliocentric orbital elements and the departure and arrival times obtained in Section V serve to establish a suitable geocentric orbit and hence to obtain the cutoff conditions.

Computation proceeds as follows:

1. The true anomaly of the reference vehicle is

$$v_s = v_{sd} + \Delta v_s$$

where v_{sd} is the angle between perihelion and the departing vehicle and Δv_s is the incremental change of the true anomaly after departure.

The time required for the vehicle to traverse the angle Δv_s can be obtained from Kepler's equation. Thus

$$\Delta t_s = \frac{1}{n_s}\left[(E_s - E_{sd}) - e_s (\sin E_s - \sin E_{sd})\right]$$

where

$$\sin E_s = \frac{\sqrt{1-e_s^2}\sin v_s}{1+e_s \cos v_s}$$

$$\sin E_{sd} = \frac{\sqrt{1-e_s^2}\sin v_{sd}}{1+e_s \cos v_{sd}}$$

and

$$n_s = k' \sqrt{\mu} \Big/ a_s^{3/2}$$

Equations (11) and (12) provide the orbital elements e_s and a_s of the reference heliocentric orbit, respectively.

By assigning different values to v_{sd} in the above equations, it should be possible to solve the problem of beginning the heliocentric orbit at points other than the perihelion point. But since the angle between the vehicle's velocity vector and the earth's velocity vector generally will increase with an increase in v_{sd}, less of the earth's velocity will be utilized. Thus larger hyperbolic excess velocities for the vehicle will be required.

Assuming that the heliocentric orbit does begin at the perihelion point at t_d, the time of departure, we have $v_{sd} = 0$ and $v_s = \Delta v_s$. The time equation is simplified to

$$\Delta t_s = \frac{1}{n_s} (E_s - e_s \sin E_s) \tag{38}$$

Now assign values to Δv_s and compute the corresponding elapsed times using Eq. (38).

2. Compute r_s for each Δv_s of Step 1 from

$$r_s = \frac{a_s(1-e_s^2)}{1+e_s \cos \Delta v_s} \tag{39}$$

3. For each date, $t_d + \Delta t_s$, obtain from an ephemeris the earth's heliocentric longitude, Λ_e, and its distance, r_e, from the sun. The change in the true anomaly of the earth's center which corresponds to Δv_s is derived from

$$\Delta v_e = \Lambda_e - \Lambda_{ed}$$

The value of Λ_{ed} is shown in Step 9 of Section V

4. Compute r_p from

$$r_p = \sqrt{r_s^2 + r_e^2 - 2V_s\, r_e\, (\cos \Delta v_s \cos \Delta v_e + \sin \Delta v_s \sin \Delta v_e \cos i_s)} \quad (40)$$

where i_s is obtained from Eq. (17). For an r_p of about 10^6 mi, use the corresponding r_s and r_e to compute the velocities

$$V_s = \sqrt{\mu(\frac{2}{r_s} - \frac{1}{a_s})} \quad (41)$$

and

$$V_e = \sqrt{\mu(\frac{2}{r_e} - \frac{1}{a_e})} \quad (42)$$

5. Compute the elevation angle of the velocity vector of the earth from

$$\gamma_e = \tan^{-1} (\frac{e_e \sin v_e}{1+e_e \cos v_e}) \quad (43)$$

and of the velocity vector of the reference vehicle from

$$\gamma_s = \tan^{-1} (\frac{e_s \sin \Delta v_s}{1+e_s \cos \Delta v_s}) \quad (44)$$

6. Compute the xyz components of $\bar{V}_s$ using

$$(V_s)_x = V_s \left[-\sin(\Delta v_s - \gamma_s) \cos \Lambda_{ed} - \cos(\Delta v_s - \gamma_s) \sin \Lambda_{ed} \cos i_s \right] \quad (45a)$$

$$(V_s)_y = V_s \left[(-\sin (\Delta v_s - \gamma_s) \sin \Lambda_{ed} + \cos(\Delta v_s - \gamma_s) \cos \Lambda_{ed} \cos i_s) \cos i - (\cos (\Delta v_s - \gamma_s) \sin i_s) \sin i \right] \quad (45b)$$

and

$$(V_s)_z = V_s \left[(-\sin (\Delta v_s - \gamma_s) \sin \Lambda_{ed} + \cos(\Delta v_s - \gamma_s) \cos \Lambda_{ed} \cos i_s) \sin i + (\cos (\Delta v_s - \gamma_s) \sin i_s) \cos i \right] \quad (45c)$$

7. Compute the xyz components of $\bar{V}_e$ from

$$(V_e)_x = V_e \left[-\sin(\Lambda_e - \gamma_e) \right] \tag{46a}$$

$$(V_e)_y = V_e \left[\cos(\Lambda_e - \gamma_e) \cos i \right] \tag{46b}$$

and

$$(V_e)_z = V_e \left[\cos(\Lambda_e - \gamma_e) \sin i \right] \tag{46c}$$

where i is the inclination of the equatorial plane toward the ecliptic plane.

8. Compute the sidereal angle θ from

$$0 = (-k \sin\theta + k_1 \cos\theta) \cos\psi + (k \sin\delta \cos\theta + k_1 \sin\delta \sin\theta - k_2 \cos\delta) \sin\psi$$

where ψ and δ are given and

$$k = (V_s)_x - (V_e)_x$$

$$k_1 = (V_s)_y - (V_e)_y$$

$$k_2 = (V_s)_z - (V_e)_z$$

9. Assign a value to γ_{co} and compute

$$e_h = \sqrt{1 - \left(2 - \frac{V_{co}^2 r_{co}}{\mu_e}\right) \frac{V_{co}^2 r_{co}}{\mu_e} \cos^2 \gamma_{co}} \tag{47}$$

$$v_\infty = \cos^{-1}\left(-\frac{1}{e_h}\right) \tag{48}$$

$$v_{co} = \cos^{-1}\left[\frac{\left(-\frac{\mu_e}{V_\infty}\right)(1-e_h^2) - r_{co}}{r_{co}\, e_h} \right] \tag{49}$$

and

$$\Delta v = V_\infty - v_{co}$$

For V_∞ see Eq. (9).

10. Compute the velocity residuals $\Delta\dot{x}$, $\Delta\dot{y}$, and $\Delta\dot{z}$ from

$$\Delta\dot{x} = k + V_\infty (-\cos \Delta v \cos \Theta \cos \delta + \sin \Delta v \sin \Theta \sin \psi + \sin \Delta v \cos \Theta \sin \delta \cos \psi) \quad (50a)$$

$$\Delta\dot{y} = k_1 + V_\infty (-\cos \Delta v \sin \Theta \cos \delta - \sin \Delta v \cos \Theta \sin \psi + \sin \Delta v \sin \Theta \sin \delta \cos \psi) \quad (50b)$$

and

$$\Delta\dot{z} = k_2 + V_\infty (-\cos \Delta v \sin \delta - \sin \Delta v \cos \delta \cos \psi) \quad (50c)$$

11. Compute

$$G = \sqrt{(\Delta\dot{x})^2 + (\Delta\dot{y})^2 + (\Delta\dot{z})^2} \quad (51)$$

If G is sufficiently small, we will have the required initial conditions Θ, ψ, V_{co}, r_{co}, and γ_{co}. But if G is too large, repeat Steps 9-11 using a new value of γ_{co} in Step 9.

12. The sidereal angle Θ of Step 11 specifies a time of day but not a date. Thus not only Θ, but also the date of cutoff must be determined. The equation for t_{co} is:

$$t_{co} = t_D + \Delta t_s - \Delta t$$

where t_D is given by Eq. (15) and Δt_s by Eq. (38). The time increment Δt is:

$$\Delta t = \frac{1}{n_h} \left[-(F - F_o) + e_h (\sinh F - \sinh F_o) \right]$$

where

$$\cosh F = \frac{|a_h| + r_p}{|a_h| e_h}$$

and

$$\cosh F_o = \frac{|a_h| + r_{co}}{|a_h| e_h}$$

13. For 0^h U.T. find the Julian date nearest to but smaller than t_{co}. Call it t'_{co}.

Now compute

Θ_g = Θ - longitude of cutoff point (where east longitude is positive). The quantity Θ_g is the Greenwich hour angle of the first point of Aries.

The Nautical Almanac will provide the Greenwich mean time (or U.T.) corresponding to Θ_g for the date t'_{co}.

The Julian date of cutoff, which now takes into account the longitude of the cutoff point and the time of day (or Θ), is

$$(J.D.)_{co} = t'_{co} + G.M.T.$$

14. The components of the initial conditions of position and velocity in the xyz coordinate system are

$$x_{co} = r_{co} \cos \delta \cos \Theta \tag{52a}$$

$$y_{co} = r_{co} \cos \delta \sin \Theta \tag{52b}$$

$$z_{co} = r_{co} \sin \delta \tag{52c}$$

and

$$\dot{x}_{co} = V_{co} (\sin \gamma_{co} \cos \Theta \cos \delta - \cos \gamma_{co} \sin \Theta \sin \psi - \cos \gamma_{co} \cos \Theta \sin \delta \cos \psi) \tag{53a}$$

$$\dot{y}_{co} = V_{co} (\sin \gamma_{co} \sin \Theta \cos \delta + \cos \gamma_{co} \cos \Theta \sin \psi - \cos \gamma_{co} \sin \Theta \sin \delta \cos \psi) \tag{53b}$$

$$\dot{z}_{co} = V_{co} (\sin \gamma_{co} \sin \delta + \cos \gamma_{co} \cos \delta \cos \psi) \tag{53c}$$

Equation (21) furnishes V_{co}.

IX. CONCLUSIONS

The method described here can be used to determine the required cutoff conditions for any interplanetary trajectory which has its heliocentric portion originating near perihelion of the heliocentric orbit.

Cutoff conditions for eleven different Earth-Mars transfer trajectories have been computed. The transfer angles ranged from 90 to 160 deg and the transfer times from 102.5 to 236.5 days. In every case the miss distance at Mars was less than 0.5 million mi. Since only slight changes in the cutoff conditions will eliminate the 0.5 million mi miss, the conditions thus obtained are approximately correct. Therefore, it is not necessary to compute the entire trajectory on a large-scale computer in order to obtain the approximate cutoff conditions.

The amount of trajectory computation necessary to correct the cutoff conditions will be much less if the conditions are determined by the outlined method rather than by solutions of the simplified, coplanar and circular orbits problem or by an "educated guess."

By solving the two-body problem repeatedly with a desk calculator and a simple trajectory "patching" technique, it is possible to obtain a close approximation of the correct cutoff conditions for an interplanetary trajectory. Since these cutoff conditions will be satisfactory for most preliminary design studies, it will not be necessary to compute the entire trajectory on a large-scale computer.

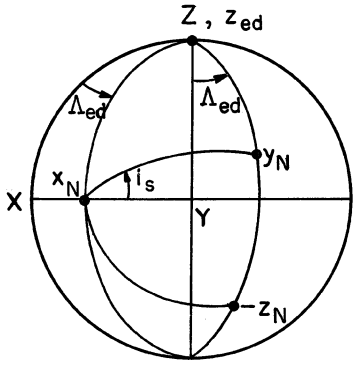
Appendix A

DETERMINATION OF THE VELOCITY VECTORS OF THE EARTH AND REFERENCE VEHICLE

In order to satisfy the equation

$$\bar{V}_p = \bar{V}_s - \bar{V}_e$$

it is necessary to determine the velocity components in the same coordinate system. The following unit spheres and associated matrices express $\bar{V}_s$ and $\bar{V}_e$ in several different coordinate systems.

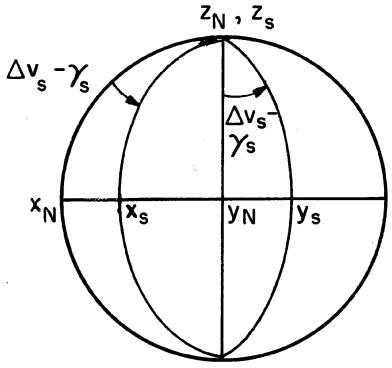


	x_N	y_N	z_N
X	$\cos \Lambda_{ed}$	$-\cos i_s \sin \Lambda_{ed}$	$\sin i_s \sin \Lambda_{ed}$
Y	$\sin \Lambda_{ed}$	$\cos i_s \cos \Lambda_{ed}$	$-\sin i_s \cos \Lambda_{ed}$
Z	0	$\sin i_s$	$\cos i_s$

(M1)

The angle Λ_{ed} is the heliocentric longitude of the line of nodes of the ecliptic plane and the reference heliocentric orbit plane defined by x_N and y_N. The angle i_s is the inclination of the reference heliocentric orbit plane to the ecliptic plane.

At t = 0 the reference vehicle and the earth's center are assumed to be at the line of nodes, x_N. At elapsed time Δt the vehicle has moved through the angle Δv_s and the earth has moved through the angle $\Delta \Lambda$. Since the perihelion point of the reference heliocentric orbit is on the line of nodes, x_N, the argument of perihelion, ω_s, is zero and $\Delta v_s = \omega_s + v_s$ reduces to $\Delta v_s = v_s$.



	x_s	y_s	z_s
x_N	$\cos(\Delta v_s - \gamma_s)$	$-\sin(\Delta v_s - \gamma_s)$	0
y_N	$\sin(\Delta v_s - \gamma_s)$	$\cos(\Delta v_s - \gamma_s)$	0
z_N	0	0	1

(M2)

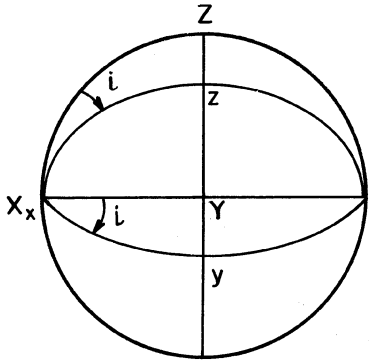
The velocity vector of the reference vehicle is

$$\bar{V}_s = V_s \, l_{y_s}$$

By using matrices (M1) and (M2) we can express $\bar{V}_s$ in the XYZ coordinate system as

$$\begin{aligned}\bar{V}_s = V_s \Big[& \left(- \sin(\Delta v_s - \gamma_s) \cos\Lambda_{ed} - \cos(\Delta v_s - \gamma_s) \sin\Lambda_{ed} \cos i_s\right) l_X \\ & + \left(- \sin(\Delta v_s - \gamma_s) \sin\Lambda_{ed} + \cos(\Delta v_s - \gamma_s) \cos\Lambda_{ed} \cos i_s\right) l_Y \\ & + \left(\cos(\Delta v_s - \gamma_s) \sin i_s\right) l_Z \Big]\end{aligned} \qquad (54)$$

By a single negative rotation of i, the inclination of the equatorial plane to the ecliptic plane, about the X axis, we obtain a matrix which relates the XYZ components of velocity to the xyz components.



	x	y	z
X	1	0	0
Y	0	cos i	sin i
Z	0	-sin i	cos i

(M3)

Now $\bar{V}_s$ becomes

$$\begin{aligned}\bar{V}_s = & (V_s)_X \, l_x + \left[(V_s)_Y \cos i - (V_s)_Z \sin i\right] l_y \\ & + \left[(V_s)_Y \sin i + (V_s)_Z \cos i\right] l_z\end{aligned}$$

where the XYZ components of $\bar{V}_s$ are obtained from Eq. (54).

The magnitude of $\bar{V}_s$ is

$$V_s = \sqrt{\mu \left(\frac{2}{r_s} - \frac{1}{a_s} \right)} \tag{55}$$

where r_s is given by

$$r_s = \frac{a_s(1-e_s^2)}{1 + e_s \cos \Delta v_s} \tag{56}$$

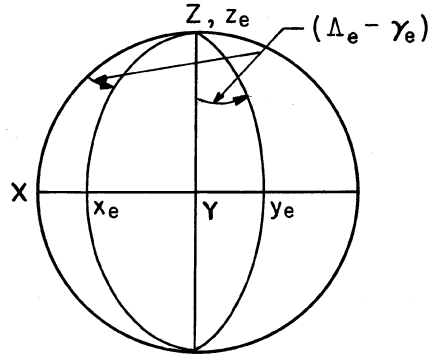
The elevation angle γ_s found from

$$\tan \gamma_s = \frac{\dot{r}}{r\dot{v}}$$

is

$$\gamma_s = \tan^{-1} \left(\frac{e_s \sin v_s}{1+e_s \cos v_s} \right) \tag{57}$$

The velocity vector of the earth's center $\bar{V}_e$ is obtained as follows:



	x_e	y_e	z_e
X	$\cos(\Lambda_e - \gamma_e)$	$-\sin(\Lambda_e - \gamma_e)$	0
Y	$\sin(\Lambda_e - \gamma_e)$	$\cos(\Lambda_e - \gamma_e)$	0
Z	0	0	1

(M4)

The angle Λ_e, the heliocentric longitude of the earth, is

$$\Lambda_e = \Lambda_{ed} + \Delta\Lambda$$

where Λ_{ed} is the earth's longitude at the departure date (at the line of nodes) and $\Delta\Lambda$ is the change in longitude during time Δt. The elevation angle of the earth's velocity vector is γ_e.

The velocity vector of the earth is

$$\bar{V}_e = V_e \, 1_{y_e}$$

In the XYZ coordinate system

$$\bar{V}_e = V_e \left[-\sin (\Lambda_e - \gamma_e) \, 1_X + \cos (\Lambda_e - \gamma_e) \, 1_Y \right] \qquad (58)$$

In the heliocentric equatorial system

$$\bar{V}_e = V_e \left[-\sin (\Lambda_e - \gamma_e) \, 1_x + \cos (\Lambda_e - \gamma_e) \cos i \, 1_y \right. \qquad (59)$$

$$\left. + \cos (\Lambda_e - \gamma_e) \sin i \, 1_z \right]$$

The magnitude of $\bar{V}_e$ is

$$V_e = \sqrt{\mu \left(\frac{2}{r_e} - \frac{1}{a_e}\right)}$$

r_e is given by Eq. (61).

The elevation angle of the earth's velocity vector is

$$\gamma_e = \tan^{-1} \left(\frac{e_e \sin v_e}{1+e_e \cos v_e}\right) \qquad (60)$$

The true anomaly of the earth, v_e, at the time $(t_D + \Delta t)$ may be computed from the standard equation

$$r_e = \frac{a_e(1-e_e^2)}{1+e_e \cos v_e} \qquad (61)$$

The value of r_e can be obtained from an ephemeris. The angle γ_e may be obtained directly from

$$\gamma_e = \pm \tan^{-1} \sqrt{\frac{r_e(2a_e - r_e)}{a_e^2 (1-e_e^2)} - 1} \qquad (62)$$

where the positive sign is used if $\dot{r}_e > 0$. (Note: $|\gamma_e|$ is always less than 2 deg.)

Since massless planets are assumed

$$V_p = V_\infty$$

The velocity V_p is the required vehicle velocity at the remote point and V_∞ is the velocity at infinity. The magnitude of V_∞ may be obtained from

$$V_\infty^2 = V_s^2 + V_e^2 - 2V_s V_e \cos \alpha$$

where $$\cos \alpha = \frac{\overline{V}_s \cdot \overline{V}_e}{V_s V_e}$$

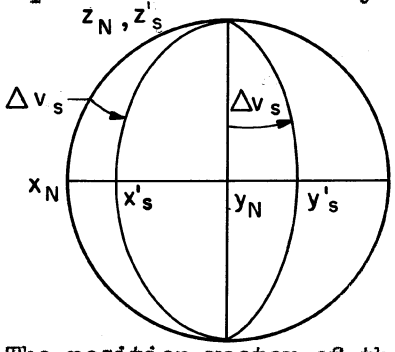
$$= \sin (\Delta v_s - \gamma_s) \sin (\Lambda_e - \gamma_e)$$

$$+ \cos (\Delta v_s - \gamma_s) \cos (\Lambda_e - \gamma_e) \cos i_s$$

Appendix B

DETERMINATION OF THE POSITION VECTORS OF THE EARTH AND REFERENCE VEHICLE

The position vector of the reference vehicle in the heliocentric equatorial coordinate system is obtained as follows:



	x'_s	y'_s	z'_s
x_N	$\cos \Delta v_s$	$-\sin \Delta v_s$	0
y_N	$\sin \Delta v_s$	$\cos \Delta v_s$	0
z_N	0	0	1

(M5)

The position vector of the reference vehicle relative to the sun is

$$\bar{r}_s = r_s \, l_{x'_s}$$

By using (M1) and (M5) we can transform $\bar{r}_s$ to the XYZ coordinate system; thus

$$\begin{aligned} \bar{r}_s = r_s \Big[& (\cos \Delta v_s \cos \Lambda_{ed} - \sin \Delta v_s \sin \Lambda_{ed} \cos i_s) \, l_X \\ & + (\cos \Delta v_s \sin \Lambda_{ed} + \sin \Delta v_s \cos \Lambda_{ed} \cos i_s) \, l_Y \\ & + \sin \Delta v_s \sin i_s \, l_Z \Big] \end{aligned} \qquad (63)$$

We can then transform $\bar{r}_s$ from the XYZ to the xyz coordinate system by using (M3). This gives

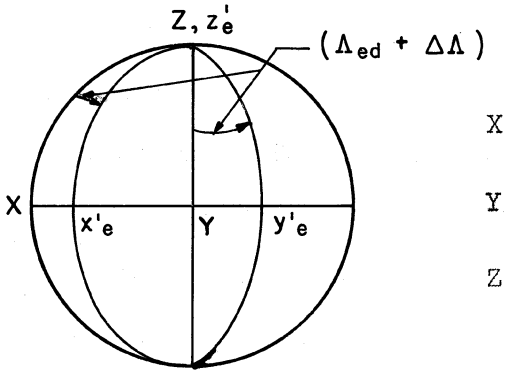
$$\begin{aligned} \bar{r}_s = & \, (r_s)_X \, l_x + \left((r_s)_Y \cos i - (r_s)_Z \sin i \right) \, l_y \\ & + \left((r_s)_Y \sin i + (r_s)_Z \cos i \right) \, l_z \end{aligned}$$

For the XYZ components of $\bar{r}_s$ see Eq. (63). The magnitude of $\bar{r}_s$ is given by Eq. (56).

The position vector of the earth is

$$\bar{r}_e = r_e \, l_{x'_e}$$

where r_e is the value given by Eq. (61).



	x'_e	y'_e	z'_e
X	$\cos(\Lambda_{ed} + \Delta\Lambda)$	$-\sin(\Lambda_{ed} + \Delta\Lambda)$	0
Y	$\sin(\Lambda_{ed} + \Delta\Lambda)$	$\cos(\Lambda_{ed} + \Delta\Lambda)$	0
Z	0	0	1

(M6)

Now

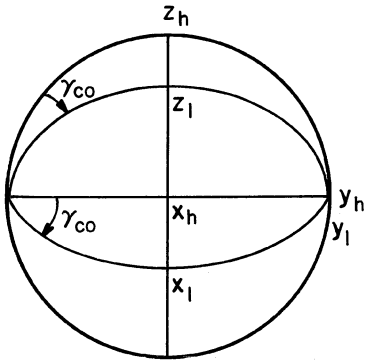
$$\bar{r}_e = r_e \left[\cos(\Lambda_{ed} + \Delta\Lambda) \, l_X + \sin(\Lambda_{ed} + \Delta\Lambda) \, l_Y \right]$$

Finally, if $\bar{r}_e$ is transformed to the xyz system by using (M3) it becomes

$$\begin{aligned} \bar{r}_e = {} & (r_e)_X \, l_x + \left((r_e)_Y \cos i - (r_e)_Z \sin i \right) l_y \\ & + \left((r_e)_Y \sin i + (r_e)_Z \cos i \right) l_z \end{aligned} \tag{64}$$

The angle ψ is the azimuth angle of the cutoff velocity vector and is measured positive clockwise from 0 deg true north. The x_h, z_h plane contains the geocentric hyperbolic orbit. For tangential take-off, i.e., take-off when the elevation angle of the velocity vector is zero, the velocity vector is

$$\bar{V}_{co} = V_{co}\, 1_{z_h}$$

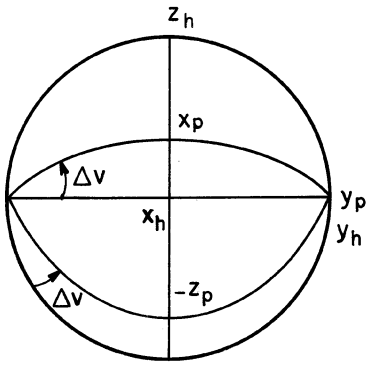


	x_h	y_h	z_h
x_l	$\cos \gamma_{co}$	0	$-\sin \gamma_{co}$
y_l	0	1	0
z_l	$\sin \gamma_{co}$	0	$\cos \gamma_{co}$

(M9)

For nontangential take-off the elevation angle of the cutoff velocity vector, γ_{co}, is not zero, and

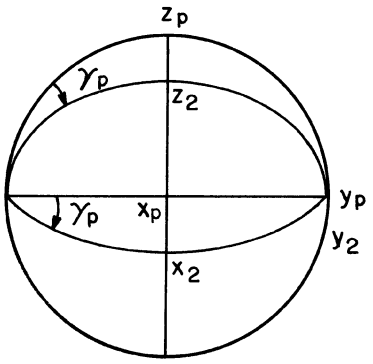
$$\bar{V}_{co} = V_{co}\, 1_{z_l}$$



	x_h	y_h	z_h
x_p	$\cos \Delta v$	0	$\sin \Delta v$
y_p	0	1	0
z_p	$-\sin \Delta v$	0	$\cos \Delta v$

(M10)

The vehicle at any point on the geocentric orbit is in the x_p axis. The angle Δv is the difference between the true anomalies of the cutoff point and the instantaneous position of the vehicle.



	x_p	y_p	z_p
x_2	$\cos \gamma_p$	0	$-\sin \gamma_p$
y_2	0	1	0
z_2	$\sin \gamma_p$	0	$\cos \gamma_p$

(M11)

The position and velocity vectors of the vehicle at a point on the geocentric hyperbolic orbit are

$$\bar{r}_p = r_p \, l_{x_p}$$

and

$$\bar{V}_p = V_p \, l_{z_2}$$

When the vehicle is at infinity, the angle $\gamma_p = 90$ deg and

$$\bar{V}_p = V_p \, l_{x_p}$$

Now the unit vector l_{z_2} can be transformed to the xyz coordinate system by obtaining the product of (M11) (M10) (M8) and (M7), which relates x_2 y_2 z_2 to xyz. By carrying out the multiplication,

$$\begin{aligned}
\bar{V}_p = {} & V_p \left[- \sin (\Delta v - \gamma_p) \cos \Theta \cos \delta - \cos (\Delta v - \gamma_p) \sin \psi \sin \Theta \right. \\
& \left. - \cos (\Delta v - \gamma_p) \cos \psi \sin \delta \cos \Theta \right] l_x \\
& + V_p \left[- \sin (\Delta v - \gamma_p) \sin \Theta \cos \delta + \cos (\Delta v - \gamma_p) \sin \psi \cos \Theta \right. \\
& \left. - \cos (\Delta v - \gamma_p) \cos \psi \sin \Theta \sin \delta \right] l_y \\
& + V_p \left[- \sin (\Delta v - \gamma_p) \sin \delta + \cos (\Delta v - \gamma_p) \cos \psi \cos \delta \right] l_z
\end{aligned} \tag{65}$$

These are the components, expressed in the equatorial plane coordinate system, of the velocity which the vehicle has at the point p on the geocentric orbit.

Since the velocity of the vehicle normal to the orbital plane is zero it is advantageous (see Section VII) to express its velocity vector in the $x_h\ y_h\ z_h$ coordinate system. This is accomplished by transforming the velocity vector

$$\bar{V}_p = V_p\, l_{z_2}$$

by using (M10) and (M11). The vector becomes

$$\bar{V}_p = V_p \left[\sin(\gamma_p - \Delta v)\, l_{x_h} + \cos(\gamma_p - \Delta v)\, l_{z_h}\right]$$

If two-body motion is assumed and the vehicle moves to infinity, then

$$\gamma_p = 90^\circ \text{ and } V_p = V_\infty$$

and

$$\bar{V}_p = \bar{V}_\infty = V_\infty \left[\cos \Delta v\, l_{x_h} + \sin \Delta v\, l_{z_h}\right] \qquad (66)$$

In a similar way the position vector may be expressed in both the $x_h\ y_h\ z_h$ and xyz coordinate systems as

$$\bar{r}_p = r_p \left[\cos \Delta v\, l_{x_h} + \sin \Delta v\, l_{z_h}\right] \qquad (67)$$

and

$$\begin{aligned}\bar{r}_p = r_p \Big[&\left(\cos \Delta v \cos \Theta \cos \delta - \sin \Delta v \left(\sin \psi \sin \Theta + \cos \psi \sin \delta \cos \Theta\right)\right) l_x \\ &+ \left(\cos \Delta v \sin \Theta \cos \delta + \sin \Delta v \left(\sin \psi \cos \Theta - \cos \psi \sin \delta \cos \Theta\right)\right) l_y \\ &+ \left(\cos \Delta v \sin \delta + \sin \Delta v \cos \psi \cos \delta\right) l_z \Big] \qquad (68)\end{aligned}$$

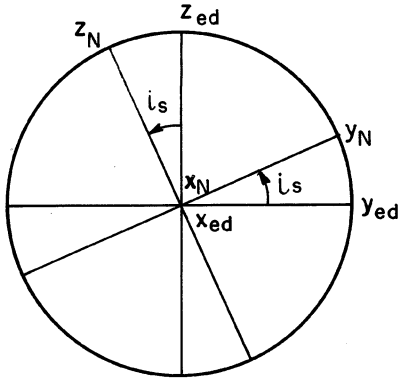
We obtain the magnitude of $\bar{r}_p$ from

$$r_p^2 = r_s^2 + r_e^2 - 2r_s r_e \cos \alpha \tag{69}$$

where

$$\cos \alpha = \frac{\bar{r}_s \cdot \bar{r}_e}{r_s r_e} = l_{x'_s} \cdot l_{x'_e}$$

and r_s is given by Eq. (56) and r_e by Eq. (61). The product $l_{x'_s} \cdot l_{x'_e}$ is most easily found if we express both the unit vectors in the $x_{ed}\ y_{ed}\ z_{ed}$ coordinate system.

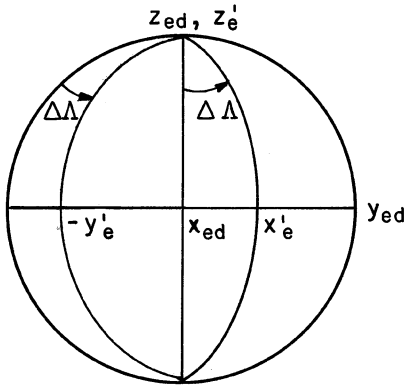


	x_N	y_N	z_N
x_{ed}	1	0	0
y_{ed}	0	$\cos i_s$	$-\sin i_s$
z_{ed}	0	$\sin i_s$	$\cos i_s$

(M12)

By using (M5) and (M12), $l_{x'_s}$ becomes

$$l_{x'_s} = \cos \Delta v_s\ l_{x_{ed}} + \sin \Delta v_s \cos i_s\ l_{y_{ed}} + \sin \Delta v_s \sin i_s\ l_{z_{ed}} \tag{70}$$



	x'_e	y'_e	z'_e
x_{ed}	$\cos \Delta\Lambda$	$-\sin \Delta\Lambda$	0
y_{ed}	$\sin \Delta\Lambda$	$\cos \Delta\Lambda$	0
z_{ed}	0	0	1

(M13)

Using (M13), $l_{x'_e}$ becomes

$$l_{x'_e} = \cos \Delta\Lambda\ l_{x_{ed}} + y_{ed} \sin \Delta\Lambda\ l_{y_{ed}} \tag{71}$$

The dot product of Eq. (70) and Eq. (71) is

$$\cos \alpha = \cos \Delta v_s \cos \Delta\Lambda + \sin \Delta v_s \sin \Delta\Lambda \cos i_s$$

The distance between the earth and the vehicle is

$$r_p = \sqrt{r_s^2 + r_e^2 - 2r_s r_e(\cos \Delta v_s \cos \Delta\Lambda + \sin \Delta v_s \sin \Delta\Lambda \cos i_s)} \qquad (72)$$

Appendix D

DETERMINATION OF THE CUTOFF TIME

The date of cutoff is determined from

$$t_{co} = t_D + \Delta t_s - \Delta t \tag{73}$$

Here t_D as given by Eq. (15) is the date of departure from the center of the earth on the reference heliocentric orbit; Δt_s, given by Eq. (38), is the time required for the reference vehicle to traverse the angle Δv_s; and Δt is the time required for the vehicle to reach the distance r_p from the earth's center.

The time of day of departure is determined from the sidereal angle θ formed by the meridian plane containing the cutoff point and the vernal equinox direction.

The time required for a vehicle to move from the cutoff point to the point P can be obtained from Kepler's equation

$$t = \frac{1}{n_h} (-F + e_h \sinh F)$$

where F is the eccentric anomaly and n_h, the mean anomaly, is

$$n_h = \frac{k' \sqrt{\mu_e}}{|a_h|^{3/2}}$$

The eccentric anomaly at a point on the hyperbolic orbit is given by

$$F = \cosh^{-1} \left(\frac{|a_h| + r}{|a_h| \; e_h} \right)$$

Thus the eccentric anomaly at the point P is

$$F_p = \cosh^{-1}\left(\frac{|a_h| + r_p}{|a_h|\ e_h}\right)$$

and the eccentric anomaly at the cutoff point is

$$F_{co} = \cosh^{-1}\left(\frac{|a_h| + r_{co}}{|a_h|\ e_h}\right)$$

Therefore the time required for a vehicle on a given geocentric hyperbolic orbit to travel between the cutoff point and the point P is

$$\Delta t = t_p - t_{co} = \frac{1}{n_h}\left[-(F_p - F_{co}) + e_h(\sin F_p - \sin F_{co})\right]$$

The date of cutoff can be found using Eq. (73). This cutoff date and the longitude of the cutoff point will then determine the corresponding sidereal angle θ' (equivalent to sidereal time).

From the geometry of the problem we find a sidereal angle θ in Step 8 of Section VIII. By this angle the meridian plane containing the cutoff point is oriented to the vernal equinox direction. Since this θ is also one of the conditions which determine the orientation of the asymptote to the geocentric hyperbolic orbit it is necessary that t_{co} and the corresponding θ' be modified so that $\theta' = \theta$.

The following example illustrates the determination of the correct t_{co}. We have given the t_{co} in Eq. (73), the longitude of the cutoff point, Λ_{co}, and the sidereal angle θ in Eq. (38). Assume that

$$t_{co} = 2437211.652 \text{ Julian date}$$

$$\Lambda_{co} = 63^{\circ}\text{E}$$

and $\Theta = 328.649^{\circ}$

The nearest, smaller Julian date for 0^h U.T. is

$$t'_{co} = 2437211.5 = 4^d\ 0^h \text{ October } 1960$$

Now

$$\theta_g = \Theta - \Lambda_{co} = 265.649^{\circ}$$

where θ_g is the Greenwich hour angle of the first point of Aries.

For the date t'_{co} the G. H. A. of the first point of Aries is listed opposite the Greenwich mean time (G.M.T.) (see page 194 of the 1960 Nautical Almanac). The nearest G.H.A. to θ_g is 268.380° and the corresponding G.M.T. is 17^h. Take the absolute difference of

$$\left| \theta_g - 268.380^{\circ} \right| = \left| -2.731^{\circ} \right| = \left| -2^{\circ}\ 43.86' \right|$$

and under the column "Aries" of the "Increments and Corrections" of the almanac (page VII) interpolate to find the correction of $10^m\ 53.8^s$. Since the G.H.A. is greater than θ_g subtract the correction $10^m\ 53.8^s$ from 17^h to get $16^h\ 49^m\ 6.2^s = 0.700766$ day.

The correct time of cutoff is now

$$t_{co} = 2437211.5 + 0.700766$$
$$= 2437212.200766 \text{ Julian date}$$

For this Julian date of cutoff the meridian plane containing the cutoff point will be oriented correctly and thus will have the sidereal angle Θ.

REFERENCES

1. Breakwell, J. V., R. W. Gillespie, and S. Ross, Researches in Interplanetary Transfer, ARS 954-59, November, 1959.

2. Moeckel, W. E., Trajectories with Constant Tangential Thrust in Central Gravitational Fields, NASA TR-R-53, 1959.

3. Moulton, F. R., An Introduction to Celestial Mechanics, Macmillan Co., 1914.