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THE THEORY OF STELLAR RADAR

Glenn W. Preston

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum is a result of RAND's continuing study of radar technology. It deals with the theoretical background of a possible exotic radar which could offer unusual operational advantages. In the light of the present study such a radar does not appear feasible, but further theoretical and experimental work on the subject is indicated. The author is a consultant to The RAND Corporation and is President of Glenn Preston Associates, of Upper Darby, Pennsylvania.

SUMMARY

Devices which can detect and track objects in space passively, without the need for active RF emanations from earth, have potential value in military operations because they give tactical information without revealing their own locations. This study considers the possibility of the passive detection and tracking of objects in space near the earth by using the radio frequency energy from the sun and from certain radio stars. The study has sought to determine whether the computed detection ranges for possible stellar radars compare with those of medium-power active radars. On the basis of available data they do not. As a surveillance device for the reasonably near future, the stellar or solar radars are apparently only useful in situations which absolutely preclude active RF emanation and where modest detection ranges are valuable.

Substantially greater ranges could be achieved if cross-correlation detectors having -80 db detection sensitivity or greater were to be developed, because in the stellar radar we escape the burden of increasing detection range against the inverse fourth power range sensitivity law under which active radar technology has labored.

ACKNOWLEDGMENT

Irving Reed of The RAND Corporation provided encouragement and the benefit of numerous comments during the preparation of this Memorandum.

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LIST OF SYMBOLS

A_i	effective antenna cross-sections $i = 1, 2$
a_s	area of sun's surface
$a_1(p)$	rms amplitude of noise from p'th element of the source which is scattered by target and received by search antenna
B_o	brightness, energy per unit area per unit solid angle per unit bandwidth
b	angular period of nulls in the antenna pattern
D	distance between receiving antennas
E	frequency spectrum of radiant energy flux at the sun's surface
F	energy flux
f	frequency
$f(\mu, \xi)$	brightness distribution law for the sun; μ, ξ is a polar coordinate system
G_d	gain in sensitivity of cross-correlator resulting from doppler compensation
G_h	detection sensitivity gained by combining the outputs of several cross-correlators operating in different frequency channels
G_i	antenna gain $i = 1, 2$; $G_i \equiv \frac{4\pi A_i}{\lambda^2}$
$H(f)$	Fourier transform of $\Phi(\tau)$
k	Boltzmann constant 1.3×10^{-23} watts per ^oK
L	incoherence loss in sensitivity which occurs because the sun is a distributed and not a point source of illumination
L_i	space losses $i = 1, 2, 3$
$L(\xi)\cos b\xi$	receiving antenna pattern
N_o	noise power per unit bandwidth
$(N_o)_S$	sky power per unit bandwidth

$(N_o)_T$	thermal power per unit bandwidth
N_1, N_2	noise voltage
$n_p(t)$	RF noise emitted from p'th element of the RF source, normalized for unit power
R	distance from sun to earth
R_s	radius of sun
R_{sr}	distance from source to receiver
R_{sT}	distance from source to target
r_{1p}	sum of distance from p'th element of source to target plus target range
r_{2p}	distance from p'th element of source to radar
S	total reference voltage received directly from source
$\left(\frac{S}{N}\right)_{out}$	signal-to-noise ratio at output of correlation detector
$S(f)$	Fourier transform of $\phi(\tau)$, the normalized power transfer function of either receiver
T	total signal voltage scattered by target
T_N	sky temperature
T_R	effective receiver noise temperature
T_s	effective temperature of the sun's surface
v	target velocity
W	equivalent noise bandwidth of either receiver
α	angular width of solar disc
α_p	angle subtended at radar by center of solar disc and the element p of sun's surface
β	$\frac{r\alpha \sin \theta}{2c}$
ϵ	doppler coefficient, $\frac{r_o\dot{\theta}_o + v(1-\cos \theta_o)}{c}$
θ	angle subtended at radar by target and center of solar disc
λ	wavelength

μ	angle subtended at center of sun by radar and the p'th element of sun's surface
ξ	azimuth angle in a polar coordinate system about the receiving antenna
ρ	minimum output signal-to-noise power ratio for reliable detection
σ	target side scattering cross-section
τ	time lag of scattered relative to direct radiation $\tau = \frac{r\ (1-\cos\Theta)}{c}$
$\Phi(\tau)$	"envelope" of ϕ
$\phi(\tau)$	normalized auto correlation function of n(t) following the power transfer characteristic S(f) of the receiver-- $\phi(\tau) = \Phi(\tau) \cos 2\pi f_o \tau$

I. INTRODUCTION

The earth and space about us are illuminated by radio frequency energy from the sun and stellar sources. In this study the possibility is considered of detecting and tracking objects in space passively by this illumination. Passive surveillance devices give tactical information without revealing their own locations; this is their foremost appeal.

The initial purpose of this study was to determine whether the computed detection ranges for state-of-the-art system parameters of stellar radars compare with those of medium-power active radars. On the basis of available radiometer data[1,2,3] this is definitely not the case, and in that respect this report presents a negative result. As a surveillance device for the reasonably near future, the stellar or solar radar is apparently only useful in situations which absolutely preclude active RF emanation and where modest detection ranges are valuable (from 30 km against 1 m^2 cross-section targets to 1000 km against 1000 m^2).

Particular attention must be directed to the "incoherence loss" of the solar radar which is due to the finite angular size of the solar disc. When this effect is ignored, grossly misleading results are obtained. In that respect an intriguing possibility raised by this report is that of optically compensating the incoherence losses. A theoretical existence proof for such a device is given below, but the matter deserves the attention of a competent if not gifted inventor of RF optical devices.

The scheme analyzed here uses coherent cross-correlation detection. For theoretical as well as practical reasons this means of detection is more sensitive (for equal dwell time and hence search rate) than radiometers. The integration losses are smaller, and the useful signal energy does not generally fall at zero frequency. Optical radiometric means (the mechanism of human vision) are out of the question at radio frequencies; the illumination is much too feeble for useful detection of targets against the ambient sky-noise. Moreover, the cross-correlation scheme gives target range while radiometry does not.

The final range equations (Eqs. (17), (18), (19), Tables 3, 4, 5) are not encouraging, except that the factors which limit the useful surveillance range are practical, not fundamental theoretical ones. Greater detection ranges than those given in these tables can be realized by utilizing the virtually unlimited bandwidth of the radiation spectrum. Appreciable energy is found in the solar spectrum at all frequencies for which we have measuring instruments (except for absorption lines), and the energy spectrum of the radio stars closely resembles that of the sky noise so that the ratio of stellar RF power to sky noise power is practically constant over the spectrum. However, the absolute stellar RF radiation power falls to minute levels in the microwave frequencies, leaving a potentially useful spectrum about 1000 Mc wide.

For this application no fundamental fact known to the author precludes cross-correlation detectors having -80 db detection sensitivity or greater. And in the stellar radar and the optically compensated solar radar we escape the burden of increasing detection range against the inverse fourth power sensitivity law under which active radar technology has labored.

The super-sensitive cross-correlation detector (-80 db and up) offers a multitude of opportunities for inventive designing. It is fertile ground for the extensive application of sequential statistical analysis,[4] for example. The applicability of super-sensitive cross-correlation and matched filter techniques in active radar merits serious immediate study. Apparently detection sensitivities of -60 db to -70 db with 20 Mc bandwidth can be obtained with state-of-the-art components.

This treatment is not general in that it restricts itself to isolated intense sources (the sun and radio stars) whereas all of the radiant energy flux scattered by a target conveys positional information. A more general theoretical treatment might give a brighter image of the future of passive ranging and detection by stellar and solar RF illumination.

The possibility of detecting targets passively at useful range from their scattered RF stellar radiation using cross-correlation techniques was suggested by C. Wiley,(5) and independently by the author and I. S. Reed (of The RAND Corporation) and others. The appealing feature of such a scheme is the absence of the radar transmitter. Not only are there situations in which silence is golden, but the elimination of the transmitter, and hence, the TR (transmit-receive function) avoids compromises in the design of the radar front end which adversely affect receiver sensitivity and contribute heavily to the cost of manufacture, installation, maintenance, and associated logistics. The known stellar radio frequency sources which are sufficiently energetic to give useful detection ranges are the sun and the radio stars Cassiopeia A, Cygnus A, possibly Centaurus A and others.

In this note the range equations for the stellar and solar radar are developed and some of the most general features of their physical embodiments are considered.

The detection sensitivity for the radio-star radar (applicable also to a form of the solar radar in which the distributed source effect is partially compensated) is an inverse square function of target range. This is in contrast to the inverse fourth power law of active radar, which is refreshing since the detectability of a target is dependent on the effective solid angle which it subtends at the receiver and not upon the solid angle subtended by the effective radar cross-section from the target. For that reason those who are unimpressed with the computed range of the stellar radar against small targets (as compared to active radar) are invited to make similar range comparisons against large targets (planets, for example). Owing to the fact that the sun is a distributed source of energy, the output signal-to-noise ratio varies inversely as the fourth power of target range. For this reason the solar radar can be equated in power to an active radar having equal search rate and antenna gain (Eq. 20). The radio stars are, in effect, presumably point sources.

II. POWER CONSIDERATIONS

NOISE

The interfering noise which limits the detection sensitivity of this type of sensor is principally from two sources; thermal receiver noise and "sky noise." If the receiver input impedance were matched to the antenna, the thermal power density $(N_o)_T$ (all through the RF spectrum) would be kT_R, where T_R is the receiver noise temperature. We may assume that brightness B_o from sky noise entering the antenna is nearly constant over the antenna pattern, in which case the received power per unit bandwidth

$$(N_o)_S = \frac{1}{2} B_o \lambda^2$$

independent of the antenna aperture (since the angular area of the receiving beam is inversely proportional to its effective cross-section). Because of this $(N_o)_S$ can be equated to a thermal source of temperature $T_N = T_N(\lambda)$ by the relationship $(N_o)_S = kT_N$.

The energy received from discrete sources is dependent on the parameters of the receiving beam and it is inappropriate to characterize these measurements by an equivalent temperature.

The total interfering noise power per unit bandwidth then is

$$N_o = k\ (T_R + T_N)$$

Maps of the sky temperature have been compiled for frequencies between 20 and 1000 Mc.[(1)] The observed temperatures of various regions of the sky cover a range of ten to one at most frequencies; roughly $(10^5)^o$ Kelvin at 20 Mc falling to 10^o Kelvin at 1000 Mc (see Table 1, page 26).

SIGNAL

If the source is sufficiently localized that the antenna pattern is reasonably constant over the source (whose total radiant energy flux per unit bandwidth is F) then the received power

$$P = \frac{1}{2} FA$$

where A is the effective cross-sectional area of the receiving antenna (the receiving antenna is responsive to only one polarization component, hence the factor $\frac{1}{2}$).

SOLAR RADIANT ENERGY FLUX

Although the solar RF spectrum as observed at the bottom of earth's atmosphere is not actually black body, the solar energy flux is characterized at various frequencies by an equivalent temperature T_s.

Since the power spectrum of solar radiation as observed on the earth is highest in the frequency band of human visibility, the wavelength at radio frequencies is comparatively so long that the spectral density of radiant energy flux at the sun's surface is very accurately given by the Rayleigh-Jeans law

$$E(f) = \frac{2\pi T_s k}{\lambda^2} \tag{1}$$

(T_s is the effective sun's temperature at frequency $f = \frac{c}{\lambda}$). The total energy spectral density from the sun is thus $2\pi\, kT_s \frac{a_s}{\lambda^2}$, where a_s is the sun's surface area. This formula gives the power per unit frequency interval Δf, whereas physicists give the energy density per increment of wave number $\Delta\left(\frac{2\pi}{\lambda}\right)$. The mean flux of solar RF energy per unit of bandwidth at the earth is then $F = \frac{2\pi k T_s}{\lambda^2}\left(\frac{\alpha}{2}\right)^2$ where α is the angular width of the solar disc, about .93 x 10^{-2}. This amounts to about 10^{-21} watts per sq m per unit bandwidth at $\lambda = 1$ m, where T_s is $(10^6)^{\circ}$ Kelvin or so.[3]

THE RADIO STARS

The radio stars (Cassiopeia A, Cygnus A for example) are astounding sources of RF energy, not only because of their great strength, but also because of their remarkable energy spectra. These spectra reach their maxima at about 10 meters wavelength and fall off rather sharply at higher frequencies (about 10 db per decade of frequency) and even faster towards the longer wavelengths.[2]

The flux density per unit bandwidth at 10 meters wavelength for the strongest measured radio star, Cassiopeia A, is 4 x 10^{-22} watts per sq m. At wavelengths longer than about 4 meters the flux from Cassiopeia A actually exceeds the thermal component of the solar flux. The measured flux density per unit bandwidth for the sun and Cassiopeia A are reproduced in Fig. 1, taken from Ref 3.

Below Cassiopeia A in power are Cygnus A (2 db weaker), Centaurus A (about 8 db), Virgo (about 10 db) and others.

SPACE LOSSES

There are three sources of space losses; L_1, the fraction of the source's radiated energy which is scattered by the target; L_2, the fraction of scattered energy which is effectively collected by the search antenna; and L_3, the fraction of the source's energy which is effectively collected by the reference antenna.

These are given by the formulas,

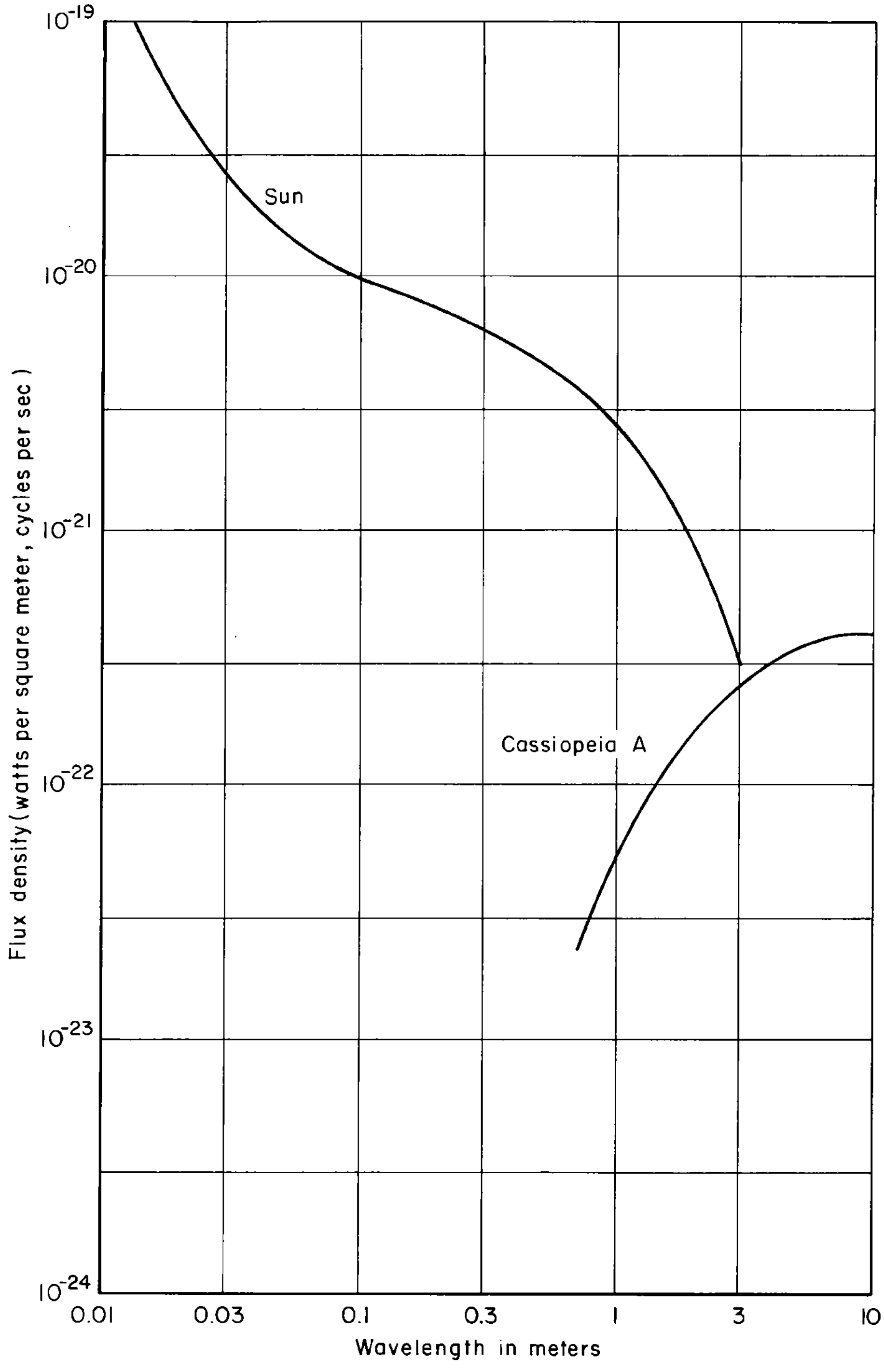

Fig. 1—Measured RF flux density from the sun and Cassiopeia A

$$\left.\begin{aligned} L_1 &= \frac{\sigma}{4\pi R_{sT}^2} \\ L_2 &= \frac{A_1}{4\pi r^2} = \frac{G_1}{(4\pi)^2} \left(\frac{\lambda}{r}\right)^2 \\ L_3 &= \frac{A_2}{4\pi R_{sr}^2} = \frac{G_2}{(4\pi)^2} \left(\frac{\lambda}{r}\right)^2 \end{aligned}\right\} \qquad (2)$$

in which σ is the target side-scatter cross-section, A_1 is the effective cross-section of the search antenna, A_2 is the effective cross-section of the reference antenna, R_{sT} is the range of target from RF source, R_{sr} is range of radar from RF source, r is target range from radar, and G_1 and G_2 are the approximate antenna gains. Clearly, $R_{sT} \cong R_{sr} = R$.

III. CROSS-CORRELATION DETECTION

THE RELATION BETWEEN TARGET GEOMETRY AND THE CROSS-CORRELATION FUNCTION

The system contemplated here is comprised of a pair of antennas and associated receiver, one of which, the reference receiver, tracks the stellar source while the other searches. The temporal cross-correlation function of the two receiver signals is computed, from which the presence of a target can be found, and its range computed. Target azimuth and elevation may be measured from that orientation of the tracking antenna at which maximum signal power is observed.

Designate by $n_p(t)$ the noise emitted by the p'th element of the stellar source--assuming the noise generated at separate elements to be statistically independent--then the noise scattered by the target and reaching the receiver is (after being filtered by the receiver)

$$T(t) = \sum_p a_1(p) n_p \left(t - \frac{r_{1p}}{c} \right) ,$$

r_{1p} being the sum of R_p (the distance from the p'th element of the source to target) and r (the target range from the receiver).

The reference signal, $S(t) = \sum_p a_2(p) n_p \left(t - \frac{r_{2p}}{c} \right)$, and the ensemble average, signified by brackets < >, of their lagged product

$$< g(t,\tau) > = S(t)T(t - \tau) = \sum_p a_p^2 \, \phi \left(\tau - \frac{r_{1p} - r_{2p}}{c} \right) \qquad (3)$$

The function ϕ is the Fourier transform of the power transfer function S(f) of the two receivers (which we assume to be the same).

Because of the variable time lags $\tau_p = \frac{1}{c} [r_{1p} - r_{2p}]$ of the signals originating from the different source elements, the corresponding cross-correlation functions--which are the useful signals--will not add up in phase generally. This lack of coherence (following the multiplier in the cross-correlator), if it cannot be compensated, seriously degrades the detection sensitivity for the solar source. There would be no coherence losses from radio star sources if they are in effect point rather than distributed sources. The question of the actual angular size of the radio stars could be settled by a fairly straightforward experiment using the theory developed herein.

The range difference $r_{1p} - r_{2p} = r\left[1 - \cos(\theta + \alpha_p)\right]$ is dependent upon the angle $\theta + \alpha_p$ subtended at the receiver by the target and the element p (θ is the angle subtended by the target and the center of the stellar RF source). However, the range to the element p does not appear, since earth's range from the sun (and a fortiori to the radio stars) is so very much greater than the maximum target range r for reliable detection.

The target velocity (but not acceleration in the case of ballistic objects or aircraft) is an important factor in the correlation process. Thus over intervals of time comparable to the integration time of the cross correlation detector $r \approx r_o + vt$, $\theta \approx \theta_o + \dot{\theta}t$ (we may ignore $\dot{\alpha}_p$) and $r\left\{1 - \cos(\theta + \alpha_p)\right\} = r_o\,(1 - \cos\theta_o) + r_o\alpha_p \sin\theta_o + \left[r_o\dot{\theta}\ \sin\theta_o + v(1 - \cos\theta_o + \alpha_p \sin\theta_o)\right]\ t$, since terms in t^2 may be neglected.

We write

$$\tau_o = \frac{r_o}{c}\,(1 - \cos\theta_o) \qquad (4a)$$

$$\tau_p = \frac{r_o}{c}\,\alpha_p \sin\theta_o \qquad (4b)$$

$$\epsilon = \frac{1}{c}\left[r_o\,\dot{\theta}\,\sin\theta_o + v(1 - \cos\theta_o)\right] \qquad (4c)$$

and

$$\epsilon_p = \frac{v}{c} \alpha_p \sin \Theta_o \tag{4d}$$

(this term will be omitted since it is small compared to ϵ) in terms of which

$$\phi\left(\tau - \frac{r_{1p} - r_{2p}}{c}\right) = \phi(\tau_o - \tau_p - \epsilon t) \tag{5}$$

Owing to the presence of ϵt in the argument (ϕ is an ensemble average) the cross-correlation process is seen to be non-ergodic. The theory of such processes is given in Ref. 7.

THE SOLAR RADAR CROSS-CORRELATION DETECTOR

We will first apply Eq. (5) to the theory of the solar radar; the theory for the radio star sources is a special case of this.

We assume the post-multiplier signal amplitude from the p'th area element of the sun's surface $a_p^2 = a_{1p}\, a_{2p}$ to be proportional to the area of that element and to be a function of the angle between the ray (to either the target or the receiver) and the surface element. To compute the summation (Eq. (3)) it is convenient to make a polar coordinate system with origin at the sun's center and polar axis orthogonal to the line from the sun's center to the receiver and in the plane of the target, radar and sun. Call the polar angle μ and the azimuth angle ζ. Then $a_p^2 = \frac{PR_s^2}{a_s}$ $f(\mu,\zeta) \sin \mu \, d\mu \, d\zeta$, where $P = \sqrt{P_1 P_2}$ and P_1 is the total power per unit area radiated within the receiver passband (of bandwidth W) collected by the reference antenna, while P_2 refers to that which is scattered from the target and received, and $f(\mu,\zeta)$ is the normalized brightness distribution. In other words, $P^2 = E^2 W^2 L_1 L_2 L_3$, where the L's are space losses and E is the radiation density function of Eq. (1). The brightness distribution at various frequencies is in theory dependent on the coronal temperature.[6] A by-product of this study is the theory for inferring the brightness distribution from cross-correlation interferometer-type observations.

We get a particularly simple formula if we take $f(\zeta,\mu) = 2$. With this assumption, making the change in index of summation from p to μ using the relation $\tau_p = \frac{r\alpha \sin \Theta}{2c} \cos \mu$ (we use the symbol β for $\frac{r\alpha \sin \Theta}{2c}$), we get for

$$< g(t,\tau) > = \frac{2\pi P R_s^2}{a_s} \int_0^\pi d\mu \sin \mu \, \phi(\epsilon t - \tau + \beta \cos \mu)$$

$$= P \frac{1}{\beta} \int_0^\beta dx \, \phi(\epsilon t - \tau + x) \qquad (6)$$

This is the useful signal term following the cross-correlator multiplier. To compute the maximum attainable signal-to-noise ratio, we next consider the optimum filter for enhancing the signal term $< g >$ with respect to the noise fluctuations $g - < g >$.

A general theoretical treatment of the optimum filtering of the correlation signal $< g(t,\tau) >$ is given in Ref. 7.

The fluctuation term $g(t,\tau) - < g(t,\tau) >$ has the power spectrum*

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*Because of the doppler coefficient ϵ, the spectrum of the signal term $g(t,\tau)$ will fall at low (audio) frequencies. On the other hand, the interfering noise following the signal multiplication is from two sources: fluctuations in the signal term $g(t,\tau)$ and the product terms containing the thermal noise voltages. Designating the thermal noise voltages in the search receiver and the reference receiver by N_1 and N_2, $<N_1^2> = <N_2^2> \equiv N_o$ the product (with lag τ in the search receiver voltage) is

$$g(t,\tau) = (T+N_1)(S+N_2),$$

and

$\sigma_g^2 = < \left[g(t,\tau) - g(t,\tau)\right]^2 > \approx N_o^2 + P_1 N_o$ where only the terms N_o^2 and $P_1 N_o$ have been retained, since our primary interest is in the maximum detection range, hence minimum-energy detectable signals--and, because of the

$(P_1N_o + N_o^2)\int_{-\infty}^{\infty} df'\, S(f')\, S(f'+f)$ whose main lobe is a passband at zero frequency, which is roughly twice the (spectral) width of the receiver passband. Over the comparatively narrow portion of the spectrum occupied by the signal $g(t,\tau)$ the fluctuation term will have the properties of white Gaussian noise, with energy density (noise power per unit bandwidth) $(P_1N_o + N_o^2)\int_{-\infty}^{\infty} df\, S^2(f)$.

The optimum filter for a signal of known form in white Gaussian noise is one having an impulse response function $h(\sigma)$ which is the shape of the signal reversed in time $h(-\sigma) = \frac{1}{\beta}\int_o^{\beta} dx\phi(\epsilon\sigma+x)$. This filter gives a peak output signal-to-noise ratio equal to the total signal energy divided by the (relatively constant) noise power per unit bandwidth. In this case, the total signal energy for any fixed lag τ is

$$\int_{-\infty}^{\infty} dt < g(t,\tau) >^2 = P_1P_2\int_{-\infty}^{\infty} dt\left[\frac{1}{\beta}\int_o^{\beta} dx\ \phi\ (\epsilon t+\tau-x)\right]^2$$

$$= P_1P_2\,\frac{1}{\epsilon}\int_{-\infty}^{\infty} df\ S^2(f)\left[2\frac{1-\cos\ 2\pi f\beta}{(2\pi f\beta)^2}\right]$$

and the peak output signal-to-noise ratio from the cross-correlation is

$$\left(\frac{S}{N}\right)_o = \frac{1}{\epsilon}\,\frac{P_1\quad P_2}{(P_1+N_o)N_o}\;\frac{\int_{-\infty}^{\infty} df\ S^2(f)\left[2\ \frac{1-\cos\ 2\pi\beta f}{(2\pi\ f\beta)^2}\right]}{\int_{-\infty}^{\infty} df\ S^2(f)}$$

The factor $L_c = \dfrac{\int_{-\infty}^{\infty} df\ S^2(f)\left[2\ \frac{1-\cos\ 2\pi\beta f}{(2\pi\ f\beta)^2}\right]}{\int_{-\infty}^{\infty} df\ S^2(f)} \leq 1$

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comparatively great detection sensitivity of correlation detection, the minimum-energy detectable signal $< T^2 > = P_2$ will be far less energetic than noise. The reference signal power, on the other hand, must be greater than N_o.

$\left(\text{since } 0 < \frac{1-\cos 2\pi f}{(2\pi f \beta)^2} \leq \frac{1}{2}\right)$ is the incoherence (or distributed source) loss. Because of the disastrous effect of the coherence loss upon the solar radar detection sensitivity, it will be discussed in a separate section. If useful detection ranges are to be obtained P_1 must be large compared to N_o, in which case

$$\left(\frac{S}{N}\right)_o = \frac{P_2}{N_o} \frac{1}{\epsilon} L_c(\beta) \tag{7}$$

Since the optimum filter is dependent on the doppler coefficient ϵ for the target (which generally will not be known a priori) an array of filters must be used corresponding to various discrete values of ϵ more or less closely spaced. Where the reflector moves along a trajectory for which $\epsilon = 0$ (the family of parabolas $r\left[1 - \cos \Theta\right] = k$) an unlimited output signal-to-noise ratio can be achieved, with the operational disadvantage of an indefinitely great integration time. Where the target trajectory is such that $\epsilon > 0$, the integration time of the estimating filter will be approximately $\frac{1}{2\ \epsilon W}$ and the output of the correlation detector will represent the target position and velocity with about that time lag. The peak cross-correlation signal for such targets will progress through a sequence of values (the argument τ_m of the transitory cross-correlation function), the lag for peak correlation changing at the time rate $\frac{d\tau_m}{dt} = \epsilon$.

DOPPLER COMPENSATED CROSS-CORRELATION

Since the rate of progression of the correlation signals through the lag values τ_m is known, it is possible in principle to achieve greater detection sensitivity by storing the rectified peak output of the ϵ, τ filter and adding it to the output from the corresponding $\epsilon, \tau + \Delta\tau$ filter taken at time $\Delta t = \frac{\Delta\tau}{\epsilon}$ later. This will result in an increase in detection sensitivity by a factor which we shall denote by G_d. Although in theory this factor can be increased without limit, there are practical limitations arising from the complexity of the data processing apparatus; moreover, the total lag in the output data is the sum of the lags in each of the ϵ filters and, speaking generally, the utility of surveillance data declines with age. On the other hand, the integration time (and hence data delay) for each filter can be reduced by enlarging the bandwidth W to its maximum practical value.

IV. THE SOLAR RADAR INCOHERENCE LOSSES

The factor L_β represents a loss in detection sensitivity of the correlation process attributable to the finite size of the source (the value 1 corresponds to $\beta = 0$, hence, also for $\alpha = 0$) due to the failure of the correlation signals corresponding to various portions of the source to add coherently following multiplication.

Some appreciation of solar radar system design problems is obtained by noting that the incoherence loss for an ideal passband receiver of bandwidth W with center frequency f_o is

$$L_\beta = \frac{1}{(2\pi\ f_o\ \beta)^2}\left[1 - \frac{\sin \pi\ \beta\ W}{\pi\ \beta\ W} \cos 2\pi\ f_o\ \right] \qquad (8)$$

provided $W \ll f_o$.

The greatest practically feasible bandwidth W (10 Mc) is the most advantageous since the data lag is then least (for a given level of doppler compensation G_d) and for target ranges of 10 km or more--where the target is not directly in the sun ($\theta_o = 0$)--$\beta\ W \sim 1$ or greater and

$$L_\beta \approx \frac{1}{(2\pi f_o\ \beta)^2} \qquad (9)$$

The incoherence losses are greater the higher the center frequency f_o, but for surveillance purposes a minimum practical center frequency would be 100 Mc or so and $L_\beta \ll 10^{-6}$.

There is, in theory at least, a physical means of greatly reducing the incoherence loss (i.e., of increasing the magnitude of the quantity L_β). This is accomplished by the use of an antenna for the reference receiver which has a fine structure pattern with periodic nulls and lobes which approximately match the nulls and lobes of the correlation function $\phi\,(\beta\cos\mu) \equiv \Phi\,(\beta\cos\mu)\cos(\omega_o\,\beta\cos\mu)$.

Designate by ξ the azimuth angle in a polar coordinate system about the receiving antenna, with the sun, target and radar defining the equatorial plane. We are assuming a main receiving lobe of the form $L(\xi)\cos(b\xi)$, G being the peak gain. (We put $L(\xi) = 1$ since the antenna cross-section A appears in the space less L_3, which we have implicitly assumed to be constant over the source.) The correspondence between ξ and μ is

$$\tan\xi = \frac{R_s}{R}\cos\mu = 1/2\,\alpha\cos\mu \approx \xi$$

If the antenna is constructed so that $b\xi = 2\pi f_o\,\beta\cos\mu$, i.e.,

$$b = 2\pi\frac{r}{\lambda}\sin\theta \tag{10}$$

the amplitude coefficient a_{p2} is changed by the factor $\cos\left[2\pi f_o\,\beta\cos\mu\right]$-- the correspondence between p and μ being given by the relation $\tau_p = \beta\cos\mu$-- and we get for the post-multiplication ensemble average in place of Eq. (6),

$$\left\langle g(t,\tau)\right\rangle = \frac{P}{2}\int_o^{\pi} d\mu\;\Phi\,(\epsilon t-\tau+\beta\cos\mu)\,\sin\mu\,\cos^2(2\pi f_o\beta\cos\mu)$$

$$= \frac{P}{4}\frac{1}{\beta}\int_o^{\beta} dx\;\Phi\,(\epsilon t-\tau+x) \tag{11}$$

The function Φ in the integrand is the envelope of the correlation function ϕ which appears in the integrand of Eq. (6).

The peak output signal-to-noise ratio for this case, computed exactly as before, is

$$\left(\frac{S}{N}\right)_o^1 = \frac{P}{2}\;\frac{1}{\epsilon}\;\frac{\int_{-\infty}^{\infty} df\;H^2(f)\;\left|\frac{1}{\beta}\int_o^{\beta} dx e^{2\pi jfx}\right|^2}{\int_{-\infty}^{\infty} df\;H^2(f)} \tag{12}$$

where H(f) is the Fourier transform of $\Phi(\tau)$.

The incoherence loss for this case

$$L'_\beta = \frac{\int_{-\infty}^{\infty} df\ H^2(f)\ \left|\frac{1}{\beta}\int_0^\beta dx e^{2\pi j f x}\right|^2}{\int_{-\infty}^{\infty} df\ H^2(f)} \tag{13}$$

differs enormously both qualitatively and quantitatively from the uncompensated incoherence loss factor, Eq. (8), since the power spectrum H(f) is greatest near f=0 (DC) and becomes relatively very small for all values of f greater than W in the following sense

$$\int_W^\infty df\ H^2(f) < \int_0^W df\ H^2(f), \text{ where}$$

$$W^2 = \frac{\int_{-\infty}^{\infty} df\ f^2 H^2(f)}{\int_{-\infty}^{\infty} df\ H^2(f)}$$

Since $L(\xi) = 1$ over the source

$$L'_\beta = 2\,\frac{\int_{-\infty}^{\infty} df\ H^2(f)\ \dfrac{1-\cos 2\pi f \beta}{(2\pi f\beta)^2}}{\int_{-\infty}^{\infty} df\ H^2(f)} \tag{14}$$

Here, in contrast to the uncompensated case, the bandwidth of W is crucial, for, if

$$(\beta W)^2 < 1, \text{ then } \frac{1-\cos 2\pi f\beta}{(2\pi f\beta)^2} \approx 1/2\left[1-\frac{1}{12}(2\pi f\beta^2\right], \text{ and}$$

$$L'_\beta \approx 1 - \frac{(2\pi)^2}{36}(W\beta)^2; \text{ whereas if}$$

η_o is the direction of sun's center) with respect to the line between the antennas is

$$\Delta\phi = \frac{2\pi}{\lambda} D \cos(\eta_o + \epsilon) = 2\pi \frac{D}{\lambda} \cos \eta_o - \left(\frac{2\pi D}{\lambda} \sin \eta_o\right) \epsilon$$

Our requirement is that

$$\frac{2\pi D}{\lambda} \sin \eta_o = 2\pi \frac{r}{\lambda} \sin \theta, \text{ or}$$

$$D = r \frac{\sin \theta}{\sin \eta_o}$$

The antenna spacing D must be comparable to the detection range. This result is not surprising. Such an arrangement would give far greater detection ranges than would otherwise be possible; however, the surveillance would be confined to the plane of the sun and the two receivers.

V. THE RANGE EQUATIONS

The received signal power from the target is

$$P_2 = \frac{1}{2} F\sigma \frac{A}{4\pi r^2} = \frac{1}{2} F\sigma \frac{G\lambda^2}{(4\pi)^2 r^2}$$

in terms of the antenna gain G.

The peak output signal-to-noise ratio from the correlator from Eq. (7) (including the doppler compensation gain G_D) is

$$\left(\frac{S}{N}\right)_o = \frac{1}{2} \frac{F\lambda^2}{k(T_N + T_r)} \frac{G}{4\pi} G_D \frac{1}{\epsilon} L_K \frac{\sigma}{4\pi r^2} \tag{16}$$

For convenience in using Eq. (16) the flux spectral density data for the sun and Cassiopeia A have been plotted in Fig. 2 in terms $F\lambda^2$. The general range equation for the passive radar follows from Eq. (16), designating by ρ the minimum output signal-to-noise ratio for reliable detection,

$$r = \sqrt{\sigma} \left[\frac{1}{32\pi^2} \frac{F\lambda^2}{k(T_N + T_r)} G G_D \epsilon^{-1} \rho^{-1} L_\beta\right]^{1/2} \tag{17}$$

The measured flux density for Cassiopeia A is retabulated in Table 2 in terms of the quantity $F\lambda^2/k$ (which is a purely fictitious equivalent temperature T_c since the angular dimensions of the source are unknown). Also tabulated are T_c/T_N maximum and T_c/T_N minimum. This ratio is reasonably constant and reaches a maximum with respect to wavelength near 3 meters.

The minimum and maximum detection ranges for the source Cassiopeia A have been computed for various λ and are given in Table 3 with the assumed values $L_\beta = 1$ and $1/\epsilon\ G\ G_D\ 1/\rho = 10^{12}$.

SOLAR RADAR RANGE EQUATION

As we have seen, for the sun

$$F = \frac{2\pi k T_s}{\lambda^2} \left(\frac{\alpha}{2}\right)^2$$

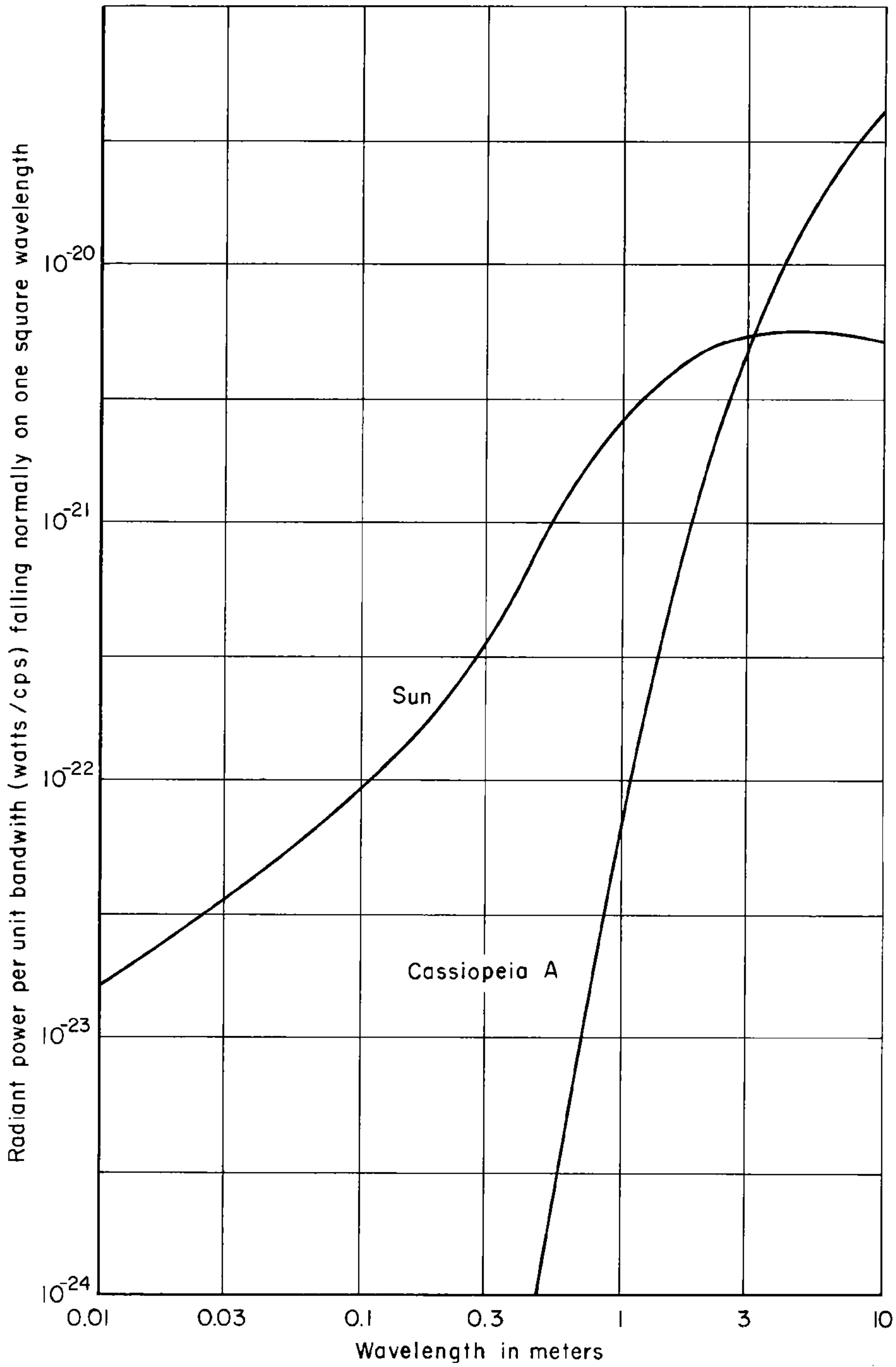

Fig. 2 — Measured RF power falling normally on surface of one square wavelength in area for the sun and Cassiopeia A

so that the effective solar temperature

$$T_s = \frac{\lambda^2 F}{2\pi k \left(\frac{\alpha}{2}\right)^2}$$

This and the minimum and maximum measured sky temperatures T_N are listed in Table 1 using data from Ref. 2. The minimum and maximum ratios T_s/T_N are also tabulated. These ratios will determine the detection sensitivity where receivers are used which have noise temperature lower than the sky temperature.

It would appear from Table 1 that 10^4 is an optimistic maximum value for $T_s/T_R + T_N$.

The output signal-to-noise ratio for the solar radar is strongly dependent on the incoherence loss L_β.

If the coherence losses are partially compensated in the manner described above, and the bandwidth made small compared to β^{-1}, then $L'\beta = 1$, and

$$\left(\frac{S}{N}\right)_o = \frac{1}{16\pi} \frac{T_s}{T_R + T_N} \left(\frac{\alpha}{2}\right)^2 G\, G_D \frac{1}{\epsilon} \frac{\sigma}{r^2}$$

Then the maximum range r with compensation for coherence losses is thus

$$r_1 = \frac{\alpha}{8\sqrt{\pi}} \left[\frac{T_s}{T_R + T_N} \frac{1}{\rho}\, G\, G_D \frac{1}{\epsilon}\right]^{1/2} \sqrt{\sigma} \qquad (18)$$

whereas, without compensation for the incoherence loss,

$$L_\beta = \frac{1}{\left(2\pi f_o \beta\right)^2}$$

and

$$r_2 = \lambda \left[\frac{1}{(4\pi)^3} \frac{T_s}{T_R + T_N} \left(\frac{1}{\rho}\, G\, G_D G_h \frac{1}{\epsilon}\right) \frac{1}{\sin^2\theta} \frac{\sigma}{\lambda^2}\right]^{1/4} \qquad (19)$$

In this latter formula, curiously, the size of the solar disc α does not appear due to the fact that the scattered power from the target varies as α^2 while the incoherence loss varies as α^{-2}.

It is noteworthy that the detection range varies as the square root of the target cross-section where the incoherence effect is negligible or is compensated, but otherwise--as for the active radar--varies as the fourth root of the target cross-section.

Equations (17), (18), and (19) are the desired range equations. In Table 4 the maximum and minimum values of $r_1/\sqrt{\sigma}$ are given for various λ using an assumed value of 10^{12} for $(1/\rho \; G \; G_D \; 1/\epsilon)$, while Table 5 gives maximum and minimum values of $r_2/_{\sigma}1/4$ for various λ using an assumed value of 10^{14} for $(1/\rho \; G \; G_D \; G_h \; 1/\epsilon)$.

The surveillance performance of the solar radar (without compensation of the coherence losses) is identical to that of an active radar which radiates an amount of energy

$$E_T = (kT_s) \frac{1}{2G} \left(\frac{G_D \; G_h}{\epsilon} \right) \frac{1}{\sin^2\theta} \tag{20}$$

during the dwell time on target. This formula vividly indicates the feeble equivalent power of the uncompensated solar radar, since $kT_s \approx 10^{-17}$ joules.(9)

Table 1

RATIO OF EFFECTIVE SOLAR TEMPERATURE (T_s) TO MINIMUM AND MAXIMUM MEASURED SKY TEMPERATURES (T_N)

(The ratio of reference solar power to noise power is $T_s/T_N\ G/16\ \alpha^2$)

λ (meters)	(T_N) max	(T_N) min	$\frac{T_s}{T_N}$ min	$\frac{T_s}{T_N}$ max
10	10^5	10^4	25	250
3	10^4	7×10^2	250	3.6×10^3
1	6.5×10^2	40	2×10^3	3.3×10^4
.3	35	1.5	5×10^3	1.2×10^5

Table 2

RATIO OF FICTITIOUS EQUIVALENT TEMPERATURE (T_c) FOR CASSIOPEIA A TO MINIMUM AND MAXIMUM MEASURED SKY TEMPERATURES (T_N)

$$\frac{F_s \lambda^2}{k} = T_c, \quad \frac{T_c}{T_{N\ max}} \quad \text{and} \quad \frac{T_c}{T_{N\ min}} \quad \text{for Cassiopeia A}$$

λ (meters)	T_c^*	$\frac{T_c}{T_N}$ max	$\frac{T_c}{T_N}$ min
10	2.9×10^3	.29	2.9×10^{-2}
3	3.2×10^2	.46	3.2×10^{-2}
1	5.1	.127	8×10^{-3}

*T_c is a fictitious equivalent star temperature. The ratio of reference signal power to noise power is $T_c/T_N\ G/8\pi$.

Table 3

MINIMUM AND MAXIMUM DETECTION RANGES FOR THE SOURCE CASSIOPEIA A FROM EQ. (17)

λ (meters)	$(r/\sqrt{\sigma})$ max (km)	$(r/\sqrt{\sigma})$ min (km)
10	30.3	9.6
3	38.2	10
1	30.1	5.1

σ is the target side-scatter cross-section in m^2

Table 4

MINIMUM AND MAXIMUM DETECTION RANGES FOR SOLAR RADAR WITH COMPENSATED INCOHERENCE LOSSES FROM EQ. (18)

λ (meters)	$(r/\sqrt{\sigma})$ min (km)	$(r/\sqrt{\sigma})$ max (km)
10	3.5	11
3	11	42
1	31.3	127
.3	49	242

σ = target side-scatter cross-section in m^2

The value 10^{12} for $(\frac{1}{\rho}\ G\ G_D\ \frac{1}{\epsilon})$ has been assumed

Table 5

MAXIMUM AND MINIMUM DETECTION RANGES FOR SOLAR RADAR
WITHOUT COMPENSATED INCOHERENCE LOSSES
FROM EQ. (19)

λ (meters)	$\frac{r}{\sigma^{1/4}}$ min (km)	$\frac{r}{\tau 1/4}$ max (km)
10	4.0	7.1
3	3.9	7.6
1	3.8	7.6
.3	2.6	5.8

σ = target side scatter cross-section in m^2

The value 10^{14} for $(\frac{1}{\rho} G\, G_D\, G_h \frac{1}{\epsilon})$ has been assumed

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