MEMORANDUM
RM-3700-PR
AUGUST 1963

SATELLITE LIFETIMES IN NEARLY CIRCULAR ORBITS FOR VARIOUS EARTH-ATMOSPHERE MODELS

L. N. Rowell

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM

RM-3700-PR

AUGUST 1963

SATELLITE LIFETIMES IN NEARLY CIRCULAR ORBITS FOR VARIOUS EARTH-ATMOSPHERE MODELS

L. N. Rowell

This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation

1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

The lifetime of an Earth satellite is affected significantly by changes in the altitude, the orientation of the orbital plane, and the orbital elements.

In studies of satellite lifetimes, especially in analytic studies, simplifying assumptions concerning the gravitational attraction of the Earth and the nature of the atmosphere are usually made. This is necessary in order to make the problem tractable.

Several studies showing the effect of various individual perturbing forces on the orbital elements and on lifetime itself have been made. This Memorandum shows how the computed satellite lifetime varies with different models of the Earth's atmosphere and in the presence of the gravitational attraction of the Moon and Sun.

SUMMARY AND CONCLUSIONS

This Memorandum investigates the effects on Earth-satellite lifetimes of altitude, orbit-plane inclination, and perturbation forces for close-in, nearly circular orbits. The lifetimes of two large, heavy satellites were studied.

Studies of orbits with initial altitudes of from 75 to 150 n mi show that satellite lifetime increases significantly as the orbit-plane inclination increases toward 90 deg, and that the ratio of change in lifetime to change in orbit-plane inclination increases as the orbital altitude increases.

Satellite lifetimes based on an oblate atmosphere model are significantly greater than those based on a spherical atmosphere model for nonequatorial satellites. The models use the 1962 ARDC Standard Atmosphere and the COSPAR International Reference Atmosphere of 1961. The lifetimes for both satellites starting at 150-n mi altitude are approximately 20 per cent longer for an orbit-plane inclination of 90 deg. This is because the satellite spends more time at higher altitudes above the oblate Earth and therefore more time in less dense atmosphere as the orbit-plane inclination increases to 90 deg. The difference in lifetimes caused by using the above atmospheres in an oblate atmosphere model is about 2 per cent.

The combined gravitational attractions of the Moon and Sun cause only negligible variations in the lifetimes of the satellites studied. The Earth's gravitational field does not directly affect lifetimes but produces orbital motions which, when combined with the atmospheric drag forces, can produce considerable changes in satellite lifetimes.

ACKNOWLEDGMENTS

The author is grateful to S. H. Dole of The RAND Corporation for his encouragement to do this study and for his useful suggestions concerning it.

The assistance of Susan Belcher in obtaining the required computer results is also appreciated. Her understanding of the computer program and its operation was a valuable asset to this study.

CONTENTS

LIST OF SYMBOLS

A_2, A_3, A_4, A_5	=	constants associated with Earth's oblateness
A, m	=	cross-sectional area and mass, respectively, of the satellite
a	=	semimajor axis
a_d	=	acceleration caused by atmospheric drag
$\bar{a}_r$	=	acceleration vector of the satellite relative to Earth
$\bar{a}_{sb}, \bar{a}_{eb}$	=	acceleration of vectors of the satellite and Earth, respectively, caused by the perturbing body
C_d	=	drag coefficient
E	=	eccentric anomaly
e	=	eccentricity
i	=	inclination of the satellite orbit plane
n	=	mean orbital angular rate
p	=	semilatus rectum
R, S, W	=	components of the disturbing acceleration in the radial direction, in the orbit plane perpendicular to the radial direction, and in the direction of increasing v, and perpendicular to the orbit plane and in the direction of the cross product of unit vectors in the directions of increasing R and S, respectively
$R(u), S(u), W(u)$	=	cosines of the angles between $\bar{r}_{eb}$ and the directions of R, S, and W, respectively
$R(\omega), S(\omega), W(\omega)$	=	components of a unit vector along an Earth radius to perigee and along a line perpendicular to the Earth radius to perigee in the direction of increasing v and perpendicular to the satellite orbit plane, respectively
r	=	distance between satellite and Earth's center
r_{sb}, r_{eb}	=	distance between the attracting body and the satellite and the attracting body and Earth's center, respectively

U = Earth's potential

u = argument of the latitude of the satellite

$\bar{V}, \bar{V}_a$ = vector velocities of the satellite and atmosphere, respectively, in inertial space

V_{rel} = velocity of the satellite relative to the atmosphere

V_r, V_s = velocity along $\bar{r}$ and perpendicular to r in the direction of increasing v, respectively

v = true anomaly

W/V = satellite weight-to-volume ratio

α = angle between Earth radii to the satellite and to the perturbing body

β = elevation angle of $\bar{V}$ relative to the local horizontal

γ = defined by Eq. (11)

δ = geocentric latitude

μ = product of the gravitational constant and the Earth's mass

μ_b = product of the gravitational constant and the mass of the perturbing body

ρ = atmospheric density

σ = a measure of the change in the time of perigee passage

Ω = argument of the ascending node relative to the vernal-equinox direction

ω = argument of perigee

ω_e = rotational rate of Earth

∇U = gradient of Earth's potential

b refers quantities to the perturbing body

d refers quantity to atmospheric drag

I. INTRODUCTION

Satellite-lifetime studies usually employ a simplified model of the Earth and its atmosphere. In general, the analytical studies assume a spherical Earth model, i.e., an Earth with a purely inverse-square gravitational-force field, and a spherical nonrotating atmosphere. Lifetime studies of an empirical nature usually use a more realistic model and require a computer. Such a model may include an oblate Earth gravitational field and a spherical nonrotating atmosphere. Except for nearly equatorial orbits the lifetimes obtained are only approximate. For either type of study the perturbations of the satellite orbit caused by the gravitational fields of the Moon and Sun are usually omitted. These results indicate that this omission is justified for low, nearly circular, satellite orbits.

The reference model of the Earth and atmosphere used for this investigation of satellite lifetimes is somewhat more realistic than either of the above because it consists of an oblate, nonrotating atmosphere. Also, the perturbations caused by the Moon and Sun are included. The results show that significant errors will be made in predicting satellite lifetimes if either of the simplified models above is used.

The atmosphere model used is the COSPAR International Reference Atmosphere of 1961. This atmosphere model was accepted at the COSPAR meeting in Florence in April 1961 and gives average values of the atmospheric properties for altitudes between 0 and 800 km. Near the end of this study the 1962 ARDC Standard Atmosphere became available, and in addition, the computer program was modified to include rotation of the atmosphere. The 1962 ARDC Atmosphere and the COSPAR Atmosphere agree very closely (see p. 16).

In order to observe the effect on satellite lifetime caused by the various perturbations and the use of different atmosphere models, some of the lifetimes were recomputed. The results appear on p. 23.

The lifetimes for two vehicles having a mass-to-volume ratio of 20 lb/ft^3 were considered. One vehicle weighed 10,000 lb, and the other, 200,000 lb. Both were assumed to be spherical. The vehicles were started on nearly circular initial orbits (e = 0.0001) at perigee altitudes of 75, 100, 125, and 150 n mi above the equator. Orbit-plane inclinations of 0.0001, 45, and 90 deg were used. (Nonzero initial values for the eccentricity and the inclination are necessary to avoid a temporary discontinuity in some of the computer equations.) Errors introduced in the lifetimes if either or both quantities start at or pass through zero are negligible because both quantities are time varying and therefore induced discontinuities are transient.

In order to assess the effect of the individual perturbations on lifetimes, several orbits with initial perigees of 150 n mi were recomputed. The separate effects of the gravitational attractions of the Moon and the Sun and variations of the atmosphere model are shown.

$$\frac{d\Omega}{dt} = \frac{r \sin u}{na^2\sqrt{1 - e^2} \sin i} W \tag{4}$$

$$\frac{di}{dt} = \frac{r \cos u}{na^2\sqrt{1 - e^2}} W \tag{5}$$

$$\begin{aligned}\frac{d\sigma}{dt} = & -\frac{1}{na}\left(\frac{2r}{a} - \frac{1 - e^2}{e} \cos v\right) R \\ & - \frac{(1 - e^2) \sin v}{nae}\left(1 + \frac{r}{p}\right) S + \frac{3}{2}\frac{nt}{a}\frac{da}{dt}\end{aligned} \tag{6}$$

where R, S, and W are components of the disturbing acceleration in the radial direction, in the orbit plane perpendicular to the radial direction and in the direction of increasing v, and perpendicular to the orbit plane and in the direction of the cross product of unit vectors in the directions of increasing R and S, respectively. The letter p denotes the semilatus rectum, v is the true anomaly, and u is the argument of the latitude of the vehicle and is given by

$$u = v + \omega \tag{7}$$

where ω is the argument of perigee. The mean orbital angular rate of the satellite, n, is given by

$$n = \frac{\sqrt{\mu}}{a^{3/2}} \tag{8}$$

where μ is the product of the gravitational constant and the mass of the Earth.

The variable σ in Eq. (6) is a measure of the change in the time of perigee passage and is given by

$$\sigma = nt - (E - e \sin E) \tag{9}$$

where E is the eccentric anomaly. Because the long-term change in the orbital elements determines fairly accurately the lifetime of a satellite, the actual position of the satellite in the orbit is not computed. Thus, Eq. (6) is used only in the derivation of an expression for the time rate of change of the true anomaly.

From Eqs. (1), (2), and (6), the time derivative of Eq. (9) and the well-known equations (see Ref. 1, p. 164 or 392) that express the true anomaly as a function of the eccentric anomaly and the eccentricity, the time rate of change of the true anomaly can be found to be

$$\frac{dv}{dt} = \frac{\mu p}{r^2} \left\{1 + \frac{r^2}{\mu e}\left[R \cos v - (1 + \frac{r}{p}) S \sin v\right]\right\} \qquad (10)$$

where the W component of acceleration comes from the last term on the right side of Eq. (3).

If we set

$$\gamma = \left\{1 + \frac{r^2}{\mu e}\left[R \cos v - (1 + \frac{r}{p}) S \sin v\right]\right\}^{-1} \qquad (11)$$

and use the relation

$$\frac{d(\ )}{dt} = \frac{d(\ )}{dv} \frac{dv}{dt} \qquad (12)$$

then Eqs. (1) through (5) can be rewritten with v as the independent variable.

$$\frac{da}{dv} = \frac{2a^2 r^2 \gamma}{\mu p}\left[Re \sin v + S(1 + e \cos v)\right] \qquad (13)$$

$$\frac{de}{dv} = \frac{r^2 \gamma}{\mu}\left\{R \sin v + S \frac{r}{p}\left[2 \cos v + e(1 + \cos^2 v)\right]\right\} \qquad (14)$$

$$\frac{d\omega}{dv} = \frac{r^2 \gamma}{\mu e}\left[-R \cos v + S(1 + \frac{r}{p}) \sin v\right] - \frac{d\Omega}{dv} \cos i \qquad (15)$$

$$\frac{d\Omega}{dv} = W \frac{r^3\gamma}{\mu p} \frac{\sin u}{\sin i} \tag{16}$$

$$\frac{di}{dv} = W \frac{r^3\gamma}{\mu p} \cos u \tag{17}$$

In the presence of atmospheric drag, the orbital element a may change significantly during one period of the satellite. Since the method assumes that the orbital elements are constants during the computation interval, i.e., during any specified integral number of orbital periods, it is desirable to use p instead of a as one of the orbital elements because its change during the computation interval is less than that of a for small eccentricities. This is demonstrated in the Appendix. The orbital element a is replaced by p as follows: From the equation $p = a(1 - e^2)$ we get

$$\frac{dp}{dv} = (1 - e^2) \frac{da}{dv} - 2ae \frac{de}{dv} \tag{18}$$

After substituting for da/dv and de/dv from Eqs. (13) and (14) and simplifying we get

$$\frac{dp}{dv} = \frac{2r^3\gamma}{\mu} S \tag{19}$$

Equations (11) and (13) through (17) are a closed set of first-order nonlinear differential equations. If the perturbing accelerations are obtained as functions of the orbital elements and the independent variable v, then the elements can be determined using the initial values of the orbital elements and the computed changes in the elements during the integration period.

Except for orbits with very small eccentricities and low altitudes, the R and S components of Eq. (11) will be approximately zero, and γ will be approximately equal to one for the nonrotating-atmosphere case. Thus, except for the model containing a rotating atmosphere, lifetimes were computed with $\gamma = 1$.

III. PERTURBATIONS CAUSED BY THE MOON AND SUN

The gravitational perturbations included in the lifetime computations are caused by the attractions of the Moon and Sun on the vehicle. The computer program will compute the perturbation caused by solar-radiation pressure; however, for close-in, heavy satellites like the ones considered in this study, this perturbation has a negligible effect on satellite lifetime and therefore is not included.

The equations for the components of the perturbing accelerations are obtained subject to the assumption that the solar system is geocentric, i.e., that the Sun as well as the Moon rotates around the Earth. The perturbations are computed explicitly by making use of the known orbits of the Moon and Earth (or in this case the Sun).

In order to determine the acceleration of the satellite relative to the Earth we must first determine the acceleration of the Earth and satellite relative to the attracting body. In vector form the acceleration of the satellite relative to the Earth is

$$\bar{a}_r = \bar{a}_{sb} - \bar{a}_{eb} \tag{20}$$

where $\bar{a}_{sb}$ and $\bar{a}_{eb}$ are the acceleration vectors of the satellite and Earth, respectively, caused by the perturbing body. The vector relationships are shown in Fig. 1.

By using the standard equation for the acceleration of a point mass in an inverse-square gravitational-force field, i.e.

$$\ddot{\bar{r}} = -\frac{\mu_b \bar{r}}{r^3}$$

where $\ddot{r}$ is the derivative with respect to time and μ_b is the product of the gravitational constant and the mass of the perturbing body, and some trigonometric relations, the equation for $\bar{a}_r$ becomes

$$\bar{a}_r = \frac{\mu_b}{r_{eb}^3}\left\{\bar{r}_{sb}\left[1 + 3\left(\frac{r}{r_{eb}}\right)\cos\alpha\right] - \bar{r}_{eb}\right\} \tag{21}$$

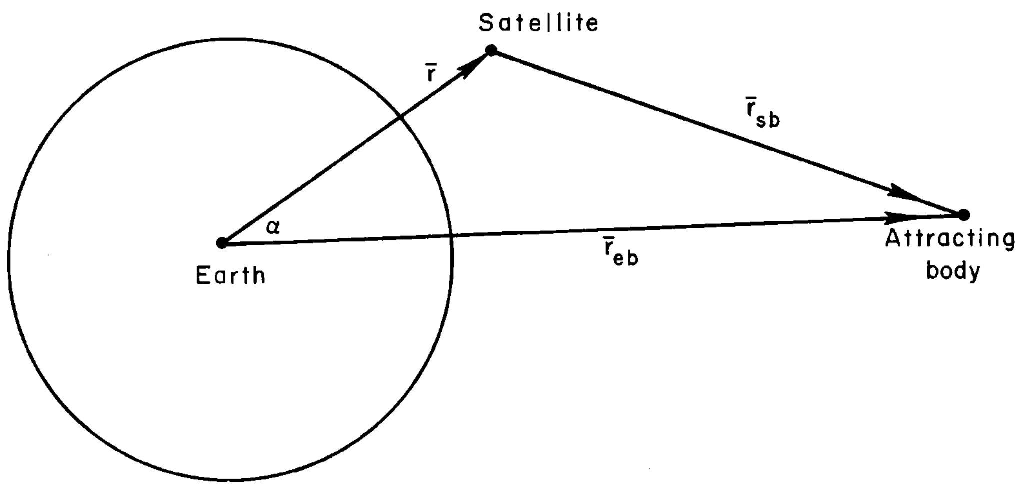

Fig. 1—Typical orientation of Earth, satellite, and attracting body

Then, using the equation

$$\bar{r}_{sb} = \bar{r}_{eb} - \bar{r}$$

we get

$$\bar{a}_r = \frac{\mu_b}{r_{eb}^3}\left\{\left[3\left(\frac{r}{r_{eb}}\right)\cos\alpha + \frac{3}{2}\left(\frac{r}{r_{eb}}\right)^2(5\cos^2\alpha - 1) + \ldots\right]\bar{r}_{eb} - \left[1 + 3\left(\frac{r}{r_{eb}}\right)\cos\alpha + \frac{3}{2}\left(\frac{r}{r_{eb}}\right)^2(5\cos^2\alpha - 1) + \ldots\right]\bar{r}\right. \tag{22}$$

The acceleration vector $\bar{a}_r$ can be resolved into the orthogonal components R, S, and W by using the elements of the orbits of the satellite and the perturbing body and the instantaneous relative orientation of the bodies. Ignoring second- and higher-order terms in (r/r_{eb}), the components of acceleration are

$$R_b = \frac{\mu_b}{r_{eb}^3}\left[3r\,R(u)\cos\alpha - r\right] \tag{23}$$

$$S_b = \frac{\mu_b}{r_{eb}^3}\left[3r\ S(u)\ \cos\ \alpha\right] \tag{24}$$

and

$$W_b = \frac{\mu_b}{r_{eb}^3}\left[3r\ W(u)\ \cos\ \alpha\right] \tag{25}$$

where R(u), S(u), and W(u) are the cosines of the angles between $\bar{r}_{eb}$ and the directions R, S, and W, respectively, and are given by

$$\begin{aligned} R(u) = & \left[\cos u \sin (\Omega - \Omega_b) + \sin u \cos (\Omega - \Omega_b) \cos i\right] \cos \omega_b t \\ & + \left[\cos u \sin (\Omega - \Omega_b) + \sin u \cos (\Omega - \Omega_b) \cos i\right] \sin \omega_b t \cos i_b \\ & + \sin u \sin i \sin \omega_b t \sin i_b \end{aligned} \tag{26}$$

$$\begin{aligned} S(u) = & \left[-\sin u \cos (\Omega - \Omega_b) - \cos u \sin (\Omega - \Omega_b) \cos i\right] \cos \omega_b t \\ & + \left[-\sin u \sin (\Omega - \Omega_b) + \cos u \cos (\Omega - \Omega_b) \cos i\right] \sin \omega_b t \cos i_b \\ & + \cos u \sin i \sin \omega_b t \sin i_b \end{aligned} \tag{27}$$

and

$$\begin{aligned} W(u) = & \sin (\Omega - \Omega_b) \sin i \cos \omega_b t \\ & -\cos (\Omega - \Omega_b) \sin i \sin \omega_b t \cos i_b + \cos i \sin \omega_b t \sin i_b \end{aligned} \tag{28}$$

The quantities Ω, i, and ω are the ascending node, the inclination of the orbit plane to the equatorial plane, and the orbital angular velocity of the bodies, respectively. The equations for R(u), S(u), and W(u) can be derived by starting with a rectangular-coordinate system with the x-axis along the vernal equinox direction, the x,y plane in the equatorial plane, and the z-axis along the polar axis. By rotating an angle Ω about the z-axis, then an angle i about the new x-axis, and finally the angle u about the new z-axis for both the satellite and the perturbing body, the acceleration caused by the perturbing body can be resolved into components along R, S, and W. In the above equations u_b has been replaced by $\omega_b t$ where ω_b is the mean orbital angular rate of the perturbing body during the time interval from the last nodal crossing to the time t.

Because the changes in the orbital elements are computed using constant orbital elements during the computation period which is an integral number of revolutions, it is convenient to carry out the integrations with respect to the true anomaly v. Therefore, it is necessary to perform a separation of variables in Eqs. (26) and (28). By using the relation u = v + ω we can write

$$R(u) = R(\omega) \cos v + S(\omega) \sin v \tag{29}$$

$$S(u) = S(\omega) \cos v - R(\omega) \sin v \tag{30}$$

and

$$W(u) = W(\omega) \tag{31}$$

where R(ω), S(ω), and W(ω) are held constant during the computation interval and are updated as are the orbital elements.

Equations (23) through (25) can now be written in their final form as

$$R_b = \frac{3\mu_b r}{r_{eb}^3}\left[R^2(\omega)\cos^2 v + 2R(\omega)S(\omega)\sin v \cos v + S^2(\omega)\sin^2 v - \frac{1}{3}\right] \tag{32}$$

$$S_b = \frac{3\mu_b r}{r_{eb}^3}\left\{R(\omega)S(\omega)\cos^2 v + \left[S^2(\omega) - R^2(\omega)\right]\sin v \cos v - R(\omega)S(\omega)\sin^2 r\right\} \tag{33}$$

IV. PERTURBATIONS OF THE ORBITAL ELEMENTS CAUSED BY THE EARTH'S BULGE

The gravitational potential of the Earth is a scalar function of position. In this case the position is given in polar coordinates r and δ, where r is the radius to the satellite and δ is the geocentric latitude of the satellite. The potential, including the fifth harmonic, is

$$U = \frac{\mu}{r}\left\{1 + \frac{A_2}{r^2}\left(\frac{1}{3} - \sin^2\delta\right) + \frac{A_3}{r^3}\left(\frac{5}{2}\sin^3\delta - \frac{3}{2}\sin\delta\right) + \frac{A_4}{r^4}\left(\frac{3}{35} + \frac{1}{7}\sin^2\delta - \frac{1}{4}\sin^2 2\delta\right) + \frac{A_5}{r^5}\left(\frac{15}{8} - \frac{35}{4}\sin^2\delta + \frac{63}{8}\sin^4\delta\right)\sin\delta + \ldots\right\} \tag{35}$$

where the constants are

$\mu = 3.9863 \times 10^5 \text{ km}^3/\text{sec}^2$

$A_2 = 6.604085 \times 10^4 \text{ km}^2$

$A_3 = 5.890588 \times 10^5 \text{ km}^3$

$A_4 = 1.522760 \times 10^{10} \text{ km}^4$

$A_5 = 2.744909 \times 10^{12} \text{ km}^5$

The gravitational acceleration of the Earth is equal to the gradient of the gravitational potential U and is of the form

$$\nabla U = \frac{\partial U}{\partial r} 1_r + \frac{\partial U}{r\partial\delta} 1_\delta \tag{36}$$

where 1_r and 1_δ are unit vectors along the radius to the satellite and along a perpendicular to the radius vector in the meridian plane, respectively.

The component of acceleration $\partial U/\partial r$ is already in the desired direction; therefore, only $\partial U/r\partial\delta$ need be transformed to components in the directions of S and W. The details of the transformation are omitted, but they can be readily obtained by starting with a rectangular set of axes with origin at the Earth's center, the x-axis along the line of nodes of the satellite orbit plane and the equatorial plane, and the x,y axes defining the equatorial plane. By first rotating the angle i about the x-axis and then the angle u about the axis perpendicular to the orbital plane, the position of the radius vector to the satellite is established. This radius vector is established in the Earth latitude-longitude system by first rotating an incremented longitude angle, determined by u and i, about the Earth's polar axis and then by rotating δ about the offset y-axis. The transformation yields

$$1_\delta = \frac{\cos u \sin i}{\cos \delta} 1_s + \frac{\cos i}{\cos \delta} 1_w \tag{37}$$

where $\sin \delta = \sin i \sin u$.

Using Eq. (37), Eq. (36) becomes

$$\nabla U = \frac{\partial U}{\partial r} 1_r + \frac{\partial U}{r\partial \delta}\left(\frac{\cos u \sin i}{\cos \delta} 1_s + \frac{\cos i}{\cos \delta} 1_w\right) \tag{38}$$

The first term on the right side of Eq. (36) is the potential for a spherical homogeneous Earth and therefore is not a part of the perturbing acceleration. The components of acceleration are found to be

$$\begin{aligned} R &= \frac{\partial U}{\partial r} - \frac{\partial}{\partial r}\left(\frac{\mu}{r}\right) \\ &= -\frac{\mu}{28r^4}\Big\{28A_2 (1 - 3 \sin^2\delta) + 56 \frac{A_3}{r} (5 \sin^2\delta - 3) \sin \delta \\ &\quad + \frac{A_4}{r^2} (12 + 20 \sin^2\delta - 35 \sin^2 2\delta) \\ &\quad + 3 \frac{A_5}{r^3} (105 - 490 \sin^2\delta + 441 \sin^4\delta) \sin \delta\Big\} \end{aligned} \tag{39}$$

$$S = \frac{\cos u \sin i}{\cos \delta} \frac{1}{r} \frac{\partial U}{\partial \delta}$$

$$= -\frac{\mu \cos u \sin i}{56r^4} \left\{ + 112 A_2 \sin \delta - \frac{84 A_3}{r} (4 - 5 \sin^2 \delta) + \frac{32 A_4}{r^2} (3 - 7 \sin^2 \delta) \sin \delta + 105 \frac{A_5}{r^3} (1 - 14 \sin^2 \delta + 21 \sin^4 \delta) \right\} \quad (40)$$

and

$$W = \frac{\cos i}{\cos \delta} \frac{1}{r} \frac{\partial U}{\partial \delta}$$

$$= \frac{-\mu \cos i}{56r^4} \left\{ 112 A_2 \sin \delta - \frac{84 A_3}{r} (4 - 5 \sin^2 \delta) + \frac{32 A_4}{r^2} (3 - 7 \sin^2 \delta) \sin \delta + 105 \frac{A_5}{r^3} (1 - 14 \sin^2 \delta + 21 \sin^4 \delta) \right\} \quad (41)$$

These components of acceleration can be substituted into the standard perturbation equations, i.e., Eqs. (13) through (17) and (19), which are integrated analytically using constant orbital elements in order to obtain the changes in the orbital elements. The limits of integration are $v = o$ to $v = 2\pi$. If the integration period is k revolutions, i.e., if orbital elements are updated every k revolutions, then the changes in the elements for one revolution are multiplied by k to find the total change for k revolutions.

V. ATMOSPHERIC-DRAG PERTURBATIONS

As mentioned earlier, the 1961 COSPAR model of the atmosphere was used. It and the 1962 ARDC atmosphere model are in close agreement, as can be seen in Fig. 2. In order to compare lifetimes using the two different atmospheres, some of the longest runs were recomputed using the ARDC atmosphere. These results are given in Table 1.

The formulas for the R, S, and W projections of the perturbing acceleration caused by a nonrotating atmosphere are given below.

The perturbing acceleration caused by drag is

$$a_d = \frac{1}{2} C_d \frac{A}{m} \rho V_{rel}^2 \tag{42}$$

where V_{rel} is the velocity of the satellite relative to the atmosphere, C_d is the drag coefficient, A is the cross-sectional area of the vehicle perpendicular to the direction of $\overline{V}_{rel}$, and ρ is the atmospheric density. The velocity vector of the vehicle relative to the atmosphere is

$$\overline{V}_{rel} = \overline{V} - \overline{V}_a$$

where $\overline{V}$ and $\overline{V}_a$ are the vector velocities of the vehicle and the atmosphere in inertial space, respectively; their magnitudes are given by

$$V = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

and

$$V_a = \omega_e r \cos \delta$$

Here, a is the semimajor axis of the satellite orbit, ω_e is the rotational rate of the atmosphere (assumed to be equal to the Earth rotational rate), and δ is the geocentric latitude of the satellite. Since a nonrotating model of the atmosphere is used, $\omega_e = 0$ and $V_{rel} = V$.

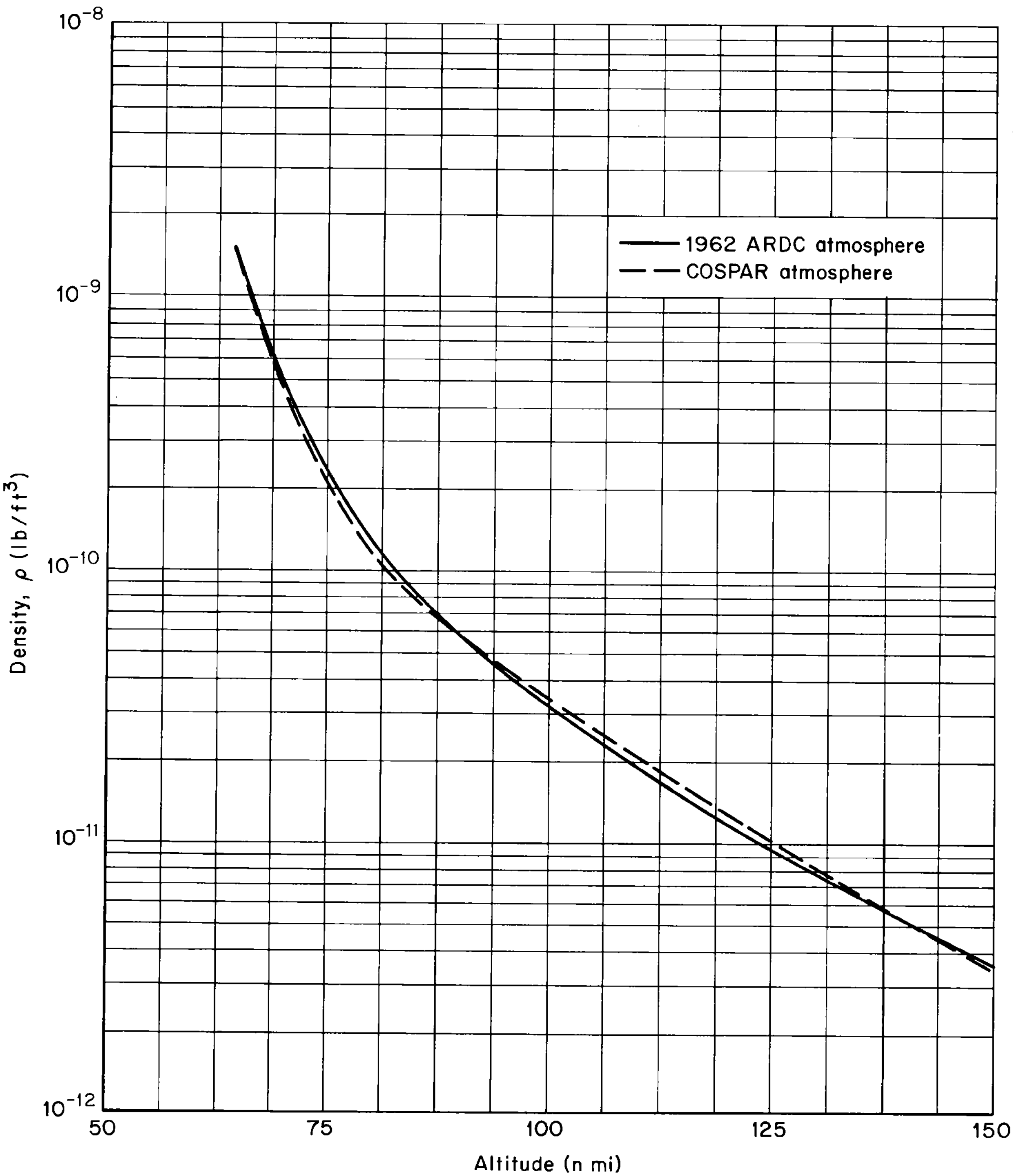

Fig. 2 — Average atmospheric densities versus altitude

Table 1

SATELLITE LIFETIMES AND INITIAL CONDITIONS[a]

Perigee Altitude (n mi)	Orbit-Plane Inclination (deg)	10,000-lb Satellite			200,000-lb Satellite		
		Orbital Lifetimes		Initial Drag Acceleration (ft/sec^2)	Orbital Lifetimes		Initial Drag Acceleration (ft/sec^2)
		Revolutions	Days		Revolutions	Days	
75	0.0001	5	0.30	-10.523	12	0.73	-3.875
	45.0	7	0.42	$\times 10^{-4}$	17	1.03	$\times 10^{-4}$
	90.0	9	0.54		22	1.39	
85	0.0001	18	1.09	-3.810	45	2.73	-1.403
	45.0	23	1.40	$\times 10^{-4}$	59	3.58	$\times 10^{-4}$
	90.0	27	1.64		71	4.31	
100	0.0001	60	3.66	-1.686	160	9.75	-0.621
	45.0	72	4.39	$\times 10^{-4}$	190	11.58	$\times 10^{-4}$
	90.0	83	5.06		221	13.47	
125	0.0001	259	15.9	-0.494	698	42.9	-0.182
	45.0	298	18.3	$\times 10^{-4}$	801	49.2	$\times 10^{-4}$
	90.0	339	20.8		905	55.6	
150	0.0001	867	53.8	-0.165	2367	147.0	-0.061
	45.0	979	60.8	$\times 10^{-4}$	2658	165.9	$\times 10^{-4}$
	90.0	1094	67.9		2980	184.7	

[a] For both satellites, $W/V = 20\ lb/ft^3$, $C_d = 2$, $\Omega_o = \omega_o = 0$, and $e_o = 0.0001$.

The components of $\overline{V}$ are in the orbital plane, and the radial and transverse components are

$$V_r = V \sin \beta = \sqrt{\frac{\mu}{p}}\, e \sin v$$

$$V_s = V \cos \beta = \sqrt{\frac{\mu}{p}}\, (1 + e \cos v)$$

where β is the elevation of $\overline{V}$ relative to the perpendicular to the radius vector, p and e are the semilatus rectums and eccentricity of the orbit, respectively, and v is the true anomaly of the satellie. Clearly

$$V = \sqrt{V_r^2 + V_s^2}$$

and the components of the perturbing acceleration $\overline{a}_d$ are

$$R_d = -\frac{1}{2} C_d \frac{A}{m} \rho V^2 \sin \beta = -\frac{1}{2} C_d \frac{A}{m} \rho V V_r$$

$$S_d = -\frac{1}{2} C_d \frac{A}{m} \rho V^2 \cos \beta = -\frac{1}{2} C_d \frac{A}{m} \rho V V_s$$

$$W_d = 0$$

These are the perturbation acceleration components required for the integration of Eqs. (13) through (17) and (19). Because ρ is a function of v, the equations are integrated numerically for $v = 0$ to $v = 2\pi$.

VI. RESULTS

The lifetimes of two satellites with a mass-to-volume ratio of 20 lb/ft^3, which is comparable to the Mercury capsule, were studied. One vehicle weighed 10,000 lb, and the other, 200,000 lb; both were assumed to be spherical. The drag force is directly proportional to the cross-sectional area, which, for a sphere, is

$$A = \pi r_s^2 = \pi \left(\frac{3 \text{ vol}}{4\pi}\right)^{2/3}$$

For the two satellites we get for the cross-sectional areas

$$A_1 = 76.1619 \text{ ft}^2$$

and

$$A_2 = 561.1656 \text{ ft}^2$$

where the subscripts 1 and 2 denote the 10,000- and 200,000-lb satellites, respectively. Table 1 is a summary of the initial conditions and the resulting lifetimes for the two satellites.

The satellites were started at perigee and on the line of nodes on nearly circular orbits (e = 0.0001) for initial altitudes of 75, 85, 100, 125, and 150 n mi. Orbit-plane inclinations of 0.0001, 45, and 90 deg were used in order to observe the effect of orbit-plane inclination on lifetime.

We see from Table 1 (and more clearly from Fig. 3) that the lifetime of a given satellite increases with inclination angle. Because of the Earth's oblateness, the altitude will increase as the satellite moves to higher latitudes even if the radial distance remains constant. This increase in the altitude is accompanied by a decrease in atmospheric density and, consequently, by an increase in the lifetime of the satellite. The ratio of the change in lifetime to the change in inclination angle gives an estimate of the derivative of lifetime with

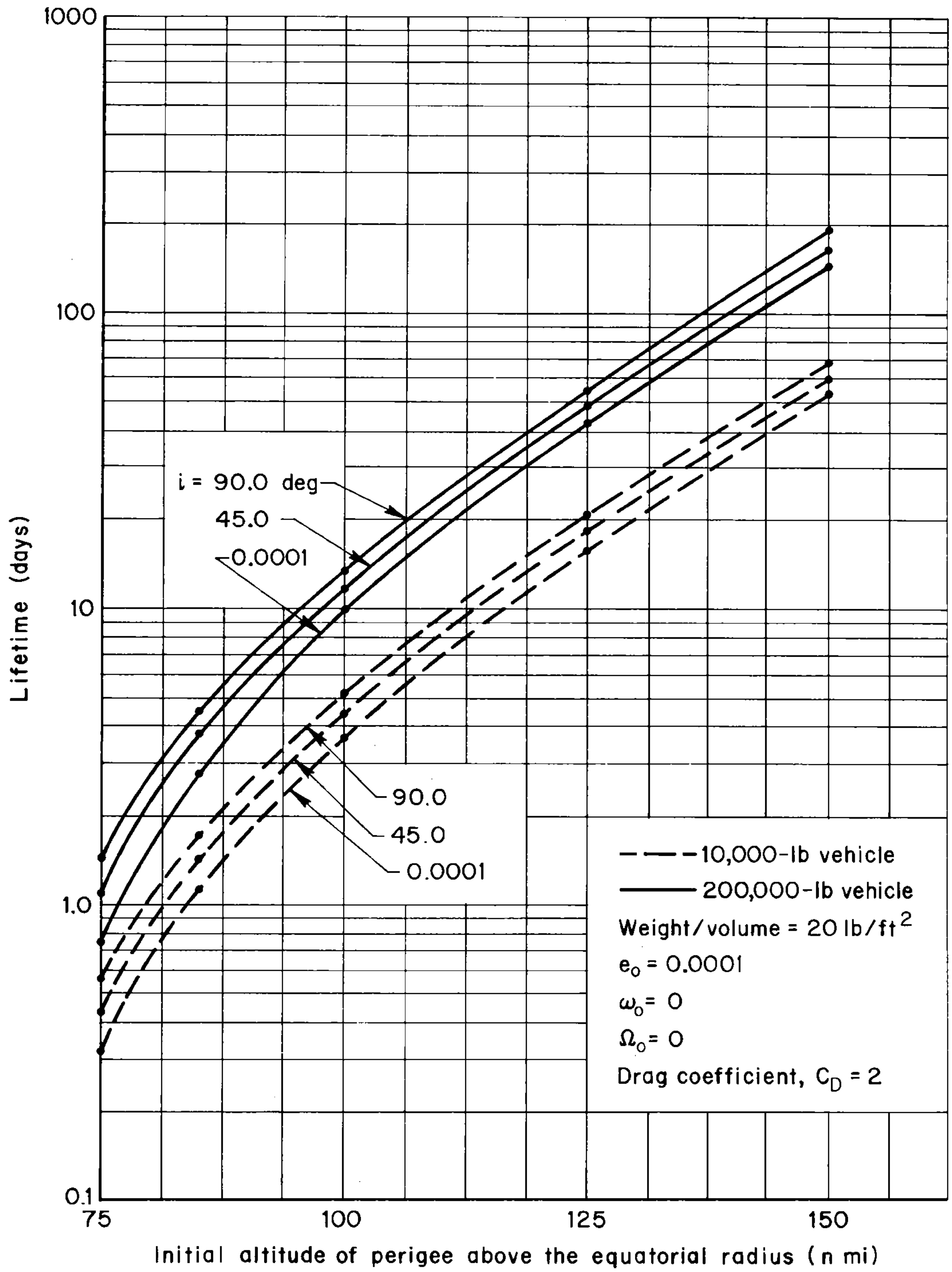

Fig. 3 — Satellite lifetimes

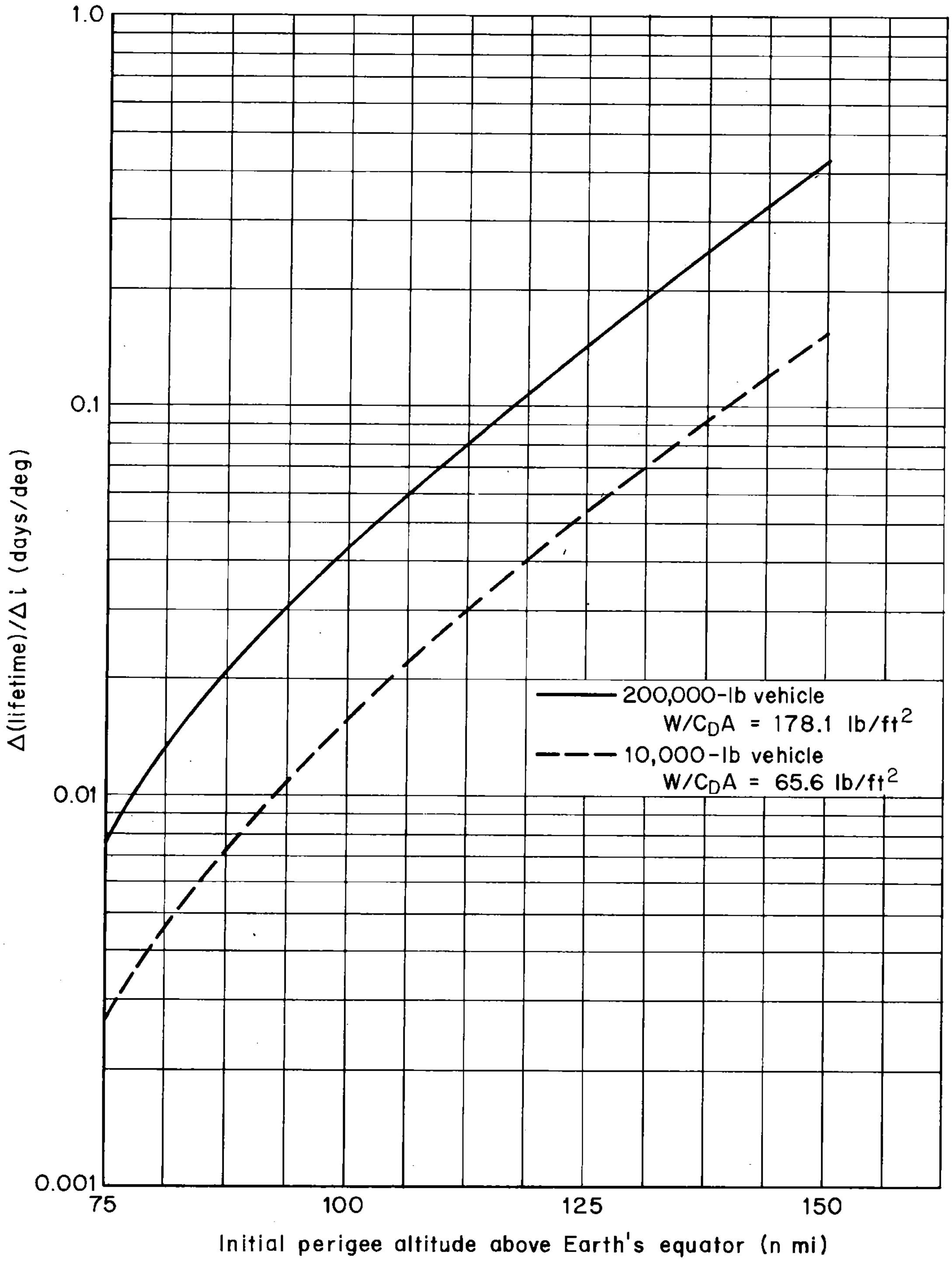

Fig. 4 — Rate of change of lifetime with inclination

altitude of the satellite and, consequently, variations in the drag acceleration. These effects vary with the inclination of the orbit plane and therefore may account for the change in lifetime with inclination angle, as indicated in Table 2.

3. Satellite lifetimes are increased when the 1961 COSPAR atmosphere is replaced by the 1962 ARDC atmosphere. This is caused by the fact that the latter atmosphere is less dense at low altitudes (see Fig. 2).

4. Because of a preference for the 1962 ARDC atmosphere model, this model, instead of the 1961 COSPAR atmosphere, was used to introduce the effect of atmosphere rotation. The effect of rotation on lifetimes using either model would be very nearly identical. The greatest change in lifetime occurs for equatorial orbits and is about 14 per cent for the direct orbits studied; this difference in lifetime appears to vary approximately with the cosine of the orbit-plane inclination angle. For polar orbits, the change in lifetime is negligible.

5. For all of the Earth-atmosphere models using an oblate Earth and oblate atmosphere, satellite lifetime increases significantly as the orbit-plane inclination increases.

VII. CONCLUSIONS

Satellite lifetime increases significantly as the orbit-plane inclination increases toward 90 deg (see Fig. 3).

The ratio of change in satellite lifetime to change in the orbit-plane inclination increases as the orbital altitude increases (see Fig. 4).

Satellite lifetimes based on oblate atmosphere models are significantly longer than those based on spherical atmosphere models for non-equatorial satellites. Table 2 shows that the lifetimes for both satellites starting at 150-n mi altitude are approximately 20 per cent longer for an orbit-plane inclination of 90 deg. This is caused by the fact that the satellite spends more time at higher altitudes above the oblate Earth and therefore more time in less dense atmosphere as the orbit-plane inclination increases to 90 deg.

The difference in lifetimes caused by using two different current models of the atmosphere (see Fig. 2) is about 2 per cent for both satellites if they are started at an altitude of 150 n mi (see Table 1).

The combined gravitational attractions of the Moon and Sun cause only negligible variations in the lifetimes of the satellites studied. The Earth's gravitational field does not directly affect lifetimes but produces orbital motions which, when combined with the atmospheric drag forces, can produce considerable changes in satellite lifetimes.

APPENDIX

In the presence of atmospheric drag, the change of the semilatus rectum, p, during the computation interval of $2k\pi$ is less than that of the semimajor axis a. This can be demonstrated as follows:

$$\frac{da}{dv} = \frac{2a^2r^2\gamma}{\mu p}\left[e \sin v\, R + (1 + e \cos v)S\right] \tag{43}$$

$$\frac{dp}{dv} = \frac{2r^3\gamma}{\mu} S \tag{44}$$

Is

$$\int_0^{2k\pi} \left(\frac{da}{dv}\right) dv > \int_0^{2k\pi} \left(\frac{dp}{dv}\right) dv \tag{45}$$

for all R and S and for small values of e?

For atmospheric drag caused by a nonrotating atmosphere

$$R = 1/2\, C_d \frac{A}{m} \rho \frac{\mu}{p} e \sin v \sqrt{1 + e^2 + 2e \cos v}$$

$$S = 1/2\, C_d \frac{A}{m} \rho \frac{\mu}{p} (1 + e \cos v) \sqrt{1 + e^2 + 2e \cos v}$$

Substituting for R and S in Eqs. (43) and (44) and rewriting Eq. (45) gives

$$\int_0^{2k\pi} \frac{2a^2r^2\gamma}{\mu p} (1 + e^2 + 2e \cos v) \sqrt{1 + e^2 + 2e \cos v}\; dv$$

$$> \int_0^{2k\pi} \frac{2r^3\gamma}{\mu} (1 + e \cos v) \sqrt{1 + e^2 + 2e \cos v}\; dv \tag{46}$$

Replace r with

$$r = \frac{p}{1 + e \cos v}$$

and simplify Eq. (46) to get

$$a^2 \int_o^{2k\pi} \frac{(1 + e^2 + 2e \cos v)^{3/2}}{(1 + e \cos v)^2} dv > p^2 \int_o^{2k\pi} \frac{(1 + e^2 + 2e \cos v)^{1/2}}{(1 + e \cos v)^2} dv \tag{47}$$

By expanding the integrands of Eq. (47) in series and neglecting terms of order e^3 and higher, Eq. (47) becomes

$$a^2 \int_o^{2k\pi} \left[(1 + \frac{3}{2} e^2) + e \cos v - \frac{15}{2} e^2 \cos^2 v\right] dv >$$

$$> p^2 \int_o^{2k\pi} \left[(1 + \frac{1}{2} e^2) - e(1 + 2e) \cos v - \frac{7}{2} e^2 \cos^2 v\right] dv \tag{48}$$

For k an integer

$$\int_o^{2k\pi} \cos v \, dv = 0$$

and

$$\int_o^{2k\pi} \cos^2 v \, dv = k\pi$$

and Eq. (48) becomes

$$a^2\left[(1 + \frac{3}{2} e^2) 2k\pi - \frac{15}{2} e^2 k\pi\right] > p^2 \left[(1 + \frac{1}{2} e^2) 2k\pi - \frac{7}{2} e^2 k\pi\right] \tag{49}$$

Now

$$p^2 = a^2(1 - e^2)^2 \approx a^2(1 - 2e^2)$$

Substituting for p^2 in Eq. (49) and simplifying we get

$$2(1 + \frac{3}{2} e^2) - \frac{15}{2} e^2 > (1 - 2e^2) \left[2(1 + \frac{1}{2} e^2) - \frac{7}{2} e^2\right]$$

or

$$(4 - 0e^2) > (4 - 13e^2)$$

which is true for all $e > o$, and therefore the inequality of Expression (45) is correct.

REFERENCES

1. Shapiro, I. I., The Prediction of Satellite Orbits, Lincoln Laboratory, Massachusetts Institute of Technology, MS-5-38, June 15, 1962.

2. Moulton, F. R., An Introduction to Celestial Mechanics, 2d rev. ed., The Macmillan Company, New York, 1914.