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A SEARCH GAME

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PREFACE

The author presents some of his recent mathematical results in the field of game theory. The mathematical theory of games has general applicability to a wide variety of conflict situations--economic, political, and military.

SUMMARY

A hide-and-seek game is formulated and partially solved. The hider chooses an integer from 1 to n; the hunter makes a guess, is told whether he is too high or too low. The process is repeated until he has guessed correctly. The payoff is the expected number of guesses.

Possible practical applications include the problem of correctly weighing an object with minimum expected number of weighings on a balance scale playing against nature.

CONTENTS

1. INTRODUCTION

The following search game was first suggested to the author by Melvin Dresher several years ago.

Blue chooses h, an integer, from the set of integers 1 to n (a region to hide). Red guesses an integer from 1 to n, is told whether he is too high or too low, and repeats until he guesses h. The payoff to Blue is one unit for each guess by Red (including the last guess h). This game and its solution are illustrated for the case $n = 3$ in [1], pages 32-35.

Progress was reported by the author in a 1958 internal RAND document which presented the solution for $n \leq 11$, using a special notational device to describe Red strategies.

Recently Gilbert [2] discussed the same problem along with related problems and gave its solution for $n \leq 6$. He stated that the calculations, even for $n = 6$, were quite lengthy. Later, in [3] Konheim computed the number of "bisecting" strategies for Red. This does not pertain directly to solving the game.

We present both the improved notational device for describing Red strategies and some recent theorems concerning necessary conditions for optimality, which greatly reduce the size of the game matrix. This makes the solutions for $n \leq 10$ quite simple. The case $n = 11$ becomes more complicated, and incidentally exhibits some qualitative features of Blue's optimal strategy which were conjectured by Gilbert in [2].

2. NOTATION

At first glance it would appear that this is a multi-move game for Red, since after each try he has new information and makes his next move accordingly. However, a better formulation is to describe a hunting strategy as a complete pattern of moves which takes care of all possible hiding places for Blue. Thus if $1 \leq j \leq n$ are the n numbers Blue can choose from, each with probability p_j, then a strategy for Red is an ordered set of n integers, denoted by $S_i = \{S_{ij}\}$. The following example will illustrate the meaning of the S_{ij}:

j	1	2	3	4	5	6	7
p_j	p_1	p_2	p_3	p_4	p_5	p_6	p_7
S_{ij}	2	3	1	3	4	2	3

Here Red guesses 3 first. If too high, he tries 1 for his second guess; if too low, he tries 6 for his second guess, etc. S_{ij} is the number of the guess when j is tried.

Red will play a mixture of these strategies S_i, each with probability t_i. The payoff, if Red plays S_i and Blue plays $\{p_j\}$, is $\sum_{j=1}^{n} S_{ij} p_j$. The value of the game is

$$V = \min_{\{t_i\}} \max_{\{p_j\}} \sum_i \sum_j S_{ij} p_j t_i = \max_{\{p_j\}} \min_{\{t_i\}} \sum_j \sum_i S_{ij} p_j t_i .$$

3. PROPERTIES OF OPTIMAL BLUE STRATEGIES

First it is clear from the definition of the game that Blue may play symmetrically about the center of the interval. Thus, we may assume

(1) $$p_j = p_{n-j} .$$

Also it is clear that p_j is positive.

Theorem 1.

(2) $$p_1 \geq p_2 .$$

Proof. If $p_1 < p_2$, then Red would always play 2 before 1 so that these two column payoffs in the game matrix could not be equal. To see this, consider any S_i where 1 is played before 2. Compare S_i with a related $\overline{S}_i$, where 2 rather than 1 is played at the k^{th} guess, and the rest of $\overline{S}_i$ matches S_i.

	p_1	p_2			
S_i	k	k+m	.	.	.
$\overline{S}_i$	k+1	k	.	.	.

Then

(3) $$\sum_j (S_{ij} - \overline{S}_{ij})p_j = - p_1 + mp_2 > 0$$

if $p_1 < p_2$.

It can be shown that $p_1 > p_2$ for $n \geq 5$. Other properties of optimal Blue strategies can be conjectured from the list of solutions in Sec. 5, but seem to be difficult to prove in general. For instance, $p_1 = p_2 + p_3$ holds for $5 \leq n \leq 11$, and $p_1 = 2p_2$ holds for $7 \leq n \leq 11$. Also, one might conjecture that $p_1 > p_j$, $j \neq 1$ for $n > 4$.

4. PROPERTIES OF OPTIMAL RED STRATEGIES

In this section we greatly reduce the list of possible optimal Red strategies against any trial Blue strategy.

Theorem 2. Suppose at a given stage that Red, playing S_i, has located h on the interval $k \leq j \leq m$, and that S_i calls for next playing at a, left of the median of the hider's frequency distribution on this interval, and if a is too small, next playing at b to the right of a. Then a necessary condition for optimality of S_i against $\{p_j\}$ is that

$$(4) \qquad \sum_{k \leq j \leq a} p_j \geq \sum_{b \leq j \leq m} p_j .$$

Proof. Assume that S_i does not satisfy (4). Then consider a related strategy $\overline{S}_i$ formed by interchanging play on numbers a and b while leaving the rest of the sequence of moves unchanged.

Thus

$$
(5)\qquad
\begin{aligned}
\overline{S}_{ij} &= S_{ij} + 1 & \quad k \le j \le a,\\
\overline{S}_{ij} &= S_{ij} & \quad a < j < b,\\
\overline{S}_{ij} &= S_{ij} - 1 & \quad b \le j \le m.
\end{aligned}
$$

The difference in payoff for playing $\overline{S}_i$ rather than S_i is

$$
(6)\qquad \sum_{j=k}^{m} \overline{S}_{ij} p_j - \sum_{j=k}^{m} S_{ij} p_j = \sum_{j=k}^{a} p_j - \sum_{j=b}^{m} p_j < 0.
$$

Therefore S_i is dominated by $\overline{S}_i$.

A similar argument holds when a is at the right of the median of Blue's frequency distribution on $k \le j \le m$.

For example, if Red's first try at 3 is too small, his second try must be $\ge n - 2$ as a consequence of (1) and Theorem 2.

<u>Theorem 3</u>. <u>At each stage Red should make his guess inside the middle third of Blue's probability distribution on the current interval of uncertainty</u>.

<u>Proof</u>. We prove a somewhat sharper result. Suppose at a given stage Red has located Blue on the interval $k \le j \le m$, and S_i calls for next playing at a to the left of the median of Blue's probability distribution on this interval; if a is too small, S_i calls for next playing at b where $b > a$, and if b is too large, next playing at c

where $a < c < b$. Suppose that subsequent play is optimal. Compare this S_i with a related $\overline{S}_i$ formed by interchanging the order of play on a and c. That is, choose c first; then if c is too large, choose a next, and if c is too small, choose b next. The order of the rest of the play in $\overline{S}_i$ matches subsequent play for S_i. The relative payoffs (each decreased by a constant) for S_i and $\overline{S}_i$ on the current interval are as follows:

	k				a				c				b				m
S_i		.	t	.	1	.	r	.	3	.	u	.	2	.	v	.	
$\overline{S}_i$		.	t+1	.	2	.	r-1	.	1	.	u-1	.	2	.	v	.	

Comparing payoffs, we have

$$\sum_{j=1}^{n} (\overline{S}_{ij} - S_{ij})p_j = \sum_{j=k}^{m} (\overline{S}_{ij} - S_{ij})p_j = \sum_{j=k}^{a} p_j - \sum_{j=a+1}^{b-1} p_j - p_c .$$

Thus if S_i is optimal against $\{p_j\}$, then

(7) $$\sum_{j=k}^{a} p_j \geq \sum_{j=a+1}^{b-1} p_j + p_c .$$

From Theorem 2, $\sum_{j=k}^{a} p_j \geq \sum_{j=b}^{m} p_j$, so that if $\sum_{j=k}^{a} p_j < \frac{1}{3}\sum_{j=k}^{m} p_j$ then $\sum_{k}^{a} p_j < \sum_{a+1}^{b-1} p_j$, violating (7). Thus Theorem 3 is proved.

5. SOLUTIONS FOR $n \leq 11$

In this section we exhibit optimal strategies and the game value for $n \leq 11$. From (1) it follows that Red can play a given S_i and its symmetric counterpart S_i' equally often. Thus we list only the strategies S_i for which $S_{ij} = 1$ for $j \leq \frac{n+1}{2}$, remembering that the probability associated with this strategy is really split equally between S_i and S_i'.

We shall list frequency distributions rather than probability distribution to further simplify the description of the solutions.

The general method of solution is as follows. For small values of n, the reduced list of Red strategies satisfying the theorems of the previous section gave a game matrix which was solved directly. As n grew longer an optimal Blue strategy was conjectured from the previous case, a list of Red strategies optimal against it was prepared, and a probability mixture was sought which gave the same expected payoff for each column for 1 to n in the game matrix.

A little trial-and-error work sufficed to find solutions for $n \leq 11$, although the last case was considerably harder. Further cases could be solved by linear programming on a computer.

The optimal mixture for Red is usually not unique. No attempt was made to find the mixture over the smallest number of Red pure strategies.

<u>n = 2 (trivial)</u>

There is just one S_1; $p_1 = p_2 = .5$; and $V(2) = \frac{3}{2}$.

<u>n = 3</u>

Here only $S_1 = (2, 1, 2)$ and $S_2 = (1, 3, 2)$, with $S_2' = (2, 3, 1)$, are undominated. By the remarks concerning symmetry, the 3-by-3 game matrix can be reduced to an equivalent 2-by-2 matrix.

Red \ Blue	p_1	p_2	p_1
S_1	2	1	2
S_2	1	3	2

becomes

Red \ Blue	2	1
3	4	1
2	3	3

Thus $P = \{p_j\} = \left\{\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right\}$, $t_1 = \frac{3}{5}$ and $t_2 = \frac{2}{5}$ split equally between S_2 and S_2', of course. $V(3) = \frac{9}{5}$.

<u>n = 4</u>

Here there are only 3 undominated strategies S_i. The reduced game matrix is 3 by 2.

Red \ Blue	p_1	p_2	p_2	p_1
S_1	2	1	2	3
S_2	2	1	3	2
S_3	1	3	4	2

Red \ Blue	1	1
0	5	3
1	4	4
0	3	7

A solution gives $p_j = \frac{1}{4}$ for each j, while S_2 and S_2' are played with probability $\frac{1}{2}$ each, $V(4) = 2$.

n = 5

The list of undominated strategies S_i and the reduced game matrix are as follows:

Red \ Blue	p_1	p_2	p_3	p_2	p_1
S_1	2	3	1	3	2
S_2	3	2	1	2	3
S_3	2	1	3	2	3
S_4	2	1	4	3	2
S_5	2	1	3	4	2
S_6	1	4	3	4	2
S_7	1	3	5	4	2

Red \ Blue	5	3	2	
4	4	6	1	40
	6	4	1	40
4	5	3	3	40
1	4	4	4	40
	4	5	3	41
	3	8	3	45
	3	7	5	46
	40	40	20	

Thus Blue plays with frequency (5, 3, 2, 3, 5), and Red plays S_1 with probability $\frac{4}{9}$; S_3 and S_3' each have $\frac{2}{9}$ probability, and S_4 and S_4' each have $\frac{1}{18}$ probability. $V(5) = 2\frac{2}{9}$.

n = 6

	p_1	p_2	p_3	p_3	p_2	p_1
S_1	2	3	1	4	3	2
.	2	3	1	3	4	2
.	2	3	1	3	2	3
.	2	3	1	2	4	3
	2	1	4	3	4	2
	2	1	5	4	3	2
	2	1	4	3	2	3
	1	4	3	5	4	2
	1	4	3	4	5	2
S_{10}	1	3	5	6	4	2

	5	3	2	
2	4	6	5	48
	4	7	4	49
6	5	5	4	48
	5	7	3	52
	4	5	7	49
	4	4	9	50
2	5	3	7	48
	3	8	8	55
	3	9	7	56
	3	7	11	56
	48	48	48	

Blue plays a frequency distribution (5, 3, 2, 2, 3, 5),
Red plays $t_1 = \frac{2}{10}$, $t_3 = \frac{6}{10}$, $t_7 = \frac{2}{10}$ and $V(6) = 2.4$.

<u>n = 7</u>

The solution for $n = 7$, giving optimal S_i only, is

Red \ Blue	2	1	1	1	1	1	2
2	3	2	3	1	3	4	2
1	3	2	3	1	3	2	3
4	2	3	1	3	4	2	3
2	2	3	1	4	3	2	3

	2	1	1	1	
2	5	6	6	1	23
1	6	4	6	1	23
4	5	5	5	3	23
2	5	5	4	4	23
	46	46	46	23	

with indicated frequency distributions for each player.
$V(7) = \frac{23}{9}$.

<u>n = 8</u>

Red \ Blue	2	1	1	1	1	1	1	2
2	2	4	3	1	4	3	2	3
2	3	2	3	1	4	3	2	3
1	2	3	1	4	3	2	4	3

	2	1	1	1	
2	5	6	6	5	27
2	6	4	6	5	27
1	5	7	3	7	27
	27	27	27	27	

with $V(8) = 2.7$.

n = 9

Red \ Blue	2	1	1	1	1	1	1	1	2
1	3	2	3	4	1	4	3	2	3
2	3	2	4	3	1	3	4	2	3
2	2	4	3	1	4	3	2	4	3
2	3	2	3	1	4	3	4	2	3
2	3	2	3	1	3	4	2	4	3
2	2	3	1	4	3	4	2	4	3

	2	1	1	1	1	
1	6	4	6	8	1	31
2	6	4	8	6	1	31
2	5	8	5	4	4	31
2	6	4	7	4	4	31
2	6	6	5	5	3	31
2	5	7	3	8	3	31
	62	62	62	62	31	

Here the number of S_i strategies is larger than necessary but suffices. $V(9) = \frac{31}{11}$.

n = 10

Here $V(10) = \frac{35}{12}$.

Red \ Blue	2	1	1	1	1	1	1	1	1	2
2	3	2	3	4	1	4	3	4	2	3
1	3	2	3	4	1	4	3	2	4	3
4	3	2	4	3	1	4	3	2	4	3
2	2	3	4	1	4	3	4	2	4	3
3	3	2	3	1	4	3	4	2	4	3

	2	1	1	1	1	
2	6	4	7	7	5	35
1	6	6	5	7	5	35
4	6	6	6	6	5	35
2	5	7	6	5	7	35
3	6	6	5	5	7	35
	70	70	70	70	70	

n = 11

$$V(11) = 3\frac{1}{62}.$$

Red \ Blue	58	29	29	34	25	22	25	34	29	29	58
18	3	4	2	3	4	1	4	3	4	2	3
2	3	4	2	3	4	1	4	3	2	4	3
29	3	2	4	3	1	4	3	4	2	4	3
6	3	2	3	1	4	5	3	4	2	4	3
2	3	2	3	1	4	3	4	2	4	3	4
5	3	2	3	1	4	3	4	2	5	4	3

	58	29	29	34	25	22	
18	6	6	6	6	8	1	1122
2	6	8	4	6	8	1	1122
29	6	6	6	7	4	4	1122
6	6	6	5	5	7	5	1122
2	7	5	7	3	8	3	1122
5	6	6	8	3	8	3	1122
	1122	1122	1122	1122	1122	561	

Note that the pattern of the hider's strategy gets more complicated here. The computation was quite lengthy, and appears to get more difficult for $n = 12$ and beyond. Nevertheless, considerable reduction in the list of Red strategies is accomplished by the techniques of this paper. For instance, if $f(n)$ is the total number of Red pure strategies, the recursion relation

$$(8) \qquad f(n) = \sum_{k=1}^{n} f(k-1)\, f(n-k)$$

with $f(0) = f(1) = 1$, etc., gives $f(11) = 58{,}786$ and $f(12) = 208{,}012$.

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