ON THE USE OF
THE CALCULUS OF VARIATIONS IN
TRAJECTORY OPTIMIZATION PROBLEMS

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PREFACE

This Memorandum was written for engineers who might welcome an introduction to the problem of the optimization of trajectories. Such optimization problems occur frequently in space studies.

This Memorandum is intended to present the methodology required in the use of the calculus of variations and to apply it to a number of trajectory problems. The relationship existing between optimization studies and perturbation studies is also discussed.
SUMMARY

The theory of the calculus of variations is reviewed, including such subjects as the Euler-Lagrange equations, the transversality condition, the problems of Bolza, Lagrange, and Mayer, the corner condition, and the Weierstrass condition.

Applications to a number of problems are given. These include that of a damped harmonic oscillator that is to be brought to rest in the shortest possible time. This problem is sufficiently simple to allow a clear demonstration of the theory. In the problem of the trajectory in a constant gravity field the range from launch to impact is to be maximized, although the formulation would be similar if the initial mass were to be minimized, or the burnout mass were to be maximized. Final mass is to be maximized in the escape trajectory problem. The vertical trajectory problem is included for demonstrating the requirement for a variable-thrust subarc.

The close relationship between the ballistic perturbation theory and the calculus of variations is explained. The perturbation theory is then applied to the case of the trajectory in a constant gravity field.
ACKNOWLEDGMENTS

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1. INTRODUCTION

Typical trajectory problems in rocketry occur in many different forms. For example, given a rocket of specified initial and final weight as well as a maximum and minimum thrust level, find the histories of the magnitude and direction of the thrust vector so as to maximize the range from launch to impact. Or it might be phrased a little differently: for a specified range, maximize the burnout weight, or minimize the total heat generated along the trajectory. Trajectory problems often involve a re-entry problem: given specified initial conditions at some altitude, how should the drag and lift be programmed so as to maximize the range, or minimize total heat? In the case of a vehicle in a circular parking orbit, how should the thrust vector be programmed so as to maximize burnout weight for an escape mission?

Two types of variables can be distinguished: state and control variables. The control variables are quantities such as thrust magnitude and direction, or lift and drag, or whatever quantities are available for controlling the trajectory. In general, the control variables are subject to constraints. The state variables are quantities such as position, velocity, and mass, for which certain initial or final conditions may often be specified.

Solving problems of this sort involves the use of concepts new to the engineer and terms such as Euler-Lagrange equations, Lagrangian multipliers, slack or dummy variables, transversality condition, Weierstrass-Erdmann corner condition, and Weierstrass condition (Pontryagin's Maximum Principle). These terms will be explained as they come up. Conditions such as Legendre's and Jacobi's, which
pertain to weak variations and the problem of conjugate points respectively, will not be pursued in this elementary treatise.

The use of the calculus of variations in trajectory optimizations is a powerful technique, though not without its limitations. Simplifying assumptions can occasionally be made that will lead to partial closed form solutions and certain general conclusions that can be of great value. Numerical integration is frequently required to solve the equations of motion and the Euler-Lagrange equations (those involving the Lagrangian multipliers, also referred to as adjoint equations), subject to certain specified end (initial and final conditions). Because some of the variables and multipliers are often either given or can be determined at one end and the remainder at the other end, a certain amount of ingenuity is required to obtain a numerical solution. There are techniques available to accomplish this, such as the "steepest descent" or "gradient" method. This subject is beyond the scope of this paper; the interested reader should consult Refs. 4 and 5. A good general discussion of optimization techniques is contained in Ref. 6.

The relationship of the calculus of variations and dynamic programming is discussed in Ref. 7. Dynamic programming offers an approach to solving optimization problems numerically.
II. THEORY

EULER-LAGRANG EQUATION

Detailed analyses of the theory used in this section are presented in Refs. 1 and 2.

It is desired to find an extremum (minimum or maximum) of an integral \( I \):\[
I = \int_{x_1}^{x_2} f(x, y, y') \, dx \quad ,
\]
where \( y' = \frac{dy}{dx} \). The particular function \( y(x) \) that gives an extremum in \( I \) is called the extremizing function. The part of the extremizing function lying between \( x_1 \) and \( x_2 \) is called the extremizing arc or extremal. It is desired to find the extremizing function \( y(x) \) that goes through \( (x_1, y_1) \) and \( (x_2, y_2) \). The function \( f \) depends on \( x, y, y' \) and should have continuous first, second, and third partial derivatives with respect to any of the three. The function \( y(x) \) must be continuous; \( y'(x) \) can be, at worst, piecewise continuous. Piecewise continuity allows a discontinuity or jump, but the quantity must be defined on either side of the jump (thus infinity is excluded). Therefore \( y(x) \) could appear as in Fig. 1. A finite number of corners,
where the slope is discontinuous, is allowed. It then follows that
\( f [ x, y(x), y'(x) ] \) will also have a finite number of discontinuities,
but the integral of \( f \) between \( x_1 \) and \( x_2 \) is, nevertheless, well defined.

The variational problem can be visualized as in Fig. 2.

It is desired to find \( y(x) \) lying between \( P_1 \) and \( P_2 \) such that some
desired function \( f(x, y, y') \) when integrated between \( P_1 \) and \( P_2 \) is
maximized or minimized. To achieve this, a comparison arc is postulated,
and the Euler-Lagrange equation can then be derived:

\[
\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0
\]

or

\[
\frac{d}{dx} \left( f_y' \right) - f_y = 0. \tag{2}
\]

It is a necessary, but not sufficient, condition for making \( I \) an
extremum.

Another form may be obtained from Ref. 1, p. 24, Eq. 27:

\[
\frac{d}{dx} \left( f - y' f_y' \right) = f_x
\]
or

\[ f - y' f_{y'} = \int_{x_1}^{x_2} f_x \, dx + C. \]  \hspace{1cm} (3)

The Euler-Lagrange equation is, in general, a second order
differential equation. Its use can be illustrated by the problem of
finding the shortest distance between two points, \( P_1 \) and \( P_2 \). Here

\[ I = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} \, dx \]

is to be minimized. Thus \( f = \sqrt{1 + (y')^2} \). Then

\[ \frac{\partial f}{\partial y'} = \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 , \]

and the Euler-Lagrange equation becomes

\[ \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = 0 , \]

which is a second order differential equation.

It follows that \( \frac{y'}{\sqrt{1 + (y')^2}} = \text{constant} \)
or \( y' = \text{constant} \).

In other words, the shortest distance between two points lies along
\( y(x) \), a straight line.

**FIRST INTEGRALS**

(Ref. 1, pp. 24-26; Ref. 2, p. 16). If \( f \) is independent of \( y \),
then \( \frac{\partial f}{\partial y'} = f_{y'} = C_1 \). If \( f \) is independent of \( y \), as well as \( x \), then
\( y' = C_2 \). If \( f \) is independent of \( x \), then from Eq. 3 it follows that
\( f - y' f_{y'} = C_3 \). If \( f \) is the total derivative of a function with
respect to $x$, that is, if $f = \frac{dg}{dx}$, then $I = g(x_2, y_2) - g(x_1, y_1)$, and $f$ must be of the form
\[ f = p(x, y) + q(x, y) y'. \] (4)

SEVERAL DEPENDENT VARIABLES

(Ref. 1, pp. 32-34). If the problem is extended to several dependent variables, $x, y, z$, and the independent variable $t$, then an extremum in $I$ is desired where
\[ I = \int_{t_1}^{t_2} f(x, y, z, t, \dot{x}, \dot{y}, \dot{z}) \, dt, \] (5)
where $\dot{x} = \frac{dx}{dt}$, etc. The corresponding Euler-Lagrange equations are
\[ \frac{d}{dt} (f_x) - f_x = 0 \]
\[ \frac{d}{dt} (f_y) - f_y = 0 \] (6)
\[ \frac{d}{dt} (f_z) - f_z = 0. \]

Another relation, not independent of these, is
\[ \frac{d}{dt} (f - \dot{x}f_x - \dot{y}f_y - \dot{z}f_z) = \frac{\partial f}{\partial t}. \] (7)

If $f$ is explicitly independent of $t$, then a first integral is
\[ f - \dot{x}f_x - \dot{y}f_y - \dot{z}f_z = C. \] (8)

HIGHER DERIVATIVES IN THE INTEGRAND

If the function to be extremized is
\[ I = \int_{x_1}^{x_2} f(x, y, y', y'') \, dx, \] (9)
the problem can be handled by introducing a side condition, namely, a differential equation involving an additional variable
\[ y' = v. \]  
(10)

Now Eq. 9 will be of the form of Eq. 5. The use of differential equations as constraints will be discussed below.

**END CONDITIONS—TRANSVERSALITY CONDITION**

(Ref. 1, pp. 36-41, and Ref. 2, p. 102). If the problem is to find an extremum in I as given in Eq. 1 above, and if at one end, the values of \( x \) and \( y \) are specified as \( x_1 \) and \( y_1 \), and on the other end \( x \) is specified as \( x_2 \) but \( y \) is left unspecified or arbitrary (Fig. 3),

![Graph showing transversality condition](image)

then it is necessary that
\[
\frac{\partial F}{\partial y'} \bigg|_{x_2} = f_{y'} \bigg|_{x_2} = 0. \]  
(11)

If \( x_1 \) and \( x_2 \) are specified, but neither \( y_1 \) nor \( y_2 \) (Fig. 4),
\[
\frac{\partial^2 F}{\partial y''} \bigg|_{x_1} = f_{y''} \bigg|_{x_2} = 0. \]  
(12)
If \( x_1 \) and \( y_1 \) are specified at one end, but the other end of the extremal is required to lie on a terminal curve \( g(x, y) = 0 \) (Fig. 5),

then it is necessary that

\[
\left[ (f - y' y') \, dx + f' y' \, dy \right] \bigg|_{x = x_2} = 0, \tag{13}
\]

where \( dx \) and \( dy \) are taken along the terminal curve \( g(x, y) = 0 \), and the other terms are related to the extremal \( y(x) \). It can be shown that Eq. 13 and Eq. 85 on page 40 of Ref. 1 are equivalent. The reason for writing Eq. 13 in the above form is that it is in the form
of the transversality condition, which will be discussed below.

Equation 13 is of great value. It can be used for either end, regardless of how things are specified. Thus it is necessary that Eq. 13 be satisfied at both ends (Fig. 6), and the two relations can be combined into one as follows

\[
\left[ (f - y'f_{y'}) \, dx + f_{y'} \, dy \right]_{x_1}^{x_2} = 0. \quad (14)
\]

![Fig. 6](image)

This is commonly referred to as the transversality condition. It deals with the manner in which the extremal and the terminal curves must intersect. It may be written

\[
(f - y'f_{y'}) \left. \frac{dx}{x_2} \right|_{x_2}^{x_1} + \left. \frac{dy}{y_2} \right|_{x_2}^{x_1} - (f - y'f_{y'}) \left. \frac{dx}{x_1} \right|_{x_1}^{x_2} - \left. f_{y'} \right|_{x_1}^{x_2} \, dy = 0. \quad (15)
\]

It is quite general and can be applied to any special situation. Suppose that \(x_1, y_1, x_2\) are specified, but \(y_2\) is left unspecified,
or arbitrary. Then the terminal curve at end 1 degenerates into a point so that \( dx_1 = dy_1 = 0 \). With \( x_2 \) specified but \( y_2 \) arbitrary, the terminal curve at end 2 degenerates into a vertical straight line: \( x = x_2 \), so that \( dx_2 = 0 \). All terms in Eq. 15 have now vanished, except the \( f_y' \) \( dy_2 \) term. In order for it to vanish it is necessary that \( f_y' = 0 \) at end 2, which is exactly the condition that is stated in Eq. 11. Similarly, if \( x_1 \) and \( x_2 \) are specified, but neither \( y_1 \) nor \( y_2 \), then \( dx_1 = dx_2 = 0 \), and Eq. 15 becomes

\[
\left. f_y' \right|_{x_2} \frac{dy_2}{dy_1} - \left. f_y' \right|_{x_1} = 0.
\]

With \( y_1 \) and \( y_2 \) arbitrary, \( dy_1 \) and \( dy_2 \) can take on any values including zero, and the only way to satisfy this relation is to have

\[
\left. f_y' \right|_{x_2} = \left. f_y' \right|_{x_1} = 0,
\]

which is identical to Eq. 12 above.

Suppose now that \( x_1 \) and \( y_1 \) are specified, making \( dx_1 = dy_1 = 0 \), but that end 2 is required to lie on a parabola: \( g(x, y) = y_2 - ax - bx_2 - cx_2^2 = 0 \). Then \( dy_2 - (b + 2cx_2) \) \( dx_2 = 0 \), and Eq. 15 becomes

\[
\left( f - y'f_y' \right) \left. \frac{dx_2}{dx_2} + \left. f_y' \right|_{x_2} (b + 2cx_2) \right|_{dx_2} = 0
\]

or

\[
\left[ f - y'f_y' + (b + 2cx) f_y' \right]_{x_2} = 0.
\]

For the case of several dependent variables, where it is desired to extremize the I of Eq. 5, it is necessary that

\[
\left[ (x - \dot{x}x_z - \dot{y}y_z - \dot{z}z_z) \right. \dot{t} + f_x dx + f_y dy + f_z dz \right]_{t_1}^{t_2} = 0, \quad (16)
\]
where again the differentials $dx, dy, dz, dt$ are taken along the terminal curves. This is the transversality condition for the case of several dependent variables.

**ISOPERIMETRIC PROBLEM**

(Ref. 1, pp. 48-53).

This introduces the assignment of constraining relations in a variational problem. The purpose here is to extremize:

$$I = \int_{x_1}^{x_2} f(x, y, y') \, dx,$$

subject to the constraint that a certain integral has a specified value $K$

$$\int_{x_1}^{x_2} g(x, y, y') \, dx = K$$

The method of Lagrange multipliers is introduced here because it is an effective method for solving maximum-minimum problems subject to constraints (Ref. 1, p. 6).

To demonstrate the use of Lagrange multipliers let us turn to a problem in the ordinary calculus. Consider the problem of finding an extreme in a function $F = F(x, y, z)$, where $x, y, z$ are constrained by the relation $G(x, y, z) = K$. The straightforward approach might be to eliminate $z$ between these two functions, and then extremize $F$:

$$G(x, y, z) = K$$

$$dG = G_x \, dx + G_y \, dy + G_z \, dz = 0$$

$$dz = \frac{G_x \, dx + G_y \, dy}{G_z}$$
\[ F = F(x, y, z) \]
\[ \frac{dF}{dx} = F_x \, dx + F_y \, dy + F_z \, dz \]
\[ = F_x \, dx + F_y \, dy - \frac{F_z}{G_z} \, (G_x \, dx + G_y \, dy) \]
\[ = (F_x - \frac{F_z}{G_z} \, G_x) \, dx + (F_y - \frac{F_z}{G_z} \, G_y) \, dy. \]

For \( F \) to be an extreme, \( dF \) must be zero, and this must be true for all values of \( x \) and \( y \). Therefore it is necessary that

\[ \frac{\partial}{\partial x} (F) = F_x - \frac{F_z}{G_z} \, G_x = 0 \quad \text{and} \quad \frac{\partial}{\partial y} (F) = F_y - \frac{F_z}{G_z} \, G_y = 0. \]

Using the method of Lagrange multipliers on this same problem would require the formation of a function \( F^* = F + \lambda \, G \), where \( \lambda \) is a Lagrange multiplier, and the simultaneous solution of the following (Ref. 1, p. 6):

\[ \frac{\partial}{\partial x} (F^*) = \frac{\partial}{\partial x} (F + \lambda \, G) = F_x + \lambda \, G_x = 0 \]
\[ \frac{\partial}{\partial y} (F^*) = \frac{\partial}{\partial y} (F + \lambda \, G) = F_y + \lambda \, G_y = 0 \]  
\[ \frac{\partial}{\partial z} (F^*) = \frac{\partial}{\partial z} (F + \lambda \, G) = F_z + \lambda \, G_z = 0 \]

\[ G(x, y, z) = K. \]

It will be seen that the third relation is the defining equation for \( \lambda : \lambda = -\frac{F_z}{G_z} \), which when substituted in the first two equations yields Eqs. 21. Thus the two approaches lead to the same result. The solution of Eq. 22 would yield one set of values of \( x, y, z, \lambda \) that make \( F \) an extreme. If there are two constraints, \( Q_1(x, y, z) = K_1 \) and \( Q_2(x, y, z) = K_2 \), then the \( F^* \) function above becomes

\[ F^* = F + \lambda_1 \, G_1 + \lambda_2 \, G_2. \]
Let us now return to the isoperimetric problem above. To extremize Eq. 17 subject to Eq. 18, an $I^*$ function is formed

$$f^* = f + \lambda g.$$  \hspace{1cm} (23)

The problem now is one of extremizing

$$I^* = \int_{x_1}^{x_2} f^* (x, y, y') \, dx,$$  \hspace{1cm} (24)

where $I^* = I + \lambda K$. The Euler-Lagrange equation is

$$\frac{d}{dx} (f_y y'^* - f_y^*) = 0.$$  \hspace{1cm} (25)

The solution of this equation contains three parameters, two constants of integration, and $\lambda$. These are determined from the specified values of $y_1$ and $y_2$ at the two ends and from Eq. 18, using the specified value of $K$.

If there are two constraining equations,

$$\int_{x_1}^{x_2} \tilde{g}_1 (x, y, y') \, dx = K_1$$

and

$$\int_{x_1}^{x_2} \tilde{g}_2 (x, y, y') \, dx = K_2,$$

then

$$f^* = \lambda_1 \tilde{g}_1 + \lambda_2 \tilde{g}_2,$$  \hspace{1cm} (26)

and Eq. 25 still applies. This can be extended to any number of constraining equations.

The transversality condition for the isoperimetric problem is

$$\left[ (f^* - y' y'^*) \, dx + f_y y'^* \, dy \right]_{x_1}^{x_2} = 0.$$  \hspace{1cm} (27)
In the case of the isoperimetric problem with several dependent variables, the Euler-Lagrange equations (6) and (7), first integral (8), and transversality condition (16) apply with \( f \) replaced by \( f^* \).

**FINITE OR DIFFERENTIAL EQUATIONS AS CONSTRAINTS**

(Ref. 1, pp. 57-63).

Whereas the isoperimetric problem was one of extremizing an integral subject to constraints in the form of integrals, we shall now look at constraints in the form of finite or differential equations.

The problem is to extremize

\[
I = \int_{t_1}^{t_2} f(x, y, \cdots, z, t, \dot{x}, \dot{y}, \cdots, \dot{z}) \, dt
\]

where the variables \( x, y, \cdots, z \) are \( n \) in number, and either specified or arbitrary at the end points, and where \( m \) constraining or side conditions (such as differential equations of motion, or certain finite equations) are imposed:

\[
\phi_j (x, y, \cdots, z, t, \dot{x}, \dot{y}, \cdots, \dot{z}) = 0 ; \quad j = 1, 2, \cdots, m
\]

and where it is necessary that \( m < n \). The Euler-Lagrange equations are

\[
\begin{align*}
\frac{d}{dt} (F_x) - F_x &= 0 \\
\frac{d}{dt} (F_y) - F_y &= 0 \\
\vdots \\
\frac{d}{dt} (F_z) - F_z &= 0
\end{align*}
\]

\( \text{n equations,} \) \hspace{1cm} (30)
where \( F = f + \sum_{j=1}^{m} \lambda_j(t) \phi_j \). \( \text{(31)} \)

Notice that the Lagrange multipliers are now functions of the independent variable \( t \). Thus there is a set of \( m + n \) equations, namely, Eqs. 29 and 30, to be solved for \( m + n \) functions: \( x, y, \cdots z, \lambda_1, \lambda_2, \cdots \lambda_m \), all, in general, functions of \( t \).

The transversality condition is

\[
\left[ (F - \dot{x}F_x - \dot{y}F_y - \cdots - \dot{z}F_z) \right] \frac{dt}{t_2} + F_x \frac{dx}{t} + F_y \frac{dy}{t} \cdots + F_z \frac{dz}{t} = 0. \tag{32}
\]

THE PROBLEMS OF BOLZA, LAGRANGE, AND MAYER

(Ref. 2, Chap. 7).

We need a further degree of generalization in the formulation of our problems, as well as the introduction of certain additional necessary conditions, before we can tackle trajectory optimizations.

The problem of Bolza is stated as follows. Given a function \( J \) to be extremized,

\[
J = g(x_1, y_1, \cdots z_1, t_1, x_2, y_2, \cdots z_2, t_2) + \int_{t_1}^{t_2} f(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}) \, dt, \tag{33}
\]

where the variables \( x, y, \cdots z \) are \( n \) in number.

The problems of Lagrange and Mayer are special cases of the problem of Bolza. If in Eq. 33 the function \( g \) does not appear, we have the problem of Lagrange; if the function \( f \) does not appear, we have the problem of Mayer. These special cases of the problem of
Bolza can therefore be handled by setting either $g$ or $f$ identically to zero.

The continuity properties described after Eq. 1 above apply here. The function $f$, which depends on $x, y, \ldots z, t, \dot{x}, \dot{y}, \ldots \dot{z}$, should have continuous first, second, and third partial derivatives with respect to any of these arguments. The functions $x(t), y(t), \ldots z(t)$ must be continuous; $\dot{x}(t), \dot{y}(t), \ldots \dot{z}(t)$ can be piecewise continuous (Fig. 7), thus allowing corners.

There are $m$ constraints, or side conditions, which generally include the equations of motion:

\[ \phi_j(x, y, \ldots z, t, \dot{x}, \dot{y}, \ldots \dot{z}) = 0; \ j = 1, 2 \ldots m \quad (34) \]

\[ m < n. \]

There are $p$ end conditions:

\[ \psi_k(x_1, y_1, \ldots z_1, t_1, x_2, y_2, \ldots z_2, t_2) = 0; \ k = 1, 2, \ldots p \]

\[ p \leq 2n + 1. \quad (35) \]
Since there are \( n \) dependent variables and one independent variable, a maximum of \((n+1)\) conditions at each of the two ends can be specified. This makes a total of \( 2n+2 \). However, one of these must be reserved for optimization in Eqs. 33 and 40. Therefore the maximum number of end conditions is \( 2n+1 \).

First, a function \( F \) is defined:

\[
F = f(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z})
+ \sum_{j=1}^{m} \lambda_j(t) \phi_j(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}).
\]  

(36)

Note that \( \Sigma \lambda_j \phi_j = 0 \), since each of the \( \phi_j \) equal zero. Then the solution is stated in terms of \( m + n \) equations, namely, the \( m \) side conditions, Eq. 34, plus a set of \( n \) Euler-Lagrange equations:

\[
\begin{align*}
\frac{d}{dt} (F_x) - F_x &= 0 \\
\frac{d}{dt} (F_y) - F_y &= 0 \\
\vdots
\end{align*}
\]

\( \vdots \)

\[
\frac{d}{dt} (F_z) - F_z = 0.
\]

(37)

These \( m+n \) equations are to be solved for \( m+n \) functions of time: \( x, y, \cdots z, \lambda_1, \cdots \lambda_m \). A relation that is not independent of Eqs. 37, but sometimes helpful, is

\[
\frac{d}{dt} (F - \dot{x} F_x - \dot{y} F_y \cdots - \dot{z} F_z) = \frac{\partial F}{\partial t}.
\]

(38)

If \( F \) is not explicitly a function of \( t \), then this becomes a first integral:

\[
(F - \dot{x} F_x - \dot{y} F_y \cdots - \dot{z} F_z) = C.
\]

(39)
The transversality condition to be satisfied is

\[
\delta g(x_1, y_1, \cdots; z_1, t_1, x_2, y_2, \cdots; z_2, t_2)
+ \left[ (F - \dot{x} \dot{F}_x - \dot{y} \dot{F}_y - \cdots - \dot{z} \dot{F}_z) \right] dt
+ F_x dx + F_y dy + \cdots + F_z dz \bigg|_{t_1}^{t_2} = 0. \tag{40}
\]

Note that the first term in the bracketed quantity is equal to \( C \) of Eq. 39. If any of the variables \( x, y, \cdots, z, t \) are specified at either end, then for those variables the differentials \( dx, dy, \cdots, dz, dt \) are zero at that end; if any of the variables are unspecified or arbitrary at either end, then for those variables the coefficients of the differentials must vanish at that end.

The Du Bois-Reymond equations (from which the Euler-Lagrange equations and Hilbert differentiability condition follow) are

\[
\begin{align*}
F_x &= \int_{t_1}^{t} F_x \, dt + C_1 \\
F_y &= \int_{t_1}^{t} F_y \, dt + C_2 \\
&\quad \vdots \\
F_z &= \int_{t_1}^{t} F_z \, dt + C_n \\
(F - \dot{x} \dot{F}_x - \dot{y} - \cdots - \dot{z} \dot{F}_z) &= \int_{t_1}^{t} \frac{\partial F}{\partial \dot{t}} \, dt + D. \tag{41}
\end{align*}
\]

\textbf{WEIERSTRASS-ERDMANN CORNER CONDITION}

(Ref. 2, p. 12).

Equations 41 and 42 lead to the Weierstrass-Erdmann corner
condition, which states that certain quantities are continuous across possible corners (Fig. 7) on the extremizing arcs \(x(t), y(t), \cdots z(t)\):

\[
\frac{F_x}{t_-} = \frac{F_x}{t_+} \quad \frac{F_y}{t_-} = \frac{F_y}{t_+} \quad \frac{F_z}{t_-} = \frac{F_z}{t_+}
\]

(43)

\[
\left[ F - \dot{x} F_x - \dot{y} F_y - \cdots - \dot{z} F_z \right]_{t_-} = \left[ F - \dot{x} F_x - \dot{y} F_y - \cdots - \dot{z} F_z \right]_{t_+}.
\]

This means that the quantities indicated converge to the same value as a corner is approached from the negative or the positive side.

For certain integrand functions this condition actually prohibits the extremizing arcs from having corners.

**WIESESTRASS CONDITION**

(Ref. 2, pp. 20-22).

Once an extremal arc has been established, in accordance with the Euler-Lagrange equations (37), the question remains whether it is a maximum, a minimum, or a stationary value. To resolve this question the Weierstrass condition is employed. It states that the Weierstrass E function must be \( \geq 0 \), provided the J function is to be minimized (if J is to be maximized \( \leq 0 \)):

\[
E = F(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}, \lambda_1, \cdots \lambda_m)
- F(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}, \lambda_1, \cdots \lambda_m)
- (\dot{x} - x) F_x(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}, \lambda_1, \cdots \lambda_m)
- (\dot{y} - y) F_y(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}, \lambda_1, \cdots \lambda_m)
- \cdots - (\dot{z} - z) F_z(x, y, \cdots z, t, \dot{x}, \dot{y}, \cdots \dot{z}, \lambda_1, \cdots \lambda_m) \geq 0 \quad (44)
\]
where the starred differentiated variables indicate admissible variations. To be admissible, the continuity requirements discussed in conjunction with Fig. 7 above must be satisfied. The Weierstrass condition may be written in abbreviated form:

\[ E = F^* - F - \sum_{i=1}^{n} (\dot{x}_i^* - \dot{x}_i) F_{\dot{x}_i} \geq 0. \] (45)

If the problem is of the Mayer type, so that \( f = 0 \), then from Eqs. 34 and 36 it will be seen that \( F = 0 \). \( F^* \) must also vanish, since the equations of motion (34) must be satisfied. Then for the Mayer problem the Weierstrass condition can be written

\[ \sum_{i=1}^{n} \dot{x}_i^* F_{\dot{x}_i} = \sum_{i=1}^{n} \dot{x}_i F_{\dot{x}_i}. \] (46)

Thus, the quantity on the left is to be maximized. Equation 46 is a special case of Pontryagin’s Maximum Principle.* The application of the Weierstrass condition will be further explained in the next section.

The Normalized Mayer Problem

Trajectory optimization problems can be put into the form of the normalized Mayer problem. This type of problem will be considered here (a distinction will be made between state and control variables) along with the use of a dummy variable to ensure constraints on a control variable.

A problem of Lagrange that requires finding the extremum of

\[ \int_{t_1}^{t_2} f(x, y, t, \dot{x}, \dot{y}) \, dt \]

*Pontryagin, however, shows that Eq. 46 is still a necessary condition if the control is restricted by inequality constraints to lie in a closed set. (Valentine also showed this in 1937, but Weierstrass did not consider the problem.)
can always be converted into a problem of Mayer that requires finding the extremum of \( g(x_1, y_1, t_1, x_2, y_2, t_2) \) merely by considering \( g \) to be the integral of \( f \) between \( t_1 \) and \( t_2 \). Thus there is no loss in generality in treating all variational problems as those of the Mayer type.

Furthermore, in most trajectory and dynamics problems the equations of motion are such that they can be written in normalized form, meaning that first derivatives can be solved for explicitly.

To simplify matters for the reader and not lose him in subscripts, summation signs, or vector or tensor notation, only two state variables, one control and one dummy variable, and one independent variable are used here. The number of first order differential equations is always the same as the number of state variables. There is always only one independent variable: usually time \( t \).

The Mayer problem, with normalized equations of motion, can then be formulated:

**Given:** state variables \( x, y \)

control variable \( z \) with constraint \( z_L \leq z \leq z_u \)

"dummy" variable \( u \), where \( z = z(u) \) to insure constraint on \( z \)

independent variable \( t \)

Here \( z(u) \) may be of the form

\[
z = \frac{z_u + z_L}{2} + \frac{z_u - z_L}{2} \cos u,
\]

(48a)

illustrated in Fig. 8;
or it may be of the form \((z_u - z)(z - z_l) - u^2 = 0\), \(\text{(48b)}\)

where \(u\) is real, illustrated in Fig. 9. Still other forms are possible.

There are two ways of handling constraints on control variables.

One is to eliminate the control variable \(z\) from the problem by introducing a dummy variable \(u\), using a form like Eq. 48a which automatically takes care of the constraint on \(z\). The other is to introduce a finite equation like Eq. 48b as an additional side condition to the equations of motion. The first approach will be used here. In those cases where the control variable constraints also involve the state variables (example: aerodynamic normal force or heating rate), the second approach must be used.
To minimize:

\[ J = g(x_1, y_1, t_1, x_2, y_2, t_2) \]

Equations of motion in normalized form must be given:

\[ \dot{\phi}_1 = \dot{x} - h_1 (x, y, z) = 0 \]
\[ \dot{\phi}_2 = \dot{y} - h_2 (x, y, z) = 0 \]

where \( z = z(u) \)

and where \( t \) does not appear explicitly.

Initial and final conditions must be given:

Conditions may be specified for each of the state variables \( x, y \) and the independent variable \( t \) initially, and the state variables \( x, y \) finally. Or, in this problem, as many as five conditions can be imposed, in any desired distribution between initial and final.

Form function \( F \):

\[ F = \lambda_1 \dot{\phi}_1 + \lambda_2 \dot{\phi}_2 \]
\[ = \lambda_1 (\dot{x} - h_1) + \lambda_2 (\dot{y} - h_2) = 0. \]

(50)

By differentiation:

\[ F_x = \lambda_1 \]
\[ F_y = \lambda_2 \]

\[ F_{\dot{x}} = -\lambda_1 \frac{\partial h_1}{\partial x} - \lambda_2 \frac{\partial h_2}{\partial x} \]
\[ F_{\dot{y}} = -\lambda_1 \frac{\partial h_1}{\partial y} - \lambda_2 \frac{\partial h_2}{\partial y} \]

Letting \( u = \dot{y} \), a differentiated variable:

\[ F_p = F_u = F_z \frac{d\zeta}{du} \]
\[ F_p = 0 \]

\[ = ( -\lambda_1 \frac{\partial h_1}{\partial \zeta} - \lambda_2 \frac{\partial h_2}{\partial \zeta} ) \frac{d\zeta}{du} \]

It is important to note that the control variable \( u \) is treated as if it were a differentiated variable \( \dot{\zeta} \) so as to allow finite
jumps, or piecewise discontinuities, in the control variable. The
undifferentiated equivalent, $p$, of the control variable will thus
never appear in the problem.

Euler-Lagrange equations:

$$\frac{d}{dt} (F_x) - F_x = 0$$

$$\frac{d}{dt} (F_y) - F_y = 0$$

$$\frac{d}{dt} (F_u) - 0 = 0 \text{ or } F_u = F_x = F_y = C_1 = 0.$$ 

The fact that $C_1 = 0$ follows from the transversality condition below.

It follows that

$$\dot{l}_1 + \lambda_1 \frac{\partial h_1}{\partial x} + \lambda_2 \frac{\partial h_2}{\partial x} = 0$$

$$\dot{l}_2 + \lambda_1 \frac{\partial h_1}{\partial y} + \lambda_2 \frac{\partial h_2}{\partial y} = 0$$

$$(\lambda_1 \frac{\partial h_1}{\partial z} + \lambda_2 \frac{\partial h_2}{\partial z}) \frac{dz}{dt} = 0.$$ 

(51)

First integral:

$$F - \dot{x} F_x - \dot{y} F_y - u F_u = C_2$$

or

$$- \lambda_1 \dot{x} - \lambda_2 \dot{y} = C_2 \text{ since } F = F_u = C_1 = 0$$ 

(52)

or

$$\lambda_1 \dot{h}_1 + \lambda_2 \dot{h}_2 = - C_2.$$ 

Transversality condition:

$$dg + \left[(F - \dot{x} F_x - \dot{y} F_y - u F_u) \ dt + F_x \ dx + F_y \ dy + F_p \ dp\right]_{t_1}^{t_2} = 0$$

or

$$dg + \left[C_2 \ dt + \lambda_1 \ dx + \lambda_2 \ dy + C_1 \ dp\right]_{t_1}^{t_2} = 0$$

or

$$dg + \left[C_2 \ dt + \lambda_1 \ dx + \lambda_2 \ dy\right]_{t_1}^{t_2} = 0.$$ 

(53)
Here \( C_1 = 0 \) since the undifferentiated equivalent, \( p \), of the control variable \( \dot{p} = u \), is not specified either initially or finally.

Corner condition:

It is necessary that \( \lambda_1, \lambda_2 \) be continuous across corners.

Weierstrass condition:

A further necessary condition for a minimum in \( J = g \left( x_1, y_1, t_1, x_2, y_2, t_2 \right) \) is

\[
\dot{x} F_x + \dot{y} F_y + u F_u \geq \dot{x}^* F_x + \dot{y}^* F_y + u^* F_u
\]

or

\[
\dot{x} F_x + \dot{y} F_y \geq \dot{x}^* F_x + \dot{y}^* F_y.
\]

Since

\[
\dot{x} = h_1 \left( x, y, u \right) \text{ and } \dot{y} = h_2 \left( x, y, u \right),
\]

then

\[
\lambda_1 h_1 \left( x, y, u \right) + \lambda_2 h_2 \left( x, y, u \right)
\]

\[
\geq \lambda_1 h_1 \left( x^*, y^*, u^* \right) + \lambda_2 h_2 \left( x^*, y^*, u^* \right).
\]

Now since the state variables \( x \) and \( y \) must be continuous in \( t \), allowing no jumps, it is necessary that \( x^* = x, y^* = y \). Of course, \( u \) can be piecewise continuous, so that generally \( u^* \neq u \), but any value within the permissible limits. This is illustrated in Fig. 10.
At any time \( t \), \( x \) and \( y \) must be continuous, but \( u \) may suddenly jump to any value within the permissible range.

We then have

\[
\lambda_1 h_1 (x, y, u) + \lambda_2 h_2 (x, y, u) \geq \lambda_1 h_1 (x, y, u^*) + \lambda_2 h_2 (x, y, u^*)
\]

or

\[ H \geq H^* . \tag{54} \]

The quantity \( \lambda_1 h_1 + \lambda_2 h_2 \) is called \( H \), the Hamiltonian of the system.

It follows that

\[
\frac{\partial H}{\partial \lambda_1} = \dot{x} \\
\frac{\partial H}{\partial \lambda_2} = \dot{y} \\
\frac{\partial H}{\partial x} = \lambda_1 \frac{\partial h_1}{\partial x} + \lambda_2 \frac{\partial h_2}{\partial x} = -\dot{\lambda}_1 \\
\frac{\partial H}{\partial y} = \lambda_1 \frac{\partial h_1}{\partial y} + \lambda_2 \frac{\partial h_2}{\partial y} = -\dot{\lambda}_2
\]

These are Hamilton's canonical equations (Ref. 8, p. 217, Eqs. 7-12).

In the first integral it is seen that \( H = C_2 \), which is proper since \( H \) is the total energy of the system, potential plus kinetic (Ref. 8, p. 54, Eqs. 2-52).

It will be seen that in order to maximize \( H \) in Eq. 54 with respect to \( u \) it is necessary that

\[
\frac{\partial H}{\partial u} = 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial u^2} < 0.
\]

The first condition may be written

\[
\frac{\partial H}{\partial z} \frac{dz}{du} = 0
\]

or

\[
(\lambda_1 \frac{\partial h_1}{\partial z} + \lambda_2 \frac{\partial h_2}{\partial z}) \frac{dz}{du} = 0,
\]
which is identical to the third Euler-Lagrange equation (51). Thus the Euler-Lagrange equations assure an extremum but do not distinguish between a maximum or minimum. It takes the Weierstrass condition to do that.
III. APPLICATIONS

BUSHAW'S PROBLEM

Let us look at an application of the above theory. Bushaw's problem is not a trajectory problem but a rather simple dynamics problem in which the principles of the calculus of variations can be clearly demonstrated. It is the problem of a harmonic oscillator (spring-mass system with either positive or negative damping) in motion; the question is, in what manner should a forcing function be applied in order to bring the system to rest in the shortest possible time?

In the following equation of motion,
\[ x + 2\zeta w x + w^2 x = z(t), \]  
(55)
x is the displacement variable, \( \zeta \) and \( w \) are constants, and \( z(t) \) is the forcing function, which is constrained to lie between -1 and +1:
\[ |z| \leq 1. \]  
(56)

Initially the system is in motion:
\[
\begin{align*}
  x(t=t_0) &= x_0 \\
  \dot{x}(t=t_0) &= \dot{x}_0.
\end{align*}
\]

Finally we wish the system to be at rest:
\[
\begin{align*}
  x(t=t_F) &= 0 \\
  \dot{x}(t=t_F) &= 0.
\end{align*}
\]

(57)

The problem then is to find \( z(t) \) such that \( t_F \) is a minimum.

First we normalize the equations of motion; that is, we reduce the second order differential equation to two first order differential equations where the first term of each is a differentiated state variable:
\[ \phi_1 = \dot{x} - y = \dot{x} - h_1(x, y, z) = 0 \]

\[ \phi_2 = \dot{y} + 2\xi wy + w^2 x - z(t) = \dot{y} - h_2(x, y, z) = 0. \quad (58) \]

Thus the first equation is merely a defining equation stating that we are letting \( \dot{x} = y \). The constraint on \( z(t) \) of Eq. 58 is included by the replacement of \( z \) by a "slack" or "dummy" variable \( u \):

\[ z(t) = \cos u(t). \quad (59) \]

Then Eqs. 58 become:

\[ \phi_1 = \dot{x} - y = 0 \]

\[ \phi_2 = \dot{y} + 2\xi wy + w^2 x - \cos u = 0. \quad (60) \]

We now have \( J = g(x_0, y_0, t_0, x_f, y_f, t_f) = t_f \) to minimize. The initial and final conditions are given in Eq. 57, the side conditions (which are actually the equations of motion) are given in Eq. 60. The state variables are \( x(t), y(t) \). The control variable \( u(t) \) is desired.

We introduce Lagrangian multipliers with the formation of function \( F \) as in Eq. 50:

\[ F = \lambda_1 \phi_1 + \lambda_2 \phi_2 = 0 \quad (51) \]

\[ F = \lambda_1 (\dot{x} - y) + \lambda_2 (\dot{y} + 2\xi wy + w^2 x - \cos u) = 0. \]

Next we find the following quantities, needed in the Euler-Lagrange equations:

\[ F_x = \lambda_1 \]
\[ F_x = w^2 \lambda_2 \]
\[ F_y = \lambda_2 \]
\[ F_y = 2\xi wy - \lambda_1 \]
\[ F_u = \lambda_2 \sin u \]

Substituting in the Euler-Lagrange equations (51) yields
\[
\frac{d}{dt} (F_x) - F_x = 0 \\
\frac{d}{dt} (F_y) - F_y = 0 \\
\frac{d}{dt} (F_u) - 0 = 0
\]

\[
\dot{\lambda}_1 - u^2 \lambda_2 = 0 \\
\dot{\lambda}_2 - 2 \omega \lambda_2 + \lambda_1 = 0
\]

(62)

For the last of Eqs. 62 to be satisfied either \( \lambda_2 = 0 \), or \( u = 0 \) or \( \pi \), that is, \( z = +1 \) or \(-1\), namely "bang-bang" control, which is an interesting conclusion.

The problem now is to solve Eqs. 60 and 62 for \( x, y, u, \lambda_1, \lambda_2 \) all as functions of time. Since \( t \) does not appear explicitly in \( F \) in Eq. 61, we have a first integral like Eq. 52:

\[
\lambda_1 \dot{x} + \lambda_2 \dot{y} = -C_2. \tag{63}
\]

Since \( g = t_f \), the transversality condition of Eq. 53 becomes

\[
\left. \frac{dt}{dF} \right|_{t_0} + \left[ C_2 \frac{dt}{dx} + \lambda_1 \frac{dx}{dt} + \lambda_2 \frac{dy}{dt} \right]_{t_0} = 0,
\]

or

\[
\frac{dt}{dx} + C_2 \frac{dt}{dx} + \lambda_1 \frac{dx}{dt} + \lambda_2 \frac{dy}{dt} - \left( + C_2 \frac{dt}{dx} + \lambda_1 \frac{dx}{dt} + \lambda_2 \frac{dy}{dt} \right) = 0. \tag{64}
\]

Initially, \( t = t_0, x = x_0, \dot{x} = y = \dot{x}_0 \) are specified; therefore \( \frac{dt}{dx} = \frac{dx}{dx} = \frac{dy}{dx} = 0 \). Finally, \( x = 0, \dot{x} = y = 0 \) are specified; therefore \( \frac{dx}{dx} = \frac{dy}{dx} = 0 \). \( t_f \), not specified, is to be minimized. The only remaining terms in Eq. 64 are those involving \( \frac{dt}{dx} \); therefore it is necessary that \( C_2 = -1 \). With this information the first integral equation (63) becomes

\[
\lambda_1 \dot{x} + \lambda_2 \dot{y} = 1. \tag{65}
\]
According to the Weierstrass-Erdmann corner condition, $\lambda_1$ and $\lambda_2$ must be continuous across possible corners.

Applying the Weierstrass condition as in Eq. 54,

$$\lambda_1 y + \lambda_2 (-2\xi wy - w^2 x + \cos u) \geq \lambda_1 y + \lambda_2 (-2\xi wy - w^2 x + \cos u)$$

or

$$\lambda_2 \cos u \geq \lambda_2 \cos u^*.$$  \hspace{1cm} (66)

Thus $\lambda_2 \cos u$ is to be maximized. Therefore, whenever $\lambda_2$ is positive, $\cos u$ should be maximum, namely $u = 0$; whenever $\lambda_2$ is negative, $\cos u$ should be minimum, namely $u = \pi$. This then is a condition to be used when integrating Eqs. 60 and 62.

When

$$\lambda_2 > 0, \ u = 0$$  \hspace{1cm} (67)

$$\lambda_2 < 0, \ u = \pi.$$  

Let us now summarize the pertinent facts.

Equations of motion:

$$\dot{x} - y = 0$$

$$\dot{y} + 2\xi wy + w^2 x - \cos u = 0.$$  \hspace{1cm} (60)

Euler-Lagrange equations:

$$\lambda_1 - w^2 \lambda_2 = 0$$

$$\lambda_2 - 2\xi w \lambda_2 + \lambda_1 = 0$$

$$\lambda_2 \sin u = 0.$$  \hspace{1cm} (62)

First integral:

$$\lambda_1 \dot{x} + \lambda_2 \dot{y} = 1$$

or

$$\lambda_1 y + \lambda_2 \dot{y} = 1.$$  \hspace{1cm} (65)

Initial and final conditions:

<table>
<thead>
<tr>
<th>[t_o]</th>
<th>[t_f]</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x_o$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>$\dot{x}_o$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(57)
Corner conditions:

\( \lambda_1 \) and \( \lambda_2 \) must be continuous across corners.

Weierstrass condition:

\[
\begin{align*}
\text{When } & \lambda_2 > 0, \ u = 0 \\
\text{and } & \lambda_2 < 0, \ u = \pi.
\end{align*}
\]

(67)

The question arises of what \( u \) should be if \( \lambda_2 = 0 \) over a finite length of time. In this case, over this time period, both \( \lambda_2 \) and \( \hat{\lambda}_2 \) would have to be zero. From Eqs. 62 it follows that \( \lambda_1 \) and \( \hat{\lambda}_1 \) would have to vanish. But then the first integral equation (65) would be violated. Therefore \( \lambda_2 \neq 0 \) over a finite length of time. Thus \( \lambda_2 \) is a switching function that can be portrayed graphically as in Fig. 11.

![Graph of \( \lambda_2 \) vs. time](image)

**Fig. 11**

An examination of Eqs. 60, 62, 65 at the final conditions (57) reveals that

\[
\begin{align*}
\lambda_{2f} \cos u_{f} &= 1 \\
\lambda_{2f} \sin u_{f} &= 0;
\end{align*}
\]

therefore, when \( u_{f} = 0 \), \( \lambda_{2f} = 1 \)

and when \( u_{f} = \pi \), \( \lambda_{2f} = -1 \).
The table (57) may then be expanded:

<table>
<thead>
<tr>
<th></th>
<th>$t_0$</th>
<th>$t_0 &lt; t &lt; t_f$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x_0$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>$y_0$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td>0</td>
<td>$\pi$</td>
<td>0</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\lambda_2(t)$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(68)

The general solution for $x, y, u, \lambda_1, \lambda_2$ must come out of the integration of Eqs. 60, 62, and 65, subject to the conditions (68). Thus from (62) we can get:

$$\ddot{\lambda}_2 - 2\xi \dot{\lambda}_2 + \omega^2 \lambda_2 = 0$$

(69)

which can be solved for $\lambda_2 = \lambda_2(t)$. Then $\lambda_1$ can be obtained from

$$\lambda_1 = -\dot{\lambda}_2 + 2\xi \lambda_2$$

(70)

From (60)

$$\ddot{x} + 2\xi \dot{x} + \omega^2 x = \cos u = \pm 1$$

(71)

depending on the sign of $\lambda_2$. These equations can now be integrated backwards in time, starting with the conditions at $t_f$ given in (68).

The choice of $u$ and $\lambda_2$ at $t_f$ that leads to satisfaction of the initial conditions can be determined by trial and error. A convenient manner of carrying out the integration, with due regard for the switching function and the necessary continuity of $\lambda_1$ and $\lambda_2$, is to set the problem up on an analog computer.

It is interesting to note that digital computation would be beset with difficulties because if the equations of motion (60) contain a positive damping force, the Euler-Lagrange equation (69) contains a negative damping force, which could result in computational instabilities.
This problem seems to exist whenever dissipative forces are under consideration. Furthermore, the problem exists whether the damping factor $\zeta$ is inherently positive or negative, or whether forward or backward integration in time is utilized.

**TRAJECTORY IN CONSTANT GRAVITY FIELD**

The problem to be considered is that of the flight of a powered rocket in a vacuum and a constant (in magnitude and direction) gravity field. The forces acting on the rocket are due to gravity $g$ and thrust $T = \beta c$, where $\beta =$ mass flow rate (inherently positive), and $c$ is the effective exhaust velocity (which is equal to the product of $g$ and specific impulse).

These forces produce accelerations $\dot{x} = \dot{\phi}$ and $\dot{y} = \ddot{\phi}$. The flow rate is to be limited: $\beta_l \leq \beta \leq \beta_u$. No limits are imposed on $\phi$, the direction of thrust. The equations of motion in normalized form then are:

\[
\begin{align*}
\phi_1 &= \dot{\phi} - \frac{\beta c}{m} \cos \phi = 0 \\
\phi_2 &= \dot{\phi} - \frac{\beta c}{m} \sin \phi + g = 0 \\
\phi_3 &= \dot{x} - p = 0 \\
\phi_4 &= \dot{y} - q = 0 \\
\phi_5 &= \ddot{\phi} + \beta = 0
\end{align*}
\] (72)
where \( \beta = \beta (\alpha) = \frac{\beta_u + \beta_L}{2} - \frac{\beta_u - \beta_L}{2} \cos \alpha \).

The last equation replaces \( \beta \) by a dummy variable \( \alpha \) to provide for the limits on \( \beta \). Of course, \( \beta_L \) could be zero. Here \( p, q, x, y, m \) are state variables; \( \phi \) and \( \alpha \) are control variables.

Fig. 13

Let us maximize the range from launch to impact. Then initial and final conditions can be stated as follows:

\[
\begin{array}{c|cc}
 & t_o & t_f \\
 p & 0 & \\
 q & 0 & \\
 x & 0 & \\
 y & 0 & 0 \\
 m & m_o & m_f \\
 \phi & \\
 \alpha & \\
\end{array}
\]

(73)

Where there are blanks the quantities are not specified; this includes \( t_f \).

We wish to minimize \( J \):

\[
J = g = -x_f.
\]
Maximizing range is equivalent to minimizing negative range. We form the function $F$:

$$F = \sum_{i=1}^{5} \lambda_i \phi_i$$

or

$$F = \lambda_1 (p - \frac{8c}{m} \cos \phi) + \lambda_2 (q - \frac{8c}{m} \sin \phi \cdot g) + \lambda_3 (\dot{x} - p) + \lambda_4 (\dot{y} - q) + \lambda_5 (\dot{n} + \beta) = 0. \quad (75)$$

The Euler-Lagrange equations are

$$\frac{d}{dt} (F_p) - F_p = 0$$
$$\frac{d}{dt} (F_q) - F_q = 0$$
$$\frac{d}{dt} (F_x) - F_x = 0$$
$$\frac{d}{dt} (F_y) - F_y = 0$$
$$\frac{d}{dt} (F_m) - F_m = 0$$
$$\frac{d}{dt} (F_\phi) - 0 = 0 \text{ or } F_\phi = C_1$$
$$\frac{d}{dt} (F_\alpha) - 0 = 0 \text{ or } F_\alpha = F_\beta \frac{d\beta}{d\alpha} = C_2$$

where $C_1 = C_2 = 0$ by the transversality condition.

Notice that we are treating the control variables as differentiated variables, for continuity reasons. Substituting in the above equations we get

$$\dot{\lambda}_1 + \lambda_3 = 0$$
$$\dot{\lambda}_2 + \lambda_4 = 0$$
$$\dot{\lambda}_3 = 0$$
$$\dot{\lambda}_4 = 0$$

$$\dot{\lambda}_5 - \frac{8c}{m^2} (\lambda_1 \cos \phi + \lambda_2 \sin \phi) = 0. \quad (76)$$
\[ \frac{6c}{m} (\lambda_1 \sin \phi - \lambda_2 \cos \phi) = 0 \]
\[ \left[ \lambda_3 - \frac{c}{m} (\lambda_1 \cos \phi + \lambda_2 \sin \phi) \right] \frac{d\theta}{d\alpha} = 0 \]

From these equations we can then deduce that
\[
\lambda_3 = a \\
\lambda_4 = b \\
\lambda_1 = -at + c \\
\lambda_2 = -bt + d
\]

where \( a, b, c, d \) are constants. From the next to the last equation it follows that
\[
\tan \phi = \frac{\lambda_2}{\lambda_1}
\]
or
\[
\tan \phi = -\frac{bt + d}{-at + c} .
\]

(77)

Thus we already have an important conclusion: the tangent of the thrust angle should consist of the quotient of two linear functions of the time. This is the case for a wide variety of trajectory optimizations in a constant-gravity field: maximizing range, or minimizing initial weight, or maximizing burnout weight (that is, payload weight), and others.

The last equation of Eqs. 76 implies that either the bracketed quantity, or \( \frac{d\theta}{d\alpha} \), or both, are zero. Of course, \( \frac{d\theta}{d\alpha} = 0 \) means that
either $\beta = \beta_u$ or $\beta = \beta_\ell$. These questions can be resolved through the use of the Weierstrass condition, which will be taken up below.

We have a first integral:

$$F - \ddot{\mathbf{P}} - \ddot{\mathbf{Q}} - \ddot{\mathbf{X}} - \ddot{\mathbf{Y}} - \ddot{\mathbf{F}} = \mathbf{C}_3$$

or

$$-\lambda_1\dot{z} - \lambda_2\dot{y} - \lambda_3\dot{x} - \lambda_4\dot{\phi} - \lambda_5\dot{\psi} = \mathbf{C}_3.$$  \hspace{1cm} (78)

The transversality condition is

$$-dx_f + \left[ C_3 \dot{t} + \lambda_1 \dot{d} + \lambda_2 \dot{q} + \lambda_3 \dot{x} + \lambda_4 \dot{y} + \lambda_5 \dot{m} \right]_{t_0}^{t_f} = 0.$$  

This may be rewritten:

$$-dx_f + C_3 \dot{t}_f + \lambda_1 \dot{d}_f + \lambda_2 \dot{q}_f + \lambda_3 \dot{x}_f + \lambda_4 \dot{y}_f + \lambda_5 \dot{m}_f$$

$$- C_3 \dot{x} - \lambda_1 \dot{d} - \lambda_2 \dot{q} - \lambda_3 \dot{x} - \lambda_4 \dot{y} - \lambda_5 \dot{m} = 0.$$  \hspace{1cm} (79)

Where quantities have been specified in Eq. 73, the differentials vanish; where the quantities are arbitrary, the coefficients must vanish; thus we conclude that

$$C_3 = 0$$

$$\lambda_{1f} = \lambda_{2f} = 0$$  \hspace{1cm} (80)

$$\lambda_{3f} = 1.$$  

The corner condition requires that $\lambda_1 \cdots \lambda_5$ be continuous across corners.
So far we can summarize as follows:

<table>
<thead>
<tr>
<th></th>
<th>( t_o = 0 )</th>
<th>( t_o &lt; t &lt; t_f )</th>
<th>( t_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( q )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>( m )</td>
<td>( m_o )</td>
<td>( m_f )</td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>( t_f - t_o )</td>
<td>( -at + c = t_f - t )</td>
<td>0</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>( b(t_f - t_o) )</td>
<td>( -bt + d = b(t_f - t) )</td>
<td>0</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>1</td>
<td>( a = 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>( b )</td>
<td>( b )</td>
<td></td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>( b )</td>
<td>( \frac{\lambda_2^2}{\lambda_1} = b )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \lambda = \sqrt{\lambda_1^2 + \lambda_2^2} \]

\[ \lambda_1 = \frac{\sqrt{1 + b^2} (t_f - t)}{(t_f - t) \sqrt{1 + b^2}} \]

Thus we are faced with the integration of the equations of motion and the Euler-Lagrange equations subject to these mixed end conditions.

We still need to determine what thrust level to use and when, and the proper quadrant for the angle \( \phi \). No limits were placed on \( \phi \).

Intuitively we would expect \( \phi \) to lie in the first quadrant. This can be resolved through the use of the Weierstrass condition, Eq. 51,

which for this problem becomes

\[
\begin{align*}
\left( \frac{\dot{\phi}}{\dot{r}} \cos \phi \right) \lambda_1 + \left( \frac{\dot{\phi}}{\dot{r}} \sin \phi - g \right) \lambda_2 + \beta \lambda_3 + q \lambda_4 - \beta \lambda_5 & = \\
\left( \frac{\dot{\phi}^*}{\dot{r}} \cos \phi^* \right) \lambda_1 + \left( \frac{\dot{\phi}^*}{\dot{r}} \sin \phi^* - g \right) \lambda_2 + \beta \lambda_3 + q \lambda_4 - \beta \lambda_5 & = \\
\end{align*}
\]
or
\[ \beta \left[ \frac{c}{m} (\lambda_1 \cos \phi + \lambda_2 \sin \phi) - \lambda_5 \right] \geq \beta^* \left[ \frac{c}{m} (\lambda_1 \cos \phi^* + \lambda_2 \sin \phi^*) - \lambda_5 \right] . \tag{82} \]

For convenience let us define \( Q \):
\[ Q = \frac{c}{m} (\lambda_1 \cos \phi + \lambda_2 \sin \phi) - \lambda_5 \]
\[ = \frac{c}{m} (\pm \lambda) - \lambda_5 \tag{83} \]
where \( \lambda = \lambda_1 \cos \phi + \lambda_2 \sin \phi = \sqrt{\lambda_1^2 + \lambda_2^2} \) as can be seen from Fig. 14. Notice that \( Q \) also appears in the last of the Euler-Lagrange equations (76). Then it is necessary that
\[ \beta Q \geq \beta^* Q^* \tag{84} \]
for all admissible variations. We can look at two cases:

Case a. When \( \beta = \beta^* \), \( Q \neq Q^* \), \( \beta Q \geq \beta^* Q^* \). Since \( \beta \) is inherently positive (or zero), it is necessary that \( Q \geq Q^* \); that is, \( Q \) must be maximized. Clearly then the \((\pm \lambda)\) must be chosen in Eq. 83:
\[ Q = \frac{c}{m} (\pm \lambda) - \lambda_5. \tag{85} \]

Case b. When \( Q = Q^* \), \( \beta \neq \beta^* \), \( \beta Q \geq \beta^* Q \). Since \( \beta^*_L \leq \beta \leq \beta^*_U \), we conclude that
\[
\text{when } Q > 0, \beta = \beta^*_U \quad Q < 0, \beta = \beta^*_L. \tag{86}
\]

A question remains: what if \( Q = 0 \) for any period of time? It can be shown that this is impossible in this problem. In fact, it turns out that \( Q(t) \) is a monotonically decreasing function. We take 85 and differentiate:
\[ \dot{Q} = \frac{c}{m} \ddot{x} - \frac{c}{m^2} \dddot{x} \lambda - \dot{\lambda}_5. \]
and substitute in it the \( \dot{\lambda}_5 \) of Eq. 76:

\[
\dot{\lambda}_5 = \frac{8c}{m^2} \lambda.
\]

Then

\[
\dot{\varepsilon} = \frac{c}{m} \dot{\lambda} + \frac{c}{m^2} \beta \lambda - \frac{8c}{m^2} \lambda,
\]

and since from Eq. 81, \( \lambda = (t_f - t) \sqrt{1 + b^2} \), we get

\[
\dot{\varepsilon} = \frac{-c \sqrt{1 + b^2}}{m(t)}.
\]  

(87)

Thus \( Q(t) \) has a negative slope, where with increasing time the magnitude of the slope increases, since mass decreases. When \( Q(t) \) crosses the axis, the flow rate must be switched from \( \beta_u \) to \( \beta_\ell \) in accordance with Eq. 86. Thus \( Q(t) \) is a switching function. This is illustrated in Fig. 15.

Let us now suppose that \( \beta_u \) is some upper value of flow rate and that \( \beta_\ell \) is zero (zero thrust; i.e., coasting flight). Then from Eq. 87:

\[
\int_0^t Q_0 \, dt = -c \sqrt{1 + b^2} \int_{t_s}^t \frac{dt}{m - \beta_u t}
\]

\[
Q_0 = \frac{c \sqrt{1 + b^2}}{\beta_u} \ln \left( \frac{m_0}{m_f} \right)
\]  

(88)

Solving Eq. 85 for \( \lambda_\ell(t_o) \):

\[
\lambda_\ell(t_o) = \frac{c}{m_0} \lambda_o - Q_0
\]

\[
= \frac{c}{m_0} (t_f - t_o) \sqrt{1 + b^2}
\]

\[
- \frac{c \sqrt{1 + b^2}}{\beta_u} \ln \left( \frac{m_0}{m_f} \right)
\]
or  
\[ \lambda_5(t_o) = c \sqrt{1 + b^2} \left[ \frac{t_f - t_o}{m_o} - \frac{1}{\beta_n} \ln \frac{m_o}{m_f} \right]. \]  

With this we now have a complete set of initial conditions in Eq. 81, allowing us to integrate the equations of motion and the Euler-Lagrange equations forward. It is a two-parameter type of solution in that values for \( b \) and \( t_f \) must still be determined. The use of the first integral will allow us to reduce this to a single parameter. Substituting the equations of motion (72) in the first integral (78) and using the known \( \lambda \)'s from 81:

\[ -\lambda_1 \left( \frac{\beta c}{m} \cos \phi \right) - \lambda_2 \left( \frac{\beta c}{m} \sin \phi - g \right) - \lambda_3 p - \lambda_4 q + \lambda_5 \beta = 0. \]

But  
\[ \lambda_1 \cos \phi + \lambda_2 \sin \phi = \lambda = (t_f - t)\sqrt{1 + b^2}. \]
Then 
\[ - \frac{\beta c}{m} (t_f - t) \sqrt{1 + \frac{b^2}{c^2}} + b(t_f - t)x_0 - p - bq + \lambda \beta = 0. \] (90)

At \( t = t_0 \) this becomes
\[ - \frac{\beta u}{m_0} (t_f - t_0) \sqrt{1 + \frac{b^2}{c^2}} + b(t_f - t_0)x_0 + \beta \lambda \beta(t_0) = 0, \]

which, upon substitution of \( \lambda \beta(t_0) \) of Eq. 89, simplifies to
\[ (t_f - t_0) = \frac{c}{\beta} \sqrt{1 + \frac{b^2}{c^2}} \ln \left( \frac{m_0}{m_f} \right). \] (91)

This relation of \( t_f \) and \( b \) reduces the problem then to one involving a single parameter. The course of action now should be to integrate Eqs. 72 and 76, estimating a value of \( b \) (\( b = 0 \); that is, \( \phi = 45^\circ \), is a good starting value), and integrating from \( t_0 \) to \( t_f \). This process should be repeated until the final conditions are satisfied. In this particular problem, the Euler-Lagrange equations for all \( \lambda \)'s, except \( \lambda \gamma \), have already been integrated; and the equations of motion are sufficiently simple to permit solution in closed form (see Eqs. 127, with allowance made for the fact that \( \phi \) is not necessarily equal to \( 45^\circ \), but a constant, nevertheless).

At \( t = t_f \), Eq. 90 becomes
\[ - p_x - bq_x = 0, \]
or
\[ b = \tan \phi = - \frac{x_x}{y_f}. \] (92)

that is, the flight path angle (direction of velocity vector) with respect to the vertical at impact is equal to the optimum value (a constant) of the thrust vector direction with respect to the horizontal during powered flight. This is of no particular help in the integration, but it is an interesting fact.
In summary, we have shown that for the maximum range problem in a constant gravity field, the thrust should be held at its maximum value until all fuel is consumed, followed by a coasting flight to impact, and that the optimum thrust vector direction is a constant during powered flight.

Let us go back now and take a broader look at trajectories in a constant gravity field. There are a total of six variables that appear in the transversality condition: five state variables and time. If one of these six is to be optimized, either initially or finally, then as many as eleven end conditions can be specified: six at one end, five at the other. In the above example six conditions were specified initially, only two finally. Without assigning end conditions, or deciding on what should be optimized, certain general conclusions can be deduced\(^{(9)}\) concerning optimum trajectories of single stage rockets in vacuum, constant gravity, flight:

1. The optimum trajectory is composed of maximum and minimum (coasting) thrust arcs only.
2. There occur at most three subarcs in a trajectory.
3. An arc is one of maximum thrust when \(Q > 0\), and of minimum thrust (coasting) when \(Q < 0\).
4. Transition between arcs takes place when \(Q = 0\).
5. Burnout \((m = m_f)\) occurs either at the junction between a thrust and final coasting arc, or during a final thrust arc.
ESCAPE TRAJECTORY

It is desired to take a space vehicle from specified initial conditions of mass, position, and velocity to escape. Using the same symbols that have been previously employed, the equations of motion are:

\[ \dot{\phi}_1 = \dot{\phi} - \frac{\beta c}{m} \cos \phi + \frac{\mu x}{r^3} = 0 \]
\[ \dot{\phi}_2 = \dot{\theta} - \frac{\beta c}{m} \sin \phi + \frac{\mu y}{r^3} = 0 \]
\[ \dot{\phi}_3 = \dot{x} - \dot{p} = 0 \]
\[ \dot{\phi}_4 = \dot{y} - \dot{q} = 0 \]
\[ \dot{\phi}_5 = \dot{m} + \beta = 0 \]

where \( \beta = \beta (\alpha) = \frac{\beta_u + \beta_v}{2} - \frac{\beta_u - \beta_v}{2} \cos \alpha \)

and \( r = \sqrt{x'^2 + y'^2} \).

Initial values of \( p, q, x, y, m \) are specified. To achieve escape it is necessary, at the final time, that

\[ \frac{V_f^2}{2} - \frac{\mu}{r_f} = 0 \]

or

\[ p_f^2 + q_f^2 = \frac{2\mu}{\sqrt{x_f^2 + y_f^2}} \].

It is desired to maximize the final mass \( m_f \), or minimize \( -m_f \):

\[ J = g = -m_f \]

Then \( F \) is

\[ F = \lambda_1 \left( \dot{\phi} - \frac{\beta c}{m} \cos \phi + \frac{\mu x}{r^3} \right) + \lambda_2 \left( \dot{\theta} - \frac{\beta c}{m} \sin \phi + \frac{\mu y}{r^3} \right) + \lambda_3 (\dot{x} - \dot{p}) + \lambda_4 (\dot{y} - \dot{q}) + \lambda_5 (\dot{m} + \beta) = 0. \]
The Euler-Lagrange equations are:

\[ \dot{\lambda}_1 + \lambda_3 = 0 \]
\[ \dot{\lambda}_2 + \lambda_4 = 0 \]
\[ \dot{\lambda}_3 - \frac{\mu}{r^5} \left[ \lambda_1 (r^2 - 3x^2) - \lambda_2 3x y \right] = 0 \]
\[ \dot{\lambda}_4 - \frac{\mu}{r^5} \left[ - \lambda_1 3x y + \lambda_2 (r^2 - 3y^2) \right] = 0 \]
\[ \dot{\lambda}_5 - \frac{8a}{m} \left( \lambda_1 \cos \phi + \lambda_2 \sin \phi \right) = 0 \]
\[ F_\phi = \frac{8a}{m} \left( \lambda_1 \sin \phi - \lambda_2 \cos \phi \right) = 0 \]

\[ F_\alpha = \left[ \lambda_5 - \frac{c}{m} \left( \lambda_1 \cos \phi + \lambda_2 \sin \phi \right) \right] \frac{d\delta}{d\alpha} = 0. \]

From the next to the last equation it follows that \( \tan \phi = \frac{\lambda_2}{\lambda_1} \).

From the last equation, either the bracketed quantity, or \( \frac{d\delta}{d\alpha} \), or both, are zero.

The first integral is

\[ - \lambda_1 \dot{x} - \lambda_2 \dot{y} - \lambda_3 \dot{x} - \lambda_4 \dot{y} - \lambda_5 \dot{z} = C_3 \]

or

\[ - \lambda_1 \left( \frac{8a}{m} \cos \phi - \frac{\mu x}{r^3} \right) - \lambda_2 \left( \frac{8a}{m} \sin \phi - \frac{\mu y}{r^3} \right) \]

\[ - \lambda_3 \rho - \lambda_4 \varphi + \lambda_5 \beta = C_3. \]

The transversality condition is

\[ - \frac{dm_f}{dt} + \left[ C_3 \frac{dt}{t} + \lambda_1 \frac{dp}{p} + \lambda_2 \frac{dq}{q} + \lambda_3 \frac{dx}{x} + \lambda_4 \frac{dy}{y} + \lambda_5 \frac{dm}{m} \right] t_f = 0 \]

\[ - \frac{dm_f}{dt} + C_3 \frac{dt}{t} + \lambda_1 \frac{dp}{p} + \lambda_2 \frac{dq}{q} + \lambda_3 \frac{dx}{x} + \lambda_4 \frac{dy}{y} + \lambda_5 \frac{dm}{m} \]

\[ - C_3 \frac{dq}{q} - \lambda_6 \frac{dq}{q} - \lambda_7 \frac{dx}{x} - \lambda_8 \frac{dy}{y} - \lambda_9 \frac{dm}{m} = 0. \]
Where quantities have been specified, the differentials have been crossed out; where they are arbitrary, the coefficients must vanish, subject, of course, to Eq. 94. We conclude that
\[
\begin{align*}
0_3 &= 0 \\
\lambda_{5f} &= 1 \\
\lambda_{1f} d\rho_f + \lambda_{2f} d\eta_f + \lambda_{3f} dx_f + \lambda_{4f} dy_f &= 0.
\end{align*}
\]
(100)

Differentiating Eq. 94 yields
\[
2p_f \frac{d\rho_f}{dx_f} + 2q_f \frac{d\eta_f}{dx_f} = -\frac{2\mu}{r_f^3} (x_f \frac{dx_f}{dx_f} + y_f \frac{dy_f}{dx_f}).
\]

Solving this for \( d\rho_f \) and substituting in (100) yields
\[
-\frac{q_f}{p_f} \lambda_{1f} d\eta_f - \frac{\mu}{p_f^2 r_f^3} (x_f \frac{dx_f}{dx_f} + y_f \frac{dy_f}{dx_f}) \lambda_{1f} \\
+ \lambda_{2f} d\eta_f + \lambda_{3f} dx_f + \lambda_{4f} dy_f = 0.
\]

Equating each of the coefficients of \( d\eta_f, dx_f, dy_f \) to zero, since there is no further constraining relation between \( q_f, x_f, y_f \), yields
\[
\lambda_{2f} = \frac{q_f}{p_f} \lambda_{1f} = 0 \\
\lambda_{3f} = \frac{\mu x_f}{p_f^2 r_f^3} \lambda_{1f} = 0 \\
\lambda_{4f} = \frac{\mu y_f}{p_f^2 r_f^3} \lambda_{1f} = 0.
\]
(101)

Thus it can be seen that \( \frac{\lambda_{2f}}{\lambda_{1f}} = \frac{q_f}{p_f} = \frac{\dot{y}_f}{\dot{x}_f} = \tan \phi_f \);

that is, at the final time \( t_f \), the thrust and velocity directions are equal. Also \( \frac{\lambda_{4f}}{\lambda_{3f}} = \frac{y_f}{x_f} \).
The use of the Weierstrass condition leads to the same conclusion as in the above problem on the trajectory in a constant gravity field, namely, Eq. 86, illustrated in Fig. 15.

We can summarize our knowledge so far as follows:

<table>
<thead>
<tr>
<th></th>
<th>$t_0$</th>
<th>$t_0 &lt; t &lt; t_f$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
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<td>$q_f$</td>
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<td>$\lambda_{1f}$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td></td>
<td></td>
<td>$\frac{q_f}{p_f} \lambda_{1f}$</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td></td>
<td></td>
<td>$\frac{\mu x_f}{p_f^2 r_f^3} \lambda_{1f}$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td></td>
<td></td>
<td>$\frac{\mu y_f}{p_f^2 r_f^3} \lambda_{1f}$</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td></td>
<td></td>
<td>$l$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\tan^{-1} \frac{\lambda_2}{\lambda_1}$ (prin. value in 1st quadrant)</td>
<td></td>
<td>$\tan^{-1} \frac{q_f}{p_f}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>if $Q &gt; 0$, $\beta = \beta_u$; if $Q &lt; 0$, $\beta = \beta_l$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If we make a guess at five of the final variables, say $p_f$, $q_f$, $x_f$, $m_f$, $\lambda_{1f}$, then the other five final variables are determined. Thus we again have the mixed-end-condition problem, in that we have
a set of ten first-order differential equations to integrate (5 equations of motion, 5 Euler-Lagrange equations) involving ten variables: \( p, q, x, y, m, \lambda_1, \ldots, \lambda_5 \), with half of the initial conditions specified, and half of the final conditions known. A further investigation of the \( Q(t) \) function and the use of the first integral at \( t_o \) and \( t_f \), much as was done in the above trajectory problem, should give additional information of value in the integration problem.

**VERTICAL TRAJECTORY**

Next we wish to examine a problem in which the \( Q(t) \) function could be identically zero for a finite length of time, implying a variable-thrust subarc.

In a sounding flight originating at the earth's surface the apogee altitude is to be maximized. We shall include effects of gravity and drag, but shall make certain simplifying assumptions to assure tractability, the purpose being to illustrate a point in the analysis.

The equations of motion are:

\[
\begin{align*}
\phi_1 &= \dot{\phi} - \frac{\beta a}{m} + \frac{u}{2} + 3 \frac{a(v)}{m} \frac{b}{2} v_2^2 = 0 \\
&= \dot{\phi} - \frac{\beta a}{m} + k_1 + k_2 \frac{v_2^2}{m} = 0 \\
\phi_2 &= \dot{r} - v = 0 \\
\phi_3 &= \dot{\alpha} + \beta = 0 \\
\text{where } &\beta = \beta(\alpha) = \frac{\beta u + \beta_1}{2} - \frac{u - \beta_2}{2} \cos \alpha \\
\text{and } &\beta_1 = 0
\end{align*}
\]

(103)

where for simplicity's sake \( k_1 \) and \( k_2 \) will be treated as constants.
Initially \( t = 0, \ v = 0, \ r = r_0, \ m = m_0; \) finally \( m = m_f \). The problem is to minimize \((-r_f)\). We have

\[
F = \lambda_1 \left( \dot{v} - \frac{\beta c}{m} + k_1 + k_2 \frac{v^2}{m} \right) + \lambda_2 (\dot{r} - v) + \lambda_3 (\dot{m} + \beta) = 0
\]

\[
\begin{aligned}
\dot{\lambda}_1 - \frac{2k_2 v}{m} \lambda_1 + \lambda_2 &= 0 \\
\dot{\lambda}_2 &= 0 \\
\dot{\lambda}_3 - \left( \frac{\beta c}{m^2} - \frac{k_2 v^2}{m^2} \right) \lambda_1 &= 0 \\
F_\alpha &= \left[-\lambda_1 \frac{c}{m} + \lambda_3 \right] \frac{d\beta}{dt} = 0.
\end{aligned}
\]  

(105)

The first integral is

\[
F = F_v \dot{v} - F_r \dot{r} - F_m \dot{m} = C_2
\]

or

\[
- \lambda_1 \left( \frac{\beta c}{m} - k_1 - k_2 \frac{v^2}{m} \right) - \lambda_2 v + \lambda_3 \beta = C_2.
\]  

(106)

From the transversality condition,

\[
- \Delta r_f + \left[ C_2 \frac{dt}{\tau} + \lambda_1 \frac{dv}{\tau} + \lambda_2 \frac{dr}{\tau} + \lambda_3 \frac{dm}{\tau} \right]_{t_f}^{t_0} = 0,
\]

it follows that

\[
C_2 = 0
\]

\[
\lambda_{1_f} = 0
\]

\[
\lambda_{2_f} = 1.
\]  

(107)

From Eq. 105 it then follows that \( \lambda_2 = 1 \). The Weierstrass condition,

Eq. 54, gives

\[
\lambda_1 \left( \frac{\beta c}{m} - k_1 - k_2 \frac{v^2}{m} \right) + \lambda_2 (v) + \lambda_3 (-\beta)
\]

\[
\geq \lambda_1 \left( \frac{\beta^* c}{m} - k_1 - k_2 \frac{v^2}{m} \right) + \lambda_2 (v) + \lambda_3 (-\beta^*)
\]
or
\[
\left( \lambda_1 \frac{c}{m} - \lambda_3 \right) \beta = \left( \lambda_1 \frac{c}{m} - \lambda_3 \right) \beta^*.
\]  
(108)

Calling \( \lambda_1 \frac{c}{m} - \lambda_3 = Q \):

\[
\begin{align*}
Q > 0, & \quad \beta = \beta_u \\
Q < 0, & \quad \beta = \beta_L.
\end{align*}
\]
(109)

By the corner condition the \( \lambda \)'s must be continuous across corners.

Let us now examine the possibility of \( Q = \dot{Q} = 0 \) for a finite time interval. Differentiating \( Q \) and substituting \( \lambda_1 \) and \( \lambda_3 \) from Eq. 105 yields

\[
\dot{Q} = \frac{c}{m} \left( \lambda_1 \frac{k_v v}{m} \left( 2 + \frac{v}{c} \right) - 1 \right).
\]
(110)

Then \( \dot{Q} = 0 \) when

\[
\lambda_1 = \frac{m}{k_v v \left( 2 + \frac{v}{c} \right)}.
\]
(111)

Furthermore, from the first integral (106):

\[
\left\{ \begin{array}{l}
- \lambda_1 \left( \frac{6c}{m} - k_v \frac{k_v^2}{m} \right) \cdot \nu + \lambda_3 \beta = 0
\end{array} \right.
\]
(112)

From the definition of \( Q \), when \( Q = 0 \),

\[
\lambda_1 \frac{6c}{m} - \lambda_3 \beta = 0.
\]

Solving for \( \lambda_1 \) when \( Q = 0 \):

\[
\lambda_1 = \frac{\nu}{k_v + \frac{k_v^2}{m}}.
\]
(113)

Equating Eqs. 111 and 113 yields

\[
\frac{m}{k_v v \left( 2 + \frac{v}{c} \right)} = \frac{\nu}{k_v + \frac{k_v^2}{m}}.
\]
or

\[ m = \frac{k_2}{k_1} \sqrt{2} (1 + \frac{v}{c}) \]  \hspace{1cm} (114)

The situation is portrayed in Fig. 16, a plot of \( m \) vs. \( v \).

![Fig. 16](image)

Curve \( C_1 \) represents \( \beta = \beta_u \); \( C_2 \) represents \( \beta = \beta_L = 0 \); \( C_3 \) represents \( \beta_u < \beta < \beta_L \), \( \beta \) being extremized. Eq. 114, that is, \( \dot{Q} = \dot{q} = 0 \). The course to be followed is to proceed from point (1) to (2) where \( \beta = \beta_u \), (2) to (3) along \( C_3 \) where \( \beta_L < \beta < \beta_u \), (3) to (4) where \( \beta = \beta_L = 0 \).

In Ref. 10 this problem is approached through the definition of a "fundamental function" \( \omega \) that is related to the integrand of the function to be extremized. The optimization problem is presented as a problem of Lagrange. If the above problem is converted to one of Lagrange, it can be shown that for the region where \( \omega = 0 \), Eq. 114 follows, meaning that \( Q = \dot{Q} = 0 \) (see Appendix).

**PERTURBATION STUDIES--ADJOINT EQUATIONS**

Refs. 11 and 12.

Ballistic perturbation theory utilizes a system of adjoint equations--equations "adjoined" to, or considered in conjunction with, the equations of motion. There is an interesting relationship between
the approach by the calculus of variations and ballistic perturbation theory that will be brought out below.

Let us look back at the problem of the trajectory in a constant gravity field. The equations of motion (72) were

\[
\begin{align*}
\dot{x} &= \frac{\delta c}{m} \cos \phi \\
\dot{y} &= \frac{\delta c}{m} \sin \phi - g \\
\dot{z} &= p \\
\dot{y} &= q \\
\dot{z} &= -\beta.
\end{align*}
\]

The linearized (considering only small perturbations) equations of variation corresponding to Eqs. 115 are

\[
\begin{align*}
\delta \ddot{x} &= \frac{d}{dt} (\delta p) = \left( -\frac{\delta c}{m^2} \cos \phi \right) \delta m + \left( \frac{c}{m} \cos \phi \right) \delta \beta + \left( -\frac{\delta c}{m} \sin \phi \right) \delta \phi \\
\delta \ddot{y} &= \frac{d}{dt} (\delta q) = \left( -\frac{\delta c}{m^2} \sin \phi \right) \delta m + \left( \frac{c}{m} \sin \phi \right) \delta \beta + \left( \frac{\delta c}{m} \cos \phi \right) \delta \phi \\
\delta \dot{x} &= \frac{d}{dt} (\delta x) = \delta p \\
\delta \dot{y} &= \frac{d}{dt} (\delta y) = \delta q \\
\delta \dot{z} &= \frac{d}{dt} (\delta m) = -\delta \beta.
\end{align*}
\]

These equations express the effect of small perturbations in the state and control variables on the differentiated state variables. The coefficients of the variations on the right side of these equations are evaluated along the nominal trajectory. Eqs. 116 may be written in matrix form:
\[
\begin{bmatrix}
\frac{d}{dt} \delta p \\
\frac{d}{dt} \delta q \\
\frac{d}{dt} \delta x \\
\frac{d}{dt} \delta y \\
\frac{d}{dt} \delta m
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{8c}{m^2} \cos \phi & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta p \\
\delta q \\
\delta x \\
\delta y \\
\delta m
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
-1
\end{bmatrix}
\begin{bmatrix}
\delta \phi
\end{bmatrix}
\] (117)

The first matrix on the right contains state variables, the second, the control variables. Next an adjoint system of equations is set up as follows:

\[
\begin{bmatrix}
\dot{\lambda}_1 \\
\dot{\lambda}_2 \\
\dot{\lambda}_3 \\
\dot{\lambda}_4 \\
\dot{\lambda}_5
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{8c}{m^2} \cos \phi & \frac{8c}{m^2} \sin \phi & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5
\end{bmatrix}
\] (118)

This matrix is the negative transpose of the state variable matrix in Eqs. 117; that is, rows are made into columns and the sign of each element is changed. Notice that Eqs. 118 are identical to the first five of Eqs. 76. Thus the adjoint equations and the Euler-Lagrange equations for the state variables are identical. Eqs. 118 have an interesting property. If we multiply the equations of variation (117) by
\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5
\end{bmatrix}
\text{, and the adjoint system (118) by}
\begin{bmatrix}
\delta p \\
\delta q \\
\delta x \\
\delta y \\
\delta m
\end{bmatrix}
\text{, and then}
\]

sum the resulting ten equations, we will find many terms cancelling
and arrive at:

\[
\frac{d}{dt} \left( \lambda_1 \delta p + \lambda_2 \delta q + \lambda_3 \delta x + \lambda_4 \delta y + \lambda_5 \delta m \right) =
\left[ \frac{c}{m} \left( \lambda_1 \cos \phi + \lambda_2 \sin \phi \right) - \lambda_5 \right] \delta \beta
\]

\[
+ \frac{2c}{m} \left[ - \lambda_1 \sin \phi + \lambda_2 \cos \phi \right] \delta \phi.
\]

(119)

Integrating from \( t_a \) to \( t_b \) yields

\[
\left( \lambda_1 \delta p + \lambda_2 \delta q + \lambda_3 \delta x + \lambda_4 \delta y + \lambda_5 \delta m \right)_{t_b} =
\left( \lambda_1 \delta p + \lambda_2 \delta q + \lambda_3 \delta x + \lambda_4 \delta y + \lambda_5 \delta m \right)_{t_a}
\]

\[
+ \int_{t_a}^{t_b} B(\beta, c, m, \phi, \lambda_1, \lambda_2, \lambda_5) \, dt,
\]

(120)

where \( B = \left[ \frac{c}{m} \left( \lambda_1 \cos \phi + \lambda_2 \sin \phi \right) - \lambda_5 \right] \delta \beta + \frac{2c}{m} \left[ - \lambda_1 \sin \phi + \lambda_2 \cos \phi \right] \delta \phi \)

\[
= B_1 \delta \beta + B_2 \delta \phi.
\]

This is known as Bliss' fundamental formula, after G. A. Bliss, who
is credited with its development. As far as the \( \lambda \)'s are concerned, if
we have a reference trajectory where \( \beta, c, m, \phi \) are known as functions of time, then Eqs. 118 can be solved for the \( \lambda' \)'s, provided certain initial or final values are assigned to these \( \lambda' \)'s. Suppose we make 
\[
\lambda_{1b} = 1 \quad \text{and} \quad \lambda_{2b} = \lambda_{3b} = \lambda_{4b} = \lambda_{5b} = 0.
\]
Then Eqs. 118 can be solved for the \( \lambda' \)'s as functions of time in the interval \( t_a \leq t \leq t_b \), consistent with these end conditions. Let us call these \( \lambda' \). The fundamental formula then becomes
\[
\delta p_b = \lambda_{1a} \delta p_a + \lambda_{2a} \delta q_a + \lambda_{3a} \delta x_a + \lambda_{4a} \delta y_a + \lambda_{5a} \delta m_a \\
+ \int_{t_a}^{t_b} \left[ B_1 \left( \lambda_{1}^2, \lambda_{2}^2, \lambda_{3}^2 \right) \delta \beta + B_2 \left( \lambda_{1}^2, \lambda_{2}^2 \right) \delta \phi \right] dt 
\]
(121)
Thus \( \delta p \) at \( t_b \) can be determined as a function of variations in the state variables at \( t_a \) as well as the integrated effect of control variables over the interval \( t_a \leq t \leq t_b \). From the form of Eq. 121 it will be seen that the \( \lambda' \)'s are sensitivity coefficients:
\[
\lambda_{1a} = \frac{\delta p_b}{\delta p_a}, \quad \lambda_{2a} = \frac{\delta p_b}{\delta q_a}, \quad \text{etc. Similarly, we can make the final}
\]
\( \lambda' \)'s: \( \lambda_{1b} = 0, \lambda_{2b} = 1, \lambda_{3b} = \lambda_{4b} = \lambda_{5b} = 0 \), and we get a set of solutions we shall call \( \lambda^2 \) (the 2 is a superscript for reference, not an exponent):
\[
\delta q_b = \lambda_{1a}^2 \delta p_a + \lambda_{2a}^2 \delta q_a + \lambda_{3a}^2 \delta x_a + \lambda_{4a}^2 \delta y_a + \lambda_{5a}^2 \delta m_a \\
+ \int_{t_a}^{t_b} \left[ B_1 \left( \lambda_{1}^2, \lambda_{2}^2, \lambda_{3}^2 \right) \delta \beta + B_2 \left( \lambda_{1}^2, \lambda_{2}^2 \right) \delta \phi \right] dt. 
\]
(122)
If we repeat this process several times, we can write the final result in matrix form:
\[
\begin{bmatrix}
\delta p_b \\
\delta q_b \\
\delta x_b \\
\delta y_b \\
\delta m_b
\end{bmatrix} = \begin{bmatrix}
\lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_2^1 & \lambda_2^2 \\
\lambda_2^1 & \lambda_2^2 & \lambda_3^2 & \lambda_3^3 & \lambda_3^4 \\
\lambda_3^1 & \lambda_3^2 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 \\
\lambda_4^1 & \lambda_4^2 & \lambda_4^3 & \lambda_5^3 & \lambda_5^5 \\
\lambda_5^1 & \lambda_5^2 & \lambda_5^3 & \lambda_5^4 & \lambda_5^5
\end{bmatrix} \begin{bmatrix}
\delta p_a \\
\delta q_a \\
\delta x_a \\
\delta y_a \\
\delta m_a
\end{bmatrix} + \int_{t_a}^{t_b} \begin{bmatrix} B \end{bmatrix} dt \quad (123)
\]

where
\[
\begin{bmatrix}
B_1^1 & B_1^2 \\
B_2^1 & B_2^2 \\
B_3^3 & B_3^3 \\
B_4^4 & B_4^5 \\
B_5^5 & B_5^5
\end{bmatrix} = \begin{bmatrix}
B_1^1 & B_1^2 \\
B_2^1 & B_2^2 \\
B_3^3 & B_3^3 \\
B_4^4 & B_4^5 \\
B_5^5 & B_5^5
\end{bmatrix}
\]

\[
B_1^k = \frac{c}{\mu} \left( \lambda_1^k \cos \phi + \lambda_2^k \sin \phi \right) - \lambda_5^k \quad (125)
\]

\[
B_2^k = \frac{8c}{\mu} \left[ - \lambda_1^k \sin \phi + \lambda_2^k \cos \phi \right].
\]

We thus have a set of equations describing the effect of (a) variations in the state variables at a given time \( t_a \) along the trajectory, and (b) variations in control variables (forcing functions such as winds, or variations in winds, could be included here) over the time interval \( t_a \leq t \leq t_b \), on the state variables at time \( t_b \). The \( \lambda \)'s turn out to be sensitivity coefficients. Times \( t_a \) and \( t_b \) could be taken as launch and impact, or any intermediate points.

Perturbation and optimization analyses are both based on the use of the equations of motion and an adjoint set of equations.
In the former the \( \lambda \)'s are given the meaning of sensitivity coefficients through the choice of the \( \lambda \)'s at \( t_b \). In the latter the \( \lambda \)'s are determined on the basis of finding a minimum of some function of the state variables initially or finally, through the determination of some of the \( \lambda \)'s initially or finally, or both, using the transversality condition. When an unconstrained optimum trajectory has been established then, no change in the control variables (\( \delta \beta \) or \( \delta \varphi \)) can result in an improvement, and therefore in this case \( B_1 = 0 \) and \( B_2 = 0 \) in Eq. 120.

Further detail on the effect of small perturbations on trajectories can be obtained from Ref. 12.

**PERTURBATION STUDY FOR TRAJECTORY IN CONSTANT GRAVITY FIELD**

Since the constant gravity field trajectory problem is amenable to closed form solution, it may be of interest to obtain the sensitivities of state variables at one time to state variables at another time by two different methods: first, from the closed form solutions, and second, from the \( \lambda \) matrix approach implied in Eq. 123.

Suppose we take a reference trajectory for the case of constant thrust \( \beta c = T = \text{const.} \), and constant \( \varphi = 45^\circ \), and integrate the equations of motion between \( t_a \) and \( t_b \) where these times lie within the powered portion of the trajectory; then:

\[
\begin{align*}
\dot{p} &= \frac{\sqrt{2}}{\mu} \frac{\beta c}{m} \\
\dot{q} &= \frac{\sqrt{2}}{\mu} \frac{\beta c}{m} - g \\
\dot{x} &= p \\
\dot{y} &= q \\
\dot{\theta} &= \beta.
\end{align*}
\]

(126)
These can be integrated to yield the state variables at \( t_b \) in terms of those at \( t_a \):

\[
\begin{align*}
\rho_b &= \rho_a + \frac{\sqrt{2}}{2} c \ln \frac{m_a}{m_b} \\
q_b &= q_a + \frac{\sqrt{2}}{2} c \ln \frac{m_a}{m_b} - g (t_b - t_a) \\
x_b &= x_a + p_a (t_b - t_a) + \frac{\sqrt{2}}{2} \frac{c}{\beta} \left[ m_b \ln \frac{m_b}{m_a} - (m_b - m_a) \right] \\
y_b &= y_a + q_a (t_b - t_a) + \frac{\sqrt{2}}{2} \frac{c}{\beta} \left[ m_b \ln \frac{m_b}{m_a} - (m_b - m_a) \right] \\
&\quad - \frac{1}{2} g (t_b - t_a)^2 \\
m_b &= m_a - \beta(t_b - t_a). 
\end{align*}
\]

By differentiation the following sensitivities can be obtained (care must be exercised in realizing that \( m_b \) is a function of \( m_a \) through the last relationship):

\[
\begin{array}{cccc}
\frac{\partial \rho_b}{\partial p_a} = 1 & \frac{\partial \rho_b}{\partial q_a} = 0 & \frac{\partial \rho_b}{\partial x_a} = 0 & \frac{\partial \rho_b}{\partial y_a} = 0 \\
\frac{\partial q_b}{\partial p_a} = 0 & \frac{\partial q_b}{\partial q_a} = 1 & \frac{\partial q_b}{\partial x_a} = 0 & \frac{\partial q_b}{\partial y_a} = 0 \\
\frac{\partial x_b}{\partial p_a} = t_b - t_a & \frac{\partial x_b}{\partial q_a} = 0 & \frac{\partial x_b}{\partial x_a} = 1 & \frac{\partial x_b}{\partial y_a} = 0 \\
\frac{\partial y_b}{\partial p_a} = 0 & \frac{\partial y_b}{\partial q_a} = t_b - t_a & \frac{\partial y_b}{\partial x_a} = 0 & \frac{\partial y_b}{\partial y_a} = 1 \\
\frac{\partial m_b}{\partial p_a} = 0 & \frac{\partial m_b}{\partial q_a} = 0 & \frac{\partial m_b}{\partial x_a} = 0 & \frac{\partial m_b}{\partial y_a} = 0 \\
\frac{\partial m_b}{\partial m_a} = 1 
\end{array}
\]
In the other approach, we solve Eqs. 118, and obtain:

\[ \lambda_1(t) = \lambda_{1b} + c_3 (t_b - t) \]

\[ \lambda_2(t) = \lambda_{2b} + c_4 (t_b - t) \]

\[ \lambda_3(t) = c_3 \]

\[ \lambda_4(t) = c_4 \]

\[ \lambda_5(t) = \lambda_{5b} + \frac{\sqrt{2}}{2} c \left\{ \left[ \lambda_{1b} + \lambda_{2b} - \frac{\lambda_{3b} + \lambda_{4b}}{\beta} \right] \left[ \frac{1}{m} - \frac{1}{m_b} \right] + \frac{\lambda_{3b} + \lambda_{4b}}{\beta} \ln \frac{m_b}{m} \right\} \] (129)

Making these solutions fit conditions at \( t_b \) such that \( \lambda_{1b} = 1 \),

\[ \lambda_{2b} = \lambda_{3b} = \lambda_{4b} = \lambda_{5b} = 0 \], yields:

\[ \lambda_1(t) = 1 \quad ; \quad \lambda_{1a} = 1 \]

\[ \lambda_2(t) = 0 \quad ; \quad \lambda_{2a} = 0 \] (130)

\[ \lambda_3(t) = 0 \quad ; \quad \lambda_{3a} = 0 \]

\[ \lambda_4(t) = 0 \quad ; \quad \lambda_{4a} = 0 \]

\[ \lambda_5(t) = \frac{\sqrt{2}}{2} c \left( \frac{1}{m} - \frac{1}{m_b} \right) \quad ; \quad \lambda_{5a} = \frac{\sqrt{2}}{2} c \left( \frac{1}{m_a} - \frac{1}{m_b} \right) \]

These values of \( \lambda_{1a} \ldots \lambda_{2a} \) are identical to the first row of sensitivities in Eqs. 128, which, as was pointed out above, they should be.

Repeating this process several times, for different end conditions, and tabulating the results:
\[ \begin{array}{c|cccccccccccc}
  \text{t}_b & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
  \text{t}_a & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
  \hline
  1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & R & 1 & 0 & 0 & 0 & R_a \\
  2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & R & 0 & 1 & 0 & 0 & R_a \\
  3 & 0 & 0 & 1 & 0 & 0 & (t_b - t) & 0 & 1 & 0 & S & (t_b - t_a) & 0 & 1 & 0 & S_a \\
  4 & 0 & 0 & 0 & 1 & 0 & 0 & (t_b - t) & 0 & 1 & S & 0 & (t_b - t_a) & 0 & 1 & S_a \\
  5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]

where

\[
\begin{align*}
R &= \frac{\sqrt{2}}{2} \, c \left( \frac{1}{m} - \frac{1}{m_b} \right); \\
R_a &= \frac{\sqrt{2}}{2} \, c \left( \frac{1}{m_a} - \frac{1}{m_b} \right) \\
S &= \frac{\sqrt{2}}{2} \, \frac{c}{\beta} \left( 1 - \frac{m_b}{m} + \ln \frac{m_b}{m} \right); \\
S_a &= \frac{\sqrt{2}}{2} \, \frac{c}{\beta} \left( 1 - \frac{m_b}{m_a} + \ln \frac{m_b}{m_a} \right). 
\end{align*}
\]

The matrix for \( t_a \) is identical with Eqs. 128. Notice that we have also obtained the \( \lambda \)'s as general functions of time.

The effect of variations in \( \beta \) and \( \phi \) on the state variables at \( t_b \) can also be determined, by evaluating \( R^k_1 \) and \( R^k_2 \) from 125 using the \( \lambda^k_1(t), \lambda^k_2(t), \lambda^k_5(t) \) of 131.

Solving for the \( B \) matrix in the manner indicated yields:

\[
\begin{bmatrix}
\frac{\sqrt{2}}{2} \, \frac{c}{m_b} & -\frac{\sqrt{2}}{2} \, \frac{\beta c}{m} \\
\frac{\sqrt{2}}{2} \, \frac{c}{m_b} & \frac{\sqrt{2}}{2} \, \frac{\beta c}{m} \\
-\frac{\sqrt{2}}{2} \, \frac{c}{\beta} \ln \frac{m_b}{m} & -\frac{\sqrt{2}}{2} \, c \left( 1 - \frac{m_b}{m} \right) \\
-\frac{\sqrt{2}}{2} \, \frac{c}{\beta} \ln \frac{m_b}{m} & \frac{\sqrt{2}}{2} \, c \left( 1 - \frac{m_b}{m_a} \right) \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta \beta \\
\delta \phi
\end{bmatrix}
\]

(132)
This matrix is generally a function of time, since $m$, $\delta \beta$, and $\delta \theta$ are time-varying. With $\delta \beta$, and $\delta \theta$ specified, the integration in (123) can be carried out. Thus (131) and (132) give us the $\lambda$ and $B$ matrices required for evaluating variations in state variables at $t_b$ due to variations in state variables at $t_a$, as well as variations in control variables from $t_a$ to $t_b$, as expressed in (123).
Appendix

ILLUSTRATION OF GREEN'S THEOREM APPROACH FOR
THE CASE OF THE VERTICAL TRAJECTORY

In Ref. 10, A. Miele proposes the use of a fundamental function \( \phi \) to be employed in combination with Green's theorem. This approach will be applied to the problem of the vertical trajectory.

It is desired to maximize apogee altitude \( r_f \):

\[
r_f = \int_0^{t_f} v \, dt.
\]

But since \( dm = -\beta \, dt \),

then

\[
r_f = \int_0^{t_f} \frac{v}{\beta} \, dm.
\]

From Eq. 103 it follows that

\[
\frac{dv}{dt} = -\beta \frac{dv}{dm} = \frac{3c}{m} - k_1 - \frac{k_2 \, v^2}{m}.
\]

Solving for \( \beta \):

\[
\beta = \frac{k_1 + k_2 \, v^2}{\frac{c}{m} + \frac{dv}{dm}}.
\]

Substituting this in the above expression for \( r_f \):

\[
r_f = \int_0^{t_f} \left[ \left( \frac{-dv}{k_1 m + k_2 \, v^2} \, dm + \frac{-mv}{k_1 m + k_2 \, v^2} \right) \, dv \right].
\]

This is of the form:

\[
r_f = \int \left[ \varphi(m,v) \, dm + \varphi(m,v) \, dv \right].
\]
The fundamental function \( w \) is defined as

\[
\omega = \frac{\partial y}{\partial m} - \frac{\partial v}{\partial v}.
\]

Then \( \omega \) becomes:

\[
\omega = \frac{\partial}{\partial m} \left( \frac{-\xi v}{k_1 m + k_2 v^2} \right) - \frac{\partial}{\partial v} \left( \frac{-cv}{k_1 m + k_2 v^2} \right)
= \frac{c \left[ k_1 m - k_2 v^2 \left( 1 + \frac{y}{c} \right) \right]}{(k_1 m + k_2 v^2)^2}
\]

It will be seen that when \( \omega = 0, \ m = \frac{k_2}{k_1} v^2 \left( 1 + \frac{y}{c} \right) \)

which is identical to Eq. 114. The behavior of the switching function \( Q \) is closely related to that of the fundamental function \( \omega \). When \( Q = \dot{Q} = 0, \) then \( \omega = 0.\)
REFERENCES


