MEMORANDUM
RM-3860-PR
JANUARY 1964

TWO CHARGED SPHERICAL CONDUCTORS IN A UNIFORM ELECTRIC FIELD: FORCES AND FIELD STRENGTH

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

This Memorandum extends and improves the treatment of the two-sphere electrostatic problem presented in the earlier publication, RM-2607-1-PR. It represents a stage in RAND's continuing research into the basic mechanisms of cloud-droplet interaction; this field of research has heretofore lacked a tool for the correct treatment of the effect of electrostatic fields on droplet pairs. New material includes the computation of the field strength between the spheres, and the series rearrangements used for efficient computation of the force coefficients. It should be noted that the nomenclature differs from that of the previous publication and that e.s.u. rather than MKS units are used. This report will be of interest to scientists engaged in research in precipitation and colloid physics.

SUMMARY

A complete solution is presented to the electrostatic boundary value problem of two charged conducting spheres in a uniform electric field. Rapidly convergent expressions are given for the forces acting on the spheres, and for the maximum field strength between them. Numerical results are presented for a number of relative sphere sizes and separations.

CONTENTS

LIST OF SYMBOLS

a	bispherical scale factor, see Fig. 1
A_n, B_n, G_n, H_n	coefficients in potential function, Eq. (15)
C_{11}, C_{12}, C_{22}	coefficients of capacitance
$\underset{\sim}{E}_o$	strength of impressed electric field
E_A	field strength at point "A" (see Fig. 1)
E_1, E_2, E_3	field strength coefficients, see Eq. (31)
f_k	see Eq. (43), Table 3
$F_z(2), F_x(2)$	force components on sphere 2
$F_1, \ldots, F_{10}$	force coefficients, see Eqs. (39), (40), (43)
K_{jk}	see Eq. (43)
P_n, P_n^1	Legendre polynomials, standard notation
p_{11}, p_{12}, p_{22}	see Eqs. (34)
P_{11}, P_{12}, P_{22}	coefficients of induction
Q_1, Q_2	total charges on the spheres
Q_1^*, Q_2^*	charges induced on the spheres if they were grounded
R_1, R_2	radii of the spheres
s	separation of the spheres
S_m, S'_m, T_m, U_m	series defined by Eqs. (20), (33), (41), (42)
v_1, v_2	defined by Eqs. (34)
V_1, V_2	potentials of the two spheres
Y_n, Z_n	coefficients in Eq. (27)

α	$= e^{2\mu_1}$ in Table 4
$\beta_j, \varkappa_j$	defined by Eq. (43), see Table 2
γ	$= \frac{1}{2}(e^{2\mu_2} + 1)$ in Table 4
ϵ	specific inductive capacity of the medium
θ	polar angle on sphere 2
$\lambda_\nu, \lambda'_\nu$	defined by Eqs. (48), (52)
μ, η, φ	bispherical coordinates
μ_1, μ_2	the two spherical surfaces, see Fig. 1
ψ	angle between E_0 and the z axis

I. INTRODUCTION

The problem of the electrostatics of two conducting spheres has a long history. It was discussed by Poisson (1) in 1811 and later by Lord Kelvin (2), A. Russell (3), and others, and is treated in a number of texts including Maxwell (4), Morse and Feshbach (5), and Buchholz (6). The earliest work was apparently motivated by a desire to check experimentally the newly developed theory of electrostatics. An advantage of the two-sphere geometry for this purpose was that it could be modeled in the laboratory with a high degree of accuracy. Recent interest centers on the influence of electrostatic forces on the motion of particles in colloidal suspension and on the effect of electric fields and charges on the coalescence of water droplets as, for instance, in warm cloud precipitation. Noteworthy in recent research is the work of S. Mason on solid particles and liquid bubbles (7,8), and of Sartor on cloud droplets (9).

The electrostatic problem can be formulated as follows. Given two conducting spheres of specified and (in general) unequal radii, separated by a specified distance and bearing given charges, in an impressed electric field that is uniform at a great distance from the conductors: find the field strength at the surfaces of the conductors and the forces acting on the conductors. The resulting expressions should permit rapid and accurate computation even for minute separations. Although a number of useful results are given by Buchholz and others, the complete solution to the problem as stated has, apparently, never heretofore been worked out.

The present study was motivated by the need in cloud-physics problems for a correct treatment of the trajectories of cloud and rain droplets under the influence of both hydrodynamic and electrostatic forces. Water droplets are here approximated in their electrostatic properties by conducting spheres. It should be possible to combine the results with hydrodynamic equations such as those given by Hocking (10). Then, by numerical integration, the relative trajectories of interacting droplets can be studied as a function of their charges and the ambient electric-field strength. The field intensification between the droplets, a by-product of this analysis, may be useful in investigations of droplet coalescence and charge transfer between droplets, an important factor in cloud electrification as pointed out by Sartor (11).

In the following sections the two-sphere boundary-value problem is solved and expressions are derived for the field strength and the force components. Numerical results are presented for a number of relative sphere sizes and separations. Details of the computational procedure are discussed in the Appendixes. Electrostatic units are used throughout.

This Memorandum, besides presenting new material, repeats in essence most of the discussion of the previous publication, Davis (13). However, it should be noted that e.s.u. are used here rather than MKS units. There are also a number of changes in nomenclature, as will be noted explicitly on page 20.

II. THE BISPHERICAL COORDINATE SYSTEM

The boundary value problem of two spherical conductors external to each other can be treated in many ways, including infinite series of images discussed by Smythe (12) and spherical harmonics centered on each sphere in turn as in Maxwell (4). However, analysis of the problem is facilitated by use of the bispherical coordinate system described by Morse and Feshbach (5), whose symbolism we will use. The coordinate transformation between this system and Cartesian coordinates is

$$z = \frac{a \sinh\mu}{\cosh\mu - \cos\eta} , \quad x = \frac{a \sin\eta \cos\phi}{\cosh\mu - \cos\eta} , \quad y = \frac{a \sin\eta \sin\phi}{\cosh\mu - \cos\eta} . \tag{1}$$

Using relations given by Morse and Feshbach (hereinafter called "M-F"), it is not difficult to show that the two spheres illustrated in Fig. 1 are characterized by

$$\mu = \mu_1 , \quad \mu = -\mu_2 ,$$

with

$$\mu_1 = \ln\left(\frac{D_1 + a}{R_1}\right) , \quad \mu_2 = \ln\left(\frac{D_2 + a}{R_2}\right) , \tag{2}$$

where

$$a = (D_1^2 - R_1^2)^{\frac{1}{2}} = (D_2^2 - R_2^2)^{\frac{1}{2}} . \tag{3}$$

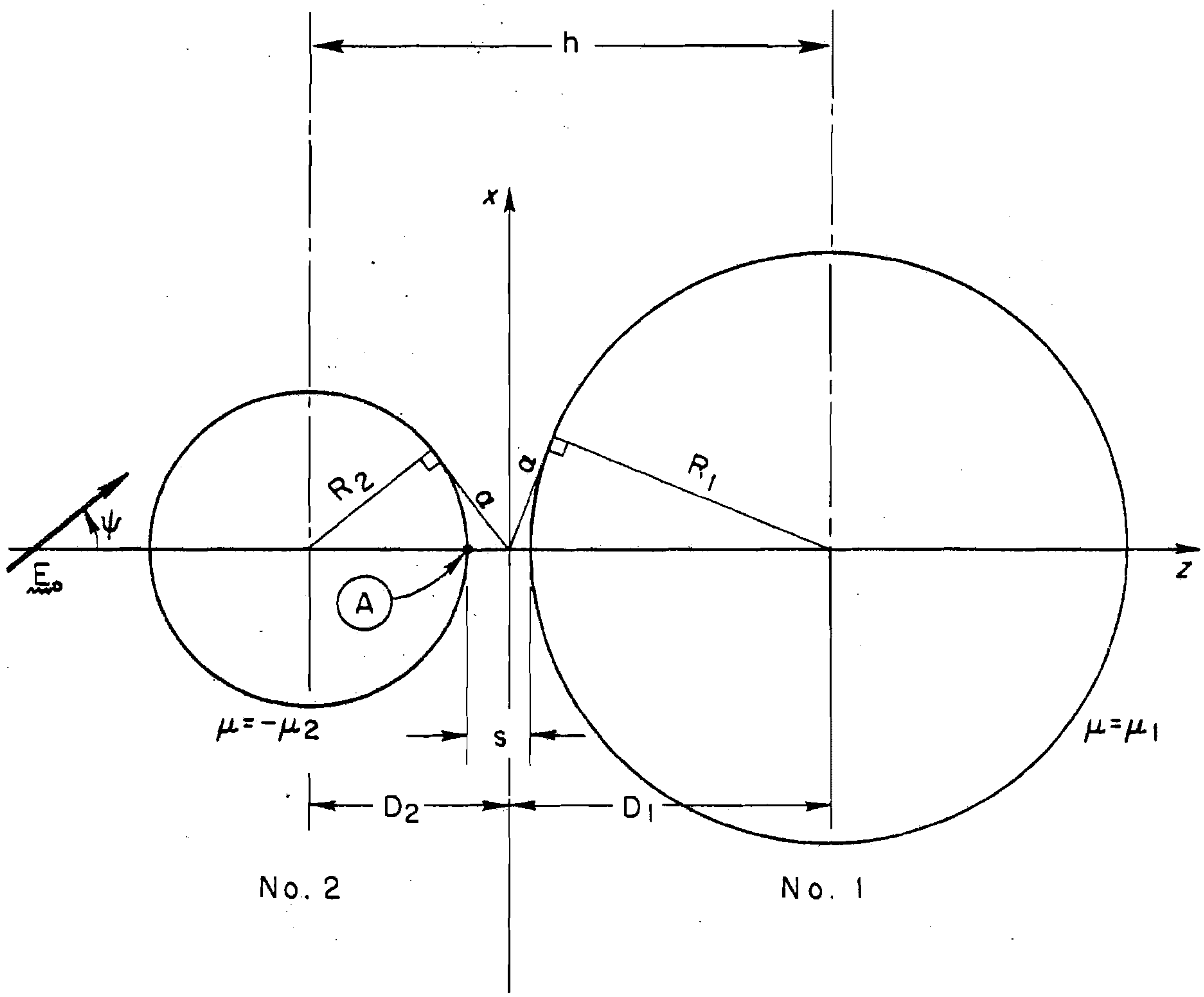

Fig. 1 The Geometry of the Problem

D_1, D_2, the distances of the centers from the origin, are given by the equations:

$$D_1 = \frac{1}{2h}\left(h^2 + R_1^2 - R_2^2\right) \quad , \tag{4}$$

$$D_2 = \frac{1}{2h}\left(h^2 + R_2^2 - R_1^2\right) \quad ,$$

where h is the distance between the centers,

$$h = R_1 + R_2 + s = D_1 + D_2 \quad .$$

The parameter a determines the linear scale.

Expressions can also be readily derived for the element of area on one of the spheres and for the sine and cosine of the polar angle. On sphere number 2,

$$dS = \frac{a^2 \sin\eta \, d\eta \, d\phi}{(\cosh\mu_2 - \cos\eta)^2} \quad , \tag{5}$$

$$\cos\theta = \frac{1 - \cosh\mu_2 \cos\eta}{\cosh\mu_2 - \cos\eta} \quad , \tag{6}$$

$$\sin\theta = \frac{\sinh\mu_2 \sin\eta}{\cosh\mu_2 - \cos\eta} \quad . \tag{7}$$

Typical solutions to Laplace's equation are

$$(\cosh\mu - \cos\eta)^{\frac{1}{2}} \, e^{\pm(n+\frac{1}{2})\mu} \, P_n^m(\cos\eta) \cos m\phi \, . \tag{8}$$

Useful expansions can be derived from the generating function for Legendre polynomials

$$(1-2u\xi+\xi^2)^{-\frac{1}{2}} = \sum_0^\infty \xi^n P_n(u) \quad . \tag{9}$$

Substituting $\xi = e^{-\mu}$, $u = \cos\eta$, this expression becomes

$$(\cosh\mu - \cos\eta)^{-\frac{1}{2}} = \sqrt{2}\sum_0^\infty e^{-(n+\frac{1}{2})|\mu|} P_n(\cos\eta) . \tag{10}$$

Then by differentiation first with respect to μ and then η, comparison with Eq. (1) gives:

$$|z| = a\sqrt{2}\,(\cosh\mu - \cos\eta)^{\frac{1}{2}}\sum_0^\infty (2n+1)\,e^{-(n+\frac{1}{2})|\mu|} P_n(\cos\eta) , \tag{11}$$

$$x = 2a\sqrt{2}\,(\cosh\mu - \cos\eta)^{\frac{1}{2}}\sum_1^\infty e^{-(n+\frac{1}{2})|\mu|} P_n^1(\cos\eta)\cos\phi . \tag{12}$$

III. METHOD

The first task is to derive the potential function, Φ, that satisfies the boundary conditions of the problem. The force acting on either sphere is then computed by integrating the electrical stress over its surface:

$$\underset{\sim}{F} \cdot \underset{\sim}{p} = \frac{\epsilon}{8\pi} \int_S \left(\frac{\partial \Phi}{\partial n}\right)^2 \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS \quad , \tag{13}$$

where $\underset{\sim}{p}$ is an arbitrary unit vector, $\underset{\sim}{n}$ is the unit vector normal to the surface element dS, and ϵ is the specific inductive capacity of the medium.

We need compute the force acting on only one of the spheres, say number $\underline{2}$, since the force on number $\underline{1}$, $\underset{\sim}{F}(1)$, is necessarily related to the force on $\underline{2}$, $\underset{\sim}{F}(2)$, by the equation

$$\underset{\sim}{F}(1) = \underset{\sim}{E}_0 (Q_1 + Q_2) - \underset{\sim}{F}(2) \quad , \tag{14}$$

where Q_1, Q_2 are the total charges on the two spheres and $\underset{\sim}{E}_0$ is the impressed field.

IV. SOLUTION TO THE BOUNDARY VALUE PROBLEM

The boundary conditions are: (1) at great distances from the spheres the potential function must correspond to that of a uniform field, and (2) the two spheres are equipotential surfaces and carry the specified total charges Q_1, Q_2. Using the general solutions to Laplace's equation in bispherical coordinates given by M-F, it is apparent from symmetry considerations that the potential function for the present problem must have the form

$$\Phi = (\cosh\mu - \cos\eta)^{\frac{1}{2}} \Big\{ \sum_0^\infty \left(A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu}\right) P_n(\cos\eta) + \cos\phi \sum_1^\infty \left(G_n e^{(n+\frac{1}{2})\mu} + H_n e^{-(n+\frac{1}{2})\mu}\right) P_n^1(\cos\eta) \Big\} - E_0 (z\cos\psi + x\sin\psi) , \qquad (15)$$

where $E_0 = |\underset{\sim}{E}_0|$.

The part representing the uniform field has been split off; the remainder represents the potential arising from the surface charge distributions on the two spheres.

The boundary value problem will be solved by superposition of the solutions to several special cases: (a) grounded spheres in a uniform field in the $\underline{z}$ direction; (b) insulated spheres (no impressed field) carrying charges that are equal in magnitude to the charges induced on the conductors in case (a) but of opposite sign; (c) insulated spheres (no impressed field) carrying the specified charges Q_1, Q_2; and (d) grounded spheres in a uniform field in the $\underline{x}$ direction. The result is easily obtained following the methods of M-F and making use of the expansions given in Eqs. (10), (11), and (12). The coefficients in Eq. (15) are

$$
\left.
\begin{aligned}
A_n &= \frac{\sqrt{2}\, a E_0 \cos\psi (2n+1)\left[e^{(2n+1)\mu_2}+1\right] + \sqrt{2}\left[V_1 e^{(2n+1)\mu_2} - V_2\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1}, \\
B_n &= \frac{-\sqrt{2}\, a E_0 \cos\psi (2n+1)\left[e^{(2n+1)\mu_1}+1\right] + \sqrt{2}\left[V_2 e^{(2n+1)\mu_1} - V_1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1}, \\
G_n &= \frac{2\sqrt{2}\, a E_0 \sin\psi \left[e^{(2n+1)\mu_2} + 1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1}, \\
H_n &= \frac{2\sqrt{2}\, a E_0 \sin\psi \left[e^{(2n+1)\mu_1} + 1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1},
\end{aligned}
\right\} \qquad (16)
$$

where V_1, V_2 are the potentials of the spheres. We must now relate V_1, V_2 to the impressed field $\underset{\sim}{E}_0$ and the specified charges Q_1, Q_2. To do this, we will first compute the total charges that would be induced on the two spheres if they were grounded; then we will derive the relation between the total charges and potentials of two insulated spheres with no impressed field. If such insulated spheres carry charges that are equal in magnitude and of opposite sign to those induced on similar grounded spheres, their potentials will correspond, by the principle of superposition, to those of two uncharged insulated spheres in the impressed field. These potentials together with the potentials of two insulated spheres with the specified charges Q_1, Q_2 (no impressed field) give, finally, the desired potentials V_1, V_2.

The normal derivative at the surface of sphere 2 is

$$\frac{\partial}{\partial n} = -\frac{1}{a}\left(\cosh\mu_2 - \cos\eta\right)\frac{\partial}{\partial\mu} . \tag{17}$$

By applying this operator to Φ and making use of expansions (11) and (11), it is possible to show that the normal derivative of Φ evaluated on sphere 2 is given by

$$\left.\frac{\partial\Phi}{\partial n}\right]_{\mu=-\mu_2} = -\frac{1}{a}\left(\cosh\mu_2 - \cos\eta\right)^{\frac{3}{2}}\left\{\sum_0^\infty (2n+1)\, e^{(n+\frac{1}{2})\mu_2} B_n P_n(\cos\eta) + \cos\phi \sum_1^\infty (2n+1)\, e^{(n+\frac{1}{2})\mu_2} H_n P_n^1(\cos\eta)\right\} . \tag{18}$$

And, similarly, on sphere 1,

$$\left.\frac{\partial\Phi}{\partial n}\right]_{\mu=\mu_1} = -\frac{1}{a}\left(\cosh\mu_1 - \cos\eta\right)^{\frac{3}{2}}\left\{\sum_0^\infty (2n+1)\, e^{(n+\frac{1}{2})\mu_1} A_n P_n(\cos\eta) + \cos\phi \sum_1^\infty (2n+1)\, e^{(n+\frac{1}{2})\mu_1} G_n P_n^1(\cos\eta)\right\} . \tag{19}$$

Now define

$$S_m(\xi) = \sum_{n=0}^\infty \frac{(2n+1)^m e^{(2n+1)\xi}}{e^{(2n+1)(\mu_1+\mu_2)} - 1} , \tag{20}$$

$$m = 0, 1, 2, 3 .$$

Many results will be expressed in terms of series of this form; a summation method is discussed in Appendix A. In every case we will consider, $\xi < (\mu_1+\mu_2)$, so the series converge.

By setting both V_1 and V_2 equal to zero in Eq. (16), we obtain the potential function for two grounded spheres in the field $\underset{\sim}{E}_o$. The charge on each sphere that would be induced by the field is found by integrating the normal derivative of the potential over the surface. Expressions for the charges are conveniently expressed in terms of S_m:

$$\left.\begin{aligned} Q_1^* &= 2\epsilon E_o a^2 \cos\psi \left[S_1(\mu_2) + S_1(0)\right] \\ Q_2^* &= -2\epsilon E_o a^2 \cos\psi \left[S_1(\mu_1) + S_1(0)\right] . \end{aligned}\right\} \qquad (21)$$

(The asterisks signify that these charges refer to the "grounded case.")

By setting $\underset{\sim}{E}_o$ equal to zero, the potential function for two insulated, freely charged spheres at potentials V_1', V_2' is obtained from Eq. (16). The total charge on each sphere in this case can be calculated by integrating the normal derivative of the potential function over the surface as before. It is then found that

$$\left.\begin{aligned} Q_1' &= C_{11} V_1' + C_{12} V_2' \\ Q_2' &= C_{12} V_1' + C_{22} V_2' \end{aligned}\right\} \qquad (22)$$

(The primes show that these charges and potentials are for the "freely charged case.") C_{11}, C_{12}, C_{22}, the coefficients of capacitance, are given by the expressions:

$$\left.\begin{aligned} C_{11} &= 2\epsilon a\, S_a(\mu_2) \quad , \\ C_{12} &= -2\epsilon a\, S_a(0) \quad , \\ C_{22} &= 2\epsilon a\, S_o(\mu_1) \quad . \end{aligned}\right\} \tag{23}$$

When Eqs. (22) are solved to give V_1', V_2' in terms of the given charges, it is found that

$$\left.\begin{aligned} V_1' &= P_{11} Q_1' + P_{12} Q_2' \\ V_2' &= P_{12} Q_1' + P_{22} Q_2' \ . \end{aligned}\right\} \tag{24}$$

where

$$P_{11} = C_{22}/\Delta \ , \ P_{12} = -C_{12}/\Delta \ , \ P_{22} = C_{11}/\Delta \ , \tag{25}$$

and

$$\Delta = C_{11}C_{22} - C_{12}^2 \quad .$$

The potentials V_1, V_2 which appear in Eqs. (16) can now be written in terms of the quantities just computed

$$\left.\begin{aligned} V_1 &= P_{11}(Q_1 - Q_1^*) + P_{12}(Q_2 - Q_2^*) \\ V_2 &= P_{12}(Q_1 - Q_1^*) + P_{22}(Q_2 - Q_2^*) \end{aligned}\right\} \tag{26}$$

The normal derivative of Φ on sphere $\underline{2}$ is now obtained by rewriting Eq. (18), using the above results:

$$\left.\frac{\partial\Phi}{\partial n}\right]_{\mu=-\mu_2} = -E_0(\cosh\mu_2-\cos\eta)^{\frac{3}{2}}\left[\cos\psi\sum_0^\infty Y_n P_n(\cos\eta) + \cos\phi\sin\psi\sum_1^\infty Z_n P_n^1(\cos\eta)\right] \quad , \tag{27}$$

where

$$Y_n = -\sqrt{2}(2n+1)e^{(n+\frac{1}{2})\mu_2}\left[\frac{(2n+1)\left(e^{(2n+1)\mu_1}+1\right)-w_2 e^{(2n+1)\mu_1}+w_1}{e^{(2n+1)(\mu_1+\mu_2)}-1}\right] \quad , \tag{28}$$

$$Z_n = 2\sqrt{2}(2n+1)e^{(n+\frac{1}{2})\mu_2}\left[\frac{e^{(2n+1)\mu_1}-1}{e^{(2n+1)(\mu_1+\mu_2)}-1}\right] \quad , \tag{29}$$

and

$$w_1 = \frac{1}{E_0 a\cos\psi}\left[P_{11}(Q_1-Q_1^*)+P_{12}(Q_2-Q_2^*)\right] \quad ,$$

$$w_2 = \frac{1}{E_0 a\cos\psi}\left[P_{12}(Q_1-Q_1^*)+P_{22}(Q_2-Q_2^*)\right] \quad .$$

(It should be noted that w_1, w_2 are dimensionless.)

V. THE FIELD INTENSIFICATION

The maximum field strength between the spheres is at point "A" (see Fig. 1). It is computed from the normal derivative:

$$E_A = - \frac{\partial \Phi}{\partial n}\Big]_{\substack{\mu = -\mu_2 \\ \eta = \pi}} .$$

Therefore, from Eq. (27),

$$E_A = E_0 \cos\psi (\cosh\mu_2 + 1)^{\frac{3}{2}} \sum_0^\infty (-1)^n Y_n . \tag{30}$$

Terms containing the impressed field and the total charges separate readily and we can write

$$E_A = \frac{1}{\epsilon R_2^2}\left(E_1 Q_1 + E_2 Q_2\right) + E_3 E_0 \cos\psi , \tag{31}$$

where, using Eq. (28),

$$\left.\begin{aligned} E_1 &= \sqrt{2}\left(\frac{R_2}{a}\right)^2 (\cosh\mu_2 + 1)^{\frac{3}{2}}\left[p_{12} S_1'(\mu_1 + \tfrac{\mu_2}{2}) - p_{11} S_1'(\tfrac{\mu_2}{2})\right] \\ E_2 &= \sqrt{2}\left(\frac{R_2}{a}\right)^2 (\cosh\mu_2 + 1)^{\frac{3}{2}}\left[p_{22} S_1'(\mu_1 + \tfrac{\mu_2}{2}) - p_{12} S_1'(\tfrac{\mu_2}{2})\right] \\ E_3 &= \sqrt{2}(\cosh\mu_2 + 1)^{\frac{3}{2}}\left[v_2 S_1'(\mu_1 + \tfrac{\mu_2}{2}) - S_2'(\mu_1 + \tfrac{\mu_2}{2}) - v_1 S_1'(\tfrac{\mu_2}{2}) - S_2'(\tfrac{\mu_2}{2})\right] . \end{aligned}\right\} \tag{32}$$

These quantities are tabulated in Table 1 for a number of relative sizes and separations.

The S'_m are alternating series closely related to S_m,

$$S'_m(\xi) = \sum_{n=0}^{\infty} \frac{(2n+1)^m e^{(2n+1)\xi} (-1)^n}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \quad , \quad m = 1, 2. \tag{33}$$

The summation of these series is discussed in Appendix A. The quantities p_{11}, p_{12}, p_{22}, v_1, v_2 in Eq. (32) are dimensionless and are defined as follows:

$$\left.\begin{aligned} &p_{11} = a P_{11} \quad , \quad p_{12} = a P_{12} \quad , \quad p_{22} = a P_{22} \quad ; \\ &v_1 = -\frac{1}{E_0 a \cos\psi}\left(P_{11} Q_1^* + P_{12} Q_2^*\right) , \\ &v_2 = -\frac{1}{E_0 a \cos\psi}\left(P_{12} Q_1^* + P_{22} Q_2^*\right) . \end{aligned}\right\} \tag{34}$$

VI. THE FORCE COMPONENTS

The force components are calculated by means of the surface integral (13), letting $\underset{\sim}{p}$ be a unit vector first in the z direction, then in the x direction. Substituting from Eq. (27), using expressions for dS, cos θ, and sin θ from Eqs. (5), (6), and (7), and omitting terms which integrate to zero, the z and x components of the force on sphere 2 may be written:

$$F_z(2) = \epsilon \frac{a^2 E_0^2}{4} \left[\cos^2\psi \int_{-1}^{1} (1 - u\cosh\mu_2) \left(\sum_0^\infty Y_n P_n(u) \right)^2 du \right.$$
$$\left. + \frac{1}{2} \sin^2\psi \int_{-1}^{1} (1 - u\cosh\mu_2) \left(\sum_1^\infty Z_n P_n^1(u) \right)^2 du \right] , \qquad (35)$$

$$F_x(2) = \epsilon \frac{a^2 E_0^2}{8} \sin 2\psi \sinh\mu_2 \int_{-1}^{1} \left(\sum_0^\infty Y_n P_n(u) \right) \left(\sum_1^\infty Z_n P_n^1(u) \right) (1-u^2)^{\frac{1}{2}} du , \qquad (36)$$

where u = cos η.

These integrals are evaluated using well-known integral properties and recursion relations for Legendre polynomials. After a considerable amount of algebraic reduction, the result can be written:

$$F_z(2) = \epsilon \frac{a^2 E_0^2}{4} \left\{ 2\cos^2\psi \sum_0^\infty \frac{Y_n}{(2n+1)} \left[Y_n - 2(\cosh\mu_2)\left(\frac{n+1}{2n+3}\right) Y_{n+1} \right] \right.$$
$$\left. + \sin^2\psi \sum_0^\infty \frac{n(n+1)}{(2n+1)} Z_n \left[Z_n - 2(\cosh\mu_2)\left(\frac{n+2}{2n+3}\right) Z_{n+1} \right] \right\} \qquad (37)$$

$$F_x(2) = \epsilon \frac{a^2 E_0^2}{4} \sin 2\psi \sinh \mu_2 \sum_0^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} \left[(n+2) Z_{n+1} Y_n - n Z_n Y_{n+1} \right] , \qquad (38)$$

where Y_n, Z_n are given by Eqs. (28), (29).

When the dependence on $\underset{\sim}{E_0}$, Q_1, and Q_2, implicit in Eqs. (37) and (38), is made explicit, it is found that the force components can be written:

$$\begin{aligned} F_z(2) = \Big\{ & \epsilon R_2^2 E_0^2 \left(F_1 \cos^2\psi + F_2 \sin^2\psi \right) + E_0 \cos\psi \left(F_3 Q_1 + F_4 Q_2 \right) \\ & + \frac{1}{\epsilon R_2^2} \left(F_5 Q_1^2 + F_6 Q_1 Q_2 + F_7 Q_2^2 \right) \Big\} + E_0 Q_2 \cos\psi , \end{aligned} \qquad (39)$$

$$\begin{aligned} F_x(2) = \Big\{ & \epsilon R_2^2 E_0^2 F_8 \sin 2\psi + E_0 \sin\psi \left(F_9 Q_1 + F_{10} Q_2 \right) \Big\} \\ & + E_0 Q_2 \sin\psi , \end{aligned} \qquad (40)$$

where the coefficients F_1, F_2, ..., F_{10} are complicated series expressions.*

The force coefficients can be expressed conveniently in terms of two types of series related to the S_m series already defined:

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*The correspondence between the F_j and the force coefficients defined in Davis (13) is given on page 20.

$$T_m(\xi) = \sum_{n=0}^{\infty} \frac{(2n+1)^m e^{(2n+1)\xi}}{\left(e^{(2n+1)(\mu_1+\mu_2)}-1\right)^2} , \qquad (41)$$

$$U_m(\xi) = \sum_{n=0}^{\infty} \frac{(2n+1)^m e^{(2n+1)\xi}}{\left(e^{(2n+1)(\mu_1+\mu_2)}-1\right)\left(e^{(2n+3)(\mu_1+\mu_2)}-1\right)} , \qquad (42)$$

$m = 0, 1, 2, 3.$

These series converge, since the condition $\xi < 2(\mu_1+\mu_2)$ holds in every case where they are used. A method for summing $T_m(\xi)$ and $U_m(\xi)$ is discussed in Appendix A.

The force coefficients are written as sums of series of these types:

$$F_j = \varkappa_j \sum_{k=1}^{24} K_{jk} f_k + \beta_j \qquad j = 1, \ldots, 10 , \qquad (43)$$

where f_k are series of type T_m or U_m, the quantities K_{jk} are relatively simple polynomial expressions, $\varkappa_j$ are simple factors, and β_j are either 0 or−1. These quantities are given explicitly in tables in Appendix B.

The coefficients F_j depend only on the <u>relative</u> geometry of the problem, i.e., on R_1/R_2 and s/R_2. Other specified parameters such as the linear scale, the applied field $\underset{\sim}{E}_0$, and the net charges Q_1, Q_2, have been separated out. Once the force coefficients have been calculated, they can be applied to a variety of physical situations.

In Eqs. (39), (40) the terms proportional to E_o^2 result from the action of the field on the induced charge distribution. If the spheres carried no net charges, these would be the only terms in the expressions for the force components. A field impressed on uncharged spheres gives rise to a force of attraction together with a force tending to align them with the field. On the other hand, if the spheres carry net charges but there is no impressed field, only the quadratic term in the charges remains. If the spheres are charged <u>and</u> a field is impressed, there are other mixed terms peculiar to this solution in addition to the terms that express the direct action of the field on the charge carried by each sphere individually.

VII. NUMERICAL RESULTS

The expressions for the force and field coefficients (32) and (43) are relatively easy to program for computation by an electronic digital computer. The heart of the program is a routine to evaluate the sums S_m, T_m, U_m and S'_m, as described in Appendix A. Representative results, computed on the IBM 7090 are presented in Table 1. The maximum error is believed to be less than 0.5 per cent.

With the coefficients F_j from Table 1 the force components on each of the spheres can be computed in an actual case by using Eqs. (39), (40) together with (14). The coefficients E_1, E_2, E_3 permit computation of the field strength at point "A" on sphere 2 by means of Eq. (31). Note that the results presented in Table 1 are consistent with lengths in centimeters, charges in electrostatic units, field strength in statvolts per centimeter, and force components in dynes. If MKS units are preferred, divide F_1, F_2, F_8 by 9×10^9; multiply F_5, F_6, F_7, E_1, E_2 by the same factor. Then the results will be consistent with lengths in meters, charges in coulombs, field strength in volts per meter, and force components in newtons.

The coefficients F_j correspond to those defined in Davis (13) as follows (in each pair, the first refers to the present publication, the second to the previous publication):

(F_1, F_1), $(F_2, -F_2)$, $(F_3, -B)$, (F_4, C), (F_5, D), $(F_6, -F)$, (F_7, G), $(F_8, -F_3)$, (F_9, H), $(F_{10}, -K)$.

TABLE 1 $(R_1 = R_2)$

s/R_2	F_1	F_2	F_3	F_4	F_5
10.0	0.00030	-0.00014	-0.00116	0.00116	8.1×10^{-6}
1.0	.0927	-0.0330	-0.0951	0.0951	0.0101
0.1	1.400	-0.0945	-0.9741	0.9741	0.1533
0.01	9.554	-0.1032	-5.994	5.994	0.9263
0.001	59.49	-0.1041	-36.39	36.39	5.552

s/R_2	F_6	F_7	F_8	F_9	F_{10}
10.0	-0.00694	8.1×10^{-6}	-0.00014	0.00058	-0.00058
1.0	-0.1146	0.0101	-0.0391	0.0407	-0.0407
0.1	-0.4532	0.1533	-0.2248	0.1736	-0.1736
0.01	-2.005	0.9263	-0.3756	0.2707	-0.2707
0.001	-11.26	5.552	-0.4652	0.3260	-0.3260

s/R_2	E_1	E_2	E_3
10.0	-0.0241	1.000	3.004
1.0	-0.6765	1.094	3.718
0.1	-4.188	4.188	14.17
0.01	-28.03	28.03	92.48
0.001	-211.7	211.7	696.7

TABLE 1 (continued)

($R_1 = 2R_2$)

s/R_2	F_1	F_2	F_3	F_4	F_5
10.0	0.00169	-0.00084	-0.00093	0.00730	5.4×10^{-6}
1.0	0.2758	-0.0759	-0.0576	0.3063	0.00227
0.1	3.136	-0.1436	-0.5370	2.128	0.0224
0.01	20.59	-0.1505	-3.356	11.97	0.1364
0.001	127.70	-0.1511	-20.59	71.46	0.8299

s/R_2	F_6	F_7	F_8	F_9	F_{10}
10.0	-0.00592	4.5×10^{-5}	-0.00084	0.00046	-0.00364
1.0	-0.0662	0.0255	-0.1033	0.0203	-0.1330
0.1	-0.2322	0.2977	-0.4048	0.0706	-0.3717
0.01	-1.029	1.685	-0.6202	0.1058	-0.5066
0.001	-5.816	9.946	-0.7478	0.1264	-0.5793

s/R_2	E_1	E_2	E_3
10.0	-0.0203	1.000	3.027
1.0	-0.3147	1.184	4.741
0.1	-1.445	4.866	18.16
0.01	-9.420	32.40	117.2
0.001	-71.05	244.9	883.4

TABLE 1 (continued)

($R_1 = 5R_2$)

s/R_2	F_1	F_2	F_3	F_4	F_5
10.0	0.0122	-0.00554	-0.00056	0.0611	1.9×10^{-6}
1.0	0.6422	-0.0927	-0.0191	0.8537	0.00014
0.1	5.310	-0.1033	-0.1436	3.617	0.00097
0.01	34.63	-0.1023	-0.9203	18.17	0.00611
0.001	213.0	-0.1022	-5.641	105.0	0.0373

s/R_2	F_6	F_7	F_8	F_9	F_{10}
10.0	-0.00391	0.00028	-0.00581	0.00026	-0.0305
1.0	-0.0220	0.0494	-0.1915	0.00525	-0.3765
0.1	-0.0603	0.4249	-0.4606	0.0124	-0.6470
0.01	-0.2532	2.232	-0.6369	0.0171	-0.7555
0.001	-1.402	12.80	-0.7431	0.0199	-0.8098

s/R_2	E_1	E_2	E_3
10.0	-0.0130	1.000	3.215
1.0	-0.0832	1.281	6.556
0.1	-0.2909	5.132	22.03
0.01	-1.819	33.25	137.5
0.001	-13.53	248.3	1022.7

TABLE 1 (continued)

($R_1 = 10R_2$)

s/R_2	F_1	F_2	F_3	F_4	F_5
10.0	0.0376	-0.0137	-0.00033	0.2162	4.9×10^{-7}
1.0	0.7689	-0.0579	-0.00546	1.322	9.6×10^{-6}
0.1	6.127	-0.0483	-0.0411	4.336	6.9×10^{-5}
0.01	41.38	-0.0465	-0.2757	20.60	0.00046
0.001	254.8	-0.0464	-1.696	116.9	0.00282

s/R_2	F_6	F_7	F_8	F_9	F_{10}
10.0	-0.00227	0.00073	-0.0163	0.00013	-0.1080
1.0	-0.00749	0.0603	-0.1930	0.00129	-0.5893
0.1	-0.0177	0.4586	-0.3535	0.00236	-0.7965
0.01	-0.0718	2.337	-0.4648	0.00310	-0.8657
0.001	-0.3924	13.22	-0.5338	0.00356	-0.8990

s/R_2	E_1	E_2	E_3
10.0	-0.00738	1.002	3.733
1.0	-0.0255	1.312	7.753
0.1	-0.0790	5.105	23.75
0.01	-0.483	32.59	145.4
0.001	-3.569	241.6	1073.0

Appendix A

SERIES SUMMATION

In this appendix we will develop a method for rapid summation of the series $S_m(\xi)$, $S'_m(\xi)$, $T_m(\xi)$, $U_m(\xi)$, defined by Eqs. (20), (33), (41), (42). Consider first $S_o(\xi)$. If k is chosen so that

$$e^{-(2k+1)(\mu_1+\mu_2)} \leq \delta \quad , \tag{44}$$

where δ is a small positive number less than unity (say 0.1), $S_o(\xi)$ can be written

$$S_o(\xi) = \sum_{n=0}^{k-1} \frac{e^{(2n+1)\xi}}{e^{(2n+1)(\mu_1+\mu_2)}-1} + \sum_{n=k}^{\infty} e^{(2n+1)(\xi-b)} \sum_{\nu=0}^{\infty} e^{-(2n+1)b\nu} \quad , \tag{45}$$

where $b = \mu_1 + \mu_2$. Then, after some rearrangement,

$$S_o(\xi) = \sum_{n=0}^{k-1} \frac{e^{(2n+1)\xi}}{e^{(2n+1)(\mu_1+\mu_2)}-1} + \sum_{\nu=0}^{\infty} \frac{\exp[(2k+1)(\xi-b\nu-b)]}{1-\exp[2(\xi-b\nu-b)]} \quad . \tag{46}$$

Because of condition (44), the second sum converges rapidly, essentially as a power series in δ. $S_o(\xi)$ can therefore be summed efficiently by adding terms until condition (44) is satisfied; then M terms of the second type complete the summation with an error of the order δ^M.

Other series of the S_m type, for m = 1, 2, 3, can be summed in a similar way; the required expressions are derived by differentiating Eq. (46) with respect to ξ, since

$$S_m(\xi) = \frac{\partial^m}{\partial \xi^m} S_0(\xi) \quad . \tag{47}$$

Write

$$S_m(\xi) = \sum_{n=0}^{k-1} \frac{(2n+1)^m e^{(2n+1)\xi}}{e^{(2n+1)b} - 1} + \sum_{\nu=0}^{\infty} \lambda_\nu(m) \quad . \tag{48}$$

Then the $\lambda_\nu(m)$ are given by the expressions

$$\left.\begin{aligned}
\lambda_\nu(0) &= w e^{Ny} \\
\lambda_\nu(1) &= \left[2tw^2 + Nw\right] e^{Ny} \\
\lambda_\nu(2) &= \left[8t^2w^3 + 4tw^2(N+1) + N^2w\right] e^{Ny} \\
\lambda_\nu(3) &= \left[48t^3w^4 + 24t^2w^3(N+2) + 2tw^2(3N^2+6N+4)\right. \\
&\qquad \left. + N^3w\right] e^{Ny} \quad ,
\end{aligned}\right\} \tag{49}$$

where $y = (\xi - b\nu - b)$, $N = 2k+1$, $t = e^{2y}$, $w = (1-t)^{-1}$.

It is not difficult to show that if we redefine the parameter y in these expressions, similar rapidly convergent expressions can be written for $T_m(\xi)$ and $U_m(\xi)$:

$$T_m(\xi) = \sum_{n=0}^{k-1} \frac{(2n+1)^m e^{(2n+1)\xi}}{\left(e^{(2n+1)b} - 1\right)^2} + \sum_{\nu=0}^{\infty} (\nu+1)\lambda_\nu(m) \quad , \tag{50}$$

$$U_m(\xi) = \sum_{n=0}^{k-1} \frac{(2n+1)^m e^{(2n+1)\xi}}{\left(e^{(2n+1)b} - 1\right)\left(e^{(2n+3)b} - 1\right)} + \sum_{\nu=0}^{\infty} \zeta_\nu \lambda_\nu(m) \quad , \tag{51}$$

where
$$\zeta_\nu = \sum_{r=1}^{\nu+1} e^{-2rb} \quad ,$$

and t, w, N as contained in $\lambda_\nu(m)$ have the same meaning as in Eq. (49) in terms of the parameter y, but $y = (\xi - b\nu - 2b)$.

The field strength coefficients lead to sums of the form $S'_m(\xi)$, which can be treated analogously. If $S'_m(\xi)$ is written as in Eq. (48), symbolizing the general terms in the second series by $\lambda'_\nu(m)$, then

$$\lambda'_\nu(1) = \left[-2tw^2 + Nw\right] e^{Ny}$$
$$\lambda'_\nu(2) = \left[8t^2w^3 - 4tw^2(N+1) + N^2w\right] e^{Ny} \quad , \tag{52}$$

where $y = (\xi - b\nu - b)$, $N = 2k+1$, $t = e^{2y}$, $w = (1+t)^{-1}$.

Appendix B

THE QUANTITIES IN EQUATION (43)

The quantities $\varkappa_j$ and β_j are listed in Table 2, f_k are series of the form T_m or U_m according to Table 3, and the coefficients K_{jk} are polynomial expressions, as tabulated in Table 4.

TABLE 2

j	$\varkappa_j$	β_j
1	$\left(\frac{a}{R_2}\right)^2$	0
2	$\frac{1}{2}\left(\frac{a}{R_2}\right)^2$	0
3	1	0
4	1	-1
5	$\left(\frac{R_2}{a}\right)^2$	0
6	$\left(\frac{R_2}{a}\right)^2$	0
7	$\left(\frac{R_2}{a}\right)^2$	0
8	$\frac{1}{8}\left(\frac{a}{R_2}\right)^2\left(e^{2\mu_2}-1\right)$	0
9	$\frac{1}{4}\left(e^{2\mu_2}-1\right)$	0
10	$\frac{1}{4}\left(e^{2\mu_2}-1\right)$	-1

TABLE 3

k	f_k	k	f_k	k	f_k
1	$T_o(2\mu_1 + \mu_2)$	9	$T_o(\mu_2)$	17	$U_o(\mu_1 + \mu_2)$
2	$T_1(2\mu_1 + \mu_2)$	10	$T_1(\mu_2)$	18	$U_1(\mu_1 + \mu_2)$
3	$T_2(2\mu_1 + \mu_2)$	11	$T_2(\mu_2)$	19	$U_2(\mu_1 + \mu_2)$
4	$T_3(2\mu_1 + \mu_2)$	12	$T_3(\mu_2)$	20	$U_3(\mu_1 + \mu_2)$
5	$T_o(\mu_1 + \mu_2)$	13	$U_o(2\mu_1 + \mu_2)$	21	$U_o(\mu_2)$
6	$T_1(\mu_1 + \mu_2)$	14	$U_1(2\mu_1 + \mu_2)$	22	$U_1(\mu_2)$
7	$T_2(\mu_1 + \mu_2)$	15	$U_2(2\mu_1 + \mu_2)$	23	$U_2(\mu_2)$
8	$T_3(\mu_1 + \mu_2)$	16	$U_3(2\mu_1 + \mu_2)$	24	$U_3(\mu_2)$

In Table 4 which follows,

$$\alpha \equiv e^{2\mu_1},$$

$$\gamma \equiv \frac{1}{2}\left(e^{2\mu_2} + 1\right) \quad .$$

Table 4

k	K_{1k} (j=1)	K_{2k} (j=2)
1	0	0
2	v_2^2	-1
3	$-2v_2$	0
4	1	1
5	0	0
6	$-2v_1v_2$	2
7	$2(v_1-v_2)$	0
8	2	-2
9	0	0
10	v_1^2	-1
11	$2v_1$	0
12	1	1
13	$-\gamma\alpha v_2(v_2-2)$	$3\gamma\alpha$
14	$-\gamma\alpha(v_2^2-4v_2+2)$	$\gamma\alpha$
15	$-\gamma\alpha(-2v_2+3)$	$-3\gamma\alpha$
16	$-\gamma\alpha$	$-\gamma\alpha$
17	$-\gamma(2v_1\alpha-v_1v_2(\alpha+1)-2v_2)$	$-3\gamma(\alpha+1)$
18	$-\gamma[v_1(3\alpha+1)-v_2(\alpha+3)+(2-v_1v_2)(\alpha+1)]$	$-\gamma(\alpha+1)$
19	$-\gamma(\alpha+1)(v_1-v_2+3)$	$3\gamma(\alpha+1)$
20	$-\gamma(\alpha+1)$	$\gamma(\alpha+1)$
21	$-\gamma v_1(v_1+2)$	3γ
22	$-\gamma(v_1^2+4v_1+2)$	γ
23	$-\gamma(2v_1+3)$	-3γ
24	$-\gamma$	$-\gamma$

Table 4 (continued)

k	K_{3k} (j=3)
1	0
2	$2p_{12}v_2$
3	$-2p_{12}$
4	0
5	0
6	$-2(p_{12}v_1+p_{11}v_2)$
7	$2(p_{11}-p_{12})$
8	0
9	0
10	$2p_{11}v_1$
11	$2p_{11}$
12	0
13	$-2\gamma\alpha p_{12}(v_2-1)$
14	$-2\gamma\alpha p_{12}(v_2-2)$
15	$2\gamma\alpha p_{12}$
16	0
17	$-\gamma[2p_{11}\alpha-(p_{12}v_1+p_{11}v_2)(\alpha+1)-2p_{12}]$
18	$-\gamma[p_{11}(3\alpha+1)-(p_{12}v_1+p_{11}v_2)(\alpha+1)-p_{12}(\alpha+3)]$
19	$-\gamma(\alpha+1)(p_{11}-p_{12})$
20	0
21	$-2\gamma p_{11}(v_1+1)$
22	$-2\gamma p_{11}(v_1+2)$
23	$-2\gamma p_{11}$
24	0

Table 4 (continued)

k	K_{4k} (j=4)
1	0
2	$2p_{22}v_2$
3	$-2p_{22}$
4	0
5	0
6	$-2(p_{22}v_1+p_{12}v_2)$
7	$2(p_{12}-p_{22})$
8	0
9	0
10	$2p_{12}v_1$
11	$2p_{12}$
12	0
13	$-2\gamma\alpha p_{22}(v_2-1)$
14	$-2\gamma\alpha p_{22}(v_2-2)$
15	$2\gamma\alpha p_{22}$
16	0
17	$-\gamma[2p_{12}\alpha-(p_{22}v_1+p_{12}v_2)(\alpha+1)-2p_{22}]$
18	$-\gamma[p_{12}(3\alpha+1)-(p_{22}v_1+p_{12}v_2)(\alpha+1)-p_{22}(\alpha+3)]$
19	$-\gamma(\alpha+1)(p_{12}-p_{22})$
20	0
21	$-2\gamma p_{12}(v_1+1)$
22	$-2\gamma p_{12}(v_1+2)$
23	$-2\gamma p_{12}$
24	0

Table 4 (continued)

k	K_{5k} (j=5)	K_{6k} (j=6)	K_{7k} (j=7)
1	0	0	0
2	p_{12}^2	$2p_{12}p_{22}$	p_{22}^2
3	0	0	0
4	0	0	0
5	0	0	0
6	$-2p_{11}p_{12}$	$-2(p_{11}p_{22}+p_{12}^2)$	$-2p_{12}p_{22}$
7	0	0	0
8	0	0	0
9	0	0	0
10	p_{11}^2	$2p_{11}p_{12}$	p_{12}^2
11	0	0	0
12	0	0	0
13	$-\gamma\alpha p_{12}^2$	$-2\gamma\alpha p_{12}p_{22}$	$-\gamma\alpha p_{22}^2$
14	$-\gamma\alpha p_{12}^2$	$-2\gamma\alpha p_{12}p_{22}$	$-\gamma\alpha p_{22}^2$
15	0	0	0
16	0	0	0
17	$\gamma p_{11}p_{12}(\alpha+1)$	$\gamma(\alpha+1)(p_{11}p_{22}+p_{12}^2)$	$\gamma p_{12}p_{22}(\alpha+1)$
18	$\gamma p_{11}p_{12}(\alpha+1)$	$\gamma(\alpha+1)(p_{11}p_{22}+p_{12}^2)$	$\gamma p_{12}p_{22}(\alpha+1)$
19	0	0	0
20	0	0	0
21	$-\gamma p_{11}^2$	$-2\gamma p_{11}p_{12}$	$-\gamma p_{12}^2$
22	$-\gamma p_{11}^2$	$-2\gamma p_{11}p_{12}$	$-\gamma p_{12}^2$
23	0	0	0
24	0	0	0

Table 4 (continued)

k	K_{8k} (j=8)	K_{9k} (j=9)	K_{10k} (j=10)
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0
10	0	0	0
11	0	0	0
12	0	0	0
13	$2\alpha(2v_2-1)$	$4\alpha p_{12}$	$4\alpha p_{22}$
14	$4\alpha(v_2-1)$	$4\alpha p_{12}$	$4\alpha p_{22}$
15	-2α	0	0
16	0	0	0
17	$-v_1(1+3\alpha)-v_2(3+\alpha)-2(1+\alpha)$	$-p_{11}(1+3\alpha)-p_{12}(3+\alpha)$	$-p_{12}(1+3\alpha)-p_{22}(3+\alpha)$
18	$-4(\alpha v_1+v_2)+2(1-\alpha)$	$-4(\alpha p_{11}+p_{12})$	$-4(\alpha p_{12}+p_{22})$
19	$(v_1-v_2+6)(1-\alpha)$	$(p_{11}-p_{12})(1-\alpha)$	$(p_{12}-p_{22})(1-\alpha)$
20	$2(1-\alpha)$	0	0
21	$2(2v_1+1)$	$4p_{11}$	$4p_{12}$
22	$4(v_1+1)$	$4p_{11}$	$4p_{12}$
23	2	0	0
24	0	0	0

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