MEMORANDUM
RM-4141-PR
NOVEMBER 1964

EXTERNALLY BAYESIAN GROUPS

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700 monitored by the Directorate of Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

One of Project RAND's activities is the gaming of complex group decision-making situations in order to learn how groups come to a rational consensus. This Memorandum proposes an axiom of rationality for a particular group decision-making situation and investigates the consequences of this axiom. The main finding is that group consensus procedures obeying this axiom may lead to dictatorship.

ACKNOWLEDGMENT

The author wishes to acknowledge the helpful stimulation of the participants in the UCLA Management Sciences Research Project Workshop, especially Jacques Dreze, Jacob Marschak, and Robert Wilson.

SUMMARY

Suppose that a group of individuals, each of whom is a Bayesian, is required to make a joint decision, and that although the individuals all agree on the utility function for the problem, they disagree on the prior distribution of the relevant states of nature. An axiom of group rationality is introduced—namely, that to an outsider the decisions of the group appear like the decision of a Bayesian—and its implications are explored when the group decision-making procedure either can or cannot be amended.

CONTENTS

EXTERNALLY BAYESIAN GROUPS

1. EXTERNALLY BAYESIAN GROUPS

Suppose that a group of n individuals is required to make a joint decision, and that although the individuals all agree on the utility function for the problem, they disagree on the prior distribution of the relevant states of nature. To introduce some notation, let θ denote the state of nature, $\mathcal{A}$ the set of actions, with $A \in \mathcal{A}$ denoting an action, $u(A,\theta)$ the utility of action A when θ is the state of nature, and $p_i^0(\theta)$ the prior probability density function of θ with respect to some measure $\mu(\theta)$ for the i-th member of the group, $i = 1, \ldots, n$.

Suppose further that each individual is a Bayesian, so that after observing a random variable X from a probability distribution with parameter θ, he redetermines his probability density function of θ, using Bayes' theorem. Let all the individuals agree that $f(x|\theta)$ is the probability density function of x, given θ. Then $p_i^1(\theta)$, the probability density function of θ after $X = x$ has been observed, is given by

$$p_i^1(\theta) = \frac{f(x|\theta)p_i^0(\theta)}{\int f(x|\theta)p_i^0(\theta)d\mu(\theta)} \quad .$$

Let A_i^t maximize $E_i^t u(A,\theta)$, where

$$E_i^t u(A,\theta) = \int u(A,\theta)p_i^t(\theta)d\mu(\theta)$$

for $i = 1, \ldots, n$, $t = 0, 1$.

Let A^t be the group decision at time t, and suppose that there exists a prior density function $\pi^t(\theta)$ such that A^t maximizes $E^t u(A,\theta)$, where

$$E^t u(A,\theta) = \int u(A,\theta)\, \pi^t(\theta) d\mu(\theta) \quad .$$

There may in fact be many densities $\pi^t(\theta)$ for which A^t maximizes $E^t u(A,\theta)$.

Let t_0, t_1 be any two points in time. Suppose that in the time interval (t_0, t_1) the random variable X has been independently observed n times, as $x_1, \ldots, x_n$. Let $f_n(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$. We will say that the group is <u>externally Bayesian</u> if, for every t_0, t_1, n, and observations $x_1, \ldots, x_n$, there exists a pair of probability densities $\pi^{t_0}(\theta)$, $\pi^{t_1}(\theta)$ for which $A^{t_0}(\theta)$, $A^{t_1}(\theta)$ maximize $E^{t_0} u(A,\theta)$, $E^{t_1} u(A,\theta)$, respectively, and which satisfy

$$\pi^{t_1}(\theta) = \frac{f_n(x|\theta)\pi^{t_0}(\theta)}{\int f_n(x|\theta)\pi^{t_0}(\theta) d\mu(\theta)} \quad ,$$

i.e., for which $\pi^{t_1}(\theta)$ is also the result of applying Bayes' theorem to $\pi^{t_0}(\theta)$.

Clearly there exist group decision rules such that the group is externally Bayesian, since in a dictatorship where the i-th individual's decision is the group's decision, we have $A^t = A_i^t$ and $\pi^t(\theta) = p_i^t(\theta)$.

If there is unanimity among indivuduals in the $p_i^t(\theta)$ or in the A_i^t, then we assume that in this case the group decision is the common A_i^t. We shall exclude this situation from consideration in the following.

Now consider a sequence of three group decisions, A^{t_0}, A^{t_1}, A^{t_2}. On the surface all we can infer from the statement that the group is externally Bayesian is that there exist four prior densities $\pi^{t_0}(\theta)$, $\pi^{t_1}(\theta)$, $\tilde{\pi}^{t_1}(\theta)$, and $\pi^{t_2}(\theta)$ such that: (1) A^{t_0} maximizes $E^{t_0}u(A,\theta)$, A^{t_2} maximizes $E^{t_2}u(A,\theta)$, and A^{t_1} maximizes both

$$E^{t_1}u(A,\theta) = \int u(A,\theta)\pi^{t_1}(\theta)d\mu(\theta)$$

and

$$\tilde{E}^{t_1}u(A,\theta) = \int u(A,\theta)\tilde{\pi}^{t_1}(\theta)d\mu(\theta) ;$$

and (2) $\pi^{t_1}(\theta)$ is the result of applying Bayes' theorem to $\pi^{t_0}(\theta)$ and, $\pi^{t_2}(\theta)$ is the result of applying Bayes' theorem to $\tilde{\pi}^{t_1}(\theta)$. But this is not enough to capture the spirit of the externally Bayesian assumption, namely, that any sequence of group decisions can be come to by some individual who is a Bayesian, who starts with a particular prior density, and who observes the sequence of x's observed by the group.

We shall now show that the above definition of "external Bayesian," involving only two decisions, implies that, for any sequence of data and decisions, there is a sequence of prior densities such that the t-th decision is optimal for the t-th density and that the (t+1)-st density is obtainable from the t-th density by Bayes' theorem, for all t. Since only the details, and not the method of proof, change from the three-time-period case to the t-time-period case, we present the proof only for the three-time-period case.

Moreover, we present the proof only for the case in which there is only one observation on X in (t_0, t_1) and in (t_1, t_2). For ease of notation we replace t_0 by 0, t_1 by 1, and t_2 by 2.

Theorem. *Let*

$$\pi^2(\theta) = \frac{f(x_2|\theta)\pi^1(\theta)}{\int f(x_2|\theta)\pi^1(\theta)d\mu(\theta)} ,$$

and let the group be externally Bayesian. Suppose x_t *is observed at time* t, $t = 1, 2,$ *and that the* x_t *are independent. Then there exists a* $\pi^0(\theta)$ *such that*

$$\pi^1(\theta) = \frac{f(x_1|\theta)\pi^0(\theta)}{\int f(x_1|\theta)\pi^0(\theta)d\mu(\theta)} .$$

Proof. Suppose there exists no such $\pi^0(\theta)$. Then for any $\pi^0(\theta)$

$$\pi^2(\theta) = \frac{f(x_2|\theta)\pi^1(\theta)}{\int f(x_2|\theta)\pi^1(\theta)d\mu(\theta)} \neq \frac{f(x_2|\theta)f(x_1|\theta)\pi^0(\theta)}{\int f(x_2|\theta)f(x_1|\theta)\pi^0(\theta)d\mu(\theta)} .$$

Now redefine time intervals so that decisions are only made at time 0 and at time 2, at the latter time with data $x = (x_1,x_2)$. But by the external Bayesian assumption, as this is only a two-time-period problem, there must exist a $\pi^0(\theta)$ such that

$$\pi^2(\theta) = \frac{f_2(x|\theta)\pi^0(\theta)}{\int f_2(x|\theta)\pi^0(\theta)d\mu(\theta)} .$$

But $f_2(x|\theta) = f(x_1|\theta)f(x_2|\theta)$, by independence of the x's. This contradiction establishes the proof.

2. GROUPS WITH FIXED CONSTITUTIONS

Let φ denote the process whereby the group comes to a decision, and suppose that φ depends on t only through the dependence of the individuals' prior probability densities on t, i.e., the nature of the decision process is time independent. Thus we write

$$A^t = \varphi(\{A_i^t\}, \{p_i^t(\theta)\}) \quad .$$

One heuristic way of describing this assumption is to think of the group of individuals as being bound by a fixed constitutional procedure in coming to a group decision, and that this immutable constitutional procedure φ calls only for the individuals' current prior probability density functions of θ $\{p_i^t(\theta)\}$ and their current suggested actions $\{A_i^t\}$.

What we shall do in this section is examine four group-decision rules discussed in the literature on the subject, and show by example that when there is a fixed constitution, these rules either will not lead to externally Bayesian groups or, if they do, they may of necessity lead to dictatorship. In the next section we shall consider the case of an amendable constitution and show that there is only one externally Bayesian group-decision rule, and that given certain conditions, this rule will lead to a dictatorship.

<u>RULE 1 (the opinion pool)</u>: This rule, due to Stone [6], takes $\pi^t(\theta) = \sum_{i=1}^{n} \lambda_i \, p_i^t(\theta)$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, and A^t is the action which maximizes $E^t u(A,\theta)$ for this $\pi^t(\theta)$. Note that although $\pi^0(\theta)$ and $\pi^1(\theta)$ so defined need not satisfy (1), there may still be other group prior probability densities for which the A^t are also optimal and which satisfy (1).

<u>RULE 2 (compromise)</u>: We assume that $\mathcal{A}$ is a convex set and that the group chooses $A^t = \sum_{i=1}^{n} \nu_i A_i^t$, where $\nu_i \geq 0$ and $\sum_{i=1}^{n} \nu_i = 1$.

Suppose θ can take on only two values, θ_1 and θ_2, and let $u(A,\theta) = -(A - \theta)^2$. Then maximizing $E_i^t u(A,\theta)$ is equivalent to minimizing

$$p_i^t(A - \theta_1)^2 + (1 - p_i^t)(A - \theta_2)^2 \ ,$$

where $p_i^t = p_i^t(\theta_1)$. This is accomplished by taking

$$A_i^t = p_i^t \theta_1 + (1 - p_i^t)\theta_2 \ ,$$

i.e., $A_i^t = E_i^t \theta$, the expected value of θ under $p_i^t(\theta)$.

Since the distribution of θ is a two-point distribution, it is completely determined by its mean. Thus if $\overline{\theta}_i^t = E_i^t \theta$, then

$$p_i^t = \frac{\bar{\theta}_i^t - \theta_2}{\theta_1 - \theta_2} \quad .$$

For this problem, then, there is a one-to-one correspondence between decision and prior distribution, so that $\pi^t(\theta)$ is unique.

In the compromise situation, then,

$$\pi^t(\theta_1) = \pi^t = \sum_{i=1}^{n} \frac{\nu_i \bar{\theta}_i^t - \theta_2}{\theta_1 - \theta_2}$$

and

$$A^t = \pi^t \theta_1 + (1 - \pi^t)\theta_2 = \sum_{i=1}^{n} \nu_i \bar{\theta}_i^t \quad .$$

Note that

$$\pi^t = \sum_{i=1}^{n} \nu_i p_i^t \quad ,$$

so that for group-decision problems with two-point distributions and this utility function, compromise and the opinion pool are equivalent.

Now

$$\pi^1 = \sum_{i=1}^{n} \nu_i p_i^1$$

$$= f(x|\theta_1) \sum_{i=1}^{n} \frac{\nu_i p_i^0}{[f(x|\theta_1) - f(x|\theta_2)]p_i^0 + f(x|\theta_2)} \quad ,$$

and

$$\frac{f(x|\theta_1)\pi^0}{f(x|\theta_1)\pi^0 + f(x|\theta_2)(1-\pi^0)}$$

$$= \frac{f(x|\theta_1)\sum_{i=1}^{n} \nu_i p_1^0}{[f(x|\theta_1) - f(x|\theta_2)]\sum_{i=1}^{n} \nu_i p_i^0 + f(x|\theta_2)} \; .$$

Thus the group is externally Bayesian if and only if

$$\sum_{i=1}^{n} \nu_i p_i^0 = \Delta \sum_{i=1}^{n} \nu_i p_i^0 + f(x|\theta_2) \sum_{i=1}^{n} \frac{\nu_i p_i^0}{\Delta p_i^0 + f(x|\theta_2)} \; ,$$

where $\Delta = f(x|\theta_1) - f(x|\theta_2)$. Let $a_i = \Delta p_i^0 + f(x|\theta_2)$. Then this equation is

$$\sum_{i=1}^{n} \nu_i p_i = \left(\sum_{i=1}^{n} \nu_i a_i\right)\left(\sum_{j=1}^{n} \frac{\nu_j p_j^0}{a_j}\right) \; .$$

Let $n = 2$. Then this is equivalent to

$$\nu_1(p_1^0 - p_2^0) + p_2^0 = [\nu_1(a_1 - a_2) + a_2]\left[\nu_1\left(\frac{p_1^0}{a_1} - \frac{p_2^0}{a_2}\right) + \frac{p_2^0}{a_2}\right]$$

$$= \nu_1^2(a_1 - a_2)\left(\frac{p_1^0}{a_1} - \frac{p_2^0}{a_2}\right) + p_2^0$$

$$+ \nu_1\left[\frac{a_2 p_1^0}{a_1} - p_2^0 + \frac{a_1 p_2^0}{a_2} - p_2^0\right] \; ,$$

or

$$p_1^0 - p_2^0 = \nu_1(a_1 - a_2)\left(\frac{p_1^0}{a_1} - \frac{p_2^0}{a_2}\right)$$

$$+ \left[\frac{a_2p_1^0}{a_1} + \frac{a_1p_2^0}{a_2} - 2p_2^0\right] .$$

Then

$$\nu_1 = \frac{p_1^0 + p_2^0 - \frac{a_2p_1^0}{a_1} - \frac{a_1p_2^0}{a_2}}{(a_1 - a_2)\left(\frac{p_1^0}{a_1} - \frac{p_2^0}{a_2}\right)} = \frac{a_1a_2(p_1^0 + p_2^0) - a_2^2p_1^0 - a_1^2p_2^0}{(a_1 - a_2)(a_2p_1^0 - a_1p_2^0)}$$

$$= \frac{a_1a_2(p_1^0 + p_2^0) - a_2^2p_1^0 - a_1^2p_2^0}{a_1a_2(p_1^0 + p_2^0) - a_1^2p_2^0 - a_2^2p_1^0}$$

$$= 1 .$$

Thus only a dictatorship is externally Bayesian in either the compromise or opinion pool for a group of size 2 for this problem.

To show that this is true for an n-person group, we proceed by induction, noting that

$$\sum_{i=1}^{n} \nu_i p_i^t = \sum_{j=1}^{n-1} \frac{\nu_1 p_i^t}{1 - \nu_n} \quad (1 - \nu_n) + \nu_n p_n^t ,$$

so that if dictatorship is the only externally Bayesian procedure for the (n-1) person group, then some ν_i, $i = 1, \ldots, n-1$, must equal $1 - \nu_n$ and we are left with a 2-person group, in which again only a dictatorship is externally Bayesian.

RULE 3 (group minimax): Savage [5, Chapt. 10] suggests that the group choose the decision A which minimizes

$$\max_{i=1,\ldots,n} E_i u(A_i,\theta) - E_i u(A,\theta) ,$$

where $E_i u(A_i,\theta) - E_i u(A,\theta)$ is the expected "regret" associated with A for individual i. Since in our example

$$\begin{aligned} - E_i u(A_i,\theta) &= p_i[p_i\theta_1 + (1-p_i)\theta_2 - \theta_1]^2 \\ &\quad + (1 - p_i)[p_i\theta_1 + (1-p_i)\theta_2 - \theta_2]^2 \\ &= p_i(1 - p_i)(\theta_1 - \theta_2)^2 , \end{aligned}$$

we see that the expected regret for individual i associated with A is

$$p_i(A - \theta_1)^2 + (1 - p_i)(A - \theta_2)^2 - p_i(1 - p_i)(\theta_1 - \theta_2)^2 .$$

Thus the group minimax procedure for n = 2 chooses A which minimizes

$$\max \begin{cases} - 2Ap_1(\theta_1 - \theta_2) - 2p_1(\theta_2^2 - \theta_1\theta_2) + p_1^2(\theta_1 - \theta_2)^2 + (A - \theta_2)^2 \\ \qquad = R(A, p_1) \\ - 2Ap_2(\theta_1 - \theta_2) - 2p_2(\theta_2^2 - \theta_1\theta_2) + p_2^2(\theta_1 - \theta_2)^2 + (A - \theta_2)^2 \\ \qquad = R(A, p_2) . \end{cases}$$

Let $\theta_1 > \theta_2$ and $p_1 > p_2$. Then $R(A,p_1) > R(A,p_2)$ for given A if and only if

$$2p_1(\theta_2 - A) + p_1^2(\theta_1 - \theta_2) > 2p_2(\theta_2 - A) + p_2^2(\theta_1 - \theta_2) \ .$$

This is so if and only if

$$A < \theta_2 + (\theta_1 - \theta_2)\,\frac{(p_1 + p_2)}{2} = \bar{p}\theta_1 + (1 - \bar{p})\theta_2 \ ,$$

where $\bar{p} = (p_1 + p_2)/2$.

Now $R(A,p_i)$ is minimized by $A_i = p_i\theta_1 + (1 - p_i)\theta_2$, and since

$$p_2\theta_1 + (1 - p_2)\theta_2 < \bar{p}\theta_1 + (1 - \bar{p})\theta_2 < p_1\theta_1 + (1 - p_1)\theta_1 \ ,$$

we see that the maximum expected regret is minimized by

$$A = \bar{p}\theta_1 + (1 - \bar{p})\theta_2 \ ,$$

corresponding to an opinion pool with weights $\nu_i = 1/2$. Thus the group minimax rule is not externally Bayesian.

<u>RULE 4 (pari-mutuel method)</u>: Eisenberg and Gale [2] have suggested a method of obtaining a consensus of subjective probabilities, but they begin with an assumption about each individual's behavior which is different from our assumption that each individual will make the decision that maximizes his expected utility. They assume that each

individual maximizes his expected utility, subject to the constraint that the actual group expected utility function will be

$$\int u(A,\theta)\pi(\theta)d\mu(\theta) \quad .$$

Thus if $\pi(\theta)$ were announced before decision-making, the individuals' decisions would be affected, and the i-th individual would choose whatever A was optimal for the θ for which $p_i(\theta)/\pi(\theta)$ is greatest. For this procedure, then, each individual's announced decision is not one that he thinks is best for the group, but rather one which may influence the group consensus to coincide with what he thinks is best for the group.

As pointed out in [2], this consensus procedure has the property that when $n = 2$, when the distribution of θ is a two-point distribution, and when $u(A,\theta) = \delta_{A\theta}$ (i.e., 1 if $A = \theta$ and 0 otherwise), if $p_1(\theta_1) = p_1(\theta_2) = 1/2$, then regardless of what $p_2(\theta_1)$ is, we have $\pi(\theta_1) = \pi(\theta_2) = 1/2$. Now let $p_2^0(\theta_1) = \alpha \neq 1/2$, $p_1^0(\theta_1) = 1/2$, and let $f(x|\theta_1) = [(1-\alpha)/\alpha]f(x|\theta_2)$. Then $p_2^1(\theta_1) = 1/2$, and so $\pi^0(\theta_1) = \pi^1(\theta_1) = 1/2$; yet

$$\frac{f(x|\theta_1)\pi^0(\theta_1)}{f(x|\theta_1)\pi^0(\theta_1) + f(x|\theta_2)\pi^0(\theta_2)} = 1 - \alpha \cdot$$

Thus this consensus procedure is not externally Bayesian.

3. GROUPS WITH AMENDABLE CONSTITUTIONS

Though each member of the group has the same utility function $u(A,\theta)$, each member has at time t a different expected utility function: $E_i^t u(A,\theta) = u_i^t(A)$, say. Thus our group decision-making problem is the same as the group decision-making problem with different utility functions among the individuals. Using the device of Luce and Raiffa [4, p. 343] of associating "individuals" with "states of nature," the group decision-making problem corresponds with the problem of finding a principle for decision-making under uncertainty. Let us impose the three conditions on a preference-ordering among actions given by Blackwell and Girshick [1, p. 116], namely: (L_1) independence of irrelevant alternatives, (L_2) Pareto optimality, and (L_3) positive association of individual values. Then by Theorem 4.4.1 of [1], it follows that in our situation either all acts are indifferent or there is a probability distribution $\{\lambda_i\}$ on the individuals such that A_1 is preferred to A_2 whenever $\sum_{i=1}^{n} \lambda_i u_i^t(A_1) > \sum_{i=1}^{n} \lambda_i u_i^t(A_2)$, i.e., whenever

$$\int u(A_1,\theta) \sum_{i=1}^{n} \lambda_i p_i^t(\theta) d\mu(\theta) > \int u(A_2,\theta) \sum_{i=1}^{n} \lambda_i p_i^t(\theta) d\mu(\theta) .$$

These conditions thus imply the existence of an opinion pool which, insofar as it is capable of doing so, mimics the group's preference-ordering of actions.

Note that if $\sum_{i=1}^{n} \lambda_i u_i^t(A_1) = \sum_{i=1}^{n} \lambda_i u_i^t(A_2)$, then the opinion pool whose existence is implied by the above three conditions gives us no information about the group preference between A_1 and A_2. The group need not be indifferent between A_1 and A_2. Thus this opinion pool only induces a partial order among n-vectors $[u_1(A), \ldots, u_n(A)] = u(A)$.

If in addition one assumes (L_4) linearity and (L_5) continuity (i.e., if $u(A_k)$ is preferred or indifferent to $u(B)$ for all k, and $\lim_{k\to\infty} u_i(A_k) = u_i(A)$ for all i, then $u(A)$ is preferred or indifferent to $u(B)$), then one can conclude [1, p. 120] that if $\sum_{i=1}^{n} \lambda_i u_i^t(A_1) = \sum_{i=1}^{n} \lambda_i u_i^t(A_2)$, then moreover A_1 is indifferent to A_2.

Now the opinion pool implied by these five conditions induces a chain order among n-vectors $[u_1(A), \ldots, u_n(A)] = u(A)$, so that the vector space of $u(A)$'s is an ordered vector space [3]. By Theorem 5.6 of [3], there exists a basis for this space such that relative to this basis the ordering of acts is lexicographic. Since a change of basis is a linear transformation, this implies that there are n n-vectors, $[b_{11}, \ldots, b_{1n}] = b_1, \ldots, [b_{n1}, \ldots, b_{nn}] = b_n$, such that A_1 is preferred to A_2 if $b_1 u'(A_1) > b_1 u'(A_2)$, or $b_1 u'(A_1) = b_1 u'(A_2)$ and $b_2 u'(A_1) > b_2 u'(A_2)$, or ... or $b_i u'(A_1) = b_i u'(A_2)$, $i = 1, \ldots, n-1$, and $b_n u'(A_1) = b_n u'(A_2)$. We can think of these vectors $b_1, \ldots, b_n$ as vectors of weights (not necessarily nonnegative), such that the preference-ordering of actions is determined by a serial dictatorship of groups of people compromising with weights given by the b vectors.

If instead of L_4 and L_5 one adds to L_1, L_2, L_3 a different condition on the preference-ordering among actions, namely, that order-equivalent arrays (arrays in which any column is changed by a strictly monotonic transformation) induce identical orderings, then one can show (as pointed out in [4]) that only a lexicographical ordering is possible. This implies that the group choice is determined by a serial dictatorship.

We have seen earlier that in case the group constitution is not amendable, then an externally Bayesian opinion pool may lead to dictatorship, i.e., some $\lambda_j = 0$ for $j \neq i$.

If we were to relax our rule as to how the group comes to a decision by allowing the constitution to be "amended" at every t, so that at each time t the weights λ_i are allowed to change, then one can always find a sequence of weights λ_i^t such that the group is externally Bayesian, as follows. Take

$$\lambda_i^t = \frac{\lambda_i^{t-1} \int f(x|\theta) p_i^{t-1}(\theta) d\mu(\theta)}{\sum_{j=1}^{n} \lambda_j^{t-1} \int f(x|\theta) p_j^{t-1}(\theta) d\mu(\theta)} .$$

Then

$$\pi^t(\theta) = \sum_{i=1}^{n} \lambda_i^t p_i^t(\theta)$$

$$= \frac{\sum_{i=1}^{n} \lambda_i^{t-1} \int f(x|\xi) p_i^{t-1}(\xi) d\mu(\xi) p_i^t(\theta)}{\sum_{j=1}^{n} \lambda_j^{t-1} \int f(x|\xi) p_j^{t-1}(\xi) d\mu(\xi)}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i^{t-1} f(x|\theta) p_i^{t-1}(\theta)}{\int f(x|\theta) \sum_{j=1}^{n} \lambda_j^{t-1} p_j^{t-1}(\xi) d\mu(\xi)}$$

$$= \frac{f(x|\theta)\pi^{t-1}(\theta)}{\int f(x|\theta)\pi^{t-1}(\theta) d\mu(\theta)} \ .$$

Conversely, letting $m_j^t(x) = \int f(x|\theta) p_j^t(\theta) d\theta$, the external Bayes axiom says that

$$\sum_{j=1}^{n} \lambda_j^{t+1} p_j^t(\theta)/m_j^t(x) = \frac{\sum_{j=1}^{n} \lambda_j^t p_j^t(\theta)}{\sum_{j=1}^{n} \lambda_j^t m_j^t(x)} \ .$$

Let λ^{t+1} by the n-vector of λ_j^{t+1}'s, p^t be the n-vector of $p_j^t(\theta)$'s, m^t be the n-vector of $m_j^t(x)$'s, and λ^{t+1}/m^t be the n-vector of $\lambda_j^{t+1}/m_j^t(x)$'s. This can be rewritten as

$$\left(\frac{\lambda^{t+1}}{m^t}\right)' p^t = \frac{(\lambda^t)' \; p^t}{(\lambda^t)' \; m^t} .$$

This must hold for all p^t, and this implies that

$$\left(\frac{\lambda^{t+1}}{m^t}\right) = \frac{\lambda^t}{(\lambda^t)' \; m^t} ,$$

i.e.,

$$\lambda_j^{t+1} = \frac{\lambda_j^t m_j^t(x)}{\sum_{j=1}^{n} \lambda_j^t m_j^t(x)} .$$

We see then that this is the only opinion-pool amendment procedure which is externally Bayesian. Since the opinion pool is the only rule which satisfies conditions $L_1 - L_5$, this amendment procedure is the only rule which satisfies $L_1 - L_5$ and the external Bayesian condition.

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