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DYNAMIC PROGRAMMING IN
MULTIPLICATIVE LATTICES

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the authors prove that optimal policies exist, and that the principle of optimality holds, in a category of dynamic-programming problems involving only a partially ordered criterion space.
SUMMARY

This Memorandum demonstrates that the methods of dynamic programming may be applied to problems involving partially ordered criterion functions; specifically, the existence of optimal policies and the principle of optimality are established for a category of problems in which the criterion space is a conditionally complete multiplicative lattice.
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1. INTRODUCTION

Dynamic programming is classically concerned with maximizing the value assumed by some real-valued return function defined over sequences of decisions, or on a state space which is dependent on sequences of decisions (see [1], pp. 81–83). The adoption of any such real-valued return function imposes an ordering in which any two policies can be compared as to desirability. As Zadeh [7] has pointed out, however, in real life there are many different factors which enter into the assessment of any system or state, and often these considerations cannot be subsumed under a single scalar-valued return. The purpose of this Memorandum is to present a version of the fundamental functional equation of dynamic programming for a category of problems in which the return space is only partially ordered. The idea of using a lattice-valued return function was, in fact, presented by Ellis [4] in an unpublished paper many years ago. In the present Memorandum, we shall show that conditionally complete multiplicative lattices provide a setting in which the partial ordering supplies enough intrinsic topological structure to provide us with natural conditions for the existence of "optimal" (i.e., nondominated) policies and that an analog of the "principle of optimality" ([1], p. 83) holds in such a setting.
2. DEFINITIONS

We shall consider return spaces which are conditionally complete associative multiplicative lattices (the lattice terminology throughout is in general taken from [3]). Recall the following definition:

**Definition.** We say that \((S, \geq, \circ)\) is a "conditionally complete associative multiplicative lattice" if \(S\) is a nonempty set on which is defined a relation \(\geq\) and a binary operation \(\circ\) satisfying the following conditions:

1. \(a \geq b\) and \(b \geq a\) if and only if \(a = b\), for any \(a, b \in S\).
2. \(a \geq b\) and \(b \geq c\) implies \(a \geq c\), for any \(a, b, c \in S\).
3. For any nonempty subset \(A\) of \(S\) which is bounded above, there exists an element \(\vee A\) (called the least upper bound of \(A\)) such that \(c \geq a\) for all \(a \in A\) if and only if \(c \geq \vee A\).
4. For any nonempty subset \(A\) of \(S\) which is bounded below, there exists an element \(\wedge A\) (called the greatest lower bound of \(A\)) such that \(c \leq a\) for all \(a \in A\) if and only if \(c \leq \wedge A\).
5. \(a \circ (b \circ c) = (a \circ b) \circ c\), for any \(a, b, c \in S\).
6. \(b \circ \vee A = V(b \circ A)\) and \((\vee A) \circ b = V(A \circ b)\), for any \(b \in S\), \(A \subseteq S\).

Any set \(S\) satisfying (1), (2), (3), (4) above is called a "conditionally complete lattice." If \(A\) is a set
consisting of just two elements (say, a and b) then we write \( VA = a \lor b \) and \( AA = a \land b \). The notation \( b \circ A \) of course means the set of all \( c \) such that \( a = b \circ a \) for some \( a \in A \). Condition (6) immediately implies that if \( a \geq b \) then \( a \circ c \geq b \circ c \) and \( c \circ a \geq c \circ b \) (see [3], p. 201).

In any conditionally complete lattice \( S \) we can define an "order topology" in the following way (see [3], pp. 59–60): we say that the directed set \( \{ x_\alpha \} \) converges to \( a \) if and only if

\[
a = \bigwedge \{ x_\alpha \} \quad \text{and} \quad a \beta \geq a \alpha \quad \text{for all } \beta \geq \alpha.
\]

or, in other words, if \( a = \lim \sup \{ x_\alpha \} = \lim \inf \{ x_\alpha \} \). It is not hard to show that \( S \) is a Hausdorff space under its order topology. We will coin the phrase "regular multiplicative lattice" to describe a conditionally complete multiplicative lattice which meets the following additional condition:

(7) \( b \circ AA = A(b \circ A) \) and \( AA \circ b = A(A \circ b) \),

for any \( b \in S, \quad A \in S \).

Condition (7) is clearly satisfied if \((S, \geq, \circ)\) satisfies (1) through (6) above and \( b \circ A \) and \( A \circ b \) are maps of \( S \) onto \( S \) (for example, if \( S \) is a group under \( \circ \)).

Conditions (6) and (7) imply that the binary operation law \( \circ \) defines a continuous map from \( S \times S \) into \( S \) (see [3], pp. 231–232), where the topology on \( S \) is the order topology and the topology on \( S \times S \) is the product of the topologies on \( S \). Now let us consider some specific examples of regular multiplicative lattices:
Example A. The set $\mathbb{R}$ of all real numbers under addition and the usual order relation forms a regular multiplicative lattice. The topology induced is the usual topology for the reals.

Example B. The $n$-dimensional Euclidean space $\mathbb{E}^n$, with vector addition as the operation and with the relation $a \geq b$ if every component of $a$ is greater than or equal to the corresponding component of $b$, is a regular multiplicative lattice. The topology induced is the usual topology of Euclidean space (see [3], p. 61, exercise 2).

Example C. The set of all real-valued functions on some fixed space, under point-wise addition and writing $f \geq g$ if $f(x) \geq g(x)$ for all $x$, is a regular multiplicative lattice. The topology is the topology of pointwise convergence. Example B above may be viewed as a special case of this example.

Example D. The set of all integer-valued functions on some fixed space, under the operation and relation of Example C, is another example. The topology induced is the relativized pointwise topology.

Let $T$ be some subset of a regular multiplicative lattice $S$. We define $\text{max } T$ to be the set of all elements in $T$ which have the property that no other elements in $T$ exceed them. Note that $\text{max } T$ will in general be a set, and may be empty. For example, consider the closed unit disk $D$ in $\mathbb{E}^2$ (Example B above). Then
\[
\text{max } D = \{(x, y) | x^2 + y^2 = 1, \ x \geq 0, \ y \geq 0\}.
\]

Note that the least upper bound of \(D\) is the point \((1, 1)\). The max of the interior of \(D\) (the open unit disk) is empty, but the open unit disk has the same least upper bound as the closed unit disk. We define loc max \(T\) to be the set of all elements \(p\) in \(T\) which have the property that in some neighborhood of \(p\) there are no elements of \(T\) which exceed \(p\). See Fig. 1 for a graphic representation of the difference between max \(T\) and loc max \(T\) in one particular case. Of course, max \(T\) is always a subset of loc max \(T\).

**Theorem 1.** Max \(T\) is nonempty if \(T\) is nonempty and compact; and if \(p \in T\), then there is a \(q \in \text{Max } T\) such that \(q \geq p\).
Fig. 1 — Definition of max T and loc max T
Proof. If $T$ is compact, then it clearly must be bounded. By the axiom of choice we find a maximal chain $C$ containing a given $p \in T$ and contained in $T$. Since $C$ is in $T$, $C$ is bounded; thus there is a least upper bound $q$ for the set $C$. The elements of $C$, indexed by themselves, form a directed set which has $q$ as limit; since $T$ is compact, it follows that $q$ is in $T$. If there were any element in $T$ greater than $q$, we could adjoin it to $C$ to get a larger chain in $T$ containing $p$. Thus, $q \in \max T$, and, of course, $q \geq p$.

The functions $\max$ and $\loc \max$ enjoy the following properties:

**Lemma 1.** If $p \in B \subseteq A$ and $p \in \max A$, then $p \in \max B$.

**Lemma 2.** If $A$ is compact, $C \subseteq B \subseteq A$, and $\max A = \max C$, then $\max B = \max A$.

**Proof:** $\max A \subseteq \max B$ by Lemma 1. If $p \in \max B$ and $p \notin \max A$, then by Theorem 1 we can find a $q \in \max A$ such that $q > p$; but then $q \in B$, contradicting the fact that $p \in \max B$.

**Lemma 3.** $\max [\max T] = \max T$.

**Proof.** Apply Lemma 1, with $B = \max T$, $A = T$.

**Lemma 4.** If $p \in B \subseteq A$ and $p \in \loc \max A$, then $p \in \loc \max B$.

**Lemma 5.** $\loc \max [\loc \max T] = \loc \max T$.

**Proof.** Apply Lemma 4, with $B = \loc \max T$, $A = T$.

3. **THE PROBLEM**

Let $\{X_j| j = 1,2,\ldots\}$ be a family of topological spaces, and let $\{g_i|i=1,2,\ldots\}$ be a family of continuous functions
such that $g_i$ has domain $X_i$ and range in some regular multiplicative lattice $S$. Denote $X_n \times X_{n-1} \times \ldots \times X_1$ by $X^n$; let the topology on $X^n$ (for all $n$) be the usual product topology. Let $Y$ be some compact nonempty subset of $X^n$. Each $X_i$ is a set of decisions; and the set $Y$ represents those sequences of decisions which are possible in some $n$ stage process.

Define

$$F_n(Y) = \{ p | p = g_n(x_n) \circ g_{n-1}(x_{n-1}) \circ \ldots \circ g_1(x_1)$$

for some $(x_n, x_{n-1}, \ldots, x_1) \in Y \}.$$

The set $F_n(Y)$ could be called the set of "feasible points" of our process; the order relation on $S$ reflects the relative "desirability" of the various sequences of decisions; $g_i(x_i)$ represents the contribution of the $i$-th decision. The problem therefore is to compute $\max[F_n(Y)].$

**Theorem 2.** For any $p \in F_n(Y)$, there exists a $q \in \max[F_n(Y)]$ such that $q \geq p.$

**Proof.** The map $g : Y \to S^n$ defined by $(x_n, x_{n-1}, \ldots, x_1) \to (g_n(x_n), \ldots, g_1(x_1))$ is clearly continuous since each of its projections is continuous. The map $h : S^n \to S$ defined by $(s_n, s_{n-1}, \ldots, s_1) \to s_n \circ s_{n-1} \circ \ldots \circ s_1$ is continuous (by induction) since the map of $S^2 \to S$ defined by $\circ$ is continuous and $\circ$ is associative. Thus $F_n(Y) = h(g(Y))$ is the continuous image of a compact set, and is hence compact, and so Theorem 1 applies.
We shall show that a "principle of optimality" holds with regard to \( \max [F_n(Y)] \), but first we must establish some special notation. Define \( P_1(Y) \) to be the projection of \( Y \) into \( X_1 \). Let \( (Y|x_n) \) be the subset of \( Y \) consisting of elements whose component in \( X_n \) is \( x_n \); let \( P^n(Y|x_n) \) be the projection of this subset into \( X^{n-1} \). For example, if each \( X_i \) is the nonnegative reals, so that \( X^n \) is the positive orthant of \( n \)-dimensional Euclidean space, and \( Y \) is the set of \( (x_n,x_{n-1},\ldots,x_1) \in X^n \) such that \( \sum_{i=1}^{n} x_i = a \), then \( (Y|b) \) is the set of \( n \)-tuples \( (b,x_{n-1},\ldots,x_1) \) such that \( b + \sum_{i=1}^{n-1} x_i = a \), and \( P^n(Y|b) \) is the set of \( (x_{n-1},\ldots,x_1) \in X^{n-1} \) such that \( \sum_{i=1}^{n-1} x_i = a - b \). Define for \( n > 1 \),

\[
G_n(Y) = \{ p | p = g_n(x_n) \circ q, \quad q \in \max [F_{n-1}(P^n(Y|x_n))], \quad x_n \in P_n(Y) \}.
\]

Our version of the "principle of optimality" is the following:

**Theorem 3.** \( \max [F_n(Y)] = \max [G_n(Y)] \).

**Proof.** If \( p \in \max [F_n(Y)] \), then \( p \in F_n(Y) \), so clearly \( p = g_n(x_n) \circ q \), where \( q \in F_{n-1}(P^n(Y|x_n)) \). Choose a maximal such \( q \) (since translation by \( g_n(x_n) \) is continuous, the set of all such \( q \) is a closed subset of a compact set, hence is compact); then \( q \in \max [F_{n-1}(P^n(Y|x_n))] \), for otherwise we could find a \( q' \) in \( F_{n-1}(P^n(Y|x_n)) \) such that \( q' > q \), and thus \( p' = g_n(x_n) \circ q' > p \), which would contradict the
assertion that \( p \in \text{max} [F_n(Y)] \). Thus \( p \in G_n(Y) \), and it is clear that \( \text{max} [F_n(Y)] \subseteq G_n(Y) \subseteq F_n(Y) \). Lemma 3 shows that Lemma 2 may be applied to complete the proof of the theorem.

It should be noted that the theorem does not say that \( g_n(x_n) \circ \ldots \circ g_1(x_1) \in \text{max}[F_n(Y)] \) implies \( g_{n-1}(x_{n-1}) \circ \ldots \circ g_1(x_1) \in \text{max}[F_{n-1}(P^n(Y|x_n))] \). However, a slight variant of the proof shows that this stronger conclusion is valid if \( q_1 > q_2 \) implies \( g_n(x_n) \circ q_1 > g_n(x_n) \circ q_2 \), and in particular if the map \( p_0 \) is one to one.

4. THE LOCAL CASE

The set-valued set function \( \text{loc max} \) has a more complicated structure than the function \( \text{max} \). Although \( \text{loc max} [F_n(Y)] \) is easily seen to be nonempty (as a direct corollary to Theorem 2), we are only able to prove a partial analogue of Theorem 3. Define for \( n > 1 \),

\[ H_n(Y) = \{ p \mid p = g_n(x_n) \circ q, q \in \text{loc max} [F_{n-1}(P^n(Y|x_n))] \}, \]

\[ x_n \in F_n(Y) \].

**Theorem 4.** \( \text{Max} [H_n(Y)] \subseteq \text{loc max} [F_n(Y)] \subseteq \text{loc max} [H_n(Y)] \).

**Proof.** Note that \( G_n(Y) \subseteq H_n(Y) \subseteq F_n(Y) \); by Theorem 3 and Lemma 2 it follows that \( \text{max} [H_n(Y)] = \text{max} [F_n(Y)] \), which proves the first inclusion above. To prove the second inclusion, assume \( p \in \text{loc max} [F_n(Y)] \); then \( p \in F_n(Y) \) and we can find a neighborhood \( V \) of \( p \) such that no element of \( V \cap F_n(Y) \) is greater than \( p \). Choose a maximal \( q \) such that \( p = g_n(x_n) \circ q, q \in F_{n-1}(P^n(Y|x_n)) \), and pick a neighborhood
U of q such that V ⊇ g_n(x_n) o U. If q is not in
loc max [F_{n-1}(P^n(Y|x_n))], then we can find a q' ∈ U ∩ F_{n-1}(P^n(X|x_n))
such that q' > q. But g_n(x_n) o q' ∈ V ∩ F_n(Y) and
g_n(x_n) o q' > g_n(x_n) o q, contradicting our hypothesis.
Thus we have proved that loc max [F_n(Y)] ⊆ H_n(Y); by
Lemma 4 this implies loc max [F_n(Y)] ⊆ loc max [H_n(Y)].

The following example shows that the second inclusion
above may be proper: Let S be the real numbers under
addition and the usual ordering. Let X_1 be the two closed
intervals [−1, −\frac{1}{2}] and [\frac{1}{2}, 1]; let X_2 be the interval
[−1,0]; let g_1(x) = g_2(x) = x for all x.

Let Y = X_2 × X_1; then P^2(Y|x_2) = X_1 and we have

$$F_1(P^2(Y|x_2)) = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1],$$

loc max [F_1(P^2(Y|x_2))] = \{− \frac{1}{2}\} \cup \{1\},

$$H_2(Y) = [-\frac{3}{2}, -\frac{1}{2}] \cup [0,1],$$

$$F_2(Y) = [-2, 1],$$

loc max [H_2(Y)] = \{− \frac{1}{2}\} \cup \{1\},

loc max [F_2(Y)] = \{1\}.
REFERENCES


