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VALUES OF LARGE GAMES–VII: A GENERAL EXCHANGE ECONOMY WITH MONEY

Lloyd S. Shapley

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

PREFACE

This Memorandum concerns an application of the mathematical theory of games to a central problem in theoretical economics. It is one of a series dealing with the techniques of evaluating games with many players. (See RM–2648, RM–2649, RM–2650–PR, RM–2651, RM–2860–PR, and RM–3158–PR.) The economic interpretation of the result will be discussed in a planned companion piece in collaboration with Prof. M. Shubik, a Yale economist and consultant to the RAND Mathematics Department, who suggested the present investigation.

SUMMARY

It is shown that the "Shapley value" solution of a certain general class of competitive markets, regarded as multiperson games, converges to the classical "competitive equilibrium" solution when the set of traders in the market is expanded homogeneously.

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VALUES OF LARGE GAMES —VII:

A GENERAL EXCHANGE ECONOMY WITH MONEY

1. INTRODUCTION

This Memorandum concerns the limiting behavior of a general class of market games as the number of players is increased. The main theorem establishes, in a general form, a principle first suggested by Martin Shubik, that the game value should converge to the classical competitive solution. This may be compared with recent results of Debreu and Scarf [3], Shapley and Shubik [8], Aumann [2], and Vind [12] concerning the cores of such games, again following an initial treatment by Shubik [11]. The confluence of these solution concepts, which are based on quite different heuristic constructs, is remarkable; the economic implications will be discussed in a forthcoming companion memorandum [10] (see also [6], [9]). The present note deals only with the mathematical description of the model and the proof of convergence.

The market model that we use here is based on the classical formulation of Walras (see e.g. [1]), but with one important modification: the assumption that utility is unrestrictedly transferable. This is tantamount to the inclusion of a kind of "money" as an additional commodity among those being traded. This feature simplifies, but does not trivialize, the determination of the classical competitive price equilibrium. More importantly, for our purpose, it makes possible the use of the well-developed mathematical theory of cooperative games "with side payments." The opposite assumption — of nontransferable utility — raises some novel technical and conceptual questions on the game theory side, and will not be considered here (but see [6], [9]).

We shall increase the number of player by replication of the model, as in [3], [8], [11]. That is, we assume a fixed number, n, of different types of players and a variable number, k, of players of each type. As k increases, the market expands in a homogeneous manner,* without disturbing the Walrasian price structure, and it becomes possible to compare solutions of different-sized games on a consistent basis. We must begin with a discussion of the case $k = 1$.

2. THE BASIC MARKET MODEL

Let there be n traders and m goods, not including "money." An allocation will be written as a double array:

$$x = \langle x^1, \ldots, x^n \rangle, \text{ where } x^i = (x^i_1, \ldots, x^i_m), \text{ all } x^i_j \geqq 0.$$

Here x^i_j is the amount of the j^{th} good attributed to the i^{th} trader. His utility function is assumed to have the following separable form:

$$U^i(x^i, \xi^i) \equiv u^i(x^i) + \xi^i, \tag{1}$$

with u^i concave and differentiable, where ξ^i represents the amount of "money" (positive or negative) in his account, normalized so that he starts with zero. Note that a transfer of "money" between individuals leaves the sum of their utilities unchanged.

The initial allocation of goods will be denoted by a. We shall assume that

$$\sum_i a^i_j > 0, \quad j = 1, \ldots, m, \tag{2}$$

i.e., that every good is present in some amount.

The economic model becomes a cooperative n-person game with side payments if we permit consenting players to transfer goods and "money" at will. Let N denote the set of all players, and S any subset of N. The potential worth of the coalition S is given by

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*Equivalently (by a change of scale) one can regard the economy itself as of fixed size, with the original n traders being progressively disaggregated into smaller, symmetrical fragments (see below, p. 6).

(3) $$v(S) = \max_x \left\{ \sum_S u^i(x^i), \ \underline{\text{given}} \ \Sigma_S x^i = \Sigma_S a^i \ \underline{\text{and}} \ x \geqq 0 \right\}.$$

This is known as the <u>characteristic function</u> of the game. That the maximum is achieved by some allocation follows from the continuity of the u^i and the compactness of the range of x. In particular, taking S = N, there will be at least one <u>optimal allocation for the whole market</u>; let us denote it by b. Thus we have

(4) $$\sum_N u^i(b^i) = v(N), \quad \text{and} \quad \sum_N b^i = \sum_N a^i.$$

Let u^i_j denote the partial derivative of u^i with respect to the j-th good. It is clear that if b^i_j is positive, then $u^i_j(b^i) \geq u^{i'}_j(b^{i'})$ for every i', for otherwise i could give (or sell) some j to i' and thereby increase the total utility. But, for each j, b^i_j is positive for at least one i, by (2). Hence we can unambiguously define the <u>competitive prices</u>:

(5) $$\pi_j = u^i_j(b^i) \text{ for all } i \text{ such that } b^i_j > 0,$$

for j = 1, . . . , m. From this it follows that

(6) $$\pi_j \geq u^i_j(b^i) \text{ for all } i \text{ such that } b^i_j = 0.$$

It can be verified without difficulty that these prices are independent of which optimal allocation b (if several exist) is used in the definition.*

If the traders use these prices to buy and sell their way from the initial allocation, a, to an optimal allocation, b, then their net monetary receipts are given by $\pi \cdot (a^i - b^i)$, i = 1, . . . , n. Their final utility levels are given by

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*Note that since we have not assumed free disposal of goods it is quite possible to have negative prices.

(7) $$\omega_i = u^i(b^i) + \pi \cdot (a^i - b^i), \quad i = 1, \ldots, n.$$

These payoffs, like the prices, are independent of the particular optimal allocation used.* We call ω the competitive payoff vector, since it corresponds to the classical competitive equilibrium solution of the (m+1)–goods exchange economy with utility functions (1) and no effective lower–bound constraint on the ξ^i.

3. DEFINITION OF THE VALUE

The value of the game, to a given player, can be described intuitively as his average marginal worth over all possible coalitions. Or we may say that it is his expected marginal worth in a coalition chosen at random. (The language of probability is convenient here, but not essential to the interpretation.) Thus, we define

(8) $$\phi^i = \underset{S \ni i}{E} \{v(S) - v(S - (i))\}.$$

The probabilities to be associated with the "expectation" symbol E are such that each coalition size from 1 to n has probability 1/n, while all coalitions of the same size are equally likely. Thus, the "E" in (8) could be replaced by a "Σ," followed by a coefficient $(s-1)!\,(n-s)!/n!$, where s denotes the number of players in S.

This value formula can be derived, without reference to probabilities, from axioms of symmetry, Pareto optimality, and additivity of the values of independent games (see [5]). These axioms, together with the postulate that the value of a game depends only on its characteristic function, form the basis for the claim that the value represents a "fair division" of the gains accruing from coalitional power.

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*Uniqueness here depends critically on differentiability of the u^i.

Two simple properties of the value are its Pareto optimality:

(9) $$\sum_N \phi^i = v(N),$$

and its individual rationality:

(10) $$\phi^i \geq v((i)), \; i = 1, \ldots, n,$$

the latter a consequence of the superadditivity of the set-function v.

To calculate the explicit values of our present market model would require sufficient knowledge of the functions u^i to carry out the maximization in (3) for every set S. For the competitive solution (7), on the other hand, only one such maximization is required.

4. THE k-FOLD REPLICATION OF THE MARKET

Now let there be n types of traders, with k traders of each type. Traders of the same type have identical utility functions and identical initial bundles, but they are not constrained to behave identically—there are nk independent players in the game. We shall continue to use the notation of the preceding sections, but with the understanding that the index "i" hereafter refers to types, not individuals.

Despite their independence of action, the traders in the k-fold market cannot improve on an optimal allocation of the original, 1-fold market, repeated k times. This follows from the concavity of the utility functions u^i, which implies that players of the same type cannot lose by pooling their holdings and dividing them equally. Hence the competitive prices and payoffs are the same as before. That is, they are independent of k, except in the sense that the competitive payoff vector is now the n-vector ω repeated k times. In short, homogeneous expansion of the market does not disturb the classical competitive solution.

[A different viewpoint, which is sometimes useful although we shall not pursue it here, is to regard the k-fold market as being derived from the original market by <u>fractionating</u>. Each of the original traders (or trading firms) is replaced by k smaller players, with initial holdings $\hat{a}^i = a^i/k$ and utility functions $\hat{u}^i(\hat{x}^i) = u^i(k\hat{x}^i)/k$. The resulting market is formally identical, except for a change of scale, with the version described above.

One attractive feature of this approach is that the convergence results—both for the core and for the value—probably remain valid even if the fractions into which the players are broken are not equal. All that seems to be required is that the largest fragment go to zero. Thus, the limit can be approached through games in which no two players are the same.

There is a close analogy here to the fractionating of votes in weighted majority games, as treated in [7].]

The value of the replicated game to a given player will depend only on his type, by symmetry, so that we may use the notation $\phi^i(k)$ without ambiguity. Moreover, symmetry and Pareto optimality imply that

$$\sum_{i=1}^{n} \phi^i(k) = v(N). \tag{11}$$

Note that N is now the set of all <u>types</u>, not players, while v is still the function defined at (3) above. (The characteristic function for the replicated game will be formulated in the next section.) The n-dimensional vectors $\phi(k)$ can therefore be meaningfully compared for different values of k.

THEOREM: In a k-fold market with transferable utility, and concave, differentiable utility functions, the value converges to the competitive solution; i.e.,

$$\phi(k) \rightarrow \omega \quad \text{as} \quad k \rightarrow \infty. \tag{12}$$

The idea of the proof, which will be given in detail in the following sections, is to show first that the marginal worth of a player in a balanced coalition (one in which all types are equally represented) is approximately equal to his competitive payoff, and then that "almost all" coalitions, when k is sufficiently large, are sufficiently nearly balanced.

5. THE EXTENDED CHARACTERISTIC FUNCTION

First we must formulate the characteristic function for the replicated game. Since the worth of a coalition depends only on the types of its players, it is expedient to replace the usual function, defined on sets, by a function $F(\mathscr{s})$ defined on n-tuples $\mathscr{s} = \mathscr{s}^1, \ldots, \mathscr{s}^n$ of nonnegative integers. Then, corresponding to (3), we have

$$(13) \qquad F(\mathscr{s}) = \max_x \left\{ \sum_N \mathscr{s}^i u^i(x^i), \ \underline{\text{given}} \ x \geqq 0 \ \underline{\text{and}} \ \sum_N \mathscr{s}^i x^i = \sum_N \mathscr{s}^i a^i \right\}.$$

The maximum is always attained for some x, by continuity and compactness.

Since nothing in (13) demands that the $\mathscr{s}^i$ be integers, we shall regard $F(\mathscr{s})$ as defined for <u>all</u> nonnegative real vectors $\mathscr{s}$. These vectors will be called "profiles."

LEMMA. (i) $F(\mathscr{s})$ is homogeneous of degree 1 (with respect to a nonnegative multiplier) and concave.

(ii) $F(\mathscr{s})$ has continuous first partial derivatives for all $\mathscr{s} > 0$, given by

$$(14) \qquad \frac{\partial}{\partial \mathscr{s}^i} F(\mathscr{s}) = u^i(b^i) + \sum_{j=1}^m (a^i_j - b^i_j) u^{i(j)}_j (b^{i(j)}),$$

where b (a function of $\mathscr{s}$) is any maximizer in (13), and i(j) is such that $b^{i(j)}_j > 0$ for each j.

<u>Proof</u> (i). Homogeneity is immediate. For concavity, let b' and b'' maximize in (13) for $\mathscr{s}'$ and $\mathscr{s}''$, respectively, and let $\mathscr{s} = \mathscr{s}' + \mathscr{s}''$.

Then we have

$$F(\mathscr{s}') + F(\mathscr{s}'') = \sum_N \left[\mathscr{s}'^i u^i(b'^i) + \mathscr{s}''^i u^i(b''^i) \right]$$

$$\leq \sum_N u^i\left(\frac{\mathscr{s}'^i b'^i + \mathscr{s}''^i b''^i}{\mathscr{s}^i} \right) \qquad \text{(by concavity of } u^i\text{)}$$

$$\leq \max_x \left\{ \sum_N \mathscr{s}^i u^i(x^i), \ \underline{\text{given}}\ x \geq 0\ \underline{\text{and}}\ \Sigma \mathscr{s}^i x^i = \Sigma \mathscr{s}'^i b'^i + \Sigma \mathscr{s}''^i b''^i \right\}$$

$$= F(\mathscr{s}).$$

Hence, for arbitrary $\alpha, \beta \geq 0$, we have

$$\alpha F(\mathscr{s}') + \beta F(\mathscr{s}'') = F(\alpha \mathscr{s}') + F(\beta \mathscr{s}'') \leq F(\alpha \mathscr{s}' + \beta \mathscr{s}''),$$

using the homogeneity.

Proof (ii). Take a fixed $\mathscr{s} > 0$ and a fixed maximizer b for $\mathscr{s}$. For each j, we have $\Sigma_N \mathscr{s}^i b^i_j = \Sigma_N \mathscr{s}^i a^i_j$, which is positive by (2) and the positivity of $\mathscr{s}$; hence we can find $i(j)$ with $b^{i(j)}_j > 0$. For $\mathscr{t}$ in a neighborhood of $\mathscr{s}$, define $c = c(\mathscr{t})$ by

$$c^{i(j)}_j = b^{i(j)}_j + \frac{\Sigma_N \mathscr{t}^i (a^i_j - b^i_j)}{\mathscr{t}^{i(j)}}$$

and

$$c^i_j = b^i_j \qquad \text{if } i \neq i(j).$$

The purpose of this construction is to get an allocation near b that is feasible for $\mathscr{t}$. We must verify, therefore, that $c \geq 0$ and that $\Sigma \mathscr{t}^i c^i = \Sigma \mathscr{t}^i a^i$. The latter follows by direct substitution, taking each j separately. The other follows, for a sufficiently small neighborhood of $\mathscr{s}$, from the fact that

$$\Sigma_N \mathscr{t}^i (a^i_j - b^i_j) = \Sigma_N (\mathscr{t}^i - \mathscr{s}^i)\ (a^i_j - b^i_j),$$

so that $c \to b$ as $\mathscr{t} \to \mathscr{s}$. (Note that $b^{i(j)}_j$ and $\mathscr{s}^{i(j)}$ are both strictly positive.)

Now define the function $G(t) = \Sigma_N t^i u^i(c^i(t))$. This function is clearly differentiable. Moreover, we have $F(t) \geq G(t)$ in a neighborhood of s, with equality at $t = s$. On the other hand, since F is concave, it is majorized by a linear support function L, with $L(s) = F(s)$. Trapped between two differentiable functions, F itself is necessarily differentiable. Moreover, for concave functions, differentiability (in a region) implies continuous differentiability.*

Finally, we must compute the derivatives of F at s. These are the same as the derivatives of $G(t)$, evaluated at $t=s$. Let us fix our attention on t^1, for notational simplicity. Then we have

$$\text{(15)} \qquad \frac{\partial}{\partial t^1} G(t) = u^1(c^1) + \Sigma_N t^i \sum_{j=1}^{m} u^i_j(c^i) \frac{\partial c^i_j}{\partial t^1} .$$

But $\partial c^i_j / \partial t^1 = 0$ if $i \neq i(j)$; hence only i, j pairs with $i = i(j)$ need be considered in the double summation. If $i(j) = 1$, we have

$$\frac{\partial c^{i(j)}_j}{\partial t^1} = \frac{a^1_j - b^1_j}{t^{i(j)}} - \frac{\Sigma_N t^i(a^i_j - b^i_j)}{(t^{i(j)})^2}$$

$$= \frac{a^1_j - b^1_j}{t^{i(j)}} - \frac{c^{i(j)}_j - b^{i(j)}_j}{t^{i(j)}} = \frac{a^1_j - c^1_j}{t^{i(j)}} .$$

If $i(j) \neq 1$, we also have

$$\frac{\partial c^{i(j)}}{\partial t^1} = \frac{a^1_j - b^1_j}{t^{i(j)}} = \frac{a^1_j - c^1_j}{t^{i(j)}} .$$

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*This can be seen geometrically. Consider a closed, convex body. The limit of any convergent sequence of supporting hyperplanes is itself a supporting hyperplane. It follows that if there is known to be a unique supporting hyperplane (i.e., a tangent) at every boundary point, then this tangent must vary continuously as a function of the boundary point.

Therefore, substituting in (15), we obtain

$$\frac{\partial}{\partial \boldsymbol{t}^1} G(\boldsymbol{t}) = u^1(c^1) + \sum_{j=1}^{m} (a_j^1 - c_j^1)\, u_j^{i(j)}(c^{i(j)}).$$

If we now set $\boldsymbol{t} = \boldsymbol{s}$, we obtain the desired result (14). This completes the proof of the lemma.

6. MARGINAL WORTH

We continue with the proof of the main theorem. Note that the $u_j^{i(j)}(b^{i(j)})$ that appears in the lemma is just the competitive price (5) of the j–th good in the market associated with the profile $\boldsymbol{s}$. The derivatives of F are therefore just the competitive payoffs of that market (compare (14) with (7)). Since they are homogeneous of degree zero, we can introduce a new notation: $\bar{\boldsymbol{s}} = \boldsymbol{s}/\Sigma \boldsymbol{s}^i$, for the normalized profile associated with $\boldsymbol{s}$ and define

(16) $$\omega^i(\bar{\boldsymbol{s}}) = \frac{\partial F}{\partial \boldsymbol{s}^i}.$$

Let $\bar{\boldsymbol{n}}$ denote the normalized profile of the all–player coalition; i.e., $\bar{\boldsymbol{n}} = (1/n, \ldots, 1/n)$. Then $\omega^i(\bar{\boldsymbol{n}}) = \omega^i$.

For any profile $\boldsymbol{s}$ such that $\boldsymbol{s}^1 \geq 1$, let us express the marginal worth of type 1 as follows:

$$D^1(\boldsymbol{s}) = F(\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^n) - F(\boldsymbol{s}^1 - 1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^n),$$

and similarly for types 2, ..., n. By (16) and the concavity of F, we have

(17) $$D^i(\boldsymbol{s}) \geq \omega^i(\bar{\boldsymbol{s}}), \quad \text{for } i = 1, \ldots, n.$$

Then, by the continuity of the $\partial F/\partial \boldsymbol{s}^i$, it is clear that given any $\epsilon > 0$ we can find a $\delta = \delta(\epsilon)$ such that

(18) $\qquad ||\bar{\mathscr{s}}-\bar{\mathscr{n}}|| < \delta$ implies $D^i(\mathscr{s}) \geqq \omega^i - \epsilon$, for all i.

(Here $||\mathscr{t}||$ denotes $\max_i |\mathscr{t}^i|$.)

In words, (18) states that the marginal worth of type i in a coalition that is nearly balanced is at worst approximately equal to the competitive payoff for that type. The opposite inequality: $D^i(\mathscr{s}) \leqq \omega^i + \epsilon$, though true, is more difficult to establish at this point; luckily there is an easy way to reverse the inequality later (see (21)).

This completes the first and longer part of the proof, as outlined in the last paragraph of Sec. 4.

7. BALANCED COALITIONS

We shall now give a precise statement of the idea that "almost all" coalitions are nearly balanced. Call a coalition "δ-balanced" if its profile $\mathscr{s}$ satisfies $\|\bar{\mathscr{s}}-\bar{\mathscr{n}}\| < \delta$. Then, given ϵ, δ as above, we can find a bound $s_o = s_o(\epsilon)$ such that for every integer $s > s_o$ the probability is greater than $1-\epsilon$ that an s-element set, formed by choosing the type of each element at random, will be δ-balanced. This is a form of the "law of large numbers."* But if the s-element set is formed instead by drawing from a finite collection, in which there are exactly k elements of each type to start with, then the probability of a δ-balanced profile will be even greater, since sampling "without replacement" uniformly tends to reproduce the original distribution more faithfully than sampling "with replacement."

Thus, for a random s-member coalition in the k-fold market, we have

*See [4], p. 191.

(19) $$\text{Prob}\ \{\ \|\bar{\mathscr{s}}-\bar{\mathscr{m}}\| < \delta\} > 1-\epsilon\ ,$$

where $s > s_o(\epsilon)$, $\delta = \delta(\epsilon)$, and, in order for the statement to make sense, $k \geq s/n$.

8. THE LIMIT OF THE VALUE

We are at last ready to estimate the value of the k–fold market game to a typical player p of type $i = i(p)$. We have, by definition,

$$\phi^i(k) = \frac{1}{nk} \sum_{s=1}^{nk} \left[\underset{S}{E} \left\{ D^i(\mathscr{s}),\ \underline{\text{given}}\ |S| = s\ \underline{\text{and}}\ S \ni p \right\} \right] .$$

Without loss of generality we may assume that $u^i(a^i) = 0$, whence $D^i(\mathscr{s}) \geqq 0$ for all $\mathscr{s}$ with $\mathscr{s}^i \geq 1$. Therefore, omitting only nonnegative terms and applying (19),* we have

$$\phi^i(k) \geqq \frac{1}{nk} \sum_{s=s_o}^{nk} \left[\underset{S}{E} \left\{ D^i(\mathscr{s}),\ \underline{\text{given}}\ |S| = s\ \underline{\text{and}}\ S \ni p \right\} \right]$$

$$\geqq \frac{(1-\epsilon)}{nk} \sum_{s=s_o}^{nk} \left[\underset{S}{E} \left\{ D^i(\mathscr{s}),\ \underline{\text{given}}\ |S| = s,\ S \ni p,\ \underline{\text{and}}\ \|\bar{\mathscr{s}}-\bar{\mathscr{m}}\| < \delta \right\} \right] .$$

Hence, by (18),

$$\phi^i(k) \geqq \frac{(1-\epsilon)}{nk} \sum_{s=s_o}^{nk} (\omega^i - \epsilon) = \frac{(1+\epsilon)(nk-s_o+1)(\omega^i-\epsilon)}{nk} .$$

Now define $k_o = k_o(\epsilon) = s_o/n\epsilon$. Then, for all $k > k_o$,

(20) $$\phi^i(k) \geqq (1-\epsilon)^2(\omega^i-\epsilon) = \omega^i - 0^i(\epsilon),$$

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*The condition $S \ni p$ makes no essential difference in applying (19). Without loss of generality we may assume that $u^i(a^i) \geq 0$, whence $D^i(\mathscr{s}) \geq 0$.

where 0^i is a function, not depending on k, which goes to zero with ϵ. The final step is to observe (see (11)) that

$$\phi^i(k) = v(N) - \sum_{h \neq i} \phi^h(k) \tag{21}$$

$$\leqq v(N) - \sum_{h \neq i} (\omega^h - 0^h(\epsilon))$$

$$= \omega^i + \sum_{h \neq i} 0^h(\epsilon).$$

Combining this inequality with that of (20), we obtain the desired result:

$$\phi^i(k) \to \omega^i \text{ as } k \to \infty.$$

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