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MINIMUM ENERGY LOSS HEADING CHANGES
FOR HYPersonic FLIGHT FROM ORBIT

Russell D. Shaver

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This Memorandum presents an analysis of minimum energy loss heading changes for a spacecraft operating in a geocentric orbit near the earth. The analysis is in large part an extension of the work previously reported in RAND Memorandum The Synergetic Plane Change for Orbiting Spacecraft, RM-3231-PR, by F. S. Nyland. However, the present Memorandum is a more detailed analysis of the energy requirements demanded for such maneuvers and includes additional information important to vehicle design. The conclusions of this Memorandum should be of interest to specialists in the fields of atmospheric reentry and trajectory analysis, designers of advanced aerospace vehicles, and persons concerned with space flight and its future utility.
Velocity requirements for minimum energy loss during orbit turning have been derived for both the aerodynamic synergetic mode of transfer and for the pure rocket bi-elliptic transfer. In this Memorandum velocity requirements obtained through an optimal energy transfer study are compared to velocity requirements for a rocket transfer. The comparison indicates that there are large areas of near-earth transfer regimes where a synergetic transfer requires substantially less fuel than the corresponding rocket transfer.

Various vehicle design problems are discussed in an examination of minimum energy loss inclination changing maneuvers. Indicated total heat, indicated maximum heat rates, and normal wing loading are included for each trajectory considered. From these curves, certain design criteria are obtained and their influence on performance is examined, particularly in relation to their influence upon the velocity requirements for the orbit transfer.

A simple control scheme is devised from the results of the optimization program. This simple control model produces near-optimal performance and it also can be modified to overcome structural difficulties. The results of this Memorandum indicate that synergetic transfers can be investigated by simple numerical schemes.
ACKNOWLEDGEMENTS

The author would like to acknowledge the invaluable aid and encouragement that Fred Nyland and Stuart Dreyfus, both of the RAND Corporation, rendered to him during the inception, performance, and writing of this Memorandum. Without their assistance the task of compiling this Memorandum would have been greatly increased.

Special thanks is given to Warren Hollis, who diligently checked the author's equations and constructed from them the computer program from which the numerical results of this paper were obtained.
CONTENTS

PREFACE ........................................ iii
SUMMARY ........................................ v
ACKNOWLEDGMENTS .................................... vii
LIST OF FIGURES ..................................... xi
SYMBOLS ........................................... xv

Section

I. INTRODUCTION .................................. 1

II. DEVELOPMENT ................................... 4

   Statement of the Problem ......................... 4
   Equations of Motion for Aerodynamic Maneuvering ... 11
   Acceleration Dosage Constraint ..................... 18
   Performance Criterion ............................. 21
   Simplified Control Routine ....................... 21

III. RESULTS ....................................... 23

   Control Profiles ................................. 23
   Velocity Requirements ............................ 31
   Vehicle Design Parameters ....................... 35
   Effect Due to Variations in the Initial Conditions
   and Vehicle Design ............................... 44
   Trajectories .................................... 55
   Comparisons with Non-optimal Simplified Control
   Results ......................................... 55

IV. CONCLUSIONS ................................... 61

Appendix

A. DERIVATION OF THE EQUATIONS OF MOTION ............ 65

B. DERIVATION OF THE VELOCITY INCREMENTS REQUIRED
   FOR AN IMPULSIVE TRANSFER ....................... 71

C. THE METHOD OF STEEPEST ASCENT .................... 79

D. TRAJECTORIES ................................... 88

References ...................................... 123
FIGURES

1. Vehicle coordinate system ........................................ 13
2. Euler angles ..................................................... 16
3. Maximum acceleration dosage versus vehicle acceleration .... 20
4. Vehicle angle of attack versus time of flight (acceleration dosage unbounded) ........................................ 25
5. Vehicle bank angle versus time of flight (acceleration dosage unbounded) ........................................ 26
6. Vehicle angle of attack versus time of flight (acceleration dosage bounded) ........................................ 28
7. Vehicle bank angle versus time of flight (acceleration dosage bounded) ........................................ 29
8. Total velocity increment requirements for synergetic plane changing ........................................ 34
9. Velocity impulse requirements versus terminal altitude (acceleration dosage bounded, \( \frac{L}{D} \)max = 2) .................. 36
10. Velocity impulse requirements versus terminal altitude (acceleration dosage bounded, \( \frac{L}{D} \)max = 3.32) ........ 37
11. Efficiency of conversion of vehicle's kinetic energy into heat energy at vehicle surface, as a function of altitude . 39
12. ITH absorbed versus inclination change ......................... 40
13. ITH rate versus given inclination change (acceleration dosage unbounded) ........................................ 42
14. ITH rate versus given inclination change (acceleration dosage bounded) ........................................ 43
15. Maximum vehicle normal acceleration loads versus inclination change (acceleration dosage unbounded) ........ 45
16. Maximum vehicle normal acceleration loads versus given inclination change (acceleration dosage bounded) ........ 46
17. Velocity impulse requirements and vehicle loads versus initial orbit altitude (acceleration dosage bounded) .... 48
<table>
<thead>
<tr>
<th>Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>Velocity impulse requirements and vehicle loads versus variation in initial deorbit velocity impulse (acceleration dosage bounded)</td>
<td>49</td>
</tr>
<tr>
<td>19</td>
<td>Velocity impulse requirements versus W/A (acceleration dosage bounded)</td>
<td>51</td>
</tr>
<tr>
<td>20</td>
<td>Total vehicle heat load versus W/A</td>
<td>52</td>
</tr>
<tr>
<td>21</td>
<td>Maximum vehicle heating rates versus W/A</td>
<td>53</td>
</tr>
<tr>
<td>22</td>
<td>Maximum vehicle normal loading versus W/A</td>
<td>54</td>
</tr>
<tr>
<td>23</td>
<td>Velocity increment requirements versus plane change</td>
<td>57</td>
</tr>
<tr>
<td>24</td>
<td>Vehicle heating loads versus inclination change</td>
<td>58</td>
</tr>
<tr>
<td>25</td>
<td>Vehicle acceleration loads versus inclination change</td>
<td>59</td>
</tr>
<tr>
<td>26</td>
<td>Vehicle orbit plane coordinates</td>
<td>69</td>
</tr>
<tr>
<td>27</td>
<td>Single impulse velocity orientation</td>
<td>73</td>
</tr>
<tr>
<td>28</td>
<td>Altitude versus velocity (acceleration dosage unbounded)</td>
<td>89</td>
</tr>
<tr>
<td>29</td>
<td>Altitude versus velocity (acceleration dosage bounded)</td>
<td>90</td>
</tr>
<tr>
<td>30</td>
<td>Terminal descent angle versus given inclination change</td>
<td>91</td>
</tr>
<tr>
<td>31</td>
<td>Minimum (terminal) altitude versus given inclination change (acceleration dosage unbounded)</td>
<td>92</td>
</tr>
<tr>
<td>32</td>
<td>Minimum (terminal) altitude versus given inclination change (acceleration dosage bounded)</td>
<td>93</td>
</tr>
<tr>
<td>33</td>
<td>Total range versus given inclination change (acceleration dosage unbounded)</td>
<td>94</td>
</tr>
<tr>
<td>34</td>
<td>Total range versus given inclination change (acceleration dosage bounded)</td>
<td>95</td>
</tr>
<tr>
<td>35</td>
<td>Terminal side range latitude versus inclination change (acceleration dosage unbounded)</td>
<td>96</td>
</tr>
<tr>
<td>36</td>
<td>Terminal side range latitude versus inclination change (acceleration dosage bounded)</td>
<td>97</td>
</tr>
<tr>
<td>37</td>
<td>Terminal downrange longitude versus inclination change (acceleration dosage unbounded)</td>
<td>98</td>
</tr>
<tr>
<td>38</td>
<td>Terminal downrange longitude versus inclination change (acceleration dosage bounded)</td>
<td>99</td>
</tr>
<tr>
<td>No.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>-----</td>
<td>---------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>39</td>
<td>Terminal time versus inclination change (acceleration dosage unbounded)</td>
<td>100</td>
</tr>
<tr>
<td>40</td>
<td>Terminal time versus inclination change (acceleration dosage bounded)</td>
<td>101</td>
</tr>
<tr>
<td>41</td>
<td>Altitude versus time (acceleration dosage bounded and unbounded)</td>
<td>102</td>
</tr>
<tr>
<td>42</td>
<td>Altitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 1))</td>
<td>103</td>
</tr>
<tr>
<td>43</td>
<td>Altitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 2))</td>
<td>104</td>
</tr>
<tr>
<td>44</td>
<td>Altitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>105</td>
</tr>
<tr>
<td>45</td>
<td>Altitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 1))</td>
<td>106</td>
</tr>
<tr>
<td>46</td>
<td>Altitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 2))</td>
<td>107</td>
</tr>
<tr>
<td>47</td>
<td>Altitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>108</td>
</tr>
<tr>
<td>48</td>
<td>Side range latitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 1))</td>
<td>109</td>
</tr>
<tr>
<td>49</td>
<td>Side range latitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 2))</td>
<td>110</td>
</tr>
<tr>
<td>50</td>
<td>Side range latitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>111</td>
</tr>
<tr>
<td>51</td>
<td>Side range latitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 1))</td>
<td>112</td>
</tr>
<tr>
<td>52</td>
<td>Side range latitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 2))</td>
<td>113</td>
</tr>
<tr>
<td>53</td>
<td>Side range latitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>114</td>
</tr>
<tr>
<td>54</td>
<td>Downrange longitude versus time (acceleration dosage bounded and unbounded)</td>
<td>115</td>
</tr>
<tr>
<td>55</td>
<td>Downrange longitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 1))</td>
<td>116</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>56.</td>
<td>Downrange longitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 2))</td>
<td>117</td>
</tr>
<tr>
<td>57.</td>
<td>Downrange longitude versus time (acceleration dosage unbounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>118</td>
</tr>
<tr>
<td>58.</td>
<td>Downrange longitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 1))</td>
<td>119</td>
</tr>
<tr>
<td>59.</td>
<td>Downrange longitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 2))</td>
<td>120</td>
</tr>
<tr>
<td>60.</td>
<td>Downrange longitude versus time (acceleration dosage bounded, ((L/D)_{\text{max}} = 3.32))</td>
<td>121</td>
</tr>
</tbody>
</table>
SYMBOLS

A = 'vehicle reference area, ft^2
a = vehicle acceleration
\( \mathbf{a} \) = acceleration (vector), ft/sec^2
B = total acceleration dose
b = constant

\( c_D \) = coefficient of drag (function of L/D max)

\( c_L \) = coefficient of lift
c = constant

\( \mathbf{D} \) = drag vector, lb

F = matrix of state coefficients
f = coefficient of efficiency of heating

G = matrix of control coefficients

\( g \) = acceleration of gravity (32.174 ft/sec^2)

h = altitude

\( h_0 \) = initial altitude

i = inclination of vehicle's orbital plane

\( \Delta i \) = desired orbit plane change

(\( i, \mathbf{j}, k \)) = unit triad defined in Fig. 1(b)

\( I_{\alpha \alpha}, I_{\phi \phi}, I_{\dot{\phi} \dot{\phi}} \) = influence integrals

L = lift vector, lb

m = vehicle mass, slugs

n = nodal angle of vehicle's orbital plane

p = planar angle

dP = "path" distance in function space

\( Q = \text{aerodynamic heating integral} = \frac{1}{2} \int_0^T f \mathbf{c}_D \rho \mathbf{V}^3 \, dt \)
\( R \) = radius of the earth = (3442 n mi)

\( r \) = distance from center of the earth, ft

\( T \) = final time of flight

\( t \) = time

\( u_\alpha \) = set of control variables

\( (u_r, u_\psi, u_\theta) \) = unit vectors defining spherical coordinate system located at vehicle position in Fig. 1(a)

\( V \) = velocity (vector), ft/sec

\( V_c \) = circular orbit velocity

\( V_0 \) = earth skimming velocity

\( V_r \) = vehicle velocity in the radial direction

\( V_\psi \) = vehicle velocity in the easterly direction

\( V_\theta \) = vehicle velocity in the northerly direction

\( \mathbf{W} \) = weight vector, lb

\( x_L \) = set of state variables

\( \alpha \) = vehicle angle of attack

\( \beta \) = inverse atmospheric scale height = 1/24,000 ft

\( \delta \) = dummy variable

\( \epsilon \) = induced drag constant

\( \zeta \) = ratio of initial orbit radial distance from the earth to the terminal orbit radial distance

\( \theta \) = vehicle latitude, deg

\( \lambda \) = Lagrange multiplier

\( \lambda_L \) = set of adjoint variables

\( \mu \) = Lagrange multiplier

\( \kappa \) = angle of northerly direction defining vehicle orbit plane counter-clockwise from an easterly direction, deg.
\( \xi = \) vehicle bank angle measured from velocity-radius vector plane, deg

\( \Pi = \) boundary function

\( \rho = \) atmospheric density

\( \rho_o = \) mean atmospheric density at sea level

\( \sigma = \frac{r_{\text{perigee}}}{r_{\text{apogee}}} \)

\( \hat{\theta} = \) optimization function

\( \varphi = \) vehicle descent angle measured from horizon, deg

\( \psi = \) vehicle longitude, deg

\( \Omega = \) stopping function

\( (') = \frac{d}{dt} = \) derivative with respect to time

\( (\cdot)^* = \) to be referred to nominal trajectory

\( (\cdot)_1 = \) evaluated at initial altitude

\( (\cdot)_2 = \) evaluated at apogee and transfer ellipse
I. INTRODUCTION

Because orbiting spacecraft travel at very high velocities, comparatively large expenditures of energy are required for seemingly modest changes in the orbital plane. If the impulses used to change the orbit plane are achieved by a rocket engine, then large expenditures of fuel are required for any substantial out-of-plane maneuver. Until recently even modest maneuvering has been considered uneconomical, since, under the constraint of a fixed payload in orbit for an available booster, the large fuel requirements, increased structural weight, and additional propulsive devices required for maneuvering result in substantial reductions in payload weight. It is only recently that several analysts\(^1,\ 2\) realized that some of the energy required for out-of-plane maneuvering could be obtained by utilizing the relatively dense atmosphere for generating aerodynamic forces to effect orbital plane changes, thus introducing the corresponding possibility of a substantial fuel saving. It was noted that velocity losses due to drag during such an aerodynamic maneuver could be kept small, and that the velocity impulse required for atmosphere reentry was equally small. These observations led to a flight profile for the orbiting spacecraft that initially slowed the spacecraft to achieve atmospheric reentry, then utilized the vehicle's lift to maneuver in the atmosphere, and finally used the vehicle's rocket engine to overcome the incurred drag losses and to restore the vehicle into the desired terminal orbit (Fig 1(a), p. 13). This combination of vehicle lifting forces and thrust forces has led to the use of the word "synergistic," meaning to work together, to describe
this maneuver. This Memorandum presents a further investigation of synergetic maneuvers, and how they affect previous notions about the high energy requirements of maneuvering. *

London and Nyland investigated synergetic maneuvers by first specifying a complete trajectory and its associated control profile, and then solving the differential equations of motion. (1, 2) While their studies demonstrated that aerodynamic maneuvers could result in substantial energy savings, they did not indicate how such maneuvers could be varied to increase these savings. Since the control profiles were essentially fixed by the initial assumptions required for hand solution of the differential equations, neighboring trajectories and other control profiles could not be investigated. This analysis, in contrast, studies a variety of trajectories with a view to minimizing the energy lost during a heading change maneuver (this is not necessarily the same as minimizing the velocity loss). Several parametric variations and their effects on the control profile, the trajectory, the vehicle design, etc., were analyzed with the aid of a digital computer. The results of this study include an accounting of the effects of a large number of parameter variations on the total velocity lost during a maneuver. Additional problems, such as heating and acceleration loads incurred during synergetic maneuvers are also considered to suggest possible trends in vehicle design and operation.

This study does not seek to define a specific control law or to design a vehicle configuration. It is, however, an attempt to

*The synergetic maneuver itself has many interesting applications, which, however, are not discussed in this paper. These applications are recognized on both sides of the Iron Curtain. The Soviet Union for instance, has stated that winged reconnaissance satellites, using a combination aerodynamic/propulsion system for efficient orbit maneuvers, are under development. (3, 4, 5)
investigate the general problem of maneuverability and optimal control under realistic constraints. The general results (i.e., the velocity requirements for the performance of the various maneuvers) do not, in fact, represent the absolute minimum velocity impulses that could be achieved. However, these results do represent a near minimum and tend to characterize the most likely control profiles for future synergistic maneuvers. As such, the trajectories obtained may be called "optimal," although the reader is warned that this description is to be used to represent minimum energy heading changes and not minimum impulsive velocity requirements.
II. DEVELOPMENT

STATEMENT OF THE PROBLEM

Before we define what exactly is to be meant by an "optimal" trajectory, let us proceed to a more complete description of the problem. In this study we will be concerned with the transfer of an orbiting vehicle from one specified circular orbit to another, and have restricted our scope to investigating the case where the second orbit has the same altitude as the first, but a different inclination with respect to the equator. The basic problem to be considered in this study is the determination of the appropriate vehicle controls for the minimum energy maneuver.

The general maneuver to be considered here can be broken into five separate steps or phases. The first step in the maneuver is a deorbit impulse, a retrothrust that slows the vehicle down sufficiently to insure reentry into the earth's atmosphere. The second phase of the maneuver is the gliding turn; this is the aerodynamic portion of the flight and it is for this regime of flight that we shall utilize the computer to determine the optimal vehicle controls (the controls in this case being the bank angle and angle of attack). The third phase is a pullup maneuver after achieving the desired heading change, in which the glider is maneuvered in-plane so that the vehicle is properly oriented for reinjection into space. In the fourth phase the vehicle is accelerated to a speed sufficient to allow it to proceed ballistically along an elliptical transfer path that intersects the desired final orbit tangentially. Then, completing the maneuver, the vehicle is given an impulse that increases its speed to orbital velocity. The total inclination change is accomplished during the second portion of the flight;
that is, during the gliding turn. No attempt has been made to incorporate into this analysis the possibility of accomplishing the desired orbit change by a combination of aerodynamic maneuvers (including additional inclination changes during the ascent phase) and terminal impulses.

Initial Conditions

One of the first problems that must be considered is the determination of the initial conditions. For simplicity we have assumed the initial orbit to be equatorial, so that we can select as initial conditions zero latitude and longitude with the velocity vector lying in the equatorial plane. Throughout most of this study we will consider 300 n mi as our initial departure altitude. This altitude is near enough to the earth to allow for the use of reasonably small deorbit velocity impulses, but sufficiently high to offer the operational advantage of negligible drag losses over long periods of flight.* The deorbit impulse chosen was one that would allow the vehicle to reenter the denser portion of the earth's atmosphere approximately 150 deg downrange, which corresponds to an initial velocity impulse of 480 ft/sec. This deorbit impulse is nearly minimal for such maneuvers and it will not be varied except when making direct comparisons with different initial velocities and altitudes. As the orbit altitude is varied, the deorbit impulse will also be varied so that reentry will occur approximately 150 deg downrange.**

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*This particular advantage is ignored in this study as we shall only be concerned with short-duration synergetic maneuvers.

**A detailed discussion of the effects of different initial velocity increments on the reentry paths of space vehicles and their reentry velocity can be found in Ref. 2.
Optimization Function

If no design constraints were to be considered, the optimization program for vehicle control should only be one that decreased the sum of velocity increments required for reorbiting. This process could then be made to correspond to a minimum fuel trajectory and would thus represent a lower bound on the velocity requirements for the synergetic maneuver. However, there are two reasons for not taking this approach. First, the optimization program that would minimize the total velocity increment would be difficult to construct and expensive to use. Secondly the inclusion of thrusting portions of the flight in the optimization program would have substantially enlarged the program.

In order to avoid the problems presented by minimizing total velocity increments, certain limitations were set. First we minimized the energy loss of only that portion of the flight consisting of maneuvers in the atmosphere. This approach enabled us to avoid the inclusion of thrusting arcs in the program, and, for computational reasons, left most of the terminal conditions unspecified. Secondly, to neglect structural considerations would not be realistic. The problems of heating, structural load factors, and pilot stamina impose physical limitations on any aerodynamic maneuver and, therefore, must be included in any realistic trajectory analysis. In order to circumvent the problem of heating we selected the maximization of the vehicle's energy at the termination of the maneuver as our optimization criterion. The velocity losses incurred through the pullup portion of the flight and through reorbiting after termination of the maneuver are included, but not optimized. The selection of this optimizing function leads to
trajectories in which both the required velocity impulses and heating loads are held to a near minimum. Thus, through an appropriate limitation on the range of the flight to be optimized, and through an appropriate choice of the function to be optimized, we reduced the problem to one that was readily computable and that took into account vehicle limitations arising from total heat loads and heat rates.

The atmospheric glide phase of the flight path was selected for optimization since most of the velocity losses are incurred during this portion of the flight. When inclination changes are large, the velocity losses incurred in the atmosphere dominate the entire maneuvering requirements. However, when inclination changes are small, then the velocity requirements for countering the drag losses are correspondingly smaller. If terminal altitude is not considered to be a boundary condition, then the resultant optimal energy loss turns are not optimal velocity increment trajectories, as we shall note later. The paths that we have indicated for small inclination changes terminate at low altitudes, and thus incur large drag losses. Consequently, the velocity requirements for reorbiting are correspondingly high.

One way to overcome this difficulty would be to set a bound on the terminal altitude, so that the terminal energy would be at a maximum, and would then correspond to the largest terminal velocity. If, in addition, the terminal path angle were specified to be slightly above the local horizon, then the velocity requirements for reorbiting would be nearly optimal. Furthermore, if we were to change the optimization criterion from minimum energy to minimum velocity requirements, i.e.,
if \( \phi \) were the criterion to be optimized, then, given circular orbit conditions at the initial altitude \( r(o) \) for reinjection,

\[
\phi = \sqrt{g \frac{R^2}{r(T)}} \left[ \sqrt{\frac{2 \cdot r(o)/r(T)}{1 + r(o)/r(T)}} \left( 1 - \frac{r(T)}{r(o)} \right) + \sqrt{\frac{r(T)}{r(o)}} - 1 \right]
\]

(2.1)

where

\( g = \) the acceleration of gravity
\( R = \) the radius of the earth
\( r = \) the distance from the center of the earth
\( T = \) the final time of flight.

This process would give us the minimum velocity increment trajectories.

We have not followed this procedure, however, for the reasons discussed above. Note, however, that for large inclination changes the results obtained by minimizing the energy losses during the second phase of the trajectory can be considered nearly optimal in terms of velocity impulse requirements for reorbiting.

**Acceleration Dosage Bound**

Minimizing the total energy lost during the maneuver does not guarantee that the acceleration of the vehicle will be within tolerable bounds. It was found that without additional measures to limit the vehicle's acceleration, the magnitude of the accelerating forces imposed on the vehicle were excessive for most optimal trajectories. In order to insure that the trajectories are admissible in terms of pilot stamina and vehicle integrity, a bound on acceleration dosage was introduced into the equations of motion. The dosage (the integral of a function
of the acceleration taken over the entire length of the gliding trajectory) was constrained to be less than a prescribed number. By modulating the terminal value of the dosage, it is possible to achieve any desired maximum acceleration load. Unfortunately, there is at present no physical justification for the employment of time- or temperature-independent acceleration dosage limitations when considering only vehicle structural loads. Structural bounds are temperature-dependent in both theory and practice. It is clear that, for a given orbital weight, there is a direct relationship between vehicle structural strength limitations and available fuel capacity: the lighter and thus weaker the structure, the heavier the fuel load permissible. It is also true that lighter (weaker) structures require gentler (more extended) turns, which in turn increase fuel requirements and reduce the payload. The appropriate tradeoff between these two considerations is not included in this study. Furthermore, as no proper structural limitations can be determined without recourse to an actual vehicle, the bounds on vehicle structure will also generally be ignored.*

While the acceleration dosage limitation does not adequately express the limitations imposed on the trajectories due to structural considerations, it does serve as a realistic limitation on the pilot. Since a pilot can withstand low accelerations for a long duration and high accelerations for only a comparatively short time, a weighted integral of the acceleration taken over the time of flight can be justified as an upper bound on the pilot's capability. Physiological

*A few pertinent comments will be included later when the effects of the constrained acceleration dosage are discussed.
curves expressing this bound are available,\textsuperscript{(6)} and, while they are not universally applicable (since pilots differ in individual physical capacities), they can be considered as a representative upper limit of acceptable tolerance levels. By setting the maximum dosage level allowable in flight slightly less than this probable upper limit,\textsuperscript{*} variations are kept below a significant level. In this study we will consider both unbounded dosage and bounded dosage trajectories so that appropriate comparisons and trends can be noted.

Technique

In determining the minimum energy trajectories we employed numerical gradient techniques proposed by Kelley\textsuperscript{(7)} and Bryson\textsuperscript{(8)}. Systematic use of this numerical procedure gives optimal values of the desired function although one must be careful to avoid solutions that are locally, but not globally, optimal.\textsuperscript{**} Furthermore, since the value of the function to be optimized is insensitive to considerable variations in a trajectory that is near-optimal, the gradient solutions may lead to trajectories that are apparently optimal but actually erroneous. For this reason the paths determined in this analysis must be considered only near-optimal ones. If true optimal trajectories are desired, then other techniques, such as the maximum principle of Pontryagin,\textsuperscript{(9)} would have to be applied. These other techniques, while excellent in determining true optimal trajectories, do impose additional numerical difficulties.

\textsuperscript{*}The maximum dosage selected in this study is one-half this value.

\textsuperscript{**}This is by no means a trivial problem when the numerical scheme employed here is utilized.
As this work is essentially a comparative performance study, and is not meant to design vehicle trajectories, the simpler gradient technique was adequate.*

EQUATIONS OF MOTION FOR AERODYNAMIC MANEUVERING

We now further qualify our study by imposing the following assumptions:

1. The earth is spherical and non-rotating.

2. The atmospheric density varies exponentially with the altitude until a set height is reached; it is then assumed to be zero.

3. The controls and measurements are deterministic functions; that is, they are not subject to random errors.

4. All flight paths that are obtained from an allowable set of controls are themselves admissible. This eliminates inadmissible regions of flight.

Under these assumptions, we can now write our equations of motion in the simplified vectorial form

\[ m \frac{dV}{dt} = L + D - mg \frac{R^2}{r^3} f \]  \hspace{1cm} (2.2)

where

- \( m \) = the vehicle mass
- \( V \) = the velocity vector

*The gradient techniques used are described in greater detail in Appendix C. However, those desiring to employ this numerical method are referred to the original work of Bryson and Denham. The actual numerical work for this paper was performed on an IBM 7090 computer by the Computer Sciences Department at The RAND Corporation.
\[ \mathbf{L} = \text{lift vector} \]
\[ \mathbf{D} = \text{drag force vector} \]
\[ g = \text{gravitational constant} \]
\[ R = \text{the radius of the earth} \]
\[ r = \text{radial distance of the vehicle from the center of the earth} \]

The lift force vector is defined as normally oriented to the vehicle's velocity vector in the direction prescribed by the vehicle's bank angle (Fig. 1(b)). If we denote this direction by the unit vector \( \mathbf{k} \), then we may write for \( \mathbf{L} \)

\[ \mathbf{L} = \frac{1}{2} \rho V^2 C_L A \alpha \mathbf{k} = L \mathbf{k} \quad (2.3) \]

where

\[ \rho = \text{the atmospheric density} \]
\[ C_L = \text{vehicle's coefficient of lift} \]
\[ A = \text{specific reference area of the vehicle} \]
\[ \alpha = \text{angle of attack} \]

This form for the lift is valid for small angles of attack only. However, since the physical limitations imposed on the vehicle will necessarily lead to a small angle of attack, it is sufficient. The drag force vector, \( \mathbf{D} \), is oriented opposite the velocity vector. Thus it is perpendicular to the lift force. If we define the direction of the velocity vector by the unit vector \( \mathbf{i} \), we may write

\[ \mathbf{v} = V \mathbf{i} \quad (2.4) \]

and our drag force then can be written
Fig. 1—Vehicle coordinate system
\[ \mathbf{D} = -D \mathbf{\dot{u}} = -\frac{1}{2} \rho V^2 C_D A (1 + \alpha) \mathbf{u} \] (2.5)*

where \( C_D A \) and \( \epsilon \) are additional vehicle parameters. The parameter \( \epsilon \) is introduced here as a measure of the induced drag that arises when the vehicle's velocity vector and its body axis are inclined to one another, and it will be chosen so that \((L/D)_{max}\) occurs when \( \alpha = 15^\circ \).**

In order to decompose one vector equation into three scalar equations, a spherical coordinate system is selected with coordinates \( r, \theta, \phi \), which will be called the radius (distance from earth center), longitude, and latitude, respectively, and three corresponding directions \( u_r, u_\theta, u_\phi \) (Fig. 1(a)). Substituting and expanding (Appendix A), we obtain

\[ m \left( \ddot{r} - r \dot{\phi}^2 \cos \theta - r \dot{\theta}^2 \right) = \]
\[ L \cos \xi \cos \phi + D \sin \phi - \frac{mg R^2}{r^2} \]
\[ m \left( 2 \dot{r} \dot{\phi} \cos \theta + r \dot{\theta}^2 \cos \theta - 2r \dot{\phi} \theta \sin \theta \right) = \]
\[ -L \left( \sin \xi \sin \kappa - \cos \xi \cos \kappa \sin \phi \right) - D \cos \kappa \cos \phi \]
\[ m \left( 2 \dot{r} + r \dot{\phi}^2 \sin \theta \cos \theta + \dot{\theta}^2 \right) = \]
\[ L \left( \sin \xi \cos \kappa + \cos \xi \cos \kappa \sin \phi \right) - D \sin \kappa \cos \phi \]

**This form of the lift and drag as functions of the angle of attack are generally appropriate for angles (in radians) where \( \frac{\pi}{4} < \alpha < \alpha \). The effect of \( \alpha^2 \) on the lift is usually very small, and the effect of a linear coefficient of \( \alpha \) on the drag can usually be neglected except for very small \( \alpha \). It will be seen later that in this problem angles of attack never cause serious violations of these restrictions. Furthermore, we have restricted \( \alpha \) to lie between 0 and \( \pi/6 \). Hence these forms for the lift and drag may be considered sufficient for the analysis considered here.

**The value of the angle of attack corresponding to \((L/D)_{max}\) was selected somewhat arbitrarily. It tends to agree with lift-to-drag profiles obtained from X-15 flight tests and theoretically is in the range predicted for future hypersonic experimental aircraft. While in both theory and practice it tends to vary slightly with velocity and altitude, for our purposes we shall consider it fixed.
where $\xi$, $\kappa$, and $\varphi$ are angles relating $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to the three unit vectors, $u_r$, $u_\psi$, and $u_\theta$. Henceforth $\xi$ is called the bank angle, $\kappa$ the angle of northerly deviation, and $\varphi$ the descent angle. (Fig. 2).

As $L$ and $D$ are functions of the magnitude of the velocity, it is convenient to introduce into the equations of motion the velocities in the directions $u_r$, $u_\psi$, $u_\theta$ (radial, longitudinal, and latitudinal). These velocities are related by the kinematic equation

$$V^2 = V_r^2 + V_\psi^2 + V_\theta^2$$  \hspace{1cm} (2.7)

Using these velocities as new dependent variables, our equations of motion become

$$\dot{r} = V_r$$

$$r \dot{\psi} \cos \Theta = V_\psi$$

$$r \dot{\Theta} = V_\Theta$$

$$\dot{V}_r = \frac{V_\psi^2 + V_\Theta^2}{r} + \frac{L}{m} \cos \xi \cos \varphi + \frac{D}{m} \sin \varphi - \dot{\xi} \frac{r^2}{r^2}$$

$$\dot{V}_\psi = -\frac{1}{r} (V_r V_\psi - V_\psi V_\Theta \tan \Theta)$$  \hspace{1cm} (2.8)

$$-\frac{L}{m} (\sin \xi \sin \kappa - \cos \xi \cos \kappa \sin \varphi) - \frac{D}{m} \cos \kappa \cos \varphi$$

$$\dot{V}_\Theta = -\frac{1}{r} (V_r V_\Theta - V_\psi^2 \tan \Theta) - \frac{D}{m} \sin \kappa \cos \varphi$$

$$+ \frac{L}{m} (\sin \xi \cos \kappa + \cos \xi \sin \kappa \sin \varphi)$$

where the angles $\kappa$ and $\varphi$ satisfy the relations

$$\kappa = \arctan \frac{V_\Theta}{V_\psi}$$

$$\varphi = -\arcsin \frac{V_r}{V}$$  \hspace{1cm} (2.9)
Fig. 2—Euler angles
It is now clear that once the two independent controls, i.e., the angle of attack ($\alpha$) and the bank angle ($\xi$), of the vehicle have been specified as a function of time or of the states (positions and velocities), then these six first-order differential equations uniquely describe the trajectory of the system, subject to certain restrictions on the forces and boundary conditions.

As it is the object of the maneuver to achieve a desired orbital inclination change, a relationship among the above variables and the inclination of the orbit is required. By simple geometry the appropriate relation becomes

$$\sin i = \left[ 1 - \cos^2 \alpha \cos^2 \theta \right]^{1/2}$$  \hspace{1cm} (2.10)

where $i$ is the inclination of the orbit. We shall consider the maneuver terminated (and the pullup ascent phase begun) when $i$ equals a specified desired number.

In order that we may compare the velocity requirements for the synergetic maneuver with those required for a pure rocket thrust maneuver, we must also determine the amount of fuel necessary for the rocket case. As we are transferring from one circular orbit to another, both of the same altitude, the expressions for the necessary velocity impulse are of a simple form. If restricted to a single impulse, then

$$\frac{\Delta V}{V_c} = 2 \sin \frac{\Delta i}{2}$$  \hspace{1cm} (2.11)

where $\Delta i$ is the desired change in inclination and $V_c$ is the velocity of the vehicle in the initial circular orbit. If, however, we allow a three impulse, or bi-elliptic transfer maneuver (that is, first a transfer from the circular orbit to a higher energy elliptical orbit,
then a change of inclination at the apogee of the elliptical orbit, and finally a reinjection into the desired circular orbit at perigee of the new elliptical orbit) then the expression for the required velocity impulse becomes

\[
\frac{\Delta V}{V_c} = 2 \left( \frac{2}{1 + \sigma} \right)^{\frac{1}{2}} (1 + \sigma \sin \frac{\Delta \epsilon}{2}) - 2
\]  

(2.12)

where \( \sigma \) represents the ratio of the Earth center-to-vehicle radius at perigee to that radius at apogee. It can be shown (Appendix B) that for \( \Delta \epsilon \leq 38.9^\circ \) the single impulse transfer is superior to the bi-elliptic transfer since it requires a smaller velocity increment. Furthermore, if \( \Delta \epsilon \geq 60^\circ \) then theory shows that a transfer ellipse to infinity (\( \sigma = 0 \)) is preferred. Between these two limits, there is a bi-elliptic transfer path better than the single impulse transfer, but one that does not need to transfer to infinity. The expression for the velocity increment for these inclination changes is

\[
\frac{\Delta V}{V_c} = 4 \left[ 2 \sin \frac{\Delta \epsilon}{2} (1 - \sin \frac{\Delta \epsilon}{2}) \right]^{\frac{1}{2}} - 2 \quad (38.9^\circ \leq \Delta \epsilon \leq 60^\circ)
\]  

(2.13)

ACCELERATION DOSAGE CONSTRAINT

The preceding equations of motion do not include a bound on the acceleration dosage. We shall now introduce an acceleration dosage constraint that, theoretically, will lead to a trajectory giving normal accelerations (i.e., accelerations perpendicular to the line of flight) that will not impair pilot performance. In order to demonstrate the effect of this constraint upon the flight trajectories, we will present trajectories with and without this constraint to
demonstrate its effect.

Introducing a direct bound upon the maximum permissible acceleration into the equations of motion would violate assumption 4 (p. 11). More importantly, a direct bound involving one or more state variables could cause extreme computational difficulties. To circumvent these problems, we have adopted Bryson's solution by including a new variable, the "dosage." (10) Since the dosage is introduced through a differential equation, it can be regarded as an additional equation of motion, and as such must satisfy certain prescribed boundary conditions. It is defined to be the integral, taken over the trajectory time of flight, of the inverse of that time through which the man or the vehicle could survive and function usefully at the resultant acceleration level. If \( B \) represents the dosage, then our new differential equation of motion is

\[
\frac{dB}{dt} = \frac{1}{\tau(a)}
\]

(2.14)

where \( \tau(a) \) is given empirically as a function of the magnitude of the acceleration (Fig. 3). From this definition it is clear that \( B = 1 \) corresponds to pilot failure. Thus we selected the terminal value of \( B \) as 0.5. Hopefully this value will insure success.\(^*\) For simplicity we have chosen this value as a fixed boundary value and have ignored those optimal trajectories in which the dosage is less than 0.5.

This limitation is generally not a serious one, as most unbounded trajectories produced considerably higher acceleration loads than those obtained when the dosage constraint was included. It should be emphasized

\(^*\)The initial value of \( B \) is zero. As the function \( \tau(a) \) is a positive-valued function everywhere, \( A \) is a monotonically increasing function with time. Thus, \( B \) needs only to be less than unity at the terminal time to insure its being less than unity at all points along the trajectory.
Fig. 3—Maximum acceleration dosage versus vehicle acceleration
again that this type of constraint does allow large acceleration levels for short periods of time and might not always prove to be a meaningful constraint on the maximum acceleration loads for which the vehicle structure is designed.

**PERFORMANCE CRITERION**

For the reasons mentioned earlier, we have chosen as the function to be optimized the total energy of the vehicle at the completion of its glide maneuver. Thus, energy lost to drag is minimized, or, equivalently, terminal kinetic plus potential energy is maximized, i.e., the maximum of $\Psi$ is found where

$$\Psi = \left[ g(r-R) + \frac{V^2}{2} \right]_{t=\text{terminal}}$$

From this equation it is clear that maximizing the terminal velocity (thus minimizing the velocity increment required for reorbiting)* is not exactly equivalent to minimizing the energy lost during the maneuver.** Since, for computational reasons, we have also chosen not to specify the terminal path angle, the velocity increments obtained for this performance criterion will generally be slightly larger than for the true optimum. The difference, however, is small.

**Simplified Control Routine**

In order to compare the optimal synergetic maneuvers with simplified trajectories obtained by very simple control laws, we have included in this study (Sec. III) several trajectories in which the

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*Assuming that the final altitude and path angle of the vehicle are specified.

**Nor is it equivalent to finding the minimum total velocity increment required for the maneuver, as has already been discussed.
angle of attack was specified to be fifteen degrees and the bank angle was held constant throughout the flight. This control scheme follows closely the optimal controls observed for various inclination changes and thus permits the comparison of a true optimal flight with a near-optimal flight where the controls are extremely simple. Since the controls are specified, no optimization program is needed and simple numerical integration of the trajectory is sufficient for determining the entire trajectory.
III. RESULTS

CONTROL PROFILES

The basic controls considered in the optimization process were the vehicle's bank angle and angle of attack. These two controls were calculated with and without an acceleration dosage for various values of L/D, W/A, and desired orbit plane change, all as a function of time.

The controls in the case without an acceleration dosage constraint are uniform over the entire spectrum of the various lift-to-drag ratios and orbit plane inclination changes. The results seem to demand that the vehicle perform its maneuver by flying at maximum (L/D) at all times while employing the lift solely for the turning of the trajectory. The angle of attack never decreases to less than fifteen degrees nor increases to more than slightly over seventeen degrees. This angle fluctuates very little over the entire trajectory. When there are variations, they appear as monotonic increases corresponding to velocity decreases. This slight change in the angle of attack is never sufficient to cause an observable change in the lift-to-drag ratio (to within three significant figures); the ratio remains at its maximum value throughout. The bank angle varies between 90 and 92.74 degrees, thus always tending to pull the vehicle downward without sacrificing any of the lateral turning lift. Because none of the lift goes into keeping the vehicle aloft, the vehicle tends to descend ballistically like a hypersonic rock. Note that for this type

*See Appendix C for the details of the computation scheme that produces the optimal controls.
of maneuver, the lift force is never used to keep the vehicle altitude above some bounding value. Rather, the vehicle seems to fly as low as possible in order to utilize more fully the denser air to complete its maneuver more rapidly. However, this type of flight path is admissible only when one disregards the acceleration and heating loads which such maneuvers impose on the vehicle.

A good approximation to such trajectories can be obtained by using the well-known, closed-form solutions for small reentry angle descents through the atmosphere. These solutions yield an altitude-velocity profile that can be used to calculate the lifting force and the component of force causing the vehicle to change its path heading. By integrating this latter force component over the entire trajectory, it is possible to obtain inclination change as a function of altitude and velocity. It would then be easy to obtain the velocity increment required for reorbiting that, for the purposes of this discussion, would be reasonably close to the optimal result. In order to clarify this discussion, Fig. 4 shows the time history of the angle of attack and Fig. 5 the time history of the bank angle for several combinations of inclination changes and lift-to-drag ratios. While there are small variations in the angle of attack, the lift-to-drag ratio is essentially constant.

Unlike the control routines for the unbounded acceleration dosage trajectories, the controls for the bounded cases continuously fluctuate around nominal values. The magnitude of these fluctuations and the nominal values vary for different inclination changes and

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*This fly-low behavior is in reality non-optimal, since the vehicle would have to fly back into space through a denser portion of the atmosphere.

**Graphs showing trajectories with constant bank angle and constant angle of attack are included in Appendix D.
Fig. 4—Vehicle angle of attack versus time of flight (acceleration dosage unbounded)
Fig. 5—Vehicle bank angle versus time of flight (acceleration cosage unbounded)
vehicle configurations. Because of the numerical spread of the results, one can only make the general observation that the fluctuations are smaller for larger path changes and larger L/D ratios, but they are not substantially altered by changes in the vehicle's weight-to-reference area ratio.

The angle of attack generally fluctuates between fifteen and seventeen degrees for larger inclination changes (Fig. 6). On the other hand, possibly due to the restraint imposed, (i.e., that the acceleration dosage attain a fixed terminal value) the variations in the angle of attack for low inclination changes (short maneuvering times) are much more pronounced, dropping to less than 14 degrees in one instance. In all of the runs, however, the L/D ratio never fluctuates more than one-half of one per cent. As in the unbounded runs, the vehicle flies at maximum lift-to-drag, compensating for slight decreases in the velocity by increasing the angle of attack to a value greater than the angle corresponding to maximum lift-to-drag.

It appears that imposing an acceleration bound does not change the angle of attack, although it does produce a different trajectory. Therefore the prime difference between the controls in both cases must be related to the bank angle. Indeed, the bank angle determines the acceleration dosage. Whenever the vehicle starts to descend too rapidly, its trajectory flattens, not by an increase in the angle of attack, but by a reorientation of the lift vector (Fig. 7). These changes in the bank angle have very little effect upon the magnitude

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\*W/A ratio is a vehicle design parameter, total vehicle weight to reference area. The effect of its variation upon vehicle performance will be discussed later.
Fig. 6—Vehicle angle of attack versus time of flight (acceleration dosage bounded)
Fig. 7—Vehicle bank angle versus time of flight (acceleration dosage bounded)
of the turning forces, since these forces are functions of the sine of the bank angle, which always stays reasonably near 90 degrees. However, these changes do add considerable force in the radial direction. With an upper bound on the minimum altitude, acceleration levels are smaller. As the trajectory is very sensitive to changes in the radial direction, there appears to be a tendency for the vehicle to take a slight skip in mid-trajectory, particularly when the inclination change is large. As higher altitude turns appear to result in greater energy losses (due to the large increase in time to complete the maneuver), the skip is quite flat; i.e., the bank angle appears to be modulated in such a way that it causes the vehicle to flatten out at some altitude and complete the turn in a near-equilibrium glide trajectory. The bank angle generally varies between 79 degrees and 90 degrees, although for one short duration maneuver it achieved a maximum angle of 115.5 degrees. The fluctuations in the bank angle, like the angle of attack, tend to be less when the inclination changes are greater and the L/D ratios higher.

In several instances the maximum fluctuation of the bank angle is less than three degrees over the entire trajectory; in magnitude it fails to reach 90 degrees at any time. Figures 6 and 7 show, respectively, the angle of attack and the bank angle plotted as a function of time for several different combinations of the L/D ratio and inclination change. 

*Since the variation in the W/A does not seem to have a substantial effect on the control routines, curves showing these variations have not been drawn.
VELOCITY REQUIREMENTS

Synergetic maneuvers offer the opportunity to save large amounts of fuel for large inclination changes. Because altitude must be decreased to reach a portion of the atmosphere sufficiently dense to support maneuvering, velocity requirements for synergetic maneuvers are proportionally larger for small inclination changes than for impulsive transfers implemented with a rocket engine. The general break-even point is a function of vehicle configuration and permissible loads, starting at approximately five degrees for an L/D ratio of 3.32 and increasing rapidly with decreasing L/D. A vehicle with an L/D of 0.5 matches the performance of the ordinary rocket.

Generally, there is little difference between the velocity increments required for the bounded and unbounded acceleration dosage runs. While the trajectories in both cases differ substantially, the lower terminal velocities of the bounded dosage runs are compensated by higher terminal altitudes and shallower path angles. Much of the advantage gained in unbounded runs is cancelled in the pullup portion of the trajectory where losses of over 1000 ft/sec are sometimes observed. As there is an interest in saving fuel, the near equivalence of the trajectories in both cases offers substantial hope that acceleration problems can be handled effectively without incurring significant losses in performance.

At very low inclination changes there is noticable difference between the velocity requirements in the two cases. This deviation apparently occurs because the dosage bound has been set at an artificial level, higher than the level actually needed. On this basis
it is most likely that the velocity requirements for the bounded dosage curves would continue to closely parallel those of the unbounded dosages.

There is a velocity loss associated with reorbiting the vehicle due to the drag incurred while flying out of the atmosphere. If we assumed no lift in equation 2.2 and further assumed that the vehicle flew in-plane, then our equations could be re-written as

\[
\frac{DV}{dt} = -g \sin \phi - \frac{D}{m} \tag{3.1}
\]

\[
\frac{u d\psi}{dt} = - (1 - \frac{y^2}{v_0^2}) \cos \phi
\]

To determine the magnitude of the drag losses incurred while leaving the atmosphere, assume that the angle of the flight path with the horizon is a constant. Then integrating the equations by eliminating time and assuming that \( h_2 \to \infty \), we have *

\[
\frac{V}{V_0} = \exp \left[ -\frac{g \rho_0}{2\bar{D}} e^{-\frac{\rho_0}{(W/C_0A)\sin \phi}} \right] \tag{3.2}
\]

Note that the velocity losses depend on the initial altitude, the \( W/C_0A \) ratio, and the assumed constant path angle. Numerical calculations of this equation for various terminal altitudes indicate that velocity losses somewhat in excess of 1000 ft/sec can be expected. However, since drag losses are a function of how the engine is employed, how the lift is programmed, and what drag coefficient is used, we shall assume that all trajectories are subject to a

*This development was shown to the author by F. S. Nyland of The RAND Corporation.
*This assumption is unrealistic since our trajectories terminate at substantially different altitudes; but a more accurate analysis will be made later.

**At 60 degrees a degradation in performance of only 5 per cent can be obtained by implementing a transfer to an apogee radius only 2.84 times as large as the perigee.
Fig. 8—Total velocity increment requirements for synergetic plane changing
increases. For example, given an inclination change of 45 degrees, a vehicle with an L/D of 2 can perform synergetic maneuvers and ascend with less velocity increment up to an altitude of approximately 3000 n mi. This result raises the question of how best to use aerodynamic maneuvering to achieve orbits with planes that do not intersect the launch point, and also offers some interesting possibilities about rendezvous.

Figures 9 and 10 compare the velocity increment required for both rocket transfers and synergetic transfers at different terminal altitudes for vehicles with different L/D ratios. In these graphs ζ is the ratio of the initial distance from the earth's center to the terminal distance. Note that there are envelopes of the maximum upper altitude at which point synergetic transfers out-perform pure rocket transfers. For terminal altitudes to the left of this envelope, a pure rocket transfer will be superior to the synergetic transfer.

**VEHICLE DESIGN PARAMETERS**

For our purposes, vehicle design parameters can be categorized as:

1. the total heat energy transferred into the vehicle,
2. the maximum rate of heat energy transferred into the vehicle, and
3. the maximum inertia load resulting from the maneuver that both the structure and pilot must withstand.

We assume that a given percentage of the total mechanical energy lost during the maneuver due to the atmospheric drag is converted into heat, which is then transmitted to the vehicle. We further assume that the fraction of total energy transferred as heat into the vehicle
Fig. 9—Velocity impulse requirements versus terminal altitude (acceleration dosage bounded $L/D_{\text{max}} = 2$)
Fig. 10—Velocity impulse requirements versus terminal altitude (acceleration dosage bounded, \((L/D)_{\text{max}} = 3.32\))
can be represented as a function of the altitude (Fig. 11). Using this percentage transmission, reliable to within 20 per cent, we obtain the "indicated total heat load" (ITH) transmitted to the vehicle as a function of the inclination change of the orbit plane. These computations can provide valuable insight into the various trends of heating rates and total heat.

The range of ITH variation is large. Those maneuvers that require the largest increment of velocity, and thus the greatest loss of kinetic energy, also must absorb the largest amounts of heat energy. Furthermore, the ITH absorbed is smaller for increasing L/D ratios and larger for increasing orbit changes. As in the case of the velocity requirements, the lift-to-drag ratio plays an important role in keeping the total heat load to a minimum.* Because total energy lost was selected as the criterion to be minimized, the indicated ITH transmitted into the vehicle appears to be reasonable, reaching about 10,000 BTU per square foot of reference area when the inclination change is 90 degrees and the L/D ratio is 2. Figure 12 shows ITH as a function of inclination change with various L/D ratios. For large inclination changes, the unbounded acceleration trajectories are superior in relation to lower ITHs. However, even the heat energy inputs in the bounded case appear reasonable.

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*This trend is influenced by our manner of obtaining ITHs absorbed (ITH = \( \int_{0}^{T} C_D f V^2 dt \)). Maximum L/D ratios were obtained by modulating the drag coefficient. As this coefficient also appears directly in the heat energy integral, its modulation should be expected to cause similar changes in the magnitude of the ITH absorbed by the vehicle. However, this does not necessarily invalidate our conclusions regarding the trends in heating.
Fig. 11—Efficiency of conversion of vehicle's kinetic energy into heat energy at vehicle surface, as a function of altitude.
Fig. 12—ITH absorbed versus inclination change
The higher ITH absorbed by the vehicle in the bounded case as compared to the unbounded case is due to the longer duration of flight in the atmosphere.

Of possibly more importance are the maximum indicated time rates of energy transferred into vehicle heat to which the vehicle is subjected during the maneuver. These rates demonstrate marked variations between the bounded and unbounded acceleration trajectories, particularly for large inclination changes. Because of the near-ballistic nature of the trajectory in the unbounded case, the indicated heating rates tend to increase for large inclination changes whenever the L/D ratio is large. The rates tend to be more constant for different inclination changes at lower L/D ratios because maximum heating rates usually occur during the terminal phase of the maneuver, when deceleration is greater. These differences in performance can be observed in Figs. 13 and 14.

In Fig. 13 note when inclination changes are greater than 65 degrees (approximately) the ITH rates for the unbounded acceleration trajectories of higher L/D ratios exceed those for lower L/Ds, reaching a peak of over 200 BTU per second per square foot of vehicle reference area. This trend might not be that which one would expect; it is the result of a complex interaction among vehicle velocity near the end of the turn (a function of L/D), the terminal flight path angle, the wing loading, and the variation of the energy conversion coefficient with altitude (Fig. 11). This trend may also be related to the steep bank angle encountered in the unbounded trajectories (about 90 deg), because the maximum heating rates are more well behaved when a bound on acceleration is imposed (Fig. 14). The maximum indicated heat input rate for bounded trajectories is only slightly above 100 BTU/sec/sq ft, when the L/D ratio is equal to 1 and
Fig. 13—ITH rate versus given inclination change (acceleration dosage unbounded)
Fig. 14—ITH rate versus given inclination change (acceleration dosage bounded)
the plane change is 15 degrees. A comparison between Figs. 13 and 14 shows that the trajectories of the bounded cases generally have lower indicated total heat input rates.

These heat input rates are not meant to be considered as design criteria in themselves. They only indicate the average rate of heat transferred into the vehicle over the reference area of the vehicle body. It is clear that the heat transfer will occur primarily in the vicinity of the nose cap and along the leading edges of the wings (if there are wings). For a specific vehicle design, a more accurate estimate of the magnitude of heating rates would have to be calculated.

Another important vehicle design parameter is the maximum permissible normal acceleration of the vehicle. Figures 15 and 16 show normal wing loading acceleration to which the vehicle would be subjected if it flew the optimal turning trajectory. The loads observed for the unbounded acceleration case are extremely high and might present a difficult design problem. Even when bounds on the acceleration dosage are introduced, the normal accelerations to which the vehicle is subjected are greater than may be desirable from a structural point of view. As the velocity requirements for synergetic trajectories are fairly insensitive to acceleration bounds, there is considerable hope that near-optimal trajectories can be found that keep the maximum normal acceleration and the maximum heating rates below some realistic bounds.*

EFFECT DUE TO VARIATIONS IN THE INITIAL CONDITIONS AND VEHICLE DESIGN

To determine the effect that the parameters (circular orbit altitude, velocity impulse, and vehicle weight-to-reference area) have on velocity

*A simplified control which has this property is presented later in this section.
Fig. 15—Maximum vehicle normal acceleration loads versus inclination change (acceleration dosage unbounded)

$h(0) = 300 \text{ n mi}$

$W/A = 30 \text{ lb/ft}^2$

$(L/D)_{max} = 3.32$
Fig. 16—Maximum vehicle normal acceleration loads versus given inclination change (acceleration dosage bounded)
requirements, heat loads, and heat rates, variations of these parameters have been made for a 30 degree plane change with an L/D of 2.* Changing initial altitude has little effect on velocity requirements, the latter varying less than 400 ft/sec while altitude ranges from 200 to 400 n mi. On the other hand, heat loads and heat rates show a marked increase when the initial altitude of 300 n mi is varied (Fig. 17). One reason for this change may be the need for somewhat higher reentry velocities with an altitude lower (or higher) than 300 n mi. These higher velocities may in turn be due on one hand to the greater loss of potential energy prior to reaching the atmosphere, while on the other hand due to the greater initial circular velocity for the lower altitude and, subsequently, the smaller velocity increment used to slow down for reentry.** In partial compensation for increases in heating, the maximum accelerations for the lower and higher altitude runs are slightly lower.***

An increase in the initial deorbit velocity increment increases the total velocity increment required (Fig. 18). An increase in the initial increment also causes the vehicle to reenter the atmosphere at a steeper angle. This change in the angle itself does not produce

*As such, the results of this section apply only to one orbit plane change. However, there is reason to believe that the trends indicated here are characteristic of this mode of orbit transfer and hence applicable to other inclinations and L/D ratios.

**It was coincidental that our initial altitude of 300 n mi should be that altitude where these two considerations are mutually compensating.

***On this basis, then, if the initial circular orbit is a parking orbit, i.e., an intermediate phase of transferring to final orbit, this initial orbit should be as low as is possible (considering drag losses) since lower altitudes mean larger orbiting vehicles for a given booster. Selection of an initial parking orbit altitude would have to include consideration of additional important factors (such as heating loads) in order to permit evaluation of the tradeoff between orbit altitude and vehicle loads.
Fig. 17—Velocity impulse requirements and vehicle loads versus initial orbit altitude (acceleration dosage bounded)
Fig. 18—Velocity impulse requirements and vehicle loads versus variation in initial deorbit velocity impulse (acceleration dosage bounded)
a substantial change in performance while the vehicle is maneuvering in the atmosphere, but the larger initial velocity increment already imposed does have an effect. Steeper reentries soon show a degradation in velocity requirement performance as the descent ranges get substantially smaller. This performance loss is not reflected in the total heat loads; the total heat input actually decreases with increased initial increment. The velocity change does considerably reduce normal vehicle loads. This figure indicates that the velocity increment selected for the initial altitude is nearly optimal. This result is not surprising as the increment was selected to allow for a low impulse, long descent range before the vehicle entered the atmosphere. This type of maneuver is clearly more nearly optimal than one that uses a greater velocity increment for shorter descent ranges but without corresponding increases in reentry velocities.

The variation in velocity requirements due to a change in the vehicle's W/A ratio is negligible—less than 50 ft/sec (Fig. 19). Although substantial changes in vehicle wing loading have little influence on velocity requirements, the same cannot be said for the heat loads and heat rates. Both show substantial increases with an increasing W/A, indicating that lighter weight structures will incur lower heat loads (Figs. 20 and 21). The increase appears nearly linear. On the other hand, the maximum normal acceleration is essentially constant over the ranges of W/A considered (Fig. 22). This effect indicates that the designs for the vehicle's structural loading are constant for different W/A factors. Thus, in order to
Fig. 19—Velocity impulse requirements versus W/A (acceleration dosage bounded)
Fig. 20—Total vehicle heat load versus W/A

\[ (L/D)_{\text{max}} = 2 \]
\[ \Delta i \ (\text{deg}) = 30 \]
\[ h(0) = 300 \text{ n mi} \]
Fig. 21—Maximum vehicle heating rates versus $W/A$
Fig. 22—Maximum vehicle normal loading versus W/A

\[(L/D)_{\text{max}} = 2\]
\[\Delta i = 30^\circ\]
\[h(0) = 300 \text{ n mi}\]
keep heating loads at a minimum, the lightest structure possible appears to be the best design, assuming that any additional weight would not change the vehicle's capacity to withstand heat loads.

TRAJECTORIES

The computations that were used to obtain the preceding results were also used to plot the trajectories for various maneuvers. These graphs appear in Appendix D, p. 88. They indicate how the various states, i.e., displacements and velocities, vary as a function of the different paths and inclination changes. In general, the graphs are self-explanatory. Note, however, that there is a similarity between the velocity-altitude curves and the ballistic descent curves in the unbounded case. There is also a similarity between the bounded acceleration dosage curves and the equilibrium glide reentry curves for the same variables. The variables that are described as a function of $T$ are the values of those variables at the termination of the trajectory.

COMPARISONS WITH NON-OPTIMAL SIMPLIFIED CONTROL RESULTS

As can be seen from the behavior of the optimal controls, particularly in the unbounded case, variations in the controls are slight. A near-optimal trajectory can be obtained by holding the angle of attack and the bank angle fixed for the entire portion of the maneuver. By changing the initial bank angles, the behavior of the trajectory can be varied to obtain different acceleration levels. In our analysis the angle of attack was held constant at 15 degrees, i.e., the angle where the lift-to-drag ratio achieved its maximum
value. The bank angle ($\xi$) was given three different values: 90 degrees, the value nearly equal to optimal for the unbounded acceleration case; 70 degrees, the value somewhat less than optimal for most of the bounded cases; and 45 degrees, a value selected for comparison. When $\xi$ is either 90 or 70 degrees, the velocity losses (Fig. 23) are nearly as low as those for the optimized case, but when $\xi$ is 45 degrees, the performance of the vehicle suffers considerably. However, when heat rate and maximum normal acceleration are considered, the results are different. When $\xi$ is 90 degrees, the heat rates are extremely high and the acceleration loads are unacceptable (Figs. 24 and 25). In contrast, when $\xi$ is 70 degrees the heating rates are lower than the minimum observed in the optimized case (bounded). Furthermore, the maximum normal acceleration never exceeds 2 gs, a marked improvement over the other cases. When $\xi$ is 45 degrees, the results are even better in relation to heating rates and acceleration loads, but the velocity losses are much larger, and the velocity increment requirements correspondingly higher. This result tends to emphasize the relation between maximum normal acceleration and performance.

Apparently the optimization routine is relatively insensitive to variations in the control. Consequently it seems possible to determine simple controls for a vehicle maneuver that are easy to implement without degrading performance. There are indications that simple controls can be found that will allow acceptable heating loads and structural loads and still be acceptable in terms of velocity losses. Earlier analyses offered no guarantee that
Fig. 23 — Velocity increment requirements versus plane change
Fig. 24—Vehicle heating loads versus inclination change
trajectories could be found, which would be acceptable from a structural point of view. However, results of the present study indicate that such trajectories do exist and, further, that they would entail a velocity performance essentially equivalent to the optimized results presented in Fig. 8, p. 34.

A variation on the simplified control scheme was recently performed by Johann Lau\((12)\) of the RAND Corporation. He also assumes a constant bank angle and constant angle of attack, and obtains velocity requirements for synergetic maneuvers by allowing the vehicle to descend into the atmosphere, perform whatever maneuver is possible before "skipping" out of the atmosphere, and then re-orbiting at the maximum altitude of the skip. This procedure allows the vehicle to perform a maneuver in the atmosphere and attain additional heading change while skipping to a substantially higher altitude prior to reinjecting into orbit. For low inclination changes, utilizing the skip reduces the velocity requirements to a level lower than that which was observed for the minimum energy maneuver. Lau's results have been included in Fig. 8 and justify our earlier statements that our impulsive velocity requirements were non-optimal for small plane changes.
IV. CONCLUSIONS

This study has presented the results of an optimization program evaluating the performance of minimum energy loss turning maneuvers for orbit plane changes. In this section we will summarize these results.

1. The velocity requirements for an orbit plane change are usually less for near-earth transfers when aerodynamic maneuvers are employed. On this basis future maneuverable space vehicles should have aerodynamic surfaces if they are to perform substantial out-of-plane maneuvers.

2. It is possible to optimize the performance of synergetic maneuvers by applying techniques suitable for calculation with digital computers. While in this study we have used the maximization of the total energy of the vehicle at the termination of its maneuver as our criterion, it is possible to employ the same techniques to optimize the terminal velocity subject to a terminal altitude restriction, or to minimize the velocity increment necessary to reinject into orbit. If the latter procedure were used, the velocity impulse requirements would probably be less than those observed in this analysis.

3. The optimal control law for synergetic maneuvers consists of a nearly constant bank angle and lift-to-drag ratio. This conclusion makes it possible for future analyses to be conducted without resorting to an optimization program, with the assurance that the selected trajectories will be nearly optimal. For trajectories with no acceleration bounds, almost all of the lift is
generated for turning (all bank angles are approximately 90 degrees and the component of lift along the local vertical is nearly zero). For flights with acceleration bounds, the lift is also employed to sustain some of the vehicle's weight as well as to turn.

4. Certain generalizations can be made in relation to future vehicle design specifications: the vehicle's lift-to-drag ratio should always be as large as possible for maximum performance; the vehicle's W/A ratio should generally be as low as possible to keep the total heat loads at a minimum. However, other factors must be considered, such as the tradeoff between total heat capacity and vehicle weight. Variations in the W/A ratio have no effect on the velocity requirements of the optimal synergetic maneuver.

5. Finally, varying the initial altitude between 200 and 400 n mi has essentially no effect on the velocity requirements of the synergetic maneuver. For this reason then, parking orbits should be designed for the lowest altitude possible.

This analysis shows that synergetic trajectories are feasible. Control routines can almost surely be found that will meet all the vehicle structural constraints and still retain near-optimal performance; that is, vehicles can be built that can implement such maneuvers and achieve the level of performance indicated in this study.* The penalty of decreasing payload weights by the increased structural weights required for aerodynamic surfaces and additional heat protection has not been discussed in this study. It appears,

*The implications of maneuverable space vehicles are many and varied. For example, having a vehicle that could maneuver during reentry would simplify recovery operations.
however, that the gains in performance substantially outweigh the disadvantages of increased structural requirements.
Appendix A

DERIVATION OF THE EQUATIONS OF MOTION

In spherical coordinates the velocity and acceleration expressions are given by the vector equations:

\[ \mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\cos\theta \mathbf{u}_\phi + r\dot{\theta}\mathbf{u}_\theta \]  
\[ \mathbf{a} = (\ddot{r} - r\dot{\theta}^2\cos^2\theta - r\ddot{\theta}\cos\theta)\mathbf{u}_r + (2r\dot{\theta}\cos\theta + r\dot{\theta}\mathbf{u}_\theta \]
\[ + r\ddot{\theta}\sin\theta)\mathbf{u}_\phi + (2\dot{r}\dot{\theta} + r\dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta})\mathbf{u}_\theta \]  

(A.1)  

(A.2)

where the unit triad \((u_r, u_\phi, u_\theta)\) is oriented as shown in Fig. 1(a), p.13. As usual the dot indicates differentiation with respect to time. The speed of the vehicle is defined to be

\[ \mathbf{v} = \left[ \mathbf{v} \cdot \mathbf{v} \right]^{1/2} \]

or

\[ \mathbf{v} = \left\{ \dot{r}^2 + r^2\dot{\theta}^2\cos^2\theta + r^2\ddot{\theta}^2 \right\}^{1/2} \]  

(A.3)

Consider now a new set of unit vectors \((\mathbf{i}, \mathbf{j}, \mathbf{k})\) fixed to the vehicle and oriented such that

\[ \mathbf{v} = V \mathbf{i} \]  
\[ \mathbf{L} = L \mathbf{k} \]  

(A.4)  

(A.5)

where \(L\) is the lift force vector. Physically, \(\mathbf{k}\) is a unit vector oriented along the velocity and \(\mathbf{i}\) is the unit vector along the lift (Fig. 1(b), p.13). The vehicle body axis lies in the plane defined by the \((\mathbf{i}, \mathbf{k})\) vectors and differs from the direction of \(\mathbf{i}\) by the angle \(\alpha^*\).

*Vehicle yaw is assumed to be zero throughout.
Henceforth $\alpha$ will be called the angle of attack. Now the orbit plane of the vehicle is determined by the vectors $(\mathbf{i}, \mathbf{u}_r)$. The angle $\xi$ (measuring the solid angle between the vehicle axis plane, the plane defined by $(\mathbf{i}, \mathbf{k})$, and the orbit plane of the vehicle) is the roll angle and will be called the bank angle. The angle $\psi$, which measures the deviation of the vehicle's velocity vector below the local horizon, will be known as the path angle. In order to obtain the proper transformation between the coordinate system $(\mathbf{u}_r, \mathbf{u}_\psi, \mathbf{u}_\theta)$ and the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, it is necessary to define an additional angular relation $\kappa$ (Fig. 2, p. 16). Consider the three independent rotations: (1) rotate about the axis $\mathbf{u}_r$ by the angle $\kappa$, (2) rotate about the axis $\mathbf{j}$ by an angle $\psi$, (3) rotate about the axis $\mathbf{k}$ through the angle $\xi$. The three angular rotations, $\kappa$, $\psi$, and $\xi$, are classical Euler angles. By means of them we obtain the relations

$$\mathbf{i} = - \sin \psi \mathbf{u}_r + \cos \kappa \cos \psi \mathbf{u}_\psi + \sin \kappa \cos \varphi \mathbf{u}_\theta$$

$$\mathbf{j} = - \sin \xi \cos \psi \mathbf{u}_r - (\sin \xi \cos \kappa \sin \varphi + \cos \xi \sin \kappa) \mathbf{u}_\psi$$
$$+ (\cos \xi \cos \kappa - \sin \xi \sin \kappa \sin \varphi) \mathbf{u}_\theta \quad \text{(A.6)}$$

$$\mathbf{k} = \cos \xi \cos \varphi \mathbf{u}_r - (\sin \xi \sin \kappa - \cos \xi \cos \varphi \sin \varphi) \mathbf{u}_\psi$$
$$+ (\sin \xi \cos \kappa + \cos \xi \sin \kappa \sin \varphi) \mathbf{u}_\theta$$

In order to better understand the physical meaning of the various angles, consider

$$\mathbf{v} \cdot \mathbf{u}_r = \dot{r} = -V \sin \varphi$$

Thus we obtain for $\varphi$

$$- \varphi = \arcsin \frac{\dot{r}}{V} \quad \text{(A.7)}$$
Similarly we obtain for \( \kappa \)

\[
\tan \kappa = \frac{\vec{\delta}}{\psi \cos \theta} \tag{A.8}
\]

Physically, the angle \( \kappa \) is a measure of the deviation of the vehicle's velocity vector toward the north from the east. At the equator it corresponds to the inclination of the orbit with respect to the equatorial plane, and at the peak latitude of the orbit it always becomes zero.

From Newton's laws we have the vector equation of motion

\[
m \ddot{\vec{a}} = L \vec{r} - D \vec{i} - mg \frac{R^2}{r^2} u_r \tag{A.9}
\]

where \( L \) and \( D \) have been defined in the main body of this paper. If we expand this equation in terms of the three unit vectors \( (u_r, u_\psi, u_\theta) \), we obtain

\[
m \left( \ddot{r} - r \ddot{\psi} \cos^2 \theta - r \ddot{\theta} \right) u_r + \left( 2 \dddot{\psi} \cos \theta + \ddot{\psi} \cos \theta \right) u_r
\]

\[
- 2 r \dddot{\psi} \sin \theta u_\psi + \left( 2 \dddot{\theta} + r \dddot{\psi} \sin \theta \cos \theta + \ddot{\theta} \right) u_\theta
\]

\[
= L \left\{ \cos \xi \cos \varphi u_r - (\sin \xi \sin \kappa - \cos \xi \cos \kappa \sin \varphi) u_\psi + (\sin \xi \cos \kappa + \cos \xi \sin \kappa \sin \varphi) u_\theta \right\} - D \left\{ \sin \varphi u_r \right\} - mg \frac{R^2}{r^2} u_r \tag{A.10}
\]

Equating the coefficients of the three orthogonal vectors, we have equations (2.6).
In order to determine the lift and drag forces it is necessary to define atmospheric density as a function of altitude. We have selected the simple relation

$$\rho = \rho_o \exp\left[-\beta(r-R)\right]$$  \hspace{1cm} (A.11)

where $\rho_o$ is the density at sea level and $\beta$ is an appropriate constant.

The relationship between orbital parameters describing the orbit of the vehicle and the spherical coordinates used above in the equations of motion can be obtained by simple geometry. Referring to Fig. 26, we obtain the following relations from the law of sines

$$\frac{\sin i}{\sin \theta} = \frac{\cos \nu}{\sin (\theta-n)} = \frac{1}{\sin p}$$  \hspace{1cm} (A.12)

where $p$ is the planar angle, $i$ is the inclination of the orbit, and $n$ represents the nodal angle of progression. From geometry we have the additional relation

$$\cos p = \cos \theta \cos(\theta-n)$$  \hspace{1cm} (A.13)

Grouping these equations properly, we obtain

$$\sin i = \left\{1 - \cos^2 \nu \cos^2 \theta\right\}^{\frac{1}{2}}$$

$$\sin p = \frac{\sin \theta}{\left\{1 - \cos^2 \nu \cos^2 \theta\right\}^{\frac{1}{2}}}$$  \hspace{1cm} (A.14)

$$\sin(\theta-n) = \frac{\cos \nu \sin \theta}{\left\{1 - \cos^2 \nu \cos^2 \theta\right\}^{\frac{1}{2}}}$$
Fig. 26—Vehicle orbit plane coordinates
These three equations define the three orbital elements in terms of the spherical coordinates. (The last two are not used in this study but have been included for the sake of completeness).
Appendix B

DERIVATION OF THE VELOCITY INCREASES REQUIRED FOR AN IMPULSIVE TRANSFER

The velocity requirements for synergetic maneuvers discussed in the main body of this paper have been compared with those required for rocket transfers with the same inclination changes. In this section we will briefly discuss these rocket transfers.

Pure impulsive orbit transfers have several properties that should be noted. The first is that, at a given altitude, a single impulsive thrust is always more efficient than several impulses. (This can be easily proven by using equation B.1 and the triangle inequality). This leads to the conclusion that the inclination change should be accomplished with a single velocity correction. With this observation we are able to discard immediately all consideration of intermediate transfer paths along a constant altitude and thus simplify our work.

Were we to allow the altitude to vary, then it can be shown that intermediate transfer orbits can, in some circumstances, achieve savings in the velocity increment for the transfer. We will only consider the classical bi-elliptic transfer orbits here (13-18) since additional transfer orbit variations lead at best to only modest savings over this simple mode of transfer. In addition, no consideration will be given to the time limitations on the transfer paths that arise when considering the bi-elliptic paths. Instead, we will present the equation for the velocity increment as a function of the radius of the transfer ellipse apogee to the initial circular altitude. Time
limitations may then be introduced by requiring the transfer ellipse
to remain beneath a certain limiting eccentricity.

The geometry of the single impulse at a given altitude is given
in Fig. 27. If \( \Delta V \) is the velocity impulse required for an inclination
change of \( \Delta i \) degrees, then by simple trigonometry
\[
\frac{\Delta V}{V_c} = 2 \sin \frac{\Delta i}{2} \quad (B.1)
\]

In this formula we are assuming that the velocity increment \( \Delta V \) pro-
duces only a change in the inclination and does not affect the mag-
nitude of the velocity of the rocket. Thus a vehicle in a circular
orbit would remain in a circular orbit even after the velocity impulse,
but the inclination of the second circular orbit would be different
from the first by an amount \( \Delta i \).

At the apogee of an elliptical orbit, the velocity of a vehicle
achieves its minimum value. This property leads to the observation
that it might be possible to achieve velocity savings for large in-
clination changes if the vehicle were first transferred into an
elliptical orbit and allowed to achieve its inclination change at the
apogee of the transfer ellipse. A transfer of this type has often
been studied in connection with orbit inclination changes and is
known as a bi-elliptic transfer path. We will present the derivation
of these bi-elliptic transfer paths and show some of their limiting
properties. A more thorough discussion of the results given in this
Appendix for these impulsive transfers can be found in Ref. 13, which
considers several transfer paths as well as a change in altitude
between the initial and final circular orbits.
Fig. 27—Single impulse velocity orientation
The basic orbital equations that we shall use in deriving the relations between the elliptic transfer orbits are given by

\[ \frac{g_1 r_1^2}{2} = \frac{g_2 r_2^2}{2} \]

\[ \frac{v_1^2}{2} - \frac{v_2^2}{2} = g_1 r_1 - g_2 r_2 \] (B.2)

\[ r_1 v_1 = r_2 v_2 \]

where the first states the gravitational law, the second the conservation of energy, and the third the conservation of angular momentum. The subscripts 1 and 2 represent the perigee and apogee of the transfer ellipse. Newton's law gives us the circular velocity at \( r_1 \) to be

\[ V_c = \sqrt{\frac{g_1 r_1}{2}} \] (B.3)

As we wish eventually to compare the bi-elliptic transfer velocity increment requirements with those of the single impulse, we shall use \( V_c \) as the comparison velocity to normalize our equations.

The difference between the circular velocity and the velocity of the vehicle at the perigee of the ellipse is given by

\[ \Delta V_1 = v_1 - V_c \] (B.4)

If we use equations B.2, and define \( \sigma = \frac{r_1}{r_2} \), then we obtain for the velocity increment required to reach \( r_2 \) along a Hohmann transfer ellipse the equation

\[ \Delta V_1 = V_c \left[ \left( \frac{2}{1+\sigma} \right)^{\frac{1}{2}} - 1 \right] \] (B.5)
The velocity of the vehicle at the apogee of the ellipse is then

\[ v_2 = \sigma v_1 = \sigma v_c \left( \frac{2}{1+\sigma} \right)^{1/2} \]  \hspace{0.5cm} (B.6)

The velocity increment required to change the orbit inclination at this apogee is obtained through use of equation B.1. Thus, in order to change inclinations, another velocity increment \( \Delta v_2 \) is required, i.e.,

\[ \frac{\Delta v_2}{v_c} = 2 \frac{v_2}{v_c} \sin \frac{\Delta i}{2} = 2\sigma \left( \frac{2}{1+\sigma} \right)^{1/2} \sin \frac{\Delta i}{2} \]  \hspace{0.5cm} (B.7)

As in the case of the circular orbit, the orbit-inclination-changing impulse does not affect the ellipticity itself, but only the orientation of the transfer orbit. As the magnitude of the velocity is unchanged at apogee, at perigee it will be equal to the velocity of the first elliptic orbit at its perigee. The velocity increment necessary for reinjection into a circular orbit will then be just equal to the increment required for injection from a circular orbit to the first elliptical orbit. Thus

\[ \frac{\Delta v_2}{v_c} = \frac{\Delta v_1}{v_c} = \left\{ \frac{2}{1+\sigma} \right\}^{1/2} - 1 \]  \hspace{0.5cm} (B.8)

If we add the three velocity increments required for this mode of orbital transfer, we obtain

\[ \frac{\Delta v_{\text{total}}}{v_c} = 2 \left\{ \frac{2}{1+\sigma} \right\}^{1/2} \left( 1 + \sigma \sin \frac{\Delta i}{2} \right) - 2 \]  \hspace{0.5cm} (B.9)

This expression for the total velocity impulse is a function of the inclination change and the ratio of the altitudes of the initial circular
orbit to the apogee of the transfer ellipse.

Two questions should now be asked; when is it better to transfer with a single impulse and when is it better to transfer to infinity before changing the orbital inclination? At infinity the velocity increment necessary for changing the inclination of the orbit about the earth goes to zero. For a single impulse \( \sigma = 1 \) and for a transfer to infinity \( \sigma \to 0 \).

TRANSFER AT THE SAME ALTITUDE

For what inclination changes is a simple rocket impulse superior to a bi-elliptic transfer? That is, when is

\[
2 \sin \frac{\Delta i}{2} \leq 2 \left( \frac{2}{1 + \sigma} \right)^{\frac{1}{2}} \left( 1 + \sigma \sin \frac{\Delta i}{2} \right) - 2
\]

Grouping \( \sin \frac{\Delta i}{2} \) on one side of the equation and letting \( \sigma \to 1 \), we obtain by L'Hospital's rule

\[
\sin \frac{\Delta i}{2} \leq \lim_{\sigma \to 1} \frac{\left( \frac{2}{1 + \sigma} \right)^{\frac{1}{2}} - 1}{1 - \sigma \left( \frac{2}{1 + \sigma} \right)^{\frac{1}{2}}} = \frac{1}{3}
\]

(B.10)*

This gives us the result that, for inclination changes of less than 38.9 degrees, the single impulsive transfer mode is superior to any bi-elliptic transfer.

TRANSFER TO INFINITY

For what inclination changes is it better to transfer to infinity?

We thus ask for what \( \Delta i \) is

\[
2 \sqrt{2} - 2 \leq 2 \left( \frac{2}{1 + \sigma} \right)^{\frac{1}{2}} \left( 1 + \sigma \sin \frac{\Delta i}{2} \right) - 2
\]

*This is valid since Eq. B.10 has at most one minimum between the values \( 0 \leq \sigma \leq 1 \).
Rearranging, and again using L'Hospital's rule to evaluate the right-hand side, we obtain

\[
\sin \frac{\Delta i}{2} \geq \lim_{c \to 0} \frac{(1+c)^{\frac{1}{2}} - 1}{c} = \frac{1}{2}
\]  

(B.11)

This gives us the additional result that, for all orbital inclination changes greater than 60 degrees, it is better to transfer to infinity in that the total transfer requires less fuel.

**OPTIMUM RATIO OF \( \sigma \) TO ACHIEVE THE SMALLEST VELOCITY INCREMENT**

Between the inclination changes of 38.9 and 60 degrees there exists some altitude for which the bi-elliptic transfer mode via that altitude is the best possible. To determine this altitude as a function of \( \Delta i \), consider

\[
\frac{d}{d\sigma} \frac{V_{\text{total}}}{V_c} = \frac{d}{d\sigma} \left[ 2 \left( \frac{2}{1+c} \right)^{\frac{1}{2}} (1 + \sigma \sin \frac{\Delta i}{2}) - 2 \right]
\]

Solving this equation, we obtain

\[
\sigma = \frac{1 - 2 \sin \frac{\Delta i}{2}}{\sin \frac{\Delta i}{2}} \quad 38.9^\circ \leq \Delta i \leq 60^\circ
\]

(B.12)

and thus, for this range of inclination changes, the total velocity increment required for the optimal bi-elliptic transfer becomes

\[
\frac{\Delta V_{\text{total}}}{V_c} = 4 \sqrt{2 \sin \frac{\Delta i}{2} \left(1 - \sin \frac{\Delta i}{2}\right)} - 2
\]

(B.13)
To summarize, we now have three formulas for the velocity increment required for changing an orbit's inclination by means of velocity increments alone, i.e.,

\[
\frac{\Delta V_{\text{total}}}{V_c} = \begin{cases} 
2 \sin \frac{\Delta i}{2} & \Delta i \leq 38.9^\circ \\
4 \sqrt{2} \sin \frac{\Delta i}{2} \left(1 - \sin \frac{\Delta i}{2}\right) - 2 & 38.9^\circ \leq \Delta i \leq 60^\circ \\
2 \left(\sqrt{2} - 1\right) & \Delta i \geq 60^\circ
\end{cases}
\]

If there is a time limit on the transfer, then this same procedure for the determination of \(\sigma\) would be followed except, when the limit due to the time restriction on \(\sigma\) was reached, then equation B.9 with the limiting value of \(\sigma\) would yield the remaining values of the velocity increments for inclination changes up to 90°.

In all of the above, we have restricted ourselves to inclination changes of 90° and less. This has been done because no optimal synergetic plane changes of inclinations greater than 90° were calculated and thus no comparisons above 90° were available.
Appendix C

THE METHOD OF STEEPEST ASCENT

An optimization problem can be approached in several different ways. Basically, these approaches may be divided into two distinct groups, one the so-called indirect method of the Calculus of Variations, and the other the direct method of which the gradient procedure is an example. The Calculus of Variations provides necessary equations that must be satisfied along an optimal curve. As they are necessary, they must be satisfied at every point and hence can be viewed as additional equations of motion. If the initial conditions are known, then the solution to an optimization problem consists only of integrating along the path of motion until the terminal conditions are reached. The basic difficulty encountered in this procedure is the classic two-point boundary value problem, i.e., the existence of boundary conditions at both ends of the trajectory. In general there is no direct manner in which to determine the entire set of initial conditions when some terminal conditions are specified. Furthermore, the trajectories in the neighborhood of the terminal point are generally unstable in the sense that small changes in the initial conditions lead to large variations in the terminal values. Because of this instability, the determination of the initial conditions is extremely difficult even when a large digital computer is available. It is this difficulty that has led to considerations of other techniques, and ultimately to the development of gradient techniques, in particular the method of steepest ascent (8).

The method of steepest ascent is a direct method for obtaining optimal trajectories. In principle, it corresponds to guessing a
possible path (i.e., a path which matches all boundary conditions) and
then using information about the effects of varying the path successively
to improve it. This approach differs from the indirect method of the
variational calculus where all paths must be optimal, and hence are
never possible paths for the given problem until the actual solution
path is determined. The steepest ascent method, or gradient technique,
has proven to be more readily adaptable to computing machines than
the indirect methods.

Our differential equations of the state variables (those variables
that define the state of the vehicle at a given time) may be written
in subscript form as

\[ \frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_7, u_1, u_2) = f_i(x, u) \quad i = 1, \ldots, 7 \]  

(C.1)

where \( x = [r, \psi, \Theta, V_r, V_\psi, V_\Theta, A] \) are the state variables and
\( u = [\alpha, \xi] \) are the control variables. If we assume a nominal control,
\( u = u^*(t) \), i.e., a control possible but not optimal, then we may inte-
grate our differential equations (assuming that the initial conditions
on the states are given) and obtain a nominal trajectory \( x_i^* = x_i^*(t) \).
The control \( u^* \) must be admissible; that is, it must satisfy all limi-
tations imposed on it. To remove the appearance of bounds on the
control vector (for each control which is bounded), let us introduce
dummy variables \( \delta_k \) such that

\[ u_k = C + b \sin \delta_k \]
implies

\[ C - b \leq u_k \leq C + b \]

Given a nominal trajectory, the value of the function that is to be optimized at the terminal time is

\[ \dot{J}(x^*(T)) \]

(C.2)

We wish to determine the effect upon this function of a variation in the state and control at a given point along the trajectory.

To do this, consider a variation in the controls

\[ u_k = u_k^* + \delta u_k \]  

(C.3)

and correspondingly a variation in the states

\[ x_i = x_i^* + \delta x_i \]  

(C.4)

Substituting these variations in the differential equations (C.1) and expanding in a Taylor Series (neglecting all but first order terms), we obtain a linear variational differential equation

\[ \frac{d}{dt}(\delta x_i) = \sum_{j=1}^{7} F_{ij} \delta x_j + \sum_{k=1}^{2} G_{ik} \delta u_k \]  

(C.5)

where

\[ F_{ij} = \left( \frac{\partial f_i}{\partial x_j} \right)^* \]

\[ G_{ik} = \left( \frac{\partial f_i}{\partial u_k} \right)^* \]
are to be evaluated along the nominal trajectory. In equivalent matrix notation, this equation becomes

\[ \delta \dot{x} = F \cdot \delta x + G \cdot \delta u \]  
(C.6)

To integrate this equation, and thus determine the variation in the terminal states due to variations in the controls along the path, consider the equation adjoint to (C.6),

\[ \frac{d\lambda}{dt} = -F' \cdot \lambda \]  
(C.7)

where \( \lambda = \{ \lambda_r, \lambda_\phi, \lambda_\theta, \lambda_V, \lambda_r', \lambda_r, \lambda_A \} \) and where ( )' indicates the transpose of the matrix. If we take the inner product of the adjoint variables with the state variables and then evaluate the time derivative of the inner product, we obtain the fundamental equation

\[ \lambda'(T) \cdot \delta x(T) = \int_0^T \lambda'(s) \cdot \mathcal{G}(s) \cdot \delta u(s) ds + \lambda'(o) \cdot \delta x(o) \]  
(C.8)

This equation relates how a change in the control along the path affects the terminal states of the trajectory. The vector \( \lambda' \cdot G \) is often called the Green's function, or influence function of the problem. As we are to assume that all of our initial states are known,

\[ \delta x_i(o) = 0 \]

Thus our fundamental equation becomes

\[ \lambda'(T) \cdot \delta x(T) = \int_0^T \lambda'(s) \cdot \mathcal{G}(s) \cdot \delta u(s) ds \]  
(C.9)
Hitherto we have not specified our boundary conditions on our adjoint variables. If we select for $\lambda$ at the terminal time

$$\lambda'(T) = \left. \frac{\partial \xi}{\partial \xi} \right|_T$$

(C.10)

and designate the functions $\lambda(t)$ that satisfy (C.7) with terminal conditions (C.10) by $\lambda^a(t)$, then we have

$$\Delta^a |_T = \left. \frac{\partial \xi}{\partial \xi} \right|_T \cdot \delta h(T) = \int_0^T \lambda^a(s) \cdot G(s) \cdot \delta w(s) ds$$

(C.11)

where $\Delta^a |_T$ is the change in the optimization function at time $T$ due to changes in the control along the path. Similarly, if we have $t$ boundary conditions

$$\Pi_p (\xi (T)) = 0 \quad p = 1, 2, \ldots, t \leq 6$$

(C.12)

and a condition that defines when the maneuver is completed

$$\Omega (\xi (T)) = 0,$$

(C.13)

where the terminal time $T$ is determined as that time when equation (C.13) is first satisfied, then we can obtain the influence of the variations upon their terminal values by solving for the adjoint variables subject to the terminal conditions

$$\lambda'^a |_T = \left. \frac{\partial \Pi}{\partial \xi} \right|_p$$

(C.14)

$$\lambda'^a |_T = \left. \frac{\partial \Omega}{\partial \xi} \right|_p$$

(C.15)

In our problem above, the only terminal condition that we have specified is the acceleration dosage, that is, $\Pi = A(T) - \frac{1}{2} = 0$. Similarly, the stopping condition is the completion of the plane change.
Thus

$$\Omega = \Delta i(T) - \sqrt{1 - \cos^2 \kappa(T) \cos^2 \theta(T)} = 0 \quad (C.16)$$

Thus far we have considered nominal trajectories which have satisfied the terminal boundary conditions. However, in general this will not be convenient. Therefore we can ask for a change in the control that will correct the boundary conditions and optimize the specified functions simultaneously. Thus, the variations in our controls will be selected such that we satisfy

$$\Delta \Pi = \int_0^T \gamma_1^I(s) \cdot G(s) \cdot \delta u(s) ds + \hat{\Pi}(T) dT \quad (C.17)$$

where we recognize that equation (C.13) may allow the terminal time $T$ to vary.

From equation (C.11) we have

$$d\Phi = \int_0^T \gamma_1^I(s) \cdot G(s) \cdot \delta u ds + \hat{\Phi} dT \quad (C.18)$$

where

$$\hat{\Phi} = \frac{\partial \Phi}{\partial x} \cdot f \quad (C.19)$$

Similarly (for our boundary condition and stopping function)

$$d\Pi = \int_0^T \gamma_1^I(s) \cdot G(s) \cdot \delta u(s) ds + \hat{\Pi} dT \quad (C.20)$$

$$d\Omega = \int_0^T \gamma_2^I(s) \cdot G(s) \cdot \delta u(s) ds + \hat{\Omega} dT \quad (C.21)$$

where

$$\hat{\Pi} = \frac{\partial \Pi}{\partial x} \cdot f \quad (C.22)$$

$$\hat{\Omega} = \frac{\partial \Omega}{\partial x} \cdot f \quad (C.23)$$
Note that \( d\Omega = 0 \) since \( \Omega \) is the criteria for terminating the maneuver.

Solving for \( dT \) from equation (C.21)

\[
dT = -\frac{1}{\Omega} \int_0^T \frac{\lambda_\Omega'}{\lambda_\Omega} (s) \cdot G(s) \cdot \delta u(s) ds
\]

(C.24)

and defining new adjoint variables

\[
\lambda_{\delta\Omega}' = \lambda_{\delta\Omega}' - \left( \frac{\delta}{\delta s} \right) \lambda_{\delta\Omega}' ; \quad \lambda_{\Pi\Omega}' = \lambda_{\Pi\Omega}' - \left( \frac{\delta}{\delta s} \right) \lambda_{\Pi\Omega}'
\]

(C.25)

we have for our variations in the boundary conditions and optimization criteria

\[
d\Phi = \int_0^T \lambda_{\delta\Omega}'(s) \cdot G(s) \cdot \delta u(s) ds
\]

(C.26)

\[
d\Pi = \int_0^T \lambda_{\Pi\Omega}'(s) \cdot G(s) \cdot \delta u(s) ds
\]

where \( dT \) has been eliminated from consideration.

The derivation of the linear variational equation and the subsequent adjoint equation solutions depend upon \( \delta u \) being small enough to preserve the linearity of the variational equations. To insure that \( \delta u \) is small, impose the additional constraint

\[
(dP)^2 = \int_0^T (\delta u' \cdot \delta u) ds
\]

(C.27)

d\( P \) being a number chosen to insure that \( \delta u(s) \) remains small.

If equations (C.26) and (C.27) are adjoined to equation (C.26) by means of undetermined lagrangian multipliers, we obtained
\[ d\psi = \int_{0}^{T} \left\{ \lambda'_{\psi}(s) \cdot \frac{\lambda'}{\lambda_{\psi}}(s) - \mu \delta u(s) \right\} \cdot \delta u(s) \, dt \]
\[ + \, t \, d\Pi + \mu(dP)^{2} \quad (C.28) \]

We desire to maximize \( d\psi \) along the entire trajectory. As we have used a dummy variable to remove any bounds upon \( \delta u \), we have as a necessary condition for the stationarity of \( d\psi \)

\[ \frac{d\psi}{d(\delta u)} = 0 \quad (C.29) \]

which leads to the equation

\[ \lambda'_{\psi}(t) \cdot \frac{\lambda'}{\lambda_{\psi}}(t) - \mu \delta u = 0 \quad (C.30) \]

Using equations (C.26), (C.27), and (C.30), we can eliminate the constant lagrange multipliers from this control law equation. Performing the necessary steps to carry this forth, we obtain

\[ \delta u(t) = \frac{G' \cdot (\lambda'_{\psi} - \lambda'_{\Pi})^{-1}}{\sqrt{\frac{(dP)^{2}}{(d\Pi)^{2}} - \frac{d\Pi}{I_{\Pi\Pi}^{-1}} \frac{d\Pi}{I_{\Pi\Pi}^{-1}}}} \]

where

\[ I_{\Pi\Pi} = \int_{0}^{T} \lambda'_{\Pi\Pi}(s) \frac{\lambda'_{\Pi\Pi}}{\lambda_{\Pi\Pi}}(s) \frac{\lambda'_{\Pi\Pi}}{\lambda_{\Pi\Pi}}(s) \, ds \]

\[ I_{\Pi\phi} = \int_{0}^{T} \lambda'_{\Pi\phi}(s) \frac{\lambda'_{\Pi\phi}}{\lambda_{\Pi\phi}}(s) \frac{\lambda'_{\Pi\phi}}{\lambda_{\Pi\phi}}(s) \, ds \quad (C.32) \]

\[ I_{\phi\phi} = \int_{0}^{T} \lambda'_{\phi\phi}(s) \frac{\lambda'_{\phi\phi}}{\lambda_{\phi\phi}}(s) \frac{\lambda'_{\phi\phi}}{\lambda_{\phi\phi}}(s) \, ds \]
are integrals evaluated along the nominal trajectory and where \(( \cdot )^{-1}\)
indicates the inverse matrix. We note here that, as the \( \lambda \)'s are known
at \( t = T \), we must integrate these integrals backwards.

Once \( \delta u \) has been obtained from this above procedure, we may construct
a new trajectory from \( u_1 = u^* + \delta u \). This new trajectory should yield
a larger value of \( J \) after integration of equation (C.1) and computation
of (C.2). If however, the new \( J \) is smaller than the former one, we
have obviously exceeded the limits of our linearity assumption and thus
must scale down \( \delta u \).

If we have satisfied our terminal conditions, that is, if we have
obtained a possible trajectory such that \( d \Pi = 0 \), then

\[
\frac{dJ}{dP} = \sqrt{I_{\ddot{J}} - I_{\dot{J}}'} {I_{\Pi}}^{-1} I_{\Pi} \quad (C.33)
\]

This is called the "gradient" in the function space. As \( J \) approaches
the optimum, this gradient will approach zero. From this we see that
near the optimum, corrections in the trajectory will become even smaller
\((dP\) may be regarded as the length of the "step" to be taken in function
space) thus leading to the likelihood that, even though \( J \) is nearly
optimal, the trajectories will still be somewhat in error.
Appendix D

TRAJECTORIES

The following graphs (Figs. 28-60) show the trajectories plotted for various maneuvers, as discussed in Section III, p. 55.
Fig. 28—Altitude versus velocity (acceleration dosage unbounded)
Fig. 29—Altitude versus velocity (acceleration dosage bounded)
Fig. 30—Terminal descent angle versus given inclination change
Fig. 31—Minimum (terminal) altitude versus given inclination change (acceleration dosage unbounded)
Fig. 32—Minimum (terminal) altitude versus given inclination change (acceleration dosage bounded)
Fig. 33—Total range versus given inclination change (acceleration dosage unbounded)

$h(0) = 300 \text{ n mi}$, $W/A = 30 \text{ lb/ft}^2$, $(L/D)_{\text{max}} = 3.32$
Fig. 34—Total range versus given inclination change
(acceleration dosage bounded)
Fig. 35—Terminal side range latitude versus inclination change (acceleration dosage unbounded)
Fig. 36—Terminal side range latitude versus inclination change (acceleration dosage bounded)
Fig. 37—Terminal downrange longitude versus inclination change (acceleration dosage unbounded)
Fig. 38—Terminal downrange longitude versus inclination change (acceleration dosage bounded)
Fig. 39—Terminal time versus inclination change (acceleration dosage unbounded)
Fig. 40—Terminal time versus inclination change
(acceleration dosage bounded)
Fig. 41— Altitude versus time (acceleration dosage bounded and unbounded)
Fig. 42—Altitude versus time (acceleration dosage unbounded, $\frac{L}{D}_{\text{max}} = 1$)

$h(0) = 300$ n mi, $W/A = 30$ lb/ft$^2$

$\Delta i (\text{deg}) = 15$

$30^\circ$, $45^\circ$, $60^\circ$, $75^\circ$, $88^\circ$
Fig. 43—Altitude versus time (acceleration dosage unbounded, \((L/D)_{\text{max}} = 2\))
Fig. 44—Altitude versus time (acceleration dosage unbounded, \((L/D)_{max} = 3.32\))
Fig. 45—Altitude versus time (acceleration dosage bounded, $(L/D)_{\text{max}} = 1$)
Fig. 46—Altitude versus time (acceleration dosage bounded, \((L/D)_{\text{max}} = 2\))
Fig. 47—Altitude versus time (acceleration dosage bounded, $(L/D)_{\text{max}} = 3.32$)
Fig. 48—Side range latitude versus time (acceleration dosage unbounded, \((L/D)_{\text{max}} = 1\))
Fig. 49—Side range latitude versus time (acceleration dosage unbounded, \((L/D)_{max} = 2\))
Fig. 50—Side range latitude versus time (acceleration dosage unbounded, $(L/D)_{\text{max}} = 3.32$)
Fig. 52—Side range latitude versus time (acceleration dosage bounded, (L/D)_{max} = 2)

h(0) = 300 nmi, W/A = 30 lb/ft^2

(Bar) θ
$h(0) = 300$ n mi, $W/A = 30$ lb/ft$^2$

Fig. 53—Side range latitude versus time (acceleration dosage bounded, $(L/D)_{max} = 3.32$)
Fig. 54—Downrange longitude versus time (acceleration dosage bounded and unbounded)

\( (L/D) = 3.32 \), \( 2, 1 \)

\[ h(0) = 300 \text{ n mi}, \quad W/A = 30 \text{ lb/ft}^2 \]
Fig. 55 — Downrange longitude versus time (acceleration dosage unbounded, \((L/D)_{max} = 1\))
Fig. 56—Downrange longitude versus time (acceleration dosage unbounded, \( (L/D)_{\text{max}} = 2 \))
Fig. 57—Downrange longitude versus time (acceleration dosage unbounded, \((L/D)_{\text{max}} = 3.32\))
Fig. 58—Downrange longitude versus time (acceleration dosage bounded, \( (L/D)_{\text{max}} = 1 \))
Fig. 59—Downrange longitude versus time (acceleration dosage bounded, \((L/D)_{\text{max}} = 2\))
REFERENCES


12. Lau, Johann, private communication.


