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A VARIABLE DENSITY SPHERICAL SHOCK WAVE PROBLEM

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ABSTRACT

A solution is found in the strong shock approximation for the propagation of a spherically symmetric blast wave in an infinite medium with a radial density variation of the form $\rho_o = bR^{2\alpha-3}$, where α and b are constants. It is found for this case that the velocity of the shock front is $\frac{dR}{dt} = AR^{-\alpha}$, where A is a constant related to b and the energy of the blast.

A VARIABLE DENSITY SPHERICAL SHOCK WAVE PROBLEM

An important problem in studying the propagation of a blast wave is to have some information as to the effect of a varying external density on the shock wave. The following discussion deals with the solution of such a problem. In particular, we shall consider a point explosion occurring in an infinite medium with a radial density variation. This problem is solved in the strong shock approximation for a family of density functions.

Formulation of the Problem

The Eulerian equations which are assumed to define the hydrodynamic flow in regions not containing the shock wave are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (1)$$

Conservation of Momentum

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0 \qquad (2)$$

Conservation of Mass

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left(\frac{p}{\rho^{\gamma}}\right) = 0 \qquad (3)$$

Conservation of Entropy

In these equations u is the particle velocity, p is the pressure, ρ is the density, t is the time, r is the Eulerian radius, and γ is a constant determined by the equation of state of the gas. In addition to these differential equations, the functions p, ρ, and u must satisfy boundary conditions at the shock front. We shall assume that the pressure discontinuity at the shock front is large and consequently we shall be concerned with approximate boundary conditions; these are

$$\frac{\rho}{\rho_o} = \frac{\gamma+1}{\gamma-1} \tag{4}$$

$$u = \frac{2}{\gamma+1}\frac{dR}{dt} \tag{5}$$

$$\frac{p}{\rho_o} = \frac{2}{\gamma+1}\left(\frac{dR}{dt}\right)^2 \tag{6}$$

where R is the instantaneous position of the shock front and ρ_o is the density of the undisturbed medium immediately in front of the shock wave.

Solution

The method used in treating the present problem is based on the similarity method of G. I. Taylor. We shall assume that the solution of our problem has a special form and then try to satisfy the differential equations (1) - (3) and the boundary conditions (4) - (6) with this form.

Let us assume a solution of the form

$$\frac{p(r,t)}{\rho_o(R)} = \frac{f(\epsilon)}{R^\beta} \tag{7}$$

$$\frac{\rho(r,t)}{\rho_o(R)} = g(\epsilon) \tag{8}$$

$$u(r,t) = \frac{h(\epsilon)}{R^\alpha} \tag{9}$$

where α and β are constants, $\epsilon = r/R$, and $R = R(t)$. It is easy to show that no generality is gained by replacing R^α and R^β in (7) and (9) by arbitrary functions of R. We now see under what conditions these solutions satisfy equations (1) - (6).

(i) Substituting p, ρ, and u into equation (1), we find

$$-\alpha h R^{-(\alpha+1)} \frac{dR}{dt} - \epsilon h' R^{-\alpha-1} \frac{dR}{dt} + hh' R^{-2\alpha-1} + R^{-\beta-1} \frac{f'}{g} = 0 \tag{10}$$

where a prime denotes differentiation with respect to the argument of the function. We now make the assumption that

$$\beta = 2\alpha \tag{11}$$

and

$$R^{\alpha} \frac{dR}{dt} = A \tag{12}$$

where A is a constant. Then equation (10) becomes

$$(\alpha h + \epsilon h')A - (hh' + \frac{f'}{g}) = 0 \ . \tag{13}$$

This is now an ordinary differential equation in the variable ϵ.

(ii) Substitution into equation (2) gives

$$g \frac{\rho_o' R}{\rho_o} R^{\alpha} \frac{dR}{dt} - g'\epsilon R^{\alpha} \frac{dR}{dt} + hg' + gh' + 2 \frac{gh}{\epsilon} = 0 \ . \tag{14}$$

Using equation (12) and requiring that

$$\frac{\rho_o' R}{\rho_o} = a \tag{15}$$

where a is a constant, we find

$$(ga - g'\epsilon)A + hg' + gh' + 2 \frac{gh}{\epsilon} = 0 \tag{16}$$

which is an ordinary differential equation in ϵ.

(iii) Equation (3) can be written for the above solution in the form

$$\frac{1}{\rho_o^\gamma g^\gamma}\left[f\rho_o' R^{-\beta}\frac{dR}{dt} - \beta\rho_o f R^{-\beta-1}\frac{dR}{dt} - \rho_o \epsilon f' R^{-\beta-1}\frac{dR}{dt}\right]$$

$$- \frac{\gamma f}{\rho_o^\gamma g^{\gamma+1}} R^{-\beta}\left[\rho_o' g \frac{dR}{dt} - \rho_o \epsilon g' R^{-1}\frac{dR}{dt}\right]$$

$$+ hR^{-\alpha}\left[\frac{1}{\rho_o^{\gamma-1} g^\gamma} f' R^{-\beta-1} - \frac{\gamma f g' g^{-\gamma-1}}{\rho_o^{\gamma-1}} R^{-\beta-1}\right] = 0 \qquad (17)$$

Making use of equations (11), (12) and (15), we find

$$\left[fa(1-\gamma) - \beta f - \epsilon f' + \gamma f \frac{g'}{g}\epsilon\right] A + hf' - \gamma fh \frac{g'}{g} = 0 \; . \qquad (18)$$

Again we obtain an ordinary differential equation.

Equations (13), (16) and (18) are a set of three simultaneous ordinary differential equations whose solution determines f, g and h. It remains to show the consistency of the solutions (7) - (9) with the strong shock boundary conditions. We see immediately the consistency by substituting solutions (7) - (9) into equations (4) - (6). Thus equation (4) becomes

$$g(1) = \frac{\gamma+1}{\gamma-1} \; , \qquad (19)$$

equation (5) becomes

$$h(1) = \frac{2}{\gamma+1} R^\alpha \frac{dR}{dt} \; , \qquad (20)$$

and equation (6) becomes

$$f(1) = \frac{2}{\gamma+1} R^{2\alpha}\left(\frac{dR}{dt}\right)^2 . \qquad (21)$$

Or finally using equation (12), we can rewrite these conditions in the form

$$\frac{f(1)}{A^2} = \frac{2}{\gamma+1} \tag{22}$$

$$g(1) = \frac{\gamma+1}{\gamma-1} \tag{23}$$

$$\frac{h(1)}{A} = \frac{2}{\gamma+1} \tag{24}$$

which are clearly consistent with the equations (13), (16) and (18).

The equations (13), (16), (18), and (22) - (24) define the solution for the propagation of a spherical shock wave into a medium of varying density. The family of allowable densities is defined by

$$\frac{\rho_o' R}{\rho_o} = a \ . \tag{25}$$

Solving for ρ_o, we find

$$\rho_o(R) = bR^a \ . \tag{26}$$

Since we are primarily concerned with blast waves, we must require that the total energy of the blast wave is constant. The total energy is the sum of two terms; these are the kinetic energy

$$E_{kin} = 4\pi \int_o^R r^2 dr \, \frac{\rho}{2} u^2 \tag{27}$$

and the thermal energy

$$E_{th} = 4\pi \int_o^R r^2 dr \, \frac{p}{\gamma-1} \ . \tag{28}$$

The total energy is therefore

$$E = 4\pi \int_o^R r^2 dr \left(\frac{\rho}{2} u^2 + \frac{p}{\gamma-1} \right) . \tag{29}$$

Substituting equations (7) - (9) and equation (26) into this expression, we find

$$E = 4\pi \int_0^R r^2 \, dr \quad \rho_o(R) \left(\frac{1}{2} g(\epsilon) \frac{h^2(\epsilon)}{R^{2\alpha}} + \frac{1}{\gamma-1} \frac{f(\epsilon)}{R^{2\alpha}} \right)$$

$$= 4\pi \frac{\rho_o(R)}{R^{2\alpha}} \int_0^R r^2 \, dr \left(\frac{1}{2} g(\epsilon) h^2(\epsilon) + \frac{1}{\gamma-1} f(\epsilon) \right).$$

Or

$$E = 4\pi \frac{\rho_o(R)}{R^{2\alpha-3}} \int_0^1 \epsilon^2 \, d\epsilon \left(\frac{1}{2} g(\epsilon) h^2(\epsilon) + \frac{1}{\gamma-1} f(\epsilon) \right) . \tag{30}$$

Clearly if E is to be a constant, then

$$\rho_o(R) = cR^{2\alpha-3} \tag{31}$$

where c is some constant. Comparison of equations (26) and (31) gives immediately that

$$a = 2\alpha - 3. \tag{32}$$

(We note that we must require $\alpha > 0$ to have a finite mass near the blast center)

This equation shows an immediate connection between the density variation and shock wave velocity. Thus for a density variation

$$\rho_o(R) = bR^{2\alpha-3} \tag{33}$$

the shock wave velocity is

$$\frac{dR}{dt} = \frac{A}{R^\alpha} , \tag{34}$$

where the connection between the constants b and A is determined by the expression (30) for E.

This completes the solution except for the integration of the differential equations for f, g, and h. A numerical integration of these equations is being undertaken. However, it is possible to get reasonably accurate approximate solutions using essentially the ideas of G. I. Taylor.

Approximate Solution for f, g, and h.

According to Taylor's solution for the case of constant density, we conclude that for ϵ small the density is very small, consistent with the known effect of shock waves in carrying mass with them. This implies from equation (13) that for ϵ small $f' = 0$ or f is constant. Using this result combined with equations (16) and (18), we find that $h(\epsilon) = \epsilon/\gamma$ for small ϵ. Following Taylor we now assume that

$$h(\epsilon) = \left(\frac{\epsilon}{\gamma} + x\epsilon^{y}\right)A \quad . \tag{35}$$

Under this approximation we proceed to determine f, g, x, and y. First we observe that equation (24) demands that

$$h(1) = \frac{2A}{\gamma + 1} = \left(\frac{1}{\gamma} + x\right)A \tag{36}$$

or

$$x = \frac{\gamma - 1}{\gamma(\gamma + 1)} \quad . \tag{37}$$

To determine y, we first eliminate g'/g from equation (18) with the aid of equation (16) This gives

$$\frac{f'}{f} = \frac{3A - \gamma h' - 2\gamma\, h/\epsilon}{h - \epsilon A} \tag{38}$$

where we have used equation (32).

Using equations (35) and (37), we find

$$\left.\frac{f'}{f}\right|_{\epsilon=1} = y + 2 \quad . \tag{39}$$

On the other hand, by combining equations (13), (16) and (18), we find

$$\frac{f'}{f} = \frac{\left[(2\alpha-3)(1-\gamma)-2\alpha\right](\epsilon A-h)A + \gamma\left[\frac{2h}{\epsilon} + A(2\alpha-3)\right](\epsilon A-h) - \gamma\alpha hA}{(\epsilon A-h)^2 - \gamma f/g} \quad . \tag{40}$$

Thus using equations (22) - (24), we have

$$\left.\frac{f'}{f}\right|_{\epsilon=1} = \frac{(2\alpha-1)\gamma^2 + (4+2\alpha)\gamma - 3}{\gamma^2 - 1} \quad . \tag{41}$$

Equating the two expressions (39) and (41), we find

$$y = \frac{(2\alpha-3)\gamma^2 + (4+2\alpha)\gamma - 1}{\gamma^2 - 1} \; . \tag{42}$$

This completes the determination of $h(\epsilon)$. To determine f, we substitute equation (35) into equation (38). Thus

$$\frac{f'}{f} = \frac{\gamma^2 x(2+y)\epsilon^{y-2}}{\gamma-1-x\gamma\epsilon^{y-1}} \quad . \tag{43}$$

Or integrating we have finally

$$\log f = \log \frac{2A^2}{\gamma+1} - \gamma\,\frac{(y+2)}{y-1}\,\log\left(\gamma-1-x\gamma\epsilon^{y-1}\right) \quad . \tag{44}$$

Similarly, we may write equation (16) in the form

$$\frac{g'}{g} = \frac{aA + h' + 2h/\epsilon}{\epsilon A - h} . \tag{45}$$

Integration of this equation gives

$$\log g = \log \frac{\gamma+1}{\gamma-1} - \left[\frac{y+2}{y-1} + \frac{(2\alpha-3)\gamma+3}{(\gamma-1)(y-1)}\right] \log \frac{\gamma-1-\gamma x \epsilon^{y-1}}{\gamma-1-\gamma x} + \frac{(2\alpha-3)\gamma+3}{\gamma-1} \log \epsilon . \tag{46}$$

The expression (46) for g must be consistent with our original assumption that g is very small for small ϵ. Clearly this requires that the exponent of ϵ in g must be positive or

$$(2\alpha-3)\gamma + 3 > 0 . \tag{47}$$

Thus

$$\alpha > \frac{3(\gamma-1)}{2\gamma} . \tag{48}$$

Under this condition our solution is consistent.

Discussion

The equations (35), (44) and (46) complete an approximate description of the blast problem. This approximate integration for the functions f, g and h is, however, not essential as a numerical integration of the differential equations (13), (16) and (18) is relatively simple. This latter is being undertaken mainly to specify the accuracy of the approximate solution, though it is anticipated that the latter is in error by only a few per cent.