MEMORANDUM
RM-4846-PR
DECEMBER 1965

MULTIPLE SCATTERING IN HOMOGENEOUS PLANE-PARALLEL ATMOSPHERES

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-1700—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This is a summary of work done on a class of mathematical problems related to the theory of the transfer of solar radiation in a planetary atmosphere. The techniques summarized are being used both in atmospheric studies and in neutron transport theory. This report was prepared for presentation at the Second Interdisciplinary Conference on Electromagnetic Scattering, University of Massachusetts, Amherst, Massachusetts, June 28--30, 1965.

ABSTRACT

Steady-state multiple-scattering problems for homogeneous plane-parallel atmospheres have been extensively studied by means of the principles of invariance of Ambartsumian and Chandrasekhar. These non-linear integral equations have been used for computing intensities of radiation on the faces of the atmosphere. Using linear integral-equation theory, we show how to compute the source function interior to an atmosphere with optical depth entering the computation only parametrically. Intensities, throughout the atmosphere, are expressed by nonlinear combinations of the computed source function, giving, on the faces, the familiar scattering and transmission functions. By a reduction developed by Sekera, this analysis is applied to Rayleigh scattering including the effects of polarization.

I. INTRODUCTION

We consider that portion of the theory of radiative transfer which is concerned with a plane-parallel atmosphere that scatters and absorbs radiation but does not reemit absorbed radiation. We consider the multiple-scattering problem in the steady state for monochromatic radiation. At least three stages are evident in the development of the mathematical theory of this subject.

The first stage deals with the formulation and study of linear equations.* The equations are linear integrodifferential equations of Boltzmann type and linear integral equations of Fredholm type. The theory of the existence and uniqueness of solutions to these mathematical equations is thoroughly discussed in the book of Hopf.[4]

The second stage deals with nonlinear integral and integrodifferential equations introduced by Ambartsumian[5] and extensively developed by him, by Chandrasekhar[6] and by Sobolev.[7] This work can be viewed as an attempt to translate existence and uniqueness results into feasible computational formulations. The derivations of these equations is often heuristic, use being made of certain principles of invariance.

The third stage is one in which a rigorous study is made of the connection between the linear and the nonlinear formulations. There is, for example, the work of Sobolev,[7] Kourganoff,[8] and Busbridge.[9] The Laplace transform provides the means for passing from linear to nonlinear equations. That this should be so is clear from the Wiener--Hopf method for solving the Milne problem for the semi-infinite atmosphere.

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*See Refs. 1, 2, 3, and 4.

II. ISOTROPIC SCATTERING

We first illustrate various methods for the simple case of isotropic scattering in a finite homogeneous atmosphere. This will also serve as a vehicle for reporting on the stage of development of the program of study outlined in our presentation[12] at the first Interdisciplinary Conference on Electromagnetic Scattering (ICES) at Potsdam, New York in 1962. We use isotropic scattering solely for simplicity of formulas, since everything we say has been extended to anisotropic scattering and to Rayleigh polarization scattering.

We begin with the steady-state transfer equation in slab geometry:

$$\mu \frac{\partial I}{\partial x}(x,\mu,\nu) + I(x,\mu,\nu) = J(x,\nu) \quad , \tag{2.1}$$

for $-1 \le \mu \le 1$ and $0 \le x \le \tau$, where x measures optical depth in the atmosphere of thickness τ, and μ is the cosine of an angle made with the x-axis. This is an equation for the intensity of <u>diffuse</u> radiation due to a steady flux of incident radiation on the face, $x = 0$, of the atmosphere from a direction ν. The boundary conditions are

$$I(0,\mu,\nu) = I(\tau,-\mu,\nu) = 0 \quad , \tag{2.2}$$

for $0 \le \mu \le 1$. The source function, J, is given by the sum of the reduced incident radiation field and of radiation scattered from other directions:

$$J(x,\nu) = \frac{\omega}{2}\int_{-1}^{1} I(x,s,\nu)ds + \exp(-x/\nu) \quad . \tag{2.3}$$

The constant ω for radiation transfer satisfies $0 \le \omega \le 1$ and is the albedo of single scattering.

Now the function $I(x,\mu,\nu)$ is given in terms of the function J by the expressions

$$I(x,\mu,\nu) = \frac{1}{\mu}\int_0^x J(y,\nu)\ \exp[(y-x)/\mu]dy$$

and (2.4)

$$I(x,-\mu,\nu) = \frac{1}{\mu}\int_x^\tau J(y,\nu)\ \exp[(x-y)/\mu]dy \quad ,$$

for $0 \le \mu \le 1$. Then from Eq. (2.3) we see that the function J satisifes the familiar integral equation,[4]

$$J(x,\nu) = \exp(-x/\nu) + \Omega(J)(x) \quad , \tag{2.5}$$

where

$$\Omega(J)(x) \equiv \frac{\omega}{2}\int_0^1\int_0^\tau \exp[-|x-y|/\mu]\ J(y)dy\ \frac{d\mu}{\mu} \quad . \tag{2.6}$$

The equation for J has a unique bounded solution[4] for $0 \le \omega \le 1$ and $0 \le \tau \le \infty$. This solution can be computed by iteration so that

$$J_{n+1} = \exp(-x/\nu) + \Omega(J_n) \tag{2.7}$$

determines a sequence of functions which converges to the solution of Eq. (2.5). As a numerical procedure, this gives $J(x,\mu)$ for each μ for a set of values x_i of x, $0 \le x_i \le \tau$ chosen in the numerical work. This method is not practical for ω near 1 and τ large. The rate of convergence is determined by the largest eigen value λ of the integral operator Ω. This is estimated very accurately by the following upper bound:[13]

$$\lambda \le \omega \left[1 - \cos^2 a + \tau/2 \left(\ln \frac{\pi}{2a} + \int_{2a}^{\pi} \frac{\cos t}{t}\, dt\right)\right] \quad ;$$

$$2a \tan a = \tau \qquad (0 \le 2a \le \pi) \quad .$$

It should also be noted that a change of the parameter ν requires a complete new computation of the function J. It is desirable for some applications to have a computational procedure for computing $J(x,\nu)$ in which the variable x can be treated as a parameter. We shall give one such later.

Now for special values of x, namely $x = 0$ and $x = \tau$, there are the nonlinear equations of Ambartsumian and Chandrasekhar.[5][6] We define scattering and transmission functions by

$$S(\mu,\nu) \equiv \mu I(0,-\mu,\nu) \quad ,$$

$$T(\mu,\nu) \equiv \mu I(\tau,\mu,\nu), \qquad (0 \le \mu \le 1) \quad . \tag{2.8}$$

Then it is easily shown[6][9] that S and T are given by

$$S(\mu,\nu) = \frac{\mu\nu}{\mu + \nu}\left[X(\mu)\, X(\nu) - Y(\mu)\, Y(\nu)\right] \quad , \tag{2.9}$$

$$T(\mu,\nu) = \frac{\mu}{\mu - \nu}\left[Y(\mu)\, X(\nu) - X(\mu)\, Y(\nu)\right] \quad ,$$

where X and Y are given[9] in terms of the function J by

$$X(\mu) \equiv J(0,\mu) \quad ,$$
$$Y(\mu) \equiv J(\tau,\mu) \quad . \qquad (2.10)$$

The functions X and Y, so defined, also can be shown to satisfy the following nonlinear integral equations:[5,6,9]

$$X(\mu) = 1 + \frac{\mu\omega}{2}\int_0^1 \frac{X(\mu)\,X(\nu) - Y(\mu)\,Y(\nu)}{\mu + \nu}\,d\nu \quad ,$$
$$Y(\mu) = \exp(-\tau/\mu) + \frac{\mu\omega}{2}\int_0^1 \frac{Y(\mu)\,X(\nu) - X(\mu)\,Y(\nu)}{\mu - \nu}\,d\nu \quad . \qquad (2.11)$$

These equations have been extensively used for computation by iteration.[14][15] The above representation of the functions S and T is, of course, very desirable since it effectively separates the variables μ and ν.

We have shown[10] that the Eqs. (2.11) never have a unique solution for isotropic scattering. For the special case $\omega = 1$, this is shown by Chandrasekhar.* We give in Ref. 10 necessary and sufficient conditions on the characteristic function Ψ for general X and Y equations to have a unique solution. Additional constraints must be added in case of non-uniqueness.[11][12] For isotropic scattering these constraints are for $\omega < 1$, given by

$$2/\omega = \int_0^1 \frac{X(\nu)d\nu}{1 \pm k\nu} + \exp(\pm\, k\tau)\int_0^1 \frac{Y(\nu)d\nu}{1 \pm k\nu} \qquad (2.12)$$

for k, the root of the equation

*See Ref. 6, Chap. 8.

$$2k = \omega \ln \frac{1+k}{1-k} \qquad (0 \le k \le 1) \qquad .$$

For $\omega = 1$ they are given by

$$1 = \frac{1}{2}\int_0^1 [X(\nu) + Y(\nu)]d\nu \qquad ;$$

$$\tau \int_0^1 Y(\nu)d\nu = \int_0^1 [X(\nu) - Y(\nu)]\nu d\nu \qquad . \qquad (2.13)$$

These constraints serve to determine arbitrary constants in the representation[11] of all solutions to the Eqs. (2.11).

We have also treated the general scalar transfer equations for anisotropic scattering in three papers.[16,17,18] We have shown that Chandrasekhar's ψ_ℓ and Φ_ℓ equations often have a multiplicity of solutions.[18] Correct constraints have been given in cases of nonuniqueness. In addition, we have derived singular linear integral equations for anisotropic scattering analogous to those given in Sections III and IV below for isotropic scattering.*

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*See Ref. 17; also 5 and 7.

III. X- AND Y-FUNCTIONS

We outlined, in our 1962 ICES paper, an alternative procedure to both of the above computational methods. The theoretical work has been completed[11] and an operational computer program now exists at The RAND Corporation, Santa Monica, California for computing X and Y functions for any characteristic function and any optical thickness. This work is summarized in a paper[19] to appear in the Astrophysical Journal Supplement Series, which will also contain tables of X and Y functions for the case of isotropic scattering.

Our method extends some work of Sobolev[7][20] and Busbridge[21] in the application of singular integral equation theory to the following equations:[9]

$$\lambda(\mu)\ X(\mu) = 1 - \frac{\mu\omega}{2}\int_0^1 \frac{X(\nu)}{\mu - \nu}\,d\nu - \frac{\mu\omega}{2}\exp(\tau/\mu)\int_0^1 \frac{Y(\nu)}{\mu + \nu}\,d\nu$$

$$\lambda(\mu)\ Y(\mu) = \exp(-\tau/\mu)\left[1 - \frac{\mu\omega}{2}\int_0^1 \frac{X(\nu)}{\mu + \nu}\,d\nu\right] - \frac{\mu\omega}{2}\int_0^1 \frac{Y(\nu)}{\mu - \nu}\,d\nu \quad . \tag{3.1}$$

The function λ is given by

$$\lambda(\mu) = 1 - \omega\mu^2\int_0^1 \frac{d\nu}{\mu^2 - \nu^2} \quad . \tag{3.2}$$

All singular integrals are computed as Cauchy principal values. These equations can be obtained from the nonlinear X and Y equations[9] or, in more general form, from the integral Eq. (2.5). We will return to this in the next section and show the connection with the Wiener--Hopf method for solving semi-infinite problems.

We simply give the final form of the solution to the Eq. (3.1) and constraints (2.12). We first take the case $\tau = \infty$ where $Y \equiv 0$ and X is customarily denoted by H. We have

$$H(\mu) = \frac{1 + \mu}{1 + k\mu} \exp[\mu \int_0^1 \frac{\theta(t)}{t(t + \mu)} dt] \qquad (3.3)$$

where the function θ is given by

$$\theta(\mu) = \frac{1}{\pi} \tan^{-1} [\frac{\pi\mu}{2 + \mu \omega \ln \frac{1 - \mu}{1 + \mu}}] \qquad (0 \le \theta \le 1) \quad .$$

The constant k satisfies the equation

$$2k = \omega \ln \frac{1 + k}{1 - k} \qquad (0 \le k \le 1) \quad , \qquad (3.4)$$

and its values are given in Table 1.

This representation of the H function is obviously equivalent to the solution obtained from Eq. (2.5) by Wiener--Hopf method of Fourier transforms.* The advantage of this form of the solution over that given by Sobolev,[7] say, is that our representation, obtained from such a representation by analytic continuation methods, does not contain any Cauchy principal value integrals; all integrals are simple quadratures. This makes computation extremely simple.

This method of solution gives results for finite τ as corrections to the solution for $\tau = \infty$. We have the result[11][19]

*See Refs. 4, 7, 9, and 22.

$$X(\mu) = H(\mu)\ [1 + f(\mu) + \frac{k^2\mu(f(\mu) - f(1/k))}{1 - \omega\ N(1/k)(1 - k\mu)} + \frac{\mu\omega}{2}\int_0^1 \frac{f(t) - f(\mu)}{H(t)\ \Delta(t)(t - \mu)}\ dt] \tag{3.5}$$

and a similar expression for $Y(\mu)$ containing a function $g(\mu)$ in place of $f(\mu)$. The functions f and g satisfy certain integral equations. We do not give the details here but refer to our papers.[11][19]

One can ask whether the mathematical gymnastics required to go from the integral Eq. (2.5) or Eq. (2.11) to another set of equations for the functions f and g is worth the effort. For three reasons this seems worthwhile. Most important is the fact that the integral equations for f and g can be solved by iteration with rapid convergence, since the convergence is geometric[11] with ratio less than $\exp(-\tau)$. The kernel function is a positive continuous function, which is nice compared to the kernel of the operator in Eq. (2.5), which has a logarithmic singularity along the diagonal. Finally, the above computation is for a range of values of μ with x as a parameter; the reverse of the situation in Eq. (2.5).

The rapid convergence of the equations for the functions f and g also provides a means for obtaining very good asymptotic formulas[19] for X and Y functions for large values of τ. For conservative scattering ($\omega = 1$) we get

$$X(\mu) = H(\mu)\ [1 - \frac{\mu}{\tau + A}] + 0(\tau e^{-\tau})$$
$$Y(\mu) = \frac{\mu\ H(\mu)}{\tau + A} + 0(\tau e^{-\tau}) \qquad , \tag{3.6}$$

where

$$A = 2(1 - \int_0^1 \theta(t)dt) = 1.42089 \quad . \tag{3.7}$$

Sobolev[23][24] has obtained similar results as well as van de Hulst.[25]

For nonconservative scattering the results are more complicated since they change in character from $1/\tau$ to $\exp(-k\tau)$. The simplest such estimates are

$$X(\mu) = H(\mu) + \frac{2k\mu C[k^2\mu(1-\omega)^{-1/2}\exp(-\tau/\mu) - C(1+k\mu)H(\mu)]}{[B^2 - C^2][1-(k\mu)^2]}$$

$$Y(\mu) = \frac{C}{B}[X(\mu) - H(\mu)] + \frac{k\mu[k(1-\omega)^{-1/2}\exp(-\tau/\mu) - 2CH(\mu)]}{B(1-k\mu)} \quad , \tag{3.8}$$

where

$$B = -\frac{k}{1-k}\exp[k\int_0^1 \frac{\theta(t)}{1-kt}dt]$$

$$C = \frac{k}{1+k}\exp[-k\tau - k\int_0^1 \frac{\theta(t)}{1+kt}dt] \quad . \tag{3.9}$$

Values of B and $C \exp(k\tau)$ are given in Table 2. Less complete results are given by Sobolev,[26] which are poor when k is near 1.

To give some idea of the rate at which these asymptotic formulas take over, we have computed, for isotropic scattering, exact X and Y values for τ ranging from 0 to the value at which differences from the asymptotic formulas are less than 10^{-6}. We give in Table 3 these values of τ.

Table 1

ω	k	ω	k
.3	.997414	.8	.710412
.4	.985624	.9	.525429
.5	.957504	.95	.379485
.6	.907332	.99	.172511
.7	.828635	1.00	0.000000

Table 2

ω	-B	C exp (kτ)
.3	76.6795	.528923
.4	19.6089	.537221
.5	8.51623	.539418
.6	4.63077	.532150
.7	2.76239	.511143
.8	1.65677	.468454
.9	.877357	.382675
.95	.528204	.299425
.99	.197025	.153812
1.00	0	0

Table 3

ω	τ asymp.
.3	3.5
.4	4.0
.5	4.5
.6	4.5
.7	5.0
.8	5.5
.9	6.0
.95	6.5
.99	7.5
1.00	3.5

IV. INTERIOR COMPUTATIONS

All of the results about singular integral equations for X and Y functions can be obtained in an easier and more direct manner from Eq. (2.5). At the same time this can be generalized to give computations interior to the atmosphere.

We take Eq. (2.5) and apply the integral operator Ω to both sides. When $\Omega \exp[-x/\mu]$ is computed, we obtain an integral equation for the function $\Omega(J)$ whose solution can be expressed in terms of integrals on the function J. We simply give the result and refer to our papers for details.[27] When $\Omega(J)$, so expressed, is combined with Eq. (2.5), we see that the unique solution J to Eq. (2.5) also satisfies the equation

$$\lambda(\mu)\, J(x,\mu) = \exp(-x/\mu) - \frac{\mu\omega}{2} \int_0^1 \frac{J(x,t)}{\mu - t}\, dt - \frac{\mu\omega}{2} \exp(-\tau/\mu) \int_0^1 \frac{J(\tau - x,t)}{\mu + t}\, dt \quad , \tag{4.1}$$

and a similar one with x replaced by $\tau - x$.

Since the function J is analytic[9] in the complex variable μ for $|\mu| > 0$, we obtain in addition to the above equations the two constraints, for $\omega < 1$,

$$\exp(\pm kx) = \frac{\omega}{2} \int_0^1 \frac{J(x,t)dt}{1 \pm kt} + \frac{\omega}{2} \exp(\pm k\tau) \int_0^1 \frac{J(\tau - x,t)}{1 \pm kt} \quad . \tag{4.2}$$

For $\omega = 1$ there are constants analogous to Eq. (2.13)

The study of this system of singular integral equations differs from that for the X and Y equations only in the fact that x is now a free parameter rather than being fixed at $x = 0$ and $x = \tau$. The X and

Y Eqs. (3.1) are the special case of the above equation for x = 0 and $x = \tau$.

Any computer program for X and Y functions can be trivially modified to compute $J(x,\mu)$ and $J(\tau - x,\mu)$ for any x, $0 \geq x \geq \tau$ and any μ, $|\mu| > 0$. We obtain results similar to those for X and Y functions: for $\omega < 1$

$$J(x,\mu) = H(\mu)[F(x,\mu) + \frac{k^2\mu}{\sqrt{1 - \omega B}} \frac{F(x,\mu) - F(x,1/k)}{1 - k\mu} + \frac{\mu\omega}{2} \int_0^1 \frac{F(x,t) - F(x,\mu)}{H(t)\ \Delta(t)\ (t - \mu)} dt] \quad . \tag{4.3}$$

The function F is determined[19] from rapidly convergent iteration of a linear Fredholm equation which differs from that for the function f of last section by containing the terms $\exp(-x/\mu)$ in place of 1 and $\exp[-(\tau - x)/\mu]$ in place of $\exp(-\tau/\mu)$.

With this reduction of the computation of the J function it is possible to give a representation of the Green's functions for homogeneous plane-parallel and spherical atmospheres.[24,26,28] For simplicity we give only the result for $\infty < 1$. For the plane parallel atmosphere

$$R(x,y) = \frac{k\omega(1 - k^2)\exp[-k|x - y|]}{\omega - 1 + k^2} + \frac{1}{2}\int_0^1 \frac{\exp[-|x - y|/t]}{\Delta(t)} \frac{dt}{t} - R_0(x,y) - R_0(\tau - x, \tau - y) \quad ,$$

where

$$R_0(x,y) = \frac{1}{2}\int_0^1 J(x,t) \left[\frac{k(1-k^2)}{\omega-1+k^2} \frac{\exp(-ky)}{1 + kt} + \frac{1}{2}\int_0^1 \frac{\exp(-y/s)}{\Delta(s)(s + t)} ds\right] dt$$

and

$$\Delta(s) = [1 - \frac{\omega s}{2} \ln \frac{1+s}{1-s}]^2 + [\frac{\pi \omega s}{2}]^2 \quad .$$

For the sphere of radius $\tau/2$ we have, for $0 \le x, y \le \tau/2$, the Green's function for the sphere expressed in terms of the Green's function for the slab by

$$R_s(x,y) = \frac{y}{x} [R(x + \tau/2, y + \tau/2) - R(x + \tau/2, \tau/2 - y)] \quad .$$

As an application, we find that the flux due to a point source of unit strength at the origin of a sphere of radius $\tau/2$ for $\omega = 1$, say, is given by

$$\rho(x) = \frac{1}{2\pi x} \{\frac{3}{4} + \frac{1}{2}\int_0^1 \frac{\exp(-x/t)}{\Delta(t)} \frac{dt}{t^2} + \frac{1}{2}\int_0^1 [J(x + \tau/2, t)$$

$$- J(\tau/2 - x, t)][\frac{3}{4} + \frac{1}{2}\int_0^1 \frac{\exp(-\tau/2s)}{\Delta(s)(s + t)} \frac{ds}{s}] dt\} \quad .$$

We can also directly use the results for the source function J for the purpose of computing intensities $I(x,\mu,\nu)$ interior to the atmosphere. The main point to be observed in the discussion below is the fact that in the final result the variable x enters solely as a parameter. Thus, interior intensities can be computed utilizing the dependence of functions on variations of angular variables rather than on variations of the position variable. This is in the spirit of the principles of invariance.[5][6] However, we do not simply state these principles, which are in themselves expressed as integral equations, but we give results which are in effect the solutions of these integral equations.

A formal statement of the result is the following. The intensities are given for $0 \le \mu \le 1$ by

$$\mu I(x,\mu,\nu) = U[J(\tau - x)]\ (\mu)\ T(\mu,\nu)$$

$$- \frac{\mu\nu}{\mu - \nu} [J(x,\nu) - J(x,\mu)\ \{[1 - U(X)(\mu)]\ X(\nu)$$

$$+ U(Y)(\mu)\ Y(\nu)\}]$$

and (4.4)

$$\mu I(x,-\mu,\nu) = U[J(x)](\mu)\ S(\mu,\nu)$$

$$+ \frac{\mu\nu}{\mu + \nu} [J(x,\nu) - J(\tau - x,\mu)\ \{U(Y)(\mu)X(\nu)$$

$$+ [1 - U(X)(\mu)]\ Y(\nu)\}],$$

where we have introduced the notation, for brevity,

$$U(J(x))(\mu) \equiv \frac{\mu\omega}{2} \int_0^1 \frac{J(x,t)}{\mu + t}\, dt$$

$$V(J(x))(\mu) \equiv \frac{\mu\omega}{2} \int_0^1 \frac{J(x,t)}{\mu - t}\, dt \qquad . \tag{4.5}$$

A proof of this is based on a use of the following well known result*

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*See page 91 of Ref. 9.

$$\frac{\partial J}{\partial x}(x,\nu) + \frac{1}{\nu} J(x,\nu) = X(\nu) \frac{\omega}{2} \int_0^1 J(x,t) \frac{dt}{t}$$
$$- Y(\nu) \frac{\omega}{2} \int_0^1 J(\tau - x,t) \frac{dt}{t} \quad . \tag{4.6}$$

We change x to y, multiply by $\exp[-(\tau - x - y)/\mu]$ and integrate on y over $0 \le y \le \tau - x$ to obtain

$$J(\tau - x,\nu) - X(\nu) \exp[-(\tau - x)/\mu] + \frac{\mu - \nu}{\mu\nu} \mu I(\tau - x,\mu,\nu) =$$
$$B(x,\mu) X(\nu) - A(x,\mu) Y(\nu) \quad . \tag{4.7}$$

The functions A and B are defined by

$$A(x,\mu) = \frac{\omega}{2} \int_0^1 \mu I(x,-\mu,t) \frac{dt}{t} \quad ,$$
$$B(x,\mu) = \frac{\omega}{2} \int_0^1 \mu I(\tau - x,\mu,t) \frac{dt}{t} \quad . \tag{4.8}$$

If we change x to y in Eq. (4.6) multiply by $\exp[-(y-x)/\mu]$ and integrate on y over $x \le y \le \tau$, we find

$$Y(\nu) \exp[-(\tau - x)/\mu] - J(x,\nu) + \frac{\mu + \nu}{\mu\nu} \mu I(x,-\mu,\nu) =$$
$$A(x,\mu) X(\nu) - B(x,\mu) Y(\nu) \quad . \tag{4.9}$$

We have simultaneous equations for A and B which we proceed to solve.

We now fix the variables x and μ. For the moment we assume that μ is outside the interval $[-1, 1]$. We multiply Eq. (4.7) by $\frac{\mu\omega}{2(\mu - \nu)}$

and Eq. (4.9) by $\frac{\mu\omega}{2(\mu + \nu)}$ change ν to t and integrate on t over $0 \le t \le 1$. We obtain the equations

$$A[1 - U(X)] + BU(Y) = U(J(x)) - U(Y) \exp[-(\tau - x)/\mu]$$

$$A\, V(Y) + B[1 - V(X)] = V(X) \exp[-(\tau - x)/\mu] - V(J(\tau - x)) \quad .$$

Now from the Eqs. (2.11) for X and Y and from the known identity[9]

$$\lambda \equiv [1 - U(X)][1 - V(X)] - U(Y)\, V(Y)$$

we obtain the solution

$$A = U(J(x))X - J(\tau - x)\, U(Y)$$

$$B = U(J(x))Y + J(\tau - x)[1 - U(X)] - \exp[-(\tau - x)/\mu] \quad .$$

When these expressions are substituted into Eqs. (4.7) and (4.9) we obtain Eqs. (4.4).

V. RAYLEIGH POLARIZATION SCATTERING

The mathematical model for polarization scattering introduced by Chandrasekhar[6] is a vector integro-differential equation for the four Stokes parameters. The basic ideas presented in previous sections for isotropic scalar transfer equations carry over to this problem. The primary difficulty is in the profusion of details rather than in any requirement of new ideas.

The difficult part of the analysis of the transfer equation for Stokes parameters is in dealing with the azimuthly independent terms. We discuss this briefly. By making use of a matrix factorization first observed by Sekera,[29] we can reduce the problem for the azimuthly independent terms to a study of the matrix equation for a 2 x 2 source matrix J

$$\underset{\sim}{J} = \underset{\sim}{I} \exp(-x/\nu) + \underset{\sim}{\Omega}(\underset{\sim}{J}) \qquad , \tag{5.1}$$

where $\underset{\sim}{I}$ is the 2 x 2 identity matrix and the operator $\underset{\sim}{\Omega}$ is defined by

$$\underset{\sim}{\Omega}(\underset{\sim}{J})(x) = \int_0^1 \underset{\sim}{\psi}(\mu) \int_0^\tau \underset{\sim}{J}(y) \exp[- |x-y|/\mu] \, dy \, \frac{d\mu}{\mu} \quad . \tag{5.2}$$

The matrix ψ is given by

$$\underset{\sim}{\psi}(\mu) = \frac{3}{8} \begin{pmatrix} 1 + \mu^4 & 2\mu^2(1 - \mu^2) \\ \mu^2(1 - \mu^2) & 2(1 - \mu^2)^2 \end{pmatrix} \tag{5.3}$$

This is the analogue of Eq. (2.5).

One can show that this matrix equation can be solved by iteration. This computational procedure has been used for this equation recently.* It becomes difficult for large values of τ. In analogy with Eq. (4.1) we can also show that

$$J(x,\mu)\ \lambda(\mu) = I\ \exp(-x/\mu) - \mu \int_0^1 \frac{\underset{\sim}{J}(x,t)\underset{\sim}{\psi}(t)}{\mu - t}\, dt - \mu \exp(-\tau/\mu) \int_0^1 \frac{J(\tau - x,t)\underset{\sim}{\psi}(t)}{\mu + t}\, dt \quad , \tag{5.3}$$

with

$$\lambda(\mu) = I - 2\mu^2 \int_0^1 \frac{\underset{\sim}{\psi}(t)dt}{\mu^2 - t^2} \quad . \tag{5.4}$$

We obtain constraints

$$\binom{2}{1} = \frac{3}{4}\int_0^1 [\underset{\sim}{J}(x,t) + \underset{\sim}{J}(\tau - x,t)] \begin{pmatrix} 1 + t^2 \\ 1 - t^2 \end{pmatrix} dt \quad ,$$
$$\left(\frac{\tau}{2} - x\right)\binom{2}{1} = \frac{3}{4}\int_0^1 (t + \tau/2)[\underset{\sim}{J}(x,t) - \underset{\sim}{J}(\tau - x,t)] \begin{pmatrix} 1 + t^2 \\ 1 - t^2 \end{pmatrix} dt \quad . \tag{5.5}$$

The solution to equations for Rayleigh polarization scattering can be expressed in terms of solutions to scalar equations.

$$\lambda^{(i)}(\mu)J^{(i)}(x,\mu) = \exp(-x/\mu) - \mu \int_0^1 \frac{J^{(i)}(x,t)\psi^{(i)}(t)dt}{\mu - t} - \mu \exp(-\tau/\mu) \int_0^1 \frac{J^{(i)}(\tau - x,t)\psi^{(i)}(t)dt}{\mu + t} \quad ; \tag{5.6}$$

i = 1,2,3,4,5, where

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*Private communication; to appear in Vol. 22, J. Atmos. Sci., 1965.

$$\psi^{(1)} = \frac{3}{8}(1-\mu^2)(1+2\mu^2) \qquad \psi^{(4)} = \frac{3}{8}(1-\mu^2)$$

$$\psi^{(2)} = \frac{3}{16}(1+\mu^2)^2 \qquad \psi^{(5)} = \frac{3}{4}(1-\mu^2) \tag{5.7}$$

$$\psi^{(3)} = \frac{3}{4}\mu^2 \quad .$$

The computation of solutions to the scalar equations follows the method outlined for isotropic scattering. Except for the characteristic function $\psi^{(4)}$ (ψ_r in Chandrasekhar's[6] notation), additional constraints have to be added to the above equations, namely for $i = 1,2,3$

$$\exp(\pm k_i x) = \int_0^1 \frac{J^{(i)}(x,t)\psi^{(i)}(t)dt}{1 \pm k_i t} + \exp(\pm k_i \tau) \int_0^1 \frac{J^{(i)}(\tau - x,t)}{1 \pm k_i t}\, dt \tag{5.8}$$

where

$$1 = 2\int_0^1 \frac{\psi^{(i)}(t)dt}{1-(k_i t)^2} \quad .$$

For the special value $x = 0$, these constraints were omitted by Chandrasekhar* in his reduction of Rayleigh polarization scattering and transmission matrices to X and Y functions. They are required for the nonlinear X and Y equations also to specify the uniquely desired solution.

For the special case of characteristic function $\psi^{(5)}$ (ψ_2 in Chandrasekhar's notation) we express solutions in terms of functions which are analytic in μ for $|\mu| > 0$. These solutions are obtained by adding the constraints

*See Chap. 10 of Ref. 6.

$$1 = \int_0^1 [J_0^{(5)}(x, \mu + J_0^{(5)}(\tau - x,\mu)] \psi^{(5)}(\mu)d\mu$$

$$\tau/2 - x = \int_0^1 (\tau/2 + \mu)[J_0^{(5)}(x,\mu) - J_0^{(5)}(\tau - x,\mu)]\psi^{(5)}(\mu)d\mu \quad .$$

The general solution then to Eq. (5.6) for i = 5 is

$$J^{(5)}(x,\mu) = J_0^{(5)}(x,\mu) + a(x)\mu \, [X^{(5)}(\mu) + Y^{(5)}(\mu)]$$

$$+ b(x)\mu \, [\gamma(X^5(\mu) - Y^{(5)}(\mu)) + \mu(X^{(5)}(\mu) + Y^{(5)}(\mu))]$$

where γ is a ratio of certain moments of the $X^{(5)}$ and $Y^{(5)}$ functions.[11] The arbitrary functions a(x) and b(x) are eventually evaluated by means of the constraints (5.5).

In the special case of x = 0, Chandrasekhar* selects standard solutions to the X and Y equations for i = 5 by adding the constraints

$$1 = \int_0^1 \psi^{(5)}(\mu)[X^*(\mu) \pm Y^*(\mu)]d\mu \quad .$$

His standard solutions X^* and Y^* are related to our $X^{(5)}$ and $Y^{(5)}$ functions by

$$X^*(\mu) = X^{(5)}\mu + \frac{\mu}{2\gamma} [X^{(5)}(\mu) + Y^{(5)}(\mu)]$$

$$Y^*(\mu) = Y^{(5)}(\mu) - \frac{\mu}{2\gamma} [X^{(5)}(\mu) + Y^{(5)}(\mu)] \quad .$$

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*See Chap. 8 of Ref. 6.

The entire computation of source matrices is reduced then to the computing of solutions to scalar equations and to the computation of various moments of these solutions. In all of these computations the optical depth can be considered as a mere parameter.

To obtain Stokes parameters interior to an atmosphere we need the analogue of Eq. (4.4). Since we are dealing with matrices, the derivation of this result is a little more subtle than that given for scalar equations. We have obtained results, which we give here for the azimuthly independent terms only. Expressions for Stokes parameters interior to an atmosphere follow from formulas which are obtained from Eqs. (4.4) if the scalar functions there are replaced by 2 x 2 matrices with the order of matrix products correctly indicated in Eq. (4.4). The operators U and V of Eq. (4.5) are replaced by matrix integral operators with the matrix $\underset{\sim}{\psi}$ of Eq. (5.3) replacing the constant $\omega/2$ but written to the right of the matrix $\underset{\sim}{J}$. The scalar functions X and Y are replaced by matrices defined by

$$X(\mu) \equiv J(0,\mu) \quad ,$$

$$Y(\mu) \equiv J(\tau,\mu) \quad .$$

The scalar functions S and T are replaced by matrices

$$\underset{\sim}{S}(\mu,\nu) = \frac{\mu\nu}{\mu+\nu}\,\underset{\sim}{\Pi}\,[\underset{\sim}{X}^t(\mu)\underset{\sim}{\Pi}^{-1}\,\underset{\sim}{X}(\nu) - \underset{\sim}{Y}^t(\mu)\underset{\sim}{\Pi}^{-1}\,\underset{\sim}{Y}(\nu)]$$

$$\underset{\sim}{T}(\mu,\nu) = \frac{\mu\nu}{\mu-\nu}\,\underset{\sim}{\Pi}\,[\underset{\sim}{Y}^t(\mu)\underset{\sim}{\Pi}^{-1}\,\underset{\sim}{X}(\nu) - \underset{\sim}{X}^t(\mu)\underset{\sim}{\Pi}^{-1}\,\underset{\sim}{Y}(\nu)] \quad ,$$

where

$$\Pi = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad ,$$

and the superscript "t" denotes matrix transposition. For the special values of x, x = 0 and x = τ, our representation for Stokes parameters interior to an atmosphere can be shown to reduce to that given by Chandrasekhar* for scattering and transmission matrices.

Thus the Stokes' parameters for Rayleigh polarization can be computed for any position within an atmosphere by simple extension of the existing computer program for scalar equations. This is similar in spirit to Chandrasekhar's reduction for scattering and transmission matrices, but by the use of linear equations for the source matrices we have been able to do interior computations as easily as those on the bounding faces. With the recent results of Sekera[(20)] concerning the matrix factorizations of general phase matrices, the same procedure can be applied to any type of polarization scattering, not just the Rayleigh. As is well known, however, the number of scalar equations and the complexity of algebraic manipulations increase rapidly as the number of terms in the expansion of the phase matrix increases.

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*See Chap. 10 of Ref. 6.

In this third stage, come existence theorems[9] for the solutions to the nonlinear X and Y equations of Chandrasekhar. The uniqueness equations for these equations were not completely answered until recent years.[10] We show[10] that, in most cases, these equations are not a complete set that determine the desired functions; rather, certain equations must be added. This had been overlooked in the heuristic derivations of the nonlinear equations. In the special case of conservative scattering, Chandrasekhar had recognized nonuniqueness of solutions* but even then he only partially determined the multiplicity of solutions.[11]

The purpose of the present paper is to report on the results obtained from a fruitful combination of the linear and nonlinear theories. This combination of the two methods gives a way of reducing the complexity of numerical computations for homogeneous atmospheres. Even for Rayleigh polarization scattering, we have a relatively simple computational procedure for computing Stokes parameters at any position <u>within</u> a finite or semifinite atmosphere. These solutions are exact for any atmospheric thickness and yield asymptotic formulas for thick atmospheres.

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*See Ref. 6, Chap. 8.

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