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REDUCTION OF THE EQUATIONS OF RADIATIVE TRANSFER FOR A PLANE-PARALLEL, PLANETARY ATMOSPHERE: Part II

Zdenek Sekera

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PREFACE

This Memorandum is a continuation of RM-4951-PR, in which the author further advances the theory of radiative transfer in plane-parallel atmospheres, already treated in several of his previous studies. In particular, in the present work, the author considers further certain mathematical techniques necessary in solving the problem uniquely. In addition, the mathematical expressions corresponding to physical quantities, such as directional radiation and total fluxes emerging from a Rayleigh scattering atmosphere, are derived.

The author is a professor of meteorology and Head of the Department of Meteorology at the University of California, Los Angeles, and a consultant to The RAND Corporation.

ABSTRACT

The equations of radiative transfer in a homogeneous atmosphere with Rayleigh scattering can be easily reduced from their matrix form to scalar equations if the singular linear integral equations for the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices, derived in Part I of this paper, are used as a starting point for this reduction. These equations can be arranged to yield scalar equations, the solutions of which either are unique, or consist of a one- or two-parameter family of solutions. From these equations, are derived linear constraints that allow a unique determination of all the unknown constants appearing in the solutions.

Moreover, the solutions for the azimuth-independent terms are expressed as linear combinations of two pairs of Chandrasekhar's X- and and Y-functions, (X_ℓ, Y_ℓ) and (X_r, Y_r). The nonunique solutions for the first pair have been specified in two ways: either as Busbridge's solutions, which then allow us to use Mullikin's computational scheme for large optical thicknesses, or as Chandrasekhar's standard solutions. By simple transformation the expressions for the first case are reduced to the form given by Chandrasekhar in Radiative Transfer. This important result not only offers a proof of the correctness of these alternative methods, but also indicates the equivalence of purely mathematical specifications in the first case with Chandrasekhar's determinations of unknown constants from the flux equation in the second.

The paper concludes with a derivation of the expressions for the average intensity and net flux of the radiations emerging from a plane-parallel Rayleigh atmosphere.

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CONTENTS

PREFACE iii

ABSTRACT v

ACKNOWLEDGMENTS vii

LIST OF SYMBOLS xi

I. INTRODUCTION 1

II. REDUCTION OF THE EQUATIONS FOR $\underset{\sim}{X}$- AND $\underset{\sim}{Y}$-MATRICES TO SCALAR FORM 4

III. LINEAR CONSTRAINTS ON THE SOLUTION OF THE REDUCED EQUATION: MOMENT RELATIONSHIPS FOR P- AND Q-FUNCTIONS .. 15

IV. EXPRESSIONS FOR THE SOLUTIONS P_i, Q_i ($i = 1, 2, 5, 6$) IN TERMS OF CHANDRASEKHAR'S X- and Y-FUNCTIONS 20

V. MOMENT RELATIONSHIPS FOR THE FUNCTIONS P_r, Q_r, P_ℓ, AND Q_ℓ .. 26

VI. DETERMINATION OF THE CONSTANTS IN THE EXPRESSIONS FOR P- AND Q-FUNCTIONS 30

VII. COMPARISON WITH CHANDRASEKHAR'S ANALYSIS 37

VIII. EXPRESSION FOR FUNCTIONS NEEDED FOR THE COMPUTATIONS OF RADIATIONS EMERGING FROM A PLANETARY ATMOSPHERE..... 42

IX. COMPUTATION OF THE AVERAGE INTENSITY AND OF THE FLUX COMPONENT OF THE EMERGING RADIATION 45

X. CONCLUDING REMARKS 55

REFERENCES 57

LIST OF SYMBOLS

a,b,a',b'	Constants of the two-parameter family of the solutions of Eqs. (78a), (79a), and (80a,b)
c,c',d,e,f,f'	Constants defined in Eqs. (95) and (127)
c_n,d_n	Chandrasekhar's notation for the n-th moments of P_r and Q_r
k_i,ℓ_i	Moments of the element K_i+K_{i+2}, L_i+L_{i+2} $(i=1,2)$
m,n,r,s	Second-order combinations of the moments α_k,β_k, α'_k,β'_k $(k=0,1)$ defined in Eq. (96)
$m_n(F),m_n(\underset{\sim}{F})$	n-th moments of the scalar function F or of the matrix $\underset{\sim}{F}$
$\bar{s}(\tau)$	First moment of the sum $s_\ell(\mu)+s_r(\mu)$
$s_\ell(\mu),s_r(\mu)$	Elements of the matrix $\underset{\sim}{s}(\mu)$
$t_\ell(\mu),t_r(\mu)$	Elements of the matrix $\underset{\sim}{t}(\mu)$
u_3,u_4,u_5	Chandrasekhar's constants defined in Eqs. (118) and (151)
A_k,A'_k,B_k,B'_k	Abbreviations for the M_k-moment of P_5,P_6,Q_5,Q_6 in Eq. (55)
B,D,C	Abbreviations for combinations of the modified moments of Q_ℓ, P_r, Q_r [Cf. Eq. (95)]
C_k,C'_k,D_k,D'_k	Abbreviations for the ordinary moments of P_5,P_6, Q_5,Q_6 in Eq. (55)
D_1,D_2,D_3	Functions appearing in the elements of the matrix $\underset{\sim}{D}(\mu)$

SYMBOLS, cont.

D_ℓ, D_r	Discriminant function to the characteristic functions ψ_ℓ, ψ_r
G	Constant defined in Eq. (108)
$H(\mu,x)$	Function defined in Eq. (24)
$J_\ell(0), J_r(0)$ $J_\ell(\tau), J_r(\tau)$	Components of the average intensity of the upward and downward emerging radiation
$L^{(\pm)}_{r,\ell}\{F\}$	Integral operator with ψ_ℓ or ψ_r in the argument, defined in Eq. (63)
$M_k(F_{\ell,r})$	The k-th modified (scalar) moment with respect to the characteristic functions ψ_ℓ and ψ_r, respectively
P_i (i=1,2,3,4)	Functions in the elements of the matrix $\underset{\sim}{P}$
P_5, P_6	Linear combination of the functions P_1, P_3 or P_2, P_4 respectively defined in Eq. (31)
Q	Chandrasekhar's constant defined in Eq. (120)
Q_i (i=1,2,3,4)	Functions appearing in the elements of the matrix $\underset{\sim}{Q}$
Q_5, Q_6	Linear combinations of the functions Q_1, Q_3 or Q_2, Q_4, equivalent to P_5, P_6
X_ℓ, Y_ℓ, X_r, Y_r	The X- and Y-functions for the characteristic functions ψ_ℓ and ψ_r

SYMBOLS, cont.

Bold face letters:

$\underset{\sim}{s}(\mu),\underset{\sim}{t}(\mu)$	One-column matrices defined in Eq. (143)
$\underset{\sim}{A}(\mu),\underset{\sim}{B}(\mu)$	Matrix operators on $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices defined in Eq. (11)
$\underset{\sim}{C}$	One-column matrix of the elements $(1,1/\sqrt{2})$
$\underset{\sim}{D}(\mu),\underset{\sim}{D}'(\mu)$	Discriminant matrix and its adjoint matrix
$\underset{\sim}{E}$	Unit matrix
$\underset{\sim}{F}^{\uparrow},\underset{\sim}{F}^{\downarrow}$	Net flux of the upward, downward emerging radiation
$\underset{\sim}{G}$	Symbol for the product $\underset{\sim}{D}' \cdot \underset{\sim}{\Psi}$
$\underset{\sim}{J}(0),\underset{\sim}{J}(\tau)$	Average intensity of the upward, downward emerging radiation
$\underset{\sim}{K},\underset{\sim}{L}$	Matrices appearing in the expressions for $\underset{\sim}{S}_0$, $\underset{\sim}{T}_0$
$\underset{\sim}{M}$	Factor in the azimuth-independent term of the phase matrix for Rayleigh scattering
$\underset{\sim}{M}_k(\underset{\sim}{F})$	k-th modified (matrix) moment for the matrix $(\underset{\sim}{F})$
$\underset{\sim}{P},\underset{\sim}{Q}$	Sum and difference of the matrices $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ [Eq. (20)]
$\underset{\sim}{S}_0,\underset{\sim}{T}_0$	Azimuth-independent terms of the reflection and transmission matrices
$\underset{\sim}{X},\underset{\sim}{Y}$	Matrices corresponding to Chandrasekhar's X- and Y-functions

SYMBOLS, cont.

$\underset{\sim}{\Gamma}(\mu)$	One-column matrix of the elements $(\gamma_\ell^{(1)}, \gamma_r^{(1)})$
$\underset{\sim}{\Psi}(\mu)$	Characteristic matrix for the azimuth-independent term of the phase matrix
$\underset{\sim}{\mathcal{E}}$	Column matrix with its elements all equal to unity

Greek letters:

$\alpha_i, \beta_i, \alpha_i', \beta_i'$	Abbreviations for combinations of the ordinary moments of P_ℓ, Q_ℓ, P_r, Q_r in Eq. (91)
γ, Γ	Constants defined in Eqs. (83) and (101)
$\gamma_\ell^{(1)}, \gamma_r^{(1)}$	Chandrasekhar's functions, related to the net flux
Δ, Δ'	Constants defined in Eq. (110e)
Δ_1, Δ_2	Chandrasekhar's constants defined in Eq. (119)
θ	Zenith distance
$\varkappa_n, \delta_n$	Chandrasekhar's notations for moments of P_ℓ and Q_ℓ
$\lambda(\mu)$	Discriminant function
μ, ν	Directional parameter (cosine of the zenith distance) and its reciprocal value
$\nu_1, \nu_2, \nu_3, \nu_4$	Chandrasekhar's constants defined in Eq. (118)
τ	Optical thickness of the entire atmosphere
ψ_1, ψ_2, ψ_3	Functions appearing in the elements of the characteristic matrix $\underset{\sim}{\Psi}(\mu)$

SYMBOLS, cont.

Ψ_ℓ, Ψ_r	Chandrasekhar's characteristic functions
$\psi, \varphi, \chi, \zeta, \Xi, \eta, \sigma, \theta$	Chandrasekhar's notation for the elements of the $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices

I. INTRODUCTION

In Part I of this paper[1] it was shown that, for certain types of scattering involving a "separable matrix," the reduction of the integro-differential equations for the reflection and transmission matrices for a homogeneous medium leads to a system of nonlinear equations for a pair of matrices, of the form completely analogous to the equations for Chandrasekhar's X- and Y-functions. Moreover, it was shown that this system of nonlinear equations can be easily transformed to a system of singular linear equations, which allows us to derive the linear constraints that determine uniquely the solution of the nonlinear system.

For Rayleigh scattering, the phase matrix is separable in all representations of the state of polarization of the scattered radiation field. The phase matrix can be easily expanded into a harmonic series in the azimuth difference of the meridians (vertical planes through the local zenith) containing the direction of observation and the direction of the external parallel irradiation. This series has only three terms, one of which is azimuth-independent and the other two, azimuth-dependent. Each of these three terms is itself a separable matrix and the corresponding characteristic matrices can be easily derived for each term. For the two azimuth-dependent terms, the characteristic matrix reduces to a unit matrix, multiplied by a scalar characteristic function. Consequently, the equations for these two terms reduce to scalar equations for Chandrasekhar's X- and Y-functions; further analysis is then simple, and need not be discussed here in more detail. The phase matrix for the azimuth-independent term reduces to a product of two 2-by-2 matrices and leads to a characteristic matrix of the same order. From the

characteristic matrix, one can easily derive the discriminant matrix that multiplies the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrix of the L.H.S. of the system of singular linear integral equations for these matrices. In a way completely analogous to the discussion by Mullikin[2] for the scalar form of these equations, this discriminant matrix offers the criterion of uniqueness of the solution of the systems; moreover, from the requirement that at the singular points of the equation the solution must remain analytic, it is easy to derive the linear constraints that allow us to select unique solutions to the equations.

Using the system of singular linear integral equations for the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices brings another advantage in the solution of these equations. After multiplication by the adjoint discriminant matrix, the equation can be reduced to a form that allows, in a rather simple way, the solution to be expressed in terms of Chandrasekhar's X_ℓ-, Y_ℓ-, X_r-, Y_r-functions. The constants appearing in this form of the solution can be determined uniquely from the linear constraints. In Section VI, the determination of the constants is based on the Busbridge solution of the equations for X_ℓ- and Y_ℓ-functions (i.e., analytical behavior for $\mu \to \infty$), since for this solution Mullikin has already derived the computational scheme for large τ. If, however, the standard solution is used for the X_ℓ- and Y_ℓ-functions, then as shown in Section VII, the expressions for the eight elements of the matrices $\underset{\sim}{K}$ and $\underset{\sim}{L}$ that are identical to Chandrasekhar's Greek-lettered functions, become identical with those given by Chandrasekhar.[3] This then proves the identity of the procedures that select the unique solution from the two-parameter family of solutions of the nonlinear systems. The requirement introduced by Busbridge[4] and Mullikin,[5] that the solution be analytic for

$\mu \to \infty$, is fully equivalent to the procedure used by Chandrasekhar in which the value of the final undetermined constant is obtained from the flux equation.

In the last section, the average intensity and the flux components are derived in terms of the constants of the solution for both the case considered by Busbridge and for the case where standard solutions for the X_ℓ- and Y_ℓ-functions are used. Again, these expressions can be easily rewritten in the forms given by Chandrasekhar.[3]

II. REDUCTION OF THE EQUATIONS FOR X- AND Y-MATRICES TO SCALAR FORM

The azimuth-independent terms of the reflection and transmission matrices which are solutions of the corresponding integro-differential equations can be written in the form[1]

$$\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right) \underset{\sim}{S}_0(\tau;\, \mu,\, \mu_0) = \frac{3}{4}\left[\underset{\sim}{K}(\mu) \cdot \widetilde{\underset{\sim}{K}}(\mu_0) - \underset{\sim}{L}(\mu) \cdot \widetilde{\underset{\sim}{L}}(\mu_0)\right]$$

$$\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \underset{\sim}{T}_0(\tau;\, \mu,\, \mu_0) = \frac{3}{4}\left[\underset{\sim}{K}(\mu) \cdot \widetilde{\underset{\sim}{L}}(\mu_0) - \underset{\sim}{L}(\mu) \cdot \widetilde{\underset{\sim}{K}}(\mu_0)\right] \tag{1}$$

where

$$\underset{\sim}{K}(\mu) = \underset{\sim}{M}(\mu) \cdot \underset{\sim}{X}(\mu), \qquad \underset{\sim}{L}(\mu) = \underset{\sim}{M}(\mu) \cdot \underset{\sim}{Y}(\mu) \tag{2}$$

$$M(\mu) \equiv \begin{bmatrix} \mu^2, & \sqrt{2}(1 - \mu^2) \\ 1, & 0 \end{bmatrix} \tag{3}$$

and the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices satisfy the system of linear singular equations of the form

$$\underset{\sim}{D}(\mu) \cdot \underset{\sim}{X}(\tau,\, \mu) = \underset{\sim}{E} + \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{X}(x)\, \frac{\mu\, dx}{x - \mu} - e^{-\tau/\mu} \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{Y}(x)\, \frac{\mu\, dx}{\mu + x}$$

$$\underset{\sim}{D}(\mu) \cdot \underset{\sim}{Y}(\tau,\, \mu) = \left[\underset{\sim}{E} - \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{X}(x)\, \frac{\mu\, dx}{x + \mu}\right] e^{-\tau/\mu} + \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{Y}(x)\, \frac{\mu\, dx}{x - \mu} \tag{4}$$

with the characteristic matrix

$$\underset{\sim}{\Psi}(x) = \frac{3}{8}\,\underset{\sim}{\tilde{M}}(x) \cdot \underset{\sim}{M}(x) \tag{5}$$

and $\underset{\sim}{D}(\mu)$, the discriminant matrix, is given by

$$\underset{\sim}{D}(\mu) = \underset{\sim}{E} - 2\int_0^1 \underset{\sim}{\Psi}(x)\,\frac{\mu^2 dx}{\mu^2 - x^2} \tag{6}$$

With respect to Eq. (3), we get for the elements of $\underset{\sim}{\Psi}$,

$$\underset{\sim}{\Psi}(x) \equiv \begin{bmatrix} \psi_1, & \sqrt{2}\psi_2 \\ \sqrt{2}\psi_2, & \psi_3 \end{bmatrix} \tag{7}$$

$$\psi_1(\mu) = \frac{3}{8}(1 + \mu^4), \quad \psi_2(\mu) = \frac{3}{8}\mu^2(1 - \mu^2), \quad \psi_3(\mu) = \frac{3}{4}(1 - \mu^2)^2 \tag{7a}$$

Writing

$$\underset{\sim}{D}(\mu) \equiv \begin{bmatrix} D_1, & \sqrt{2}D_2 \\ \sqrt{2}D_2, & D_3 \end{bmatrix} \tag{8}$$

we then get, after substitution of Eqs. (7) and (7a) in Eq. (8),

$$D_1(\mu) = 1 + \frac{1}{4}\mu^2 + \frac{3}{4}\mu^4 + \mu\psi_1(\mu)\,\ell n(1 - \mu)/(1 + \mu) \tag{9a}$$

$$D_2(\mu) = \frac{1}{2}\mu^2(1 - \frac{3}{2}\mu^2) + \mu\psi_2(\mu)\,\ell n(1 - \mu)/(1 + \mu) \tag{9b}$$

$$D_3(\mu) = (1 - \mu^2)(1 - \frac{3}{2}\mu^2) + \mu\psi_3(\mu)\,\ell n(1 - \mu)/(1 + \mu) \tag{9c}$$

It may be pointed out that the discriminant matrix satisfies the identity [Cf. Sekera,[1] Appendix A],

$$\underset{\sim}{E} - \underset{\sim}{D}(\mu) = [\underset{\sim}{E} - \underset{\sim}{A}(+\mu)][\underset{\sim}{E} - \underset{\sim}{\tilde{A}}(-\mu)] - \underset{\sim}{B}(+\mu) \cdot \underset{\sim}{\tilde{B}}(-\mu) \tag{10}$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ represent the following matrix operators

$$\underset{\sim}{A}(\mu) = \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{X}(x) \frac{\mu dx}{x + \mu}, \quad \underset{\sim}{B}(\mu) = \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{Y}(x) \frac{\mu dx}{x + \mu} \tag{11}$$

The operators in Eq. (11) satisfy also another identity

$$\underset{\sim}{B}(\mu) \cdot [\underset{\sim}{E} - \widetilde{\underset{\sim}{A}}(\mu)] = [\underset{\sim}{E} - \underset{\sim}{A}(\mu)] \cdot \widetilde{\underset{\sim}{B}}(\mu) \tag{12}$$

The reduction of the system in Eq. (4) into a scalar form can be facilitated if we multiply the equations of the system in Eq. (4) by the adjoint matrix

$$\underset{\sim}{D}'(\mu) \equiv \begin{bmatrix} D_3, & -\sqrt{2}D_2 \\ -\sqrt{2}D_2, & D_1 \end{bmatrix} \tag{13}$$

The L.H.S. of these equations will contain the factor

$$\underset{\sim}{D}'(\mu) \cdot \underset{\sim}{D}(\mu) = \lambda(\mu)\underset{\sim}{E} \tag{14}$$

with

$$\lambda(\mu) = D_1 D_3 - 2D_2^2 = D_\ell(\mu)\, D_r(\mu) \tag{15}$$

where D_ℓ and D_r are given by the scalar forms of Eq. (6) for the characteristic functions

$$\psi_\ell(\mu) = \frac{3}{4}(1 - \mu^2)$$

and (16)

$$\psi_r(\mu) = \frac{3}{8}(1 - \mu^2)$$

respectively, i.e.,

$$D_\ell = 1 - \frac{3}{2}\mu^2 + \mu\psi_\ell(\mu)\,\ell n(1-\mu)/(1+\mu)$$
$$D_r = 1 - \frac{3}{4}\mu^2 + \mu\psi_r(\mu)\,\ell n(1-\mu)/(1+\mu) \qquad (17)$$

Multiplication of the equations in the system of Eq. (4) by $\underset{\sim}{D}'(\mu)$ leads with respect to Eq. (14) to the system

$$\lambda(\mu)\,\underset{\sim}{X}(\mu) = \underset{\sim}{D}'(\mu) - \mu\int_0^1 \underset{\sim}{D}'(\mu)\cdot\underset{\sim}{\Psi}(x)\left[\frac{\underset{\sim}{X}(x)}{\mu - x} + e^{-\tau/\mu}\,\frac{\underset{\sim}{Y}(x)}{\mu + x}\right]dx$$
$$\lambda(\mu)\,\underset{\sim}{Y}(\mu) = \underset{\sim}{D}'(\mu)\,e^{-\tau/\mu} - \mu\int_0^1 D'(\mu)\cdot\underset{\sim}{\Psi}(x)\left[\frac{\underset{\sim}{Y}(x)}{\mu - x} + e^{-\tau/\mu}\,\frac{\underset{\sim}{X}(x)}{\mu + x}\right]dx \qquad (18)$$

The system of the coupled equations for $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ in Eq. (18) can be easily transformed by addition and subtraction into the uncoupled equations

$$\lambda(\mu)\,\underset{\sim}{P}(\mu) = (1 + e^{-\tau/\mu})\,\underset{\sim}{D}'(\mu) - \int_0^1 \underset{\sim}{G}(\mu; x)\cdot\underset{\sim}{P}(x)\left[\frac{\mu}{\mu - x} + e^{-\tau/\mu}\,\frac{\mu}{\mu + x}\right]dx$$
$$\lambda(\mu)\,\underset{\sim}{Q}(\mu) = (1 - e^{-\tau/\mu})\cdot\underset{\sim}{D}'(\mu) - \int_0^1 \underset{\sim}{G}(\mu; x)\cdot\underset{\sim}{Q}(x)\left[\frac{\mu}{\mu - x} - e^{-\tau/\mu}\,\frac{\mu}{\mu + x}\right]dx \qquad (19)$$

where

$$\underset{\sim}{P}(\mu) = \underset{\sim}{X}(\mu) + \underset{\sim}{Y}(\mu), \quad \underset{\sim}{Q}(\mu) = \underset{\sim}{X}(\mu) - \underset{\sim}{Y}(\mu) \qquad (20)$$

and

$$\underset{\sim}{G}(\mu; x) = \underset{\sim}{D}'(\mu)\cdot\underset{\sim}{\Psi}(x)$$

If we denote the elements of any of the matrices $\underset{\sim}{N} \equiv (\underset{\sim}{P}, \underset{\sim}{Q}, \underset{\sim}{G})$ as

$$N_{11} \equiv N_1, \quad N_{12} \equiv \sqrt{2}N_2, \quad N_{21} \equiv \sqrt{2}N_3, \quad N_{22} \equiv N_4 \tag{21}$$

and substitute in Eq. (19), then after performing the multiplication, the system in Eq. (19) splits into four pairs of equations for the following pairs of functions

$$(P_1, P_3), \quad (P_2, P_4), \quad (Q_1, Q_3), \quad (Q_2, Q_4)$$

which are of the form

$$\lambda(\mu)\begin{bmatrix} P_1(\mu) \\ P_3(\mu) \end{bmatrix} = (1 + e^{-\tau/\mu})\begin{bmatrix} -D_2(\mu) \\ -D_2(\mu) \end{bmatrix} - \int_0^1 \begin{bmatrix} G_1P_1(x) + 2G_2P_3(x) \\ G_3P_1(x) + G_4P_3(x) \end{bmatrix} H(\mu, x)\, dx \tag{22}$$

$$\lambda(\mu)\begin{bmatrix} P_2(\mu) \\ P_4(\mu) \end{bmatrix} = (1 + e^{-\tau/\mu})\begin{bmatrix} -D_2(\mu) \\ D_1(\mu) \end{bmatrix} - \int_0^1 \begin{bmatrix} G_1P_2(x) + G_2P_4(x) \\ 2G_3P_2(x) + G_4P_4(x) \end{bmatrix} H(\mu, x)\, dx \tag{23}$$

where

$$H(\mu, x) = \frac{\mu}{\mu - x} + e^{-\tau/\mu}\, \frac{\mu}{\mu + x} \tag{24}$$

The systems for (Q_1, Q_3) and (Q_2, Q_4) are identical to that in Eqs. (22) and (23), except that the terms with $e^{-\tau/\mu}$ are then multiplied by -1.

Further simplifications can be obtained by considering the forms of the functions D_i in Eqs. (9a-c) and D_ℓ and D_r in Eq. (17) as well as the explicit expressions for $G_i (i = 1, 2)$. With respect to Eqs. (7a) and (16), it is easy to see that

$$D_2(\mu) = \frac{1}{2}\mu^2 D_\ell(\mu), \quad D_3(\mu) = (1 - \mu^2) D_\ell(\mu) \tag{25}$$

Furthermore, from Eqs. (13), (7), and (21) it follows that

$$G_1(\mu; x) = D_3(\mu)\psi_1(x) - 2D_2(\mu)\psi_2(x)$$
$$G_2(\mu; x) = D_3(\mu)\psi_3(x) - D_2(\mu)\psi_2(x) \tag{26}$$

and with respect to Eq. (25),

$$G_1(\mu; x) = D_\ell(\mu)\left[\psi_r(\mu) + \frac{3}{8}(x^2 - \mu^2)x^2\right]$$
$$G_2(\mu; x) = D_\ell(\mu)\frac{3}{8}(1 - x^2)(x^2 - \mu^2) \tag{26a}$$

When the expressions for G_1 and G_2 from Eq. (26) and for D_2 and D_3 from Eq. (25) are substituted in Eq. (22) we see that $D_\ell(\mu)$ is a factor on both sides of the equations for P_1 and P_2. Hence, if $D_\ell(\mu) \neq 0$, we get for P_1 and P_2 the following equations:

$$D_r(\mu)P_1(\mu) = (1 + e^{-\tau/\mu})(1 - \mu^2) - \int_0^1 \psi_r(\mu)P_1(x)H(\mu, x)\,dx$$
$$- \frac{3}{8}\int_0^1 (x^2 - \mu^2)[x^2P_1(x) + 2(1 - x^2)P_3(x)]H(\mu, x)\,dx \tag{27}$$

$$D_r(\mu)P_2(\mu) = (1 + e^{-\tau/\mu})(-\frac{1}{2}\mu^2) - \int_0^1 \psi_r(\mu)P_2(x)H(\mu, x)\,dx$$
$$- \frac{3}{8}\int_0^1 (x^2 - \mu^2)[x^2P_2(x) + (1 - x^2)P_4(x)]H(\mu, x)\,dx \tag{28}$$

The expressions for G_3 and G_4 do not contain D_r or D_ℓ as simple factors. Therefore, the simplification of the second equations in the systems in Eqs. (22) and (23) has to be achieved in some other way. This can be done if we turn our attention to the matrices $\underset{\sim}{K}$ and $\underset{\sim}{L}$. Since the matrices $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are proportional to the sum and to the differences of

$\underset{\sim}{P}$ and $\underset{\sim}{Q}$ respectively, the elements of the matrices $\underset{\sim}{K}$ and $\underset{\sim}{L}$ can be denoted according to the scheme in Eq. (21). Performing the multiplication of the product of $\underset{\sim}{M}(\mu) \cdot \underset{\sim}{X}(\mu)$, we see that the elements of the matrix

$$\underset{\sim}{K} = \underset{\sim}{M}(\mu) \cdot \underset{\sim}{X}(\mu) \equiv \begin{bmatrix} K_1(\mu), & \sqrt{2}K_2(\mu) \\ K_3(\mu), & \sqrt{2}K_4(\mu) \end{bmatrix} \tag{29}$$

have the form

$$\begin{aligned} K_1(\mu) &= \mu^2 X_1 + 2(1 - \mu^2)X_3, & K_2(\mu) &= \mu^2 X_2 + (1 - \mu^2)X_4 \\ K_3(\mu) &= X_1, & K_4(\mu) &= X_2 \end{aligned} \tag{30}$$

The elements of the matrix $\underset{\sim}{L} = \underset{\sim}{M} \cdot \underset{\sim}{Y}$ have the identical forms to those in Eq. (30) except that X_i is replaced by Y_i. Then by adding the expression for K_i and $L_i (i = 1, 2)$, we get

$$\begin{aligned} P_5(\mu) &= K_1(\mu) + L_1(\mu) = \mu^2 P_1 + 2(1 - \mu^2)P_3 \\ P_6(\mu) &= K_2(\mu) + L_2(\mu) = \mu^2 P_2 + (1 - \mu^2)P_4 \end{aligned} \tag{31}$$

The equations for the functions P_5 and P_6 can be easily obtained from the systems in Eqs. (22) and (23). Multiplying the first equation in Eq. (22) by μ^2 and the second equation by $2(1 - \mu^2)$ and adding, we get

$$\begin{aligned} \lambda(\mu)P_5(\mu) = {} & (1 - e^{-\tau/\mu})[\mu^2 D_3 - 2(1 - \mu^2)D_2] \\ & - \int_0^1 \{[\mu^2 G_1 + 2(1 - \mu^2)G_3]P_1(x) + 2[\mu^2 G_2 + (1 - \mu^2)G_4]P_3(x)\} \\ & \cdot H(\mu, x)\, dx \end{aligned} \tag{32}$$

Since, with respect to Eq. (7a)

$$\mu^2\psi_3 = 2(1 - \mu^2)\psi_2$$

it follows from Eqs. (9b, c) that

$$\mu^2 D_3 - 2(1 - \mu^2)D_2 = 0$$

Furthermore, from the definition of $\underset{\sim}{G} = \underset{\sim}{D}'(\mu) \cdot \underset{\sim}{\Psi}(x)$, it follows that

$$G_3(\mu; x) = -D_2(\mu)\psi_1(x) + D_1(\mu)\psi_2(x)$$

$$G_4(\mu; x) = -2D_2(\mu)\psi_2(x) + D_1(\mu)\psi_3(x)$$

Then with respect to Eq. (26)

$$\begin{aligned} \mu^2 G_1 + 2(1 - \mu^2)G_3 &= [\mu^2 D_3 - 2(1 - \mu^2)D_2]\psi_1(x) \\ &\quad + [-2\mu^2 D_2 + 2(1 - \mu^2)D_1]\psi_2(x) \\ \mu^2 G_2 + (1 - \mu^2)G_4 &= [\mu^2 D_3 - 2(1 - \mu^2)D_2]\psi_2(x) \\ &\quad + [-\mu^2 D_2 + (1 - \mu^2)D_1]\psi_3(x) \end{aligned} \tag{34}$$

With respect to Eqs. (9a, b) and (17)

$$(1 - \mu^2)D_1 - \mu^2 D_2 = D_r(\mu)$$

which when substituted into Eq. (34), gives

$$\mu^2 G_1 + 2(1 - \mu^2)G_3 = D_r(\mu)\,\frac{3}{4}\,x^2(1 - x^2) \tag{35a}$$

$$\mu^2 G_2 + (1 - \mu^2)G_4 = D_r(\mu)\,\frac{3}{4}\,(1 - x^2)^2 \tag{35b}$$

Multiplying Eq. (35a) by $P_1(x)$ and Eq. (35b) by $2P_3(x)$, we get for the bracketed terms in the integrand of the R.H.S. of Eq. (32)

$$D_r(\mu)\,\frac{3}{4}\,(1 - x^2)[x^2P_1(x) + 2(1 - x^2)P_3(x)] = D_r(\mu)\psi_\ell(x)P_5(x)$$

Substituting this expression in Eq. (32), we get with respect to Eqs. (15) and (33b) for $D_r(\mu) \neq 0$,

$$D_\ell(\mu)P_5(\mu) = -\int_0^1 \psi_\ell(x)P_5(x)H(\mu, x)\,dx \tag{36}$$

Similarly, we get from Eq. (23)

$$D_\ell(\mu)P_6(\mu) = 1 + e^{-\tau/\mu} - \int_0^1 \psi_\ell(x)P_6(x)H(\mu, x)\,dx \tag{37}$$

With respect to Eq. (31) the equation for P_1 and P_2 in Eqs. (27) and (28) can be written as

$$D_r(\mu)P_1(\mu) = (1 + e^{-\tau/\mu})(1 - \mu^2) - \frac{3}{8}\int_0^1 (x^2 - \mu^2)P_5(x)H(\mu, x)\,dx - \int_0^1 \psi_r(\mu)P_1(x)H(\mu, x)\,dx \tag{38}$$

$$D_r(\mu)P_2(\mu) = -\frac{1}{2}\mu^2(1 + e^{-\tau/\mu}) + \frac{3}{8}\int_0^1 (\mu^2 - x^2)P_6(x)H(\mu, x)\,dx - \int_0^1 \psi_r(\mu)P_2(x)H(\mu, x)\,dx \tag{39}$$

It is easy to bring these equations to the forms equivalent to those in Eqs. (36) and (37) by merely replacing $\psi_r(\mu)$ in the integrand of the last terms in Eqs. (38) and (39) by $\psi_r(x) + [\psi_r(\mu) - \psi_r(x)]$. Then with respect to Eq. (16), we have

$$\int_0^1 \psi_r(\mu)P_k(x)H(\mu, x)\,dx = \int_0^1 \psi_r(x)P_k(x)H(\mu, x)\,dx$$
$$+ \frac{3}{8}\int_0^1 (x^2 - \mu^2)P_k(x)H(\mu, x)\,dx \qquad (k = 1, 2) \tag{40}$$

where the last term gives, after integration, the expressions (see Eq. (24))

$$\frac{3}{8}\mu^2 m_0(P_k)(1 + e^{-\tau/\mu}) + \frac{3}{8}\mu m_1(P_k)(1 - e^{-\tau/\mu}) \qquad (k = 1, 2)$$

where $m_k(F)$ has been used to denote the k^{th} moment of the function F, i.e.,

$$m_k(F) = \int_0^1 F(x)x^k\,dx \tag{41}$$

Substituting in Eqs. (38) and (39) we get the equations for P_1 and P_2 in the form

$$D_r(\mu)P_1(\mu) = (1 - \mu^2)(1 + e^{-\tau/\mu}) + \frac{3}{8}[\mu^2 m_0(P_1 + P_5)(1 + e^{-\tau/\mu})$$
$$+ \mu m_1(P_1 + P_5)(1 - e^{-\tau/\mu})] - \int_0^1 \psi_r(x)P_1(x)H(\mu, x)\,dx \tag{42}$$

$$D_r(\mu)P_2(\mu) = -\frac{1}{2}\mu^2(1 + e^{-\tau/\mu}) + \frac{3}{8}[\mu^2 m_0(P_2 + P_6)(1 + e^{-\tau/\mu})$$
$$+ \mu m_1(P_2 + P_6)(1 - e^{-\tau/\mu})] - \int_0^1 \psi_r(x)P_2(x)H(\mu, x)\,dx \tag{43}$$

The system of equations for $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices in Eq. (4) is hence reduced to four systems of linear singular integral equations for the four pairs of functions

$$(P_1, P_5),\ (P_2, P_6),\ (Q_1, Q_5),\ (Q_2, Q_6) \tag{44}$$

They satisfy Eqs. (41), (36) and (42), (37) and the corresponding equations for the Q-functions obtained from the equations for P-function by multiplying the terms containing $e^{-\tau/\mu}$ by -1. When these equations are solved, then in terms of Eqs. (30) and (31), the elements of the $\underset{\sim}{K}$-and $\underset{\sim}{L}$-matrices are given by

$$\begin{aligned} K_1 &= \tfrac{1}{2}(P_5 + Q_5) & L_1 &= \tfrac{1}{2}(P_5 - Q_5) \\ K_2 &= \tfrac{1}{2}(P_6 + Q_6) & L_2 &= \tfrac{1}{2}(P_6 - Q_6) \\ K_3 &= \tfrac{1}{2}(P_1 + Q_1) & L_3 &= \tfrac{1}{2}(P_1 - Q_1) \\ K_4 &= \tfrac{1}{2}(P_2 + Q_2) & L_4 &= \tfrac{1}{2}(P_2 - Q_2) \end{aligned} \tag{45}$$

As mentioned earlier, the solutions of singular integral equations have the advantage of being able to specify the constraints for the nonunique solutions and determine the unique solutions in terms of the constants in these constraints. Furthermore, the computational scheme developed by Mullikin[2] can then be used for large optical thicknesses with even greater ease than for moderate values of the optical thickness.

The next two sections discuss the nonuniqueness problem and the unique determination of all the constants appearing in the final expressions for these solutions.

It should also be mentioned here that the reduction applied to the equations for $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices can be applied equally well to the equations for $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices. The procedure, however, is somewhat more complicated because of the asymmetry of the $\underset{\sim}{M}$-matrix that replaces the symmetric characteristic matrix $\underset{\sim}{\Psi}$ in the integrand of the second and third terms and the identity matrix $\underset{\sim}{E}$ in the first term of Eq. (4).

III. LINEAR CONSTRAINTS ON THE SOLUTION OF THE REDUCED EQUATION: MOMENT RELATIONSHIPS FOR P- AND Q-FUNCTIONS

From the definition of the scalar functions D_ℓ and D_r

$$D_{\ell,r}(\mu) = 1 - 2\mu^2 \int_0^1 \psi_{\ell,r}(x)\, dx/(\mu^2 - x^2) \tag{46}$$

it follows immediately that for $\mu \to \infty$

$$\lim_{\mu \to \infty} D_r(\mu) = 1 - 2\int_0^1 \psi_r(x)\, dx = 1/2$$

$$\lim_{\mu \to \infty} D_\ell(\mu) = 1 - 2\int_0^1 \psi_\ell(x)\, dx = 0$$

Whereas $D_r(\mu)$ remains positive for all real $\mu \geq 0$, $D_\ell(\mu)$ has one root in the interval $0, 1$ and a double root for $\mu \to \infty$. The solutions of Eq. (19) must satisfy the constraints given by the conditions that, for $\mu \to \infty$, the R.H.S. of Eq. (19) must vanish. They can be written in the form

$$\begin{aligned} &\lim_{\mu \to \infty} \{\underset{\sim}{D}'(\mu) \cdot [(1 + e^{-\tau/\mu}) \cdot \underset{\sim}{E} - \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{P}(x) \cdot H^{(+)}(\mu, x)\, dx]\} = 0 \\ &\lim_{\mu \to \infty} \{\underset{\sim}{D}'(\mu) \cdot [(1 - e^{-\tau/\mu}) \cdot \underset{\sim}{E} - \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{Q}(x) \cdot H^{(-)}(\mu, x)\, dx]\} = 0 \end{aligned} \tag{47}$$

where $H^{(\pm)}$ denotes the function H in Eq. (24) with $e^{-\tau/\mu}$ preceded by a plus or minus sign. Since

$$\begin{aligned} 1 + e^{-\tau/\mu} &= 2 - \tau/\mu + 0(1/\mu^2) \\ 1 - e^{-\tau/\mu} &= \tau/\mu + 0(1/\mu^2) \\ H^{(+)}(\mu, x) &= 2 - \tau/\mu + 0(1/\mu^2) \\ H^{(-)}(\mu, x) &= (2x + \tau)/\mu + 0(1/\mu^2) \end{aligned} \tag{48}$$

we get the following relations from Eq. (47).

$$\underset{\sim}{D}'(\infty) \cdot \{\underset{\sim}{E} - \underset{\sim}{M}_0(P)\} = 0 \tag{49a}$$

$$\underset{\sim}{D}'(\infty) \cdot \{2\underset{\sim}{M}_1(Q) - \tau[\underset{\sim}{E} - \underset{\sim}{M}_0(Q)]\} = 0 \tag{49b}$$

where

$$\underset{\sim}{M}_i(\underset{\sim}{F}) \equiv \int_0^1 \underset{\sim}{\Psi}(x) \cdot \underset{\sim}{F}(x)x^i \, dx \tag{50}$$

With the values for ψ_i from Eq. (7a) we get for the elements of the matrix $\underset{\sim}{\Psi}(x) \cdot \underset{\sim}{P}(x)$

$$\psi_1(x)P_1(x) + 2\psi_2(x)P_3(x) = \frac{3}{8}(P_1 + P_5) - \psi_r(x)P_5(x)$$

$$\psi_2(x)P_1(x) + \psi_3(x)P_3(x) = \psi_r(x)P_5(x)$$

$$\psi_1(x)P_2(x) + \psi_2(x)P_4(x) = \frac{3}{8}(P_2 + P_6) - \psi_r(x)P_6(x)$$

$$2\psi_2(x)P_2(x) + \psi_3(x)P_4(x) = 2\psi_r(x)P_6(x)$$

The same expressions are obtained for Q_i (i = 1, 2, 3, 4, 5, 6). Then, with the notation introduced in Eq. (41)

$$\underset{\sim}{M}_i(\underset{\sim}{F}) \equiv \begin{bmatrix} \frac{3}{8} m_i(F_1 + F_5) - M_i(F_5), & \sqrt{2}\left[\frac{3}{8} m_i(F_2 + F_6) - M_i(F_6)\right] \\ \sqrt{2}\, M_i(F_5) & 2M_i(F_6) \end{bmatrix}_{(F = P, Q)} \tag{51}$$

where

$$M_i(F) = \int_0^1 \psi_r(x)F(x)x^i \, dx \qquad (F = P_5, P_6, Q_5, Q_6)$$

are modified moments for the function F, weighted by ψ_r. Since

$$D'(\infty) \equiv \frac{1}{10}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

the substitution from Eq. (51) into Eqs. (48) and (49) and the multiplication of the matrices lead to the linear constraints that the functions P_i and Q_i (i = 1, 2, 5, 6) have to satisfy

$$\begin{aligned} 1 - \frac{3}{8} m_0(P_1 + P_5) &= 0 \\ 1 - \frac{3}{4} m_0(P_2 + P_6) &= 0 \end{aligned} \tag{52}$$

$$\begin{aligned} \frac{3}{4} m_1(Q_1 + Q_5) - \tau[1 - \frac{3}{8} m_0(Q_1 + Q_5)] &= 0 \\ \frac{3}{4} m_1(Q_2 + Q_6) - \tau[1 - \frac{3}{4} m_0(Q_2 + Q_6)] &= 0 \end{aligned} \tag{53}$$

In these constraints only the ordinary moments of the solutions of the systems in Eqs. (36,37,42,43) and of the corresponding equations for Q_i (i = 1, 2, 5, 6) appear. Relationships between the modified moments of these functions can be easily derived from the identities in Eqs. (10) and (12), when we let $\mu \to \infty$. From the definition of the operators in Eq. (11) it follows that

$$\lim_{\mu \to \infty} \underset{\sim}{A}(\pm\mu) = \underset{\sim}{M}_0(\underset{\sim}{X}), \qquad \lim_{\mu \to \infty} \underset{\sim}{B}(\pm\mu) = \underset{\sim}{M}_0(\underset{\sim}{Y})$$

If we add Eq. (10) and Eq. (12) and let $\mu \to \infty$, we get the moment relationship

$$\begin{aligned} \underset{\sim}{E} - 2m_0(\underset{\sim}{\Psi}) &= [\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{X})][\underset{\sim}{E} - \underset{\sim}{\tilde{M}}_0(\underset{\sim}{X})] + [\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{X})]\underset{\sim}{\tilde{M}}_0(\underset{\sim}{Y}) \\ &\quad - \underset{\sim}{M}_0(\underset{\sim}{Y})[\underset{\sim}{E} - \underset{\sim}{\tilde{M}}_0(\underset{\sim}{X})] - \underset{\sim}{M}_0(\underset{\sim}{Y}) \cdot \underset{\sim}{M}_0(\underset{\sim}{Y}) \\ &= [\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{P})] \cdot [\underset{\sim}{E} - \underset{\sim}{\tilde{M}}_0(\underset{\sim}{Q})] \end{aligned} \tag{54}$$

If, for simplification, we write

$$\left.\begin{aligned} A_i &= M_i(P_5), \quad B_i = M_i(Q_5), \quad (i = 0, 1, 2 \ldots) \\ A_0' &= 1 - 2M_0(P_6), \quad B_0' = 1 - 2M_0(Q_6) \\ A_i' &= M_i(P_6), \quad B_i' = M_i(Q_6), \quad (i = 1, 2 \ldots) \end{aligned}\right\} \tag{55a}$$

$$\left.\begin{aligned} C_i &= \frac{3}{8} m_i(P_1 + P_5), \quad C_i' = \frac{3}{8} m_i(P_2 + P_6), \quad (i = 1, 2 \ldots) \\ D_0 &= 1 - \frac{3}{8} m_0(Q_1 + Q_5), \quad D_0' = 1 - \frac{3}{4} m_0(Q_2 + Q_6) \\ D_i &= \frac{3}{8} m_i(Q_1 + Q_5), \quad D_i' = \frac{3}{8} m_i(Q_2 + Q_6), \quad (i = 1, 2 \ldots) \end{aligned}\right\} \tag{55b}$$

then, with respect to the constraints in Eq. (52), we have from Eq. (51)

$$\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{P}) \equiv \begin{bmatrix} A_0, & -A_0'/\sqrt{2} \\ -\sqrt{2}\,A_0, & A_0' \end{bmatrix} \tag{56}$$

$$\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{Q}) \equiv \begin{bmatrix} D_0 + B_0, & (D_0' - B_0')/\sqrt{2} \\ -\sqrt{2}\,B_0, & B_0' \end{bmatrix} \tag{57}$$

Since

$$\underset{\sim}{E} - 2m_0(\underset{\sim}{\Psi}) \equiv \begin{bmatrix} 1/10 & -\sqrt{2}/10 \\ -\sqrt{2}/10 & 1/5 \end{bmatrix}$$

we get, after multiplication of the various matrices and comparison of corresponding elements on both sides of Eq. (54), four relationships. Among these, the relationships corresponding to the elements in the same column are identical. As a result we obtain only the two relationships (diagonal elements of the matrix)

$$\frac{1}{10} = A_0(D_0 + B_0) - \frac{1}{2} A'_0(D'_0 - B'_0)$$

$$\frac{1}{5} = 2A_0B_0 + A'_0B'_0$$

which, after subtraction, give two independent relationships

$$\frac{1}{5} = 2A_0B_0 + A'_0B'_0 \tag{58a}$$

$$0 = 2A_0D_0 - A'_0D'_0 \tag{58b}$$

Since with the notation introduced in Eq. (55) the constraints in Eq. (53) assume the form

$$\begin{aligned} 2D_1 - \tau D_0 &= 0 \\ 4D'_1 - \tau D'_0 &= 0 \end{aligned} \tag{59}$$

substitution from Eq. (59) in Eq. (58b) yields another form of this relationship, namely

$$0 = A_0D_1 - A'_0D'_1 \tag{60}$$

The relationships for higher moments can be derived from Eq. (10) by developing both sides of this equation in powers of $\nu = 1/\mu$ and by comparing corresponding coefficients in this development. It can be found that the comparison of the coefficients of ν leads to identities, whereas, the comparison of coefficients of ν^2 leads to the relationship

$$\begin{aligned} 2m_2(\underset{\sim}{\Psi}) = [\underset{\sim}{E} - \underset{\sim}{M}_0(\underset{\sim}{P})] \cdot \underset{\sim}{\tilde{M}}_2(\underset{\sim}{Q}) + \underset{\sim}{M}_1(\underset{\sim}{P}) \cdot \underset{\sim}{\tilde{M}}_1(\underset{\sim}{Q}) \\ + \underset{\sim}{M}_2(\underset{\sim}{P}) \cdot [\underset{\sim}{E} - \underset{\sim}{\tilde{M}}_0(\underset{\sim}{Q})] \end{aligned} \tag{51}$$

This relationship will be used in Section V.

IV. EXPRESSIONS FOR THE SOLUTIONS P_i, Q_i (i = 1, 2, 5, 6) IN TERMS OF CHANDRASEKHAR'S X- AND Y-FUNCTIONS

As was shown by Busbridge[4] and Mullikin,[5] the behavior of the solutions of Eqs. (36,37,42,43) for $\mu \to \infty$ is closely related to the functions $D_\ell(\mu)$ or $D_r(\mu)$. However, these two functions are also the discriminants of the equations for X_ℓ, Y_ℓ, X_r, and Y_r. This suggests that the solutions P_i and Q_i (i = 1, 2, 5, 6) and the functions X_i and Y_i (i = ℓ, r) are so related that we may be able to express the solutions P_i and Q_i in terms of X_ℓ, Y_ℓ, X_r, and Y_r. This can be done in the following way.

The functions X_r and Y_r satisfy the scalar forms of the system in Eq. (4), namely

$$D_r(\mu)X_r(\mu) = 1 + \int_0^1 \psi_r(x)X_r(x)\,\frac{\mu\,dx}{x-\mu} - e^{-\tau/\mu}\int_0^1 \psi_r(x)Y_r(x)\,\frac{\mu\,dx}{x+\mu}$$

$$D_r(\mu)Y_r(\mu) = e^{-\tau/\mu} + \int_0^1 \psi_r(x)Y_r(x)\,\frac{\mu\,dx}{x-\mu} - e^{-\tau/\mu}\int_0^1 \psi_r(x)Y_r(x)\,\frac{\mu\,dx}{x+\mu}$$

Adding and subtracting these two equations, we get the uncoupled system for

$$P_r(\mu) = X_r(\mu) + Y_r(\mu), \quad Q_r(\mu) = X_r(\mu) - Y_r(\mu)$$

that can be written in the form

$$\{D_r + L_r^{(+)}\}(P_r) = 1 + e^{-\tau/\mu} \tag{62a}$$

$$\{D_r + L_r^{(-)}\}(Q_r) = 1 - e^{-\tau/\mu} \tag{62b}$$

where $L_r^{(\pm)}$ denotes the operator

$$L_r^{(\pm)}\{F(x)\} = \int_0^1 H^{(\pm)}(\mu, x)\psi_r(x)F(x)\,dx \tag{63}$$

With the use of the identities

$$\frac{\mu}{\mu - x} = 1 + \frac{x}{\mu - x}, \qquad \frac{\mu}{\mu + x} = 1 - \frac{x}{\mu + x}$$

it is easy to prove from Eq. (24) that

$$\mu L_r^{(\pm)}\{F(x)\} = \mu(1 \pm e^{-\tau/\mu})M_0(F) + L_r^{(\mp)}\{xF(x)\} \tag{64}$$

and

$$\mu^2 L_r^{(\pm)}\{F(x)\} = \mu^2(1 \pm e^{-\tau/\mu})M_0(F) + \mu(1 \mp e^{-\tau/\mu})M_1(F) + L_r^{(\pm)}\{x^2F(x)\} \tag{65}$$

Multiplying Eq. (62a) by μ, we get with respect to Eq. (64)

$$D_r(\mu)\mu P_r(\mu) + L_r^{(-)}\{xP_r(x)\} = \mu(1 + e^{-\tau/\mu})[1 - M_0(P_r)]$$

or

$$\{D_r + L_r^{(-)}\}\left(\frac{xP_r(x)}{1 - M_0(P_r)}\right) = \mu(1 + e^{-\tau/\mu}) \tag{66}$$

Similarly we have from Eq. (62b)

$$\{D_r + L_r^{(+)}\}\left(\frac{xQ_r(x)}{1 - M_0(Q_r)}\right) = \mu(1 - e^{-\tau/\mu}) \tag{67}$$

Multiplying Eq. (62a) by μ^2, we get from Eqs. (65-67)

$$\{D_r + L_r^{(+)}\}[x^2P_r(x)] = \mu^2(1 + e^{-\tau/\mu})[1 - M_0(P_r)] - \mu(1 - e^{-\tau/\mu})M_1(P_r)$$

$$= \mu^2(1 + e^{-\tau/\mu})[1 - M_0(P_r)] - M_1(P_r)\{D_r + L_r^{(+)}\}\left(\frac{xQ_r(x)}{1 - M_0(Q_r)}\right)$$

or

$$\{D_r + L_r^{(+)}\}\left[\frac{x^2 P_r(x)}{1 - M_0(P_r)} + \frac{M_1(P_r)}{1 - M_0(P_r)}\ \frac{xQ_r(x)}{1 - M_0(Q_r)}\right] = \mu^2(1 + e^{-\tau/\mu}) \tag{68}$$

Similarly,

$$\{D_r + L_r^{(-)}\}\left[\frac{x^2 Q_r(x)}{1 - M_0(Q_r)} + \frac{M_1(Q_r)}{1 - M_0(P_r)}\ \frac{xP_r(x)}{1 - M_0(Q_r)}\right] = \mu^2(1 - e^{-\tau/\mu}) \tag{69}$$

With the notation introduced in Eq. (63), Eqs. (42) and (43) can be written in the form

$$\{D_r + L_r^{(+)}\}[P_1(x)] = (1 + e^{-\tau/\mu}) + \frac{3}{8} m_1(P_1 + P_5)\mu(1 - e^{-\tau/\mu}) - \mu^2(1 + e^{-\tau/\mu})[1 - \frac{3}{8} m_0(P_1 + P_5)] \tag{70}$$

$$\{D_r + L_r^{(+)}\}[P_2(x)] = \frac{3}{8} m_1(P_2 + P_6)\mu(1 - e^{-\tau/\mu}) - \frac{1}{2}\mu^2(1 + e^{-\tau/\mu}) \cdot [1 - \frac{3}{4} m_0(P_2 + P_6)] \tag{71}$$

The equations for Q_1 and Q_2 are obtained from Eqs. (70, 71) by replacing P by Q and by reversing the sign in front of the term $e^{-\tau/\mu}$; hence

$$\{D_r + L_r^{(-)}\}[Q_1(x)] = (1 - e^{-\tau/\mu}) + \frac{3}{8} m_1(Q_1 + Q_5)\mu(1 + e^{-\tau/\mu}) - \mu^2(1 - e^{-\tau/\mu})[1 - \frac{3}{8} m_0(Q_1 + Q_5)] \tag{72}$$

$$\{D_r + L_r^{(-)}\}[Q_2(x)] = \frac{3}{8} m_1(Q_2 + Q_6)\mu(1 + e^{-\tau/\mu}) - \frac{1}{2}\mu^2(1 - e^{-\tau/\mu}) \cdot [1 - \frac{3}{4} m_0(Q_2 + Q_6)] \tag{73}$$

Substituting from Eqs. (62a, 66, 67) into Eqs. (70) and (71), we get with respect to the contraints in Eq. (52) the equations for P_1 and P_2

in the form

$$\{D_r + L_r^{(+)}\}\left[P_1(x) - P_r(x) - \frac{3}{8} m_1 (P_1 + P_5) \frac{xQ_r(x)}{1 - M_0(Q_r)}\right] = 0$$

$$\{D_r + L_r^{(+)}\}\left[P_2(x) - \frac{3}{8} m_1 (P_2 + P_6) \frac{xQ_r(x)}{1 - M_0(Q_r)}\right] = 0$$

These two equations are fully equivalent to the system of homogeneous equations discussed by Mullikin[5] (p. 628). Since $D_r(\mu)$ has no zeros for $0 \leq \mu$, these equations have only one solution that identically vanishes, i.e.,

$$P_1(\mu) = P_r(\mu) + \frac{3}{8} \frac{m_1 (P_1 + P_5)}{1 - M_0(Q_r)} \mu Q_r(\mu) \tag{74}$$

$$P_2(\mu) = \frac{3}{8} \frac{m_1 (P_2 + P_6)}{1 - M_0(Q_r)} \mu Q_r(\mu) \tag{75}$$

Using the same procedure as above for Eqs. (72) and (73), we get

$$\begin{aligned} Q_1(\mu) &= Q_r(\mu) + \frac{3}{8} \frac{m_1 (Q_1 + Q_5)}{1 - M_0(P_r)} \mu P_r(\mu) \\ &- \frac{1 - 3/8\, m_0 (Q_1 + Q_5)}{[1 - M_0(P_r)][1 - M_0(Q_r)]} \{M_1(Q_r)\mu P_r(\mu) + [1 - M_0(P_r)]\mu^2 Q_r(\mu)\} \end{aligned} \tag{76}$$

$$\begin{aligned} Q_2(\mu) &= \frac{3}{8} \frac{m_1 (Q_2 + Q_6)}{1 - M_0(P_r)} \mu P_r(\mu) \\ &- \frac{1}{2} \frac{1 - 3/4\, M_0 (Q_2 + Q_5)}{[1 - M_0(P_r)][1 - M_0(Q_r)]} \{M_1(Q_r)\mu P_r(\mu) + [1 - M_0(P_r)]\mu^2 Q_r(\mu)\} \end{aligned} \tag{77}$$

The equations for P_5 and P_6 in Eqs. (36) and (37) can be written

$$\{D_\ell + L_\ell^{(+)}\}[P_5(x)] = 0 \tag{78a}$$

$$\{D_\ell + L_\ell^{(+)}\}[P_6(x)] = 1 + e^{-\tau/\mu} \tag{78b}$$

when $L_\ell^{(+)}$ is given by the operator in Eq. (63), in which ψ_r is replaced by ψ_ℓ. The equations for Q_5 and Q_6 are again given by Eqs. (78a,b), but they have the form

$$\{D_\ell + L_\ell^{(-)}\}[Q_5(x)] = 0 \tag{79a}$$

$$\{D_\ell + L_\ell^{(-)}\}[Q_6(x)] = 1 - e^{-\tau/\mu} \tag{79b}$$

The functions P_ℓ and Q_ℓ evidently satisfy the system in Eqs. (62a,b) with D_ℓ and L_ℓ replacing D_r and L_r. Subtracting these equations for P_ℓ and Q_ℓ from Eqs. (78b) and (79b), we get

$$\{D_\ell + L_\ell^{(+)}\}[P_6(x) - P_\ell(x)] = 0 \tag{80a}$$

$$\{D_\ell + L_\ell^{(-)}\}[Q_6(x) - Q_\ell(x)] = 0 \tag{80b}$$

Since $2\int_0^1 \psi_\ell(x)\,dx = 1$, we have the conservative case where the solutions of Eqs. (78a, 80a) and (79a, 80b) are not unique but are given by a two-parameter family of functions. For the X- and Y-functions, corresponding to P_5, Q_5, P_6, and Q_6 we have then expressions [see Mullikin[2] Eq. (2.12)]

$$\begin{aligned} X_5(\mu) &= a\mu[X_\ell + Y_\ell] + b\mu[\gamma X_\ell + \mu(X_\ell + Y_\ell)] \\ Y_5(\mu) &= -a\mu[X_\ell + Y_\ell] - b\mu[\gamma Y_\ell - \mu(X_\ell + Y_\ell)] \end{aligned} \tag{81}$$

$$\begin{aligned} X_6(\mu) &= X_\ell + a'\mu[X_\ell + Y_\ell] + b'\mu[\gamma X_\ell + \mu(X_\ell + Y_\ell)] \\ Y_6(\mu) &= Y_\ell - a'\mu[X_\ell + Y_\ell] - b'\mu[\gamma Y_\ell - \mu(X_\ell + Y_\ell)] \end{aligned} \tag{82}$$

where a, b, c', and b' are constants and

$$\gamma = \frac{M_1(X_\ell + Y_\ell)}{M_0(Y_\ell)} = \frac{2M_1(P_\ell)}{M_0(P_\ell) - M_0(Q_\ell)} \tag{83}$$

with

$$M_k(F_\ell) = \int_0^1 \psi_\ell(x) F_\ell(x) x^k \, dx \tag{83a}$$

Adding and subtracting the equations in Eqs. (81) and (82), we get expressions of the functions P_5, P_6, Q_5, and Q_6 in terms of $P_\ell(\mu)$ and $Q_\ell(\mu)$ in the form

$$\begin{aligned} P_5(\mu) &= b\mu[\gamma Q_\ell(\mu) + 2\mu P_\ell(\mu)] \\ Q_5(\mu) &= (2a + b\gamma)\mu P_\ell(\mu) \end{aligned} \tag{84a}$$

$$\begin{aligned} P_6(\mu) &= P_\ell(\mu) + b'\mu[\gamma Q_\ell(\mu) + 2\mu P_\ell(\mu)] \\ Q_6(\mu) &= Q_\ell(\mu) + (2a' + b'\gamma)\mu P_\ell(\mu) \end{aligned} \tag{84b}$$

In Eqs. (74-77) and (84a,b) the function $P_k(\mu)$ $Q_k(\mu)$ (k = 1, 2, 5, 6) are expressed in terms of the P_r-, Q_r-, P_ℓ-, and Q_ℓ-functions. In addition to the zero and first moments of both kinds of the functions P_k and Q_k, these equations contain four constants a, b, a', and b'. These constants can be uniquely determined by the requirement that the functions P_k and Q_k as solutions of the system of Eqs. (36,39) satisfy the constraints in Eqs. (52) and (53). Computations of these constants, as well as the elimination of the various moments of the functions $P_1 + P_5$, $Q_1 + Q_5$, etc., can be simplified with the use of some relationships among the various moments of the functions P_r, Q_r, P_ℓ, and Q_ℓ that will be derived in the following section.

V. MOMENT RELATIONSHIPS FOR THE FUNCTIONS P_r, Q_r, P_ℓ, AND Q_ℓ

While P_r and Q_r represent the unique solutions of Eqs. (62a,b), the functions P_ℓ and Q_ℓ, which are nonunique solutions of the singular integral equations, have to be specified by additional conditions. We may require that these solutions represent Busbridge's solutions of the auxiliary equations, and thus be analytic functions for $\mu \to \infty$. Since these functions satisfy the equations

$$\begin{aligned} \{D_\ell + L_\ell^{(+)}\}(P_\ell) &= 1 + e^{-\tau/\mu} \\ \{D_\ell + L_\ell^{(-)}\}(Q_\ell) &= 1 - e^{-\tau/\mu} \end{aligned} \tag{85}$$

and $D_\ell = 0$ for $\nu = 1/\mu \to 0$, the functions P_ℓ and Q_ℓ have to satisfy the constraints given by the conditions

$$\begin{aligned} \lim_{\nu \to 0} \{L_\ell^{(+)}(P_\ell) - 1 - e^{-\nu\tau}\} &= 0 \\ \lim_{\nu \to 0} \{L_\ell^{(-)}(Q_\ell) - 1 + e^{-\nu\tau}\} &= 0 \end{aligned} \tag{86}$$

From Eqs. (63) and (48) and from the fact that $\nu = 0$ is a double root of the equation $D_\ell(\nu) = 0$, the condition in Eq. (86) can be written in the form

$$\begin{aligned} \lim_{\nu \to 0} \{2M_0(P_\ell) - 2 + \nu\tau[1 - M_0(P_\ell)] + 0(\nu^2)\} &= 0 \\ \lim_{\nu \to 0} \{2M_1(Q_\ell) - \tau[1 - M_0(Q_\ell)] + 0(\nu)\} &= 0 \end{aligned}$$

which gives the conditions for the modified moments of P_ℓ and Q_ℓ as

$$\begin{aligned} 1 - M_0(P_\ell) &= 0 \\ \text{and} \qquad 2M_1(Q_\ell) - \tau[1 - M_0(Q_\ell)] &= 0 \end{aligned} \tag{87}$$

Another set of relationships for the modified moments can be easily derived from Eq. (59) by the substitutions

$$\underset{\sim}{P} = P_{r,\ell}(\mu) \cdot \underset{\sim}{E}, \quad \underset{\sim}{Q} = Q_{r,\ell}(\mu) \cdot \underset{\sim}{E}, \quad \underset{\sim}{\Psi} = \psi_{r,\ell}(\mu) \cdot \underset{\sim}{E}$$

since the X_r, Y_r, X_ℓ, and Y_ℓ satisfy the scalar form of Eq. (4). Hence we get the two relations

$$1 - 2\int_0^1 \psi_{r,\ell}(x)\, dx = [1 - M_0(P_{r,\ell})][1 - M_0(Q_{r,\ell})]$$

With respect to Eq. (87), however, this leads to only one additional relation for the subscript r, namely

$$1/2 = [1 - M_0(P_r)][1 - M_0(Q_r)] \tag{87a}$$

Similarly we get from Eq. (61)

$$1/10 = M_2(P_r)[1 - M_0(Q_r)] + M_1(P_r)M_1(Q_r) + [1 - M_0(P_r)]M_2(Q_r) \tag{88}$$

and with respect to Eq. (87)

$$1/5 = M_2(P_\ell)[1 - M_0(Q_\ell)] + M_1(P_1)M_1(Q_\ell) \tag{89}$$

The reader will have no difficulty in verifying that these moment relations can be reduced to the forms given by Busbridge[4] (p. 98) and Chandrasekhar[3] (p. 188).

The relations for ordinary moments of the functions P_ℓ, Q_ℓ, P_r, and Q_r can be derived from the nonlinear equations for the corresponding X- and Y-functions. For a characteristic function of the form

$$\psi(\mu) = a + b\mu^2$$

Chandrasekhar[3] (p. 189) has shown that

$$m_0(X) = 1 + \frac{1}{2}\, a[m_0^2(X) - m_0^2(Y)] + \frac{1}{2}\, b[m_1^2(X) - m_1^2(Y)]$$

In terms of P- and Q-functions, this relation becomes

$$m_0(P) + m_0(Q) = 2 + a m_0(P) m_0(Q) + b m_1(P) m_1(Q) \tag{90}$$

If we introduce the following abbreviations

$$\begin{aligned}
&\alpha_0 = 1 - \frac{3}{4}\, m_0(P_\ell), && \beta_0 = 1 - \frac{3}{4}\, m_0(Q_\ell) \\
&\alpha_i = \frac{3}{4}\, m_i(P_\ell), && \beta_i = \frac{3}{4}\, m_i(Q_\ell) \\
&\alpha_0' = 1 - \frac{3}{8}(P_r), && \beta_0' = 1 - \frac{3}{8}\, m_0(Q_r) \\
&\alpha_i' = \frac{3}{8}\, m_i(P_r), && \beta_i' = \frac{3}{8}\, m_i(Q_r) \qquad (i = 1, 2 \ldots)
\end{aligned} \tag{91}$$

then we get for ψ_ℓ ($a = -b = 3/4$), after multiplying Eq. (90) by 3/4

$$-1/2 = \alpha_0\beta_0 - \alpha_1\beta_1 \tag{92}$$

and for ψ_r ($a = -b = 3/8$), after multiplying Eq. (90) by 3/8

$$1/4 = \alpha_0'\beta_0' - \alpha_1'\beta_1' \tag{93}$$

The connection between the ordinary moments (α_i, β_i, etc.) and the modified moments $[M_i(P), M_i(Q)]$ can be readily established by substitution of ψ_ℓ and ψ_r in the definition of the modified moments in Eqs. (51) and (83a). This results in the following relations:

$$\begin{aligned}
&1 - M_0(P_\ell) = \alpha_0 + \alpha_2, && 1 - M_0(Q_\ell) = \beta_0 + \beta_2 \\
&M_i(P_\ell) = \alpha_i - \alpha_{i+2}, && M_i(Q_\ell) = \beta_i - \beta_{i+2} \\
&1 - M_0(P_r) = \alpha_0' + \alpha_2', && 1 - M_0(Q_r) = \beta_0' + \beta_2' \\
&M_i(P_r) = \alpha_i' - \alpha_{i+2}', && M_i(Q_r) = \beta_i' - \beta_{i+2}' \qquad (i = 1, 2, \ldots)
\end{aligned} \tag{94}$$

It is, furthermore, convenient to introduce the following notation for the combination of these moments that will substantially simplify further analysis:

$$\begin{aligned} d &= 1 - M_0(Q_\ell), & B &= 2[1 - M_0(P_r)] \\ D &= 1 - M_0(Q_r), & C &= 2M_1(Q_r) - \tau[1 - M_0(Q_r)] \end{aligned} \tag{95}$$

and

$$\begin{aligned} m &= \alpha_0\beta_0' - \alpha_1\beta_1', & r &= \alpha_0\alpha_1' - \alpha_1\alpha_0' \\ n &= \alpha_0'\beta_0 - \alpha_1'\beta_1, & s &= \beta_0\beta_1' - \beta_1\beta_0' \end{aligned} \tag{96}$$

With respect to Eqs. (87) and (87a)

$$\alpha_0 = -\alpha_2 \qquad BD = 1 \tag{97}$$

VI. DETERMINATION OF THE CONSTANTS IN THE EXPRESSIONS FOR P- AND Q-FUNCTIONS

In terms of the notation introduced in Eqs. (55) and (95) the expressions for the P- and Q-functions in Eqs. (74-77) and (89a,b) can be written in condensed form as follows:

$$\begin{aligned} P_1(\mu) &= P_r(\mu) + (C_1/D)\mu Q_r(\mu) \\ P_5(\mu) &= b[\gamma\mu Q_\ell(\mu) + 2\mu^2 P_\ell(\mu)] \end{aligned} \tag{98a}$$

$$\begin{aligned} P_2(\mu) &= (C_1'/D)\mu Q_r(\mu) \\ P_6(\mu) &= P_\ell(\mu) + b'[\gamma\mu Q_\ell(\mu) + 2\mu^2 P_\ell(\mu)] \end{aligned} \tag{98b}$$

$$\begin{aligned} Q_1(\mu) &= Q_r(\mu) - D_0[C\mu P_r(\mu) + B\mu^2 Q_r(\mu)] \\ Q_5(\mu) &= (2a + b\gamma)\mu P_\ell(\mu) \end{aligned} \tag{98c}$$

$$\begin{aligned} Q_2(\mu) &= -1/2\, D_0'[C\mu P_r(\mu) + B\mu^2 Q_r(\mu)] \\ Q_6(\mu) &= Q_\ell(\mu) + (2a' + b'\gamma)\mu P_\ell(\mu) \end{aligned} \tag{98d}$$

Eqs. (98a,b) are obtained by simple substitution; in determination of Eqs. (98c,d) the constraints in Eq. (59) were applied. As the first step in evaluation of the constants in the equations of the systems (98a-d), the moments of the functions P_k and Q_k ($k = 1, 2, 5, 6$) can be computed. Multiplying Eq. (98a) by $3/8\, d\mu$ or $3/8\, \mu d\mu$ and integrating from 0 to 1, we get

$$1 - \frac{3}{8} m_0(P_1) = 1 - \frac{3}{8} m_0(P_r) - \frac{C_1}{D}\frac{3}{8} m_1(Q_r) = \alpha_0' - \beta_0' \frac{C_1}{D} \tag{100a}$$

$$\frac{3}{8} m_0(P_5) = b[\gamma \frac{3}{8} m_1(Q_\ell) + \frac{3}{4} m_2(P_\ell)] = b[\beta_1\Gamma + \alpha_2] \tag{100b}$$

$$\frac{3}{8}\, m_1(P_1) = \frac{3}{8}\, m_1(P_r) + \frac{C_1}{D}\,\frac{3}{8}\, m_2(Q_r) = \alpha_1' + \beta_2'\,\frac{C_1}{D} \tag{100c}$$

$$\frac{3}{8}\, m_1(P_5) = b[\gamma\,\frac{3}{8}\, m_2(Q_\ell) + \frac{3}{4}\, m_3(P_\ell)] = b[\beta_2\Gamma + \alpha_3] \tag{100d}$$

where the notation introduced in Eq. (91) is used and with respect to Eqs. (83) and (87)

$$\Gamma = \frac{1}{2}\,\gamma = \frac{M_1(P_\ell)}{1 - M_0(Q_\ell)} = \frac{\alpha_1 - \alpha_3}{d} \tag{101}$$

Since from Eqs. (94), (95), and (101)

$$\alpha_2 = -\alpha_0, \qquad d = \beta_0 + \beta_2, \qquad \alpha_3 = \alpha_1 - d\Gamma$$

we may write

$$\begin{aligned} \beta_1\Gamma + \alpha_2 &= \beta_1\Gamma - \alpha_0 \\ \beta_2\Gamma + \alpha_3 &= d\Gamma - \beta_0\Gamma + \alpha_1 - d\Gamma = \alpha_1 - \beta_0\Gamma \end{aligned} \tag{101a}$$

Similarly, multiplying Eq. (98b) by 3/4 $d\mu$ or 3/4 $\mu d\mu$ and integrating from 0 to 1, we get

$$1 - \frac{3}{4}\, m_0(P_2) = 1 - 2\,\frac{C_1'}{D}\,\beta_1' \tag{102a}$$

$$\frac{3}{4}\, m_0(P_6) = 1 - \alpha_0 + 2b'[\beta_1\Gamma - \alpha_0] \tag{102b}$$

$$\frac{3}{4}\, m_1(P_2) = 2\beta_2'\,\frac{C_1'}{D} \tag{102c}$$

$$\frac{3}{4}\, m_1(P_6) = \alpha_1 + 2b'[\Gamma\beta_2 + \alpha_3] = \alpha_1 + 2b'[\alpha_1 - \beta_0\Gamma] \tag{102d}$$

Substituting Eqs. (100a,b) and (102a,b) in the constraints in Eq. (52), and realizing that with respect to Eq. (55)

$$C_1 = \frac{3}{8} m_1(P_1 + P_5), \quad 2C_1' = \frac{3}{4} m_1(P_2 + P_6)$$

we get, after simple computations, two systems of linear equations

$$\begin{aligned}
\beta_1'(C_1/D) + b[\beta_1\Gamma - \alpha_0] &= \alpha_0' \\
\beta_0'(C_1/D) + b[\beta_0\Gamma - \alpha_1] &= \alpha_1' \\
\beta_1' \, 2(C_1'/D) + 2b'[\beta_1\Gamma - \alpha_0] &= \alpha_0 \\
\beta_0' \, 2(C_1'/D) + 2b'[\beta_0\Gamma - \alpha_1] &= \alpha_1
\end{aligned}$$

that yield the expressions for the constants in the system (98a,b)

$$\begin{aligned}
C_1/D &= \frac{r + n\Gamma}{m + s\Gamma} \qquad & b &= -\frac{1}{4}\frac{1}{m + s\Gamma} \\
2C_1'/D &= -\frac{1}{2}\frac{\Gamma}{m + s\Gamma} \qquad & 2b' &= -\frac{m}{m + s\Gamma}
\end{aligned} \tag{103}$$

in terms of the combinations of the various moments of X_r, Y_r, X_ℓ, and Y_ℓ given in Eq. (96).

In order to compute the constant $D_0 = 1 - 3/8\, M_0(Q_1 + Q_5)$ in Eq. (55), we multiply Eq. (98c) by $3/8\, d\mu$ and integrate from 0 to 1. Then adding these expressions, we get with respect to Eq. (91)

$$D_0 = \beta_0' + D_0[C\alpha_1' + B\beta_2'] - (2a + b\gamma)\frac{1}{2}\alpha_1$$

or

$$D_0[1 - C\alpha_1' - B\beta_2'] + \frac{1}{2}\alpha_1(2a + b\gamma) = \beta_0'$$

Since from Eqs. (94) and (95) $D = \beta_0' + \beta_2'$, we have with respect to Eq. (97)

$$1 - C\alpha_1' - B\beta_2' = B(D - \beta_2') - C\alpha_1' = B\beta_0' - C\alpha_1' \tag{104}$$

and hence

$$(\beta_0'B - \alpha_1'C)D_0 + \frac{1}{2}\alpha_1(2a + b\gamma) = \beta_0' \tag{105}$$

From Eq. (55)

$$2D_1 = \frac{3}{4} m_1 (Q_1 + Q_5)$$

Multiplying Eq. (98c) by 3/4 $\mu d\mu$ and integrating from 0 to 1, we get with respect to Eq. (91)

$$2D_1 = 2\beta_1' - 2D_0[C\alpha_2' + B\beta_3'] + (2a + b\gamma)\alpha_2$$

However, with respect to Eqs. (94), (95), and (97)

$$C\alpha_2' + B\beta_3' = [2(\beta_1' - \beta_3') - \tau D][1/2\ B - \alpha_0'] + B\beta_3'$$

$$= -\frac{\tau}{2} - \alpha_0' C + \beta_1' B \qquad (106)$$

which, substituted above, yields with respect to the constraint in Eqs. (59) and (97)

$$2D_1 - \tau D_0 = 0 = 2\beta_1' + 2D_0[\alpha_0' C - \beta_1' B] - (2a + b\gamma)\alpha_0$$

or

$$(\beta_1' B - \alpha_0' C) D_0 + \frac{1}{2} \alpha_0 (2a + b\gamma) = \beta_1' \qquad (107)$$

From Eqs. (105) and (107) we get with respect to Eq. (96)

$$D_0 = \frac{m}{mB - rC}, \qquad 2a + b\gamma = \frac{1}{2} \frac{C}{mB - rC}$$

or, if we write $G = \frac{C}{B} = CD$

$$D_0 B = \frac{m}{m - rG}, \qquad D_0 C = \frac{mG}{m - rG}, \qquad 2a + b\gamma = \frac{1}{2} \frac{G}{m - rG} \qquad (108)$$

Similarly, multiplying Eqs. (98d) by 3/4 $d\mu$ or 3/4 $\mu d\mu$ and integrating from 0 to 1, we get

$$D_0' = 1 - \frac{3}{4} m_0(Q_2 + Q_6) = \beta_0 + D_0'[\alpha_1'C + \beta_2'B] - (2a' + b'\gamma)\alpha_1$$

$$2D_1' = \frac{3}{4} m_1(Q_2 + Q_6) = - D_0'[\alpha_2'C + \beta_3'B] + \beta_1 + (2a' + b'\gamma)\alpha_2$$

Using Eqs. (104) and (106) and the constraints in Eq. (54), the equations

$$D_0'(\beta_0'B - \alpha_1'C) + (2a' + b'\gamma)\alpha_1 = \beta_0$$

$$D_0'(\beta_1'B - \alpha_0'C) + (2a' + b'\gamma)\alpha_0 = \beta_1$$

lead to the expressions

$$D_0' = - \frac{1}{2(mB - rC)}, \qquad 2a' + b'\gamma = - \frac{sB - nC}{mB - rC}$$

or

$$D_0'B = - \frac{1}{2(m - rG)}, \quad D_0'C = - \frac{G}{2(m - rG)}, \quad 2a' + b'\gamma = - \frac{s + nG}{m - rG} \tag{109}$$

With the values of the coefficients in Eqs. (103), (108), and (109) we can write the equations for the functions P_k and Q_k ($k = 1, 5, 2, 6$) in the form

$$\begin{aligned} P_1(\mu) &= P_r(\mu) + 2(r + n\Gamma)\Delta\mu Q_r(\mu) \\ P_5(\mu) &= - \Delta[\Gamma\mu Q_\ell(\mu) + \mu^2 P_\ell(\mu)] \end{aligned} \tag{110a}$$

$$\begin{aligned} P_2(\mu) &= - 1/2\ \Delta\Gamma\mu Q_r(\mu) \\ P_6(\mu) &= P_\ell(\mu) - 2m\Delta[\Gamma\mu Q_\ell(\mu) + \mu^2 P_\ell(\mu)] \end{aligned} \tag{110b}$$

$$\begin{aligned} Q_1(\mu) &= Q_r(\mu) - 2m\Delta'[G\mu P_r(\mu) + \mu^2 Q_r(\mu)] \\ Q_5(\mu) &= G\Delta'\mu P_\ell(\mu) \end{aligned} \tag{110c}$$

$$\begin{aligned} Q_2(\mu) &= 1/2\ \Delta'[G\mu P_r(\mu) + \mu^2 Q_r(\mu)] \\ Q_6(\mu) &= Q_\ell(\mu) + 2(-s + nG)\Delta'\mu P_\ell(\mu) \end{aligned} \tag{110d}$$

where

$$2\triangle = \frac{1}{m + s\Gamma}, \qquad 2\triangle' = \frac{1}{m - rG} \tag{110e}$$

and the constants m, n, r, s, and G can be computed from the ordinary moment of the P- and Q-functions for the characteristic functions ψ_ℓ and ψ_r.

With the known values of various constants in the equations for P_5, P_6, Q_5, and Q_6, we can now check the validity of the relationships between the modified moments of these functions in Eqs. (58a,b). To get these modified moments, we have to multiply, for example, Eqs. (84a,b) by $\psi_r(\mu)$ and integrate from 0 to 1. On the R.H.S. of these equations we obtain terms like

$$\int_0^1 \psi_r(\mu) P_\ell(\mu) \mu^k \, d\mu, \qquad \int_0^1 \psi_r(\mu) Q_\ell(\mu) \mu^k \, d\mu$$

which can be easily transformed in the moments defined in Eq. (83a), if it is realized that

$$\psi_r(\mu) = \frac{1}{2} \psi_\ell(\mu)$$

With respect to the notation in Eqs. (55) and (83a), we get, from Eqs. (84a,b) or (98a-d)

$$\begin{aligned}
A_0 &= M_0(P_5) = b[\tfrac{1}{2} \gamma M_1(Q_\ell) + M_2(P_\ell)] \\
A_0' &= 1 - 2M_0(P_6) = 1 - M_0(P_\ell) - 2b'[\tfrac{1}{2} \gamma M_1(Q_\ell) + M_2(P_\ell)] \\
B_0 &= M_0(Q_5) = (2a + b\gamma) \tfrac{1}{2} M_1(P_\ell) \\
B_0' &= 1 - 2M_0(Q_6) = 1 - M_0(Q_\ell) - (2a' + b'\gamma) M_1(P_\ell)
\end{aligned} \tag{111}$$

When we substitute for γ from Eq. (101), we see that with respect to Eq. (89)

$$\frac{1}{2}\gamma M_1(Q_\ell) + M_2(P_\ell) = \frac{M_1(P_\ell)M_1(Q_\ell) + M_2(P_\ell)[1 - M_0(Q_\ell)]}{d} = \frac{1}{5d} \qquad (112)$$

Since from Eq. (82) $1 - M_0(P_\ell) = 0$, we get

$$2A_0B_0 + A_0'B_0' = \frac{b}{5d}(2a + b\gamma)M_1(P_\ell) - \frac{2b'}{5d}[d - (2a' + b'\gamma)M_1(P_\ell)]$$

$$= \frac{1}{5}\Gamma[b(2a + b\gamma) + 2b'(2a' + b'\gamma)] - \frac{1}{5}2b'$$

or after substitution from Eqs. (103), (108), and (109)

$$2A_0B_0 + A_0'B_0' = \frac{1}{5}\frac{\Gamma[ms - G(1/8 + mn)] + m[m - rG]}{(m + s\Gamma)(m - rG)} = \frac{1}{5}$$

since from the definition of m, n, r, and s in Eq. (96) and from Eqs. (92) and (93) it is easy to prove that $rs = (1/8) + mn$. The relationship in Eq. (59a) is, therefore, satisfied.

Since from Eq. (111)

$$2A_0 = \frac{2b}{5d}, \qquad A_0' = -\frac{2b'}{5d}$$

the relationship in Eq. (59b) reduces to

$$2A_0D_0 - A_0'D_0' = \frac{2}{5d}[bD_0 + b'D_0']$$

Substituting for the terms inside the brackets from Eqs. (103), (108) and (109), we see that

$$bD_0 = -\frac{1}{4}\frac{1}{m + s\Gamma}\frac{m}{mB - rC}, \qquad b'D_0' = +\frac{1}{4}\frac{m}{m + s\Gamma}\frac{1}{mB - rC}$$

and hence

$$2A_0D_0 - A_0'D_0' = 0$$

The relationship in Eq. (59b) is, therefore, satisfied identically.

VII. COMPARISON WITH CHANDRASEKHAR'S ANALYSIS

In Chapter X of his treatise,[3] Chandrasekhar expresses the azimuth-independent terms for the reflection and transmission matrices in terms of eight Greek-lettered functions. For these eight functions he obtains a simultaneous system of integral equations of order eight. The solutions of these equations are expressed in terms of X_ℓ-, Y_ℓ-, X_r-, and Y_r-functions. The details of determination of the constants in the linear combinations of these X- and Y-functions are given in an extensive paper by Chandrasekhar.[6]

As the analysis in this paper is rather involved, we may try to derive the expressions for the eight Greek-lettered functions from the forms of the solution in Eqs. (98) or (110).

Chandrasekhar's analysis is based on X_ℓ- and Y_ℓ-functions, defined as the <u>standard</u> solution of the X- and Y-equations for the conservative case $\psi_\ell = (3/4)(1 - \mu^2)$. This standard solution is defined by the moment relations

$$M_0(X_\ell) = 1, \qquad M_0(Y_\ell) = 0$$

or (113)

$$M_0(P_\ell) = 1, \qquad M_0(Q_\ell) = 1$$

This means that in addition to Eq. (97) ($\alpha_0 + \alpha_2 = 0$) we have also $\beta_0 + \beta_2 = 0$ or $d = 0$.

From Eq. (101) it follows that $1/\Gamma \to 0$ and $\Delta \to 0$, but $2\Delta\Gamma \to 1/s$; hence the system of Eqs. (110a,b) reduces to the form

$$
\begin{aligned}
P_1(\mu) &= P_r(\mu) + \frac{n}{s}\mu Q_r(\mu) \\
P_5(\mu) &= -\frac{1}{2s}\mu Q_\ell(\mu) \\
P_2(\mu) &= -\frac{1}{4s}\mu Q_r(\mu) \\
P_6(\mu) &= P_\ell(\mu) - \frac{m}{s}\mu Q_\ell(\mu)
\end{aligned}
\tag{114}
$$

while the equations for Q_i (i = 1, 2, 5, 6) remain unchanged.

The Greek-lettered functions, as they are introduced by Chandrasekhar[3] (Eq. (175), p. 265), are identical to the elements of the $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices, namely

$$
\begin{aligned}
K_1 &\equiv \psi, \quad K_2 \equiv \Phi, \quad K_3 \equiv \chi, \quad K_4 \equiv \zeta \\
L_1 &\equiv \xi, \quad L_2 \equiv \eta, \quad L_3 \equiv \sigma, \quad L_4 \equiv \theta
\end{aligned}
\tag{115}
$$

Hence, from Eq. (45)

$$
\begin{aligned}
P_1 &= K_3 + L_3 = \chi + \sigma, \qquad P_2 = K_4 + L_4 = \zeta + \theta \\
P_5 &= K_1 + L_1 = \psi + \xi, \qquad P_6 = K_2 + L_2 = \Phi + \eta
\end{aligned}
\tag{116}
$$

and the Q_i's (i = 1, 2, 5, 6) are given by the differences of the functions in the R.H.S. of Eq. (116). If we substitute in Eq. (116) the expressions for the Greek-lettered functions[3] (Eq. (183), p. 269), we get the equations for P_i and Q_i of the form

$$
\begin{aligned}
P_1 &= P_r(\mu) - (u_3 + u_4)\mu Q_r(\mu) \\
P_5 &= -(\nu_1 + \nu_2)\mu Q_\ell(\mu)
\end{aligned}
\tag{117a}
$$

$$
\begin{aligned}
P_2 &= -1/2\ (\nu_1 + \nu_2)\mu Q_r(\mu) \\
P_6 &= P_\ell(\mu) + (\nu_3 + \nu_4)\mu Q_\ell(\mu)
\end{aligned}
\tag{117b}
$$

$$
\begin{aligned}
Q_1(\mu) &= Q_r(\mu) + (u_3 - u_4)[\mu P_r(\mu) - 2\mu^2 Q_r(\mu)] \\
Q_5(\mu) &= (\nu_1 - \nu_2)\mu P_\ell(\mu)
\end{aligned}
\tag{117c}
$$

$$Q_2(\mu) = 1/2\ (\nu_1 - \nu_2)[\mu P_r(\mu) - 2Q\mu^2 Q_r(\mu)]$$
$$Q_6(\mu) = Q_\ell(\mu) + (\nu_4 - \nu_3)\mu P_\ell(\mu) \qquad (117d)$$

Chandrasekhar's solution is given by expressions completely analogous to those in Eqs. (119) and (110c,d). To prove that these solutions are identical, we have to transform the coefficients on the R.H.S. of Eq. (117) into the form of the coefficients in Eqs. (114) and (110c,d). The coefficients in Eq. (117) are given by Chandrasekhar,[3] (Eqs. (184-186), p. 269) as follows:

$$\nu_2 + \nu_1 = 2\Delta_1(\varkappa_1\delta_1 - \varkappa_2\delta_2), \quad \nu_2 - \nu_1 = 2\Delta_2(\varkappa_1\delta_1 - \varkappa_2\delta_2)$$
$$\nu_4 + \nu_3 = \Delta_1(d_1\varkappa_1 - d_0\varkappa_2), \quad \nu_4 - \nu_3 = \Delta_2[c_1\delta_1 - c_0\delta_2 - 2Q(d_0\delta_1 - d_1\delta_2)]$$
$$u_4 + u_3 = \Delta_1(c_1\delta_1 - c_0\delta_2), \quad u_4 - u_3 = \Delta_2(d_1\varkappa_1 - d_0\varkappa_2) \qquad (118)$$

where

$$\frac{1}{\Delta_1} = d_0\delta_1 - d_1\delta_2, \quad \frac{1}{\Delta_2} = c_0\varkappa_1 - c_1\varkappa_2 - 2Q(d_1\varkappa_1 - d_0\varkappa_2) \qquad (119)$$

$$Q = \frac{c_0 - c_2}{(d_0 - d_2)\tau + 2(d_1 - d_3)} \qquad (120)$$

$$c_0 = A_0 + B_0 - 8/3 \qquad d_0 = A_0 - B_0 - 8/3$$
$$c_n = A_n + B_n \qquad d_n = A_n - B_n$$
$$\varkappa_n = \bar{\alpha}_n + \bar{\beta}_n \qquad \delta_n = \bar{\alpha}_n - \bar{\beta}_n$$

and $\bar{\alpha}_n$, $\bar{\beta}_n$, A_n, and B_n are the moments of the order n of the functions X_ℓ, Y_ℓ, X_r, and Y_r respectively. If we introduce the notation defined above in Eq. (91), we have

$$c_0 = m_0(P_r) - 8/3 = -\frac{8d_0'}{3}, \qquad d_0 = m_0(Q_r) - 8/3 = -\frac{8\beta_0'}{3}$$

$$c_n = m_n(P_r) = \frac{8\alpha_n'}{3} \qquad d_n = m_n(Q_r) = \frac{8\beta_n'}{3} \tag{123}$$

$$\kappa_n = m_n(P_r) = \frac{4\alpha_n}{3} \qquad \delta_n = m_n(Q_r) = \frac{4}{3}\beta_n$$

Substitution from Eq. (123) in Eqs. (119) and (120) leads to the relations

$$\frac{1}{\Delta_1} = -\frac{32}{9}(\beta_0'\beta_1 + \beta_1'\beta_2) = \frac{32}{9}s$$

$$\frac{1}{\Delta_2} = \frac{32}{9}[-\alpha_0'\alpha_1 - \alpha_1'\alpha_2 - 2Q(\beta_1'\alpha_1 + \beta_0'\alpha_2)] = \frac{32}{9}(r + 2Qm) \tag{123a}$$

$$2Q = \frac{2(\alpha_0' + \alpha_2')}{(\beta_0' + \beta_2')\tau - 2(\beta_1' - \beta_3')} = -\frac{B}{C} = -\frac{1}{G}$$

where with respect to Eq. (113) we put $\beta_2 = -\beta_0$, $\alpha_2 = -\alpha_0$ and use the notation introduced in Eqs. (95) and (96). Finally, substituting in Eq. (118) from Eq. (123), we get

$$\nu_2 + \nu_1 = \frac{32}{9}\Delta_1(\alpha_1\beta_1 - \alpha_0\beta_0) = \frac{16}{9}\Delta_1 = \frac{1}{2s}$$

$$\nu_4 + \nu_3 = \frac{32}{9}\Delta_1(\alpha_1'\beta_1 - \alpha_0\beta_0') = -\frac{32}{9}\Delta_1 m = -\frac{m}{s}$$

$$u_4 + u_3 = \frac{32}{9}\Delta_1(\alpha_1'\beta_1 - \alpha_0'\beta_0) = -\frac{32}{9}\Delta_1 n = -\frac{n}{s}$$

$$\nu_2 - \nu_1 = \frac{32}{9}\Delta_2(\alpha_1\beta_1 - \alpha_0\beta_0) = \frac{32}{9}\frac{\Delta_2}{-2} = \frac{1}{2}\frac{1}{r + 2Qm} = -\Delta'G \tag{124}$$

$$\nu_4 - \nu_3 = \frac{32}{9}\Delta_2[\alpha_1'\beta_1 - \alpha_0'\beta_0 + 2Q(\beta_0'\beta_1 - \beta_0\beta_1')]$$

$$= -\frac{n + 2Qs}{r + 2Qm} = -\frac{s + nG}{m - rG} = 2\Delta'(-s + nG)$$

$$u_4 - u_3 = \frac{32}{9}\Delta_2(\beta_1'\alpha_1 - \beta_0'\alpha_0) = -\frac{m}{r + 2Qm} = \frac{mG}{m - rG} = 2m\Delta'G$$

When these expressions are substituted in Eq. (117), these equations reduce to the form of Eqs. (114) and (110c,d). This completes the proof that Chandrasekhar's solution follows from the solution given in Eq. (110) by introducing the condition that the X_ℓ- and Y_ℓ-functions are the standard solutions of the corresponding equations for X- and Y-functions.

The complete agreement between Chandrasekhar's solution and the solution derived in this paper has great significance. It not only offers a proof of correctness of these two quite independent methods but also indicates that the pure mathematical requirements of the regularity of solutions at infinity is fully equivalent with Chandrasekhar's procedure where the undetermined constants are determined, not by requirements to satisfy the linear constants, but by requirements for the solutions to satisfy the flux equation. Comparing the procedure followed by Chandrasekhar with that developed in this paper, we see that the latter is much more easily followed; the number of the constants is given by the form of the equations and can be determined rather easily from the definition of the various moments by using the linear constraints that these solutions have to satisfy.

Furthermore, comparing the form of Eq. (110) with that of Eq. (117), we see that for the solution with X_ℓ- and Y_ℓ-functions, selected as Busbridge solutions, we need to compute the constants m, n, r, s, Γ, and G, while for the solution with the standard solution for the X_ℓ- and Y_ℓ-functions, the number is reduced by one since the constant Γ disappears from the equations. The reader can also recognize that the expressions for the constants in Eq. (124) are simpler than those given in Chandrasekhar;[3] they reduce to simpler form since the relations $\alpha_2 = -\alpha_0$, $\beta_2 = -\beta_0$ allow us to use the relation in Eq. (92), slightly simplifying the final expressions.

VIII. EXPRESSION FOR FUNCTIONS NEEDED FOR THE COMPUTATIONS OF RADIATIONS EMERGING FROM A PLANETARY ATMOSPHERE

The ultimate goal of the analysis in this paper is to get expressions for the Stokes parameters of the radiations emerging from a planetary atmosphere. The azimuth-independent terms in these expressions can be obtained from Eq. (1), and if $\pi\underset{\sim}{F}$ is the matrix for the net flux of the parallel external radiation, the azimuth-independent terms of the intensity matrices for the radiations emerging from the top and bottom of the atmosphere are given as

$$\underset{\sim}{I}^{(0)}(0, \mu) = \frac{3}{16} \frac{\mu_0}{\mu + \mu_0} [\underset{\sim}{K}(\mu) \cdot \underset{\sim}{\tilde{K}}(\mu_0) - \underset{\sim}{L}(\mu) \cdot \underset{\sim}{\tilde{L}}(\mu_0)] \cdot \underset{\sim}{F}$$

$$\underset{\sim}{I}^{(0)}(\tau, -\mu) = \frac{3}{16} \frac{\mu_0}{\mu_0 - \mu} [\underset{\sim}{K}(\mu) \cdot \underset{\sim}{\tilde{L}}(\mu_0) - \underset{\sim}{L}(\mu) \cdot \underset{\sim}{\tilde{K}}(\mu_0)] \cdot \underset{\sim}{F} \tag{125}$$

For the neutral (unpolarized) external illumination, the matrix $\underset{\sim}{F}$ has the form

$$\underset{\sim}{F} = \frac{1}{2} F_0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} F_0 \cdot \underset{\sim}{\mathcal{E}}$$

and when the matrix multiplication in Eq. (125) is performed, the expressions for the Stokes parameters $I = I_\ell + I_r$ and $Q = I_\ell - I_r$ contain the elements of $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices either in the sum or in the difference of the elements, K_i and K_{i+2}, L_i and L_{i+2} $(i = 1, 2)$. For numerical computation we need, therefore, to compute the individual elements of the $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices. These are given by Eq. (45). When we substitute in these expressions from Eqs. (98) or (110), we can arrive at the expressions in terms of the X- and Y-functions of the form

$$
\begin{aligned}
K_1(\mu) &= \psi(\mu) = (c + b\mu)\mu X_\ell + (a + b\mu)\mu Y_\ell \\
L_1(\mu) &= \xi(\mu) = (-a + b\mu)\mu X_\ell + (-c + b\mu)\mu Y_\ell \\
K_2(\mu) &= \Phi(\mu) = (1 + c'\mu + b'\mu^2)X_\ell + (a' + b'\mu)\mu Y_\ell \\
L_2(\mu) &= \eta(\mu) = (-a' + b'\mu)\mu X_\ell + (1 - c'\mu + b'\mu^2)Y_\ell \\
K_3(\mu) &= \chi(\mu) = (1 + e\mu - f\mu^2)X_r + (-g + f\mu)\mu Y_r \\
L_3(\mu) &= \sigma(\mu) = (g + f\mu)\mu X_r + (1 - e\mu - f\mu^2)Y_r \\
2K_4(\mu) &= 2\zeta(\mu) = (c + f'\mu)\mu X_r + (a - f'\mu)\mu Y_r \\
2L_4(\mu) &= 2\theta(\mu) = -(a + f'\mu)\mu X_r + (-c + f'\mu)\mu Y_r
\end{aligned}
\tag{126}
$$

where the coefficients on the R.H.S. of Eq. (126) can be expressed in terms of the previously introduced constants m, n, r, s, G, Γ, Δ, and Δ' as follows:

$$
\begin{aligned}
b &= -\frac{1}{2}\Delta & b' &= -m\Delta = 2mb \\
f' &= \frac{1}{2}\Delta' & f &= m\Delta' = 2mf' \\
a &= -\Gamma b + Gf' & a' &= -b'\Gamma + 2(-s + nG)f' \\
c &= \Gamma b + Gf' & c' &= b'\Gamma + 2(-s + nG)f' \\
e &= -2(r + n\Gamma)b - Gf, & g &= -2(r + n\Gamma)b + Gf
\end{aligned}
\tag{127}
$$

If we use for X_ℓ and Y_ℓ the standard solution, then as $\Delta \to 0$, $b\Gamma \to -\frac{1}{4s}$ and $b'\Gamma \to -\frac{m}{2s}$; hence

$$
\begin{aligned}
(a)_{\Delta=0} &= \frac{1}{4s} + Gf' = \nu_1 \\
(a')_{\Delta=0} &= \frac{m}{2s} + 2(-s + nG)\Delta' = -\nu_3 \\
(c)_{\Delta=0} &= -\frac{1}{4s} + Gf' = -\nu_2 \\
(c')_{\Delta=0} &= -\frac{m}{2s} + 2(-s + nG)\Delta' = \nu_4 \\
(e)_{\Delta=0} &= \frac{n}{2s} - Gf = -u_4 \\
(g)_{\Delta=0} &= \frac{n}{2s} + Gf = -u_3
\end{aligned}
\tag{128}
$$

where we have substituted from Eq. (124) and used the relations

$$Gf = \frac{u_4 - u_3}{2}, \qquad Gf' = \frac{\nu_1 - \nu_2}{2} \tag{129}$$

or

$$f = -Q(u_4 - u_3), \qquad f' = Q(\nu_1 - \nu_2) \tag{130}$$

Substituting from Eqs. (128) and (130) into Eq. (126) for $\Delta \to 0$, we get the expressions for the Greek-lettered functions given by Chandrasekhar[3] (Eq. (183) p. 269). This gives another proof of the complete identity of Chandrasekhar's solution with the solution derived above from Busbridge's solution under proper consideration of the linear constraints.

IX. COMPUTATION OF THE AVERAGE INTENSITY AND OF THE FLUX COMPONENT OF THE EMERGING RADIATION

In several applications, quantities are needed in addition to those given above, such as the components of the average intensity and of the flux of the radiation emerging from a planetary atmosphere. These quantities are defined by the following integrals of the intensity matrices; the average intensity of the upward and downward radiation are given as

$$\begin{aligned} \underset{\sim}{J}(0) &= \frac{1}{4\pi}\int_0^1\int_0^{2\pi} \underset{\sim}{I}(0;\ \mu,\ \varphi)\ d\mu\ d\varphi \\ \underset{\sim}{J}(\tau) &= \frac{1}{4\pi}\int_0^1\int_0^{2\pi} \underset{\sim}{I}(\tau;\ -\mu,\ \varphi)\ d\mu\ d\varphi \end{aligned} \tag{131}$$

and the net flux as

$$\begin{aligned} \pi\underset{\sim}{F}^{\uparrow} &= \int_0^1\int_0^{2\pi} \underset{\sim}{I}(0;\ \mu,\ \varphi)\mu\, d\mu\ d\varphi \\ \pi\underset{\sim}{F}^{\downarrow} &= \int_0^1\int_0^{2\pi} \underset{\sim}{I}(\tau;\ -\mu,\ \varphi)\mu\, d\mu\ d\varphi \end{aligned} \tag{132}$$

After integration with respect to φ, only the azimuth-independent terms will remain, so that

$$\begin{aligned} \underset{\sim}{J}(0) &= \frac{1}{2}\int_0^1 \underset{\sim}{I}^{(0)}(0;\ \mu)\ d\mu \\ \underset{\sim}{J}(\tau) &= \frac{1}{2}\int_0^1 \underset{\sim}{I}^{(0)}(\tau;\ -\mu)\ d\mu \\ \pi\underset{\sim}{F}^{\uparrow} &= 2\pi\int_0^1 \underset{\sim}{I}^{(0)}(0;\ \mu)\mu\, d\mu \\ \pi\underset{\sim}{F}^{\downarrow} &= 2\pi\int_0^1 \underset{\sim}{I}^{(0)}(\tau;\ -\mu)\mu\, d\mu \end{aligned} \tag{133}$$

From the definition of the reflection and transmission matrices[1] and from Eq. (1), we have

$$\underset{\sim}{J}(0) = \frac{3}{32}\int_0^1 [\underset{\sim}{K}(\mu)\cdot\widetilde{\underset{\sim}{K}}(\mu_0) - \underset{\sim}{L}(\mu)\cdot\widetilde{\underset{\sim}{L}}(\mu_0)]\frac{\mu_0 d\mu}{\mu+\mu_0}\cdot\underset{\sim}{F}$$
$$\underset{\sim}{J}(\tau) = \frac{3}{32}\int_0^1 [\underset{\sim}{K}(\mu)\cdot\widetilde{\underset{\sim}{L}}(\mu_0) - \underset{\sim}{L}(\mu)\cdot\widetilde{\underset{\sim}{K}}(\mu_0)]\frac{\mu_0 d\mu}{\mu-\mu_0}\cdot\underset{\sim}{F} \qquad (134)$$

Substituting the expressions for the elements of the $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices in Eq. (134), we get the components of the average intensity J_ℓ, and J_r. If we want to compute the total average intensity, we can proceed in the following way. From the nonlinear integral equations for the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices, it is easy to derive the corresponding equations for the $\underset{\sim}{K}$- and $\underset{\sim}{L}$-matrices in the form

$$\underset{\sim}{K}(\mu) = \underset{\sim}{M}(\mu) + \frac{3}{8}\mu\int_0^1 [\underset{\sim}{K}(\mu)\cdot\widetilde{\underset{\sim}{K}}(\mu') - \underset{\sim}{L}(\mu)\cdot\widetilde{\underset{\sim}{L}}(\mu')]\cdot\underset{\sim}{M}(\mu')\frac{d\mu'}{\mu'+\mu}$$
$$\underset{\sim}{L}(\mu) = \underset{\sim}{M}(\mu)e^{-\tau/\mu} + \frac{3}{8}\mu\int_0^1 [\underset{\sim}{K}(\mu)\cdot\widetilde{\underset{\sim}{L}}(\mu') - \underset{\sim}{L}(\mu)\cdot\widetilde{\underset{\sim}{K}}(\mu')]\cdot\underset{\sim}{M}(\mu')\frac{d\mu'}{\mu'-\mu} \qquad (135)$$

Performing the transposition of the matrices in Eq. (135), we get

$$\widetilde{\underset{\sim}{K}}(\mu) = \widetilde{\underset{\sim}{M}}(\mu) + \frac{3}{8}\mu\int_0^1 \widetilde{\underset{\sim}{M}}(\mu')\cdot[\underset{\sim}{K}(\mu')\cdot\widetilde{\underset{\sim}{K}}(\mu) - \underset{\sim}{L}(\mu')\cdot\widetilde{\underset{\sim}{L}}(\mu)]\frac{d\mu'}{\mu'+\mu}$$
$$\widetilde{\underset{\sim}{L}}(\mu) = \widetilde{\underset{\sim}{M}}(\mu)e^{-\tau/\mu} + \frac{3}{8}\mu\int_0^1 \widetilde{\underset{\sim}{M}}(\mu')\cdot[\underset{\sim}{K}(\mu')\cdot\widetilde{\underset{\sim}{L}}(\mu) - \underset{\sim}{L}(\mu')\cdot\widetilde{\underset{\sim}{K}}(\mu)]\frac{d\mu'}{\mu'-\mu} \qquad (136)$$

Multiplying the intensity matrices for emerging radiations in Eq. (125) by $\widetilde{\underset{\sim}{M}}(\mu)d\mu$, integrating for 0 to 1, and substituting from Eq. (136), we get

$$\int_0^1 \widetilde{\underset{\sim}{M}}(\mu)\cdot\underset{\sim}{I}^{(0)}(0;\mu)\,d\mu = \frac{3}{16}\mu_0\int_0^1 \widetilde{\underset{\sim}{M}}(\mu)\cdot[\underset{\sim}{K}(\mu)\cdot\widetilde{\underset{\sim}{K}}(\mu_0) - \underset{\sim}{L}(\mu)\cdot\widetilde{\underset{\sim}{L}}(\mu_0)]\cdot\underset{\sim}{F}\frac{d\mu}{\mu+\mu_0} = \frac{1}{2}[\widetilde{\underset{\sim}{K}}(\mu_0) - \widetilde{\underset{\sim}{M}}(\mu_0)]\cdot\underset{\sim}{F} \qquad (137)$$

and

$$\int_0^1 \underset{\sim}{\tilde{M}}(\mu)\cdot\underset{\sim}{I}^{(0)}(\tau;\ -\mu)\ d\mu = \frac{1}{2}\left[\underset{\sim}{\tilde{L}}(\mu_0) - \underset{\sim}{\tilde{M}}(\mu_0)e^{-\tau/\mu}\right]\cdot\underset{\sim}{F}$$

For neutral, unpolarized, external illumination of the net flux F_0

$$\underset{\sim}{F} = \frac{1}{2}F_0\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}F_0\cdot\underset{\sim}{\mathcal{E}} \tag{138}$$

After performing the matrix multiplications in Eq. (137), we have with respect to Eq. (138) and the definition of the moments in Eq. (41)

$$m_2\left[I_\ell^{(0)}(0,\ \mu)\right] + m_0\left[I_r^{(0)}(0,\ \mu)\right] = \frac{F_0}{4}\left[K_1(\mu_0) + K_3(\mu_0) - (1+\mu_0^2)\right]$$

$$m_0\left[I_\ell^{(0)}(0,\ \mu)\right] - m_2\left[I_\ell^{(0)}(0,\ \mu)\right] = \frac{F_0}{4}\left[K_2(\mu_0) + K_4(\mu_0) - (1-\mu_0^2)\right] \tag{(139)}$$

Since the total average intensity is

$$J_\ell(0) + J_r(0) = m_0\left[I_\ell^{(0)}(0,\ \mu)\right] + m_0\left[I_r^{(0)}(0,\ \mu)\right]$$

it follows from Eq. (139) that

$$J_\ell(0) + J_r(0) = \frac{F_0}{4}\left[\sum_{k=1}^{4} K_k(\mu_0) - 2\right] \tag{140}$$

Similarly, we have for the total average intensity of the downward radiation

$$J_\ell(\tau) + J_r(\tau) = \frac{F_0}{4}\left[\sum_{i=0}^{4} L_i(\mu_0) - 2e^{-\tau/\mu_0}\right] \tag{141}$$

Equation (139) and the corresponding equations for the downward radiation can be used as the basic equations for the inversion problem in a Rayleigh atmosphere.[7]

If the intensity matrices in the expressions for the fluxes in Eq. (133) are expressed in terms of the reflection and transmission matrices, then

$$\begin{aligned} \underset{\sim}{F}^{\uparrow} &= \frac{1}{2}\int_0^1 \underset{\sim}{S}_0(\tau;\ \mu,\ \mu_0)\cdot\underset{\sim}{F}\ d\mu \\ \underset{\sim}{F}^{\downarrow} &= \frac{1}{2}\int_0^1 \underset{\sim}{T}_0(\tau;\ \mu,\ \mu_0)\cdot\underset{\sim}{F}\ d\mu \end{aligned} \tag{142}$$

The components of the flux can be conveniently expressed by four functions $s_\ell(\mu)$, $s_r(\mu)$, $t_\ell(\mu)$, and $t_r(\mu)$, defined by Chandrasekhar[3] (p. 276) as the elements of the matrices

$$\begin{aligned} \underset{\sim}{s}(\mu) &= \begin{bmatrix} s_\ell(\mu) \\ s_r(\mu) \end{bmatrix} = \frac{1}{2}\int_0^1 \underset{\sim}{S}_0(\tau;\ \mu,\ \mu')\cdot\underset{\sim}{\mathcal{E}}\ d\mu' \\ \underset{\sim}{t}(\mu) &= \begin{bmatrix} t_\ell(\mu) \\ t_r(\mu) \end{bmatrix} = \frac{1}{2}\int_0^1 \underset{\sim}{T}_0(\tau;\ \mu,\ \mu')\cdot\underset{\sim}{\mathcal{E}}\ d\mu' \end{aligned} \tag{143}$$

Substituting from Eq. (1), we have

$$\underset{\sim}{s}(\mu) = \frac{3}{8}\int_0^1 \left[\underset{\sim}{K}(\mu)\cdot\underset{\sim}{\tilde{K}}(\mu') - \underset{\sim}{L}(\mu)\cdot\underset{\sim}{\tilde{L}}(\mu')\right]\frac{\mu\mu'\,d\mu'}{\mu+\mu'}\cdot\underset{\sim}{\mathcal{E}}$$

or since

$$\frac{\mu'}{\mu+\mu'} = 1 - \frac{\mu}{\mu+\mu'},$$

$$\begin{aligned} \underset{\sim}{s}(\mu) = {} & \mu\underset{\sim}{K}(\mu)\cdot\frac{3}{8}\,m_0(\underset{\sim}{\tilde{K}})\cdot\underset{\sim}{\mathcal{E}} - \mu\underset{\sim}{L}(\mu)\cdot\frac{3}{8}\,m_0(\underset{\sim}{\tilde{L}})\cdot\underset{\sim}{\mathcal{E}} \\ & - \frac{3}{8}\mu\int_0^1 \left[\underset{\sim}{K}(\mu)\cdot\underset{\sim}{\tilde{K}}(\mu') - \underset{\sim}{L}(\mu)\cdot\underset{\sim}{\tilde{L}}(\mu')\right]\cdot\underset{\sim}{\mathcal{E}}\frac{\mu\,d\mu'}{\mu+\mu'} \end{aligned}$$

However,

$$\underset{\sim}{\mathcal{E}} = \underset{\sim}{M}(\mu)\cdot\underset{\sim}{C}, \quad \text{where} \quad \underset{\sim}{C} = \begin{bmatrix} 1 \\ 1/\sqrt{2} \end{bmatrix} \tag{144}$$

Substituting in Eq. (144), we get, with respect to Eq. (135)

$$\underset{\sim}{s}(\mu) = \mu \underset{\sim}{K}(\mu) \cdot \frac{3}{8} m_0[\widetilde{\underset{\sim}{K}}] \cdot \underset{\sim}{\mathcal{E}} - \mu \underset{\sim}{L}(\mu) \cdot \frac{3}{8} \underset{\sim}{m}_0[\widetilde{\underset{\sim}{L}}] \cdot \underset{\sim}{\mathcal{E}}$$
$$-\mu[\underset{\sim}{K}(\mu) - \underset{\sim}{M}(\mu)] \cdot \underset{\sim}{C}$$

If we write

$$k_1 = \frac{3}{8} m_0[K_1 + K_3], \quad k_2 = \frac{3}{8} m_0[K_2 + K_4]$$
$$\ell_1 = \frac{3}{8} m_0[L_1 + L_3], \quad \ell_2 = \frac{3}{8} m_0[L_2 + L_4]$$

then

$$\underset{\sim}{s}(\mu) = \mu \underset{\sim}{K}(\mu) \begin{bmatrix} k_1 \\ \sqrt{2}\, k_2 \end{bmatrix} - \mu \underset{\sim}{L}(\mu) \begin{bmatrix} \ell_1 \\ \sqrt{2}\, \ell_2 \end{bmatrix} - \mu \underset{\sim}{K}(\mu) \cdot \underset{\sim}{C} + \mu \underset{\sim}{\mathcal{E}}$$

or

$$\begin{aligned} s_\ell(\mu) &= \mu \{ 1 - \ell_1[K_1(\mu) + L_1(\mu)] - 2\ell_2[K_2(\mu) + L_2(\mu)] \} \\ s_r(\mu) &= \mu \{ 1 - \ell_1[K_3(\mu) + L_3(\mu)] - 2\ell_2[K_4(\mu) + L_4(\mu)] \} \end{aligned} \tag{145}$$

since from Eqs. (45) and (52)

$$\begin{aligned} k_1 + \ell_1 &= \frac{3}{8} m_0[P_1 + P_5] = 1 \\ k_2 + \ell_2 &= \frac{3}{8} m_0[P_2 + P_6] = 1/2 \end{aligned} \tag{146}$$

Furthermore, with respect to Eqs. (45), (52), and (55)

$$\begin{aligned} \ell_1 &= \frac{3}{8} m_0[\frac{1}{2}(P_1 + P_5) - \frac{1}{2}(Q_1 + Q_5)] = \frac{1}{2}[1 - \frac{3}{8} m_0(Q_1 + Q_5)] = \frac{1}{2} D_0 \\ 2\ell_2 &= \frac{1}{2}[1 - \frac{3}{4} m_0(Q_2 + Q_6)] = \frac{1}{2} D_0' \end{aligned} \tag{147}$$

From Eq. (145) it follows immediately that

$$\bar{s} = \int_0^1 [s_\ell(\mu) + s_r(\mu)]\, d\mu = 1 - \ell_1 m_1 [K_1 + L_1 + K_3 + L_3]$$
$$- 2\ell_2 m_1 [K_2 + L_2 + K_4 + L_4] = 1 - \frac{4}{3} D_0 C_1 - \frac{4}{3} D_0' C_1'$$

Substituting from Eqs. (103), (108), and (109), we get

$$\bar{s} = 1 - \frac{4}{3} \left[\frac{m}{mB - rC} \cdot \frac{r + n\Gamma}{m + s\Gamma} \cdot D + \frac{1}{2(mB - rC)} \cdot \frac{1}{4} \frac{\Gamma}{m + s\Gamma} \cdot D \right]$$
$$= 1 - \frac{4}{3} \frac{D}{(mB - rC)(m + s\Gamma)} [m(r + n\Gamma) + \frac{1}{8}\Gamma]$$
$$= 1 - \frac{4}{3} \frac{rD}{mB - rC}$$

since, as mentioned above, $mn = rs - 1/8$.

With the values of the constants defined in Eqs. (110e) and (127)

$$\bar{s} = 1 - \frac{8}{3} \frac{D}{B} r\Delta' = 1 - \frac{8}{3} \frac{\beta_0' + \beta_2'}{\alpha_0' + \alpha_2'} f'r \qquad (148)$$

Similarly, we have

$$\underset{\sim}{t}(\mu) = \frac{3}{8} \int_0^1 [\underset{\sim}{K}(\mu) \cdot \underset{\sim}{\tilde{L}}(\mu') - \underset{\sim}{L}(\mu) \cdot \underset{\sim}{\tilde{K}}(\mu')] \frac{\mu\mu' d\mu'}{\mu' - \mu} \cdot \underset{\sim}{\mathcal{E}}$$

and since

$$\frac{\mu'}{\mu' - \mu} = 1 + \frac{\mu}{\mu' - \mu} ,$$

$$\underset{\sim}{t}(\mu) = \mu \underset{\sim}{K}(\mu) \cdot \frac{3}{8} m_0 [\underset{\sim}{\tilde{L}}] \cdot \underset{\sim}{\mathcal{E}} - \mu \underset{\sim}{L}(\mu) \frac{3}{8} m_0 [\underset{\sim}{\tilde{K}}] \cdot \underset{\sim}{\mathcal{E}}$$
$$+ \frac{3}{8} \mu \int_0^1 [\underset{\sim}{K}(\mu) \cdot \underset{\sim}{\tilde{L}}(\mu') - \underset{\sim}{\tilde{L}}(\mu) \cdot \underset{\sim}{\tilde{K}}(\mu')] \frac{\mu d\mu'}{\mu' - \mu} \cdot$$

Introducing the matrix $\underset{\sim}{C}$, we get, after substituting from Eq. (135)

$$\underset{\sim}{t}(\mu) = \mu\underset{\sim}{K}(\mu)\cdot\frac{3}{8}\,m_0[\underset{\sim}{\tilde{L}}]\cdot\underset{\sim}{\mathcal{E}} - \mu\underset{\sim}{L}(\mu)\cdot\frac{3}{8}\,m_0[\underset{\sim}{\tilde{K}}]\cdot\underset{\sim}{\mathcal{E}} + \mu[\underset{\sim}{L}(\mu) - \underset{\sim}{M}(\mu)e^{-\tau/\mu}]\cdot\underset{\sim}{C}$$

$$= \mu\underset{\sim}{K}(\mu)\begin{bmatrix}\ell_1\\ \sqrt{2}\,\ell_2\end{bmatrix} - \mu\underset{\sim}{L}(\mu)\begin{bmatrix}k_1\\ \sqrt{2}\,k_2\end{bmatrix} + \mu\underset{\sim}{L}(\mu)\cdot\underset{\sim}{C} - \mu e^{-\tau/\mu}\cdot\underset{\sim}{\mathcal{E}}$$

$$= \mu\underset{\sim}{K}(\mu)\begin{bmatrix}\ell_1\\ \sqrt{2}\,\ell_2\end{bmatrix} + \mu\underset{\sim}{L}(\mu)\begin{bmatrix}1-k_1\\ \frac{1}{\sqrt{2}}(1-2k_2)\end{bmatrix} - \mu e^{-\tau/\mu}\cdot\underset{\sim}{\mathcal{E}}$$

and with respect to Eq. (146)

$$\underset{\sim}{t}(\mu) = \mu[\underset{\sim}{K}(\mu) + \underset{\sim}{L}(\mu)]\begin{bmatrix}\ell_1\\ \sqrt{2}\,\ell_2\end{bmatrix} - \mu e^{-\tau/\mu}\cdot\underset{\sim}{\mathcal{E}} \tag{149}$$

From this equation, the expressions for Chandrasekhar's function[3] (p. 276) $\gamma_\ell^{(1)}$ and $\gamma_r^{(1)}$ follow immediately, since

$$\underset{\sim}{\Gamma}(\mu) = \begin{bmatrix}\gamma_\ell^{(1)}(\mu)\\ \gamma_r^{(1)}(\mu)\end{bmatrix} = e^{-\tau/\mu}\,\underset{\sim}{\mathcal{E}} + \frac{1}{\mu}\,\underset{\sim}{t}(\mu) = [\underset{\sim}{K}(\mu) + \underset{\sim}{L}(\mu)\begin{bmatrix}\ell_1\\ \sqrt{2}\,\ell_2\end{bmatrix}$$

or in component forms

$$\begin{aligned}\gamma_\ell^{(1)}(\mu) &= \ell_1[K_1(\mu) + L_1(\mu)] + 2\ell_2[K_2(\mu) + L_2(\mu)]\\ \gamma_r^{(1)}(\mu) &= \ell_1[K_3(\mu) + L_3(\mu)] + 2\ell_2[K_4(\mu) + L_4(\mu)]\end{aligned} \tag{150}$$

With respect to Eqs. (45) and (147), we have

$$\begin{aligned}\gamma_\ell^{(1)}(\mu) &= \tfrac{1}{2}\,D_0P_5(\mu) + \tfrac{1}{2}\,D_0'P_0(\mu)\\ \gamma_r^{(1)}(\mu) &= \tfrac{1}{2}\,D_0P_1(\mu) + \tfrac{1}{2}\,D_0'P_2(\mu)\end{aligned} \tag{150a}$$

Substituting for P_1 and P_5 from Eqs. (98a,b) we get

$$\gamma_\ell^{(1)}(\mu) = \frac{1}{2} D_0' P_\ell(\mu) + (D_0 b + D_0' b_0')[\Gamma \mu Q_\ell(\mu) + \mu^2 P_\ell(\mu)]$$

$$\gamma_r^{(1)}(\mu) = \frac{1}{2} D_0 P_r(\mu) + \frac{1}{2D} [D_0 C_1 + D_0' C_1') \mu Q_r(\mu)$$

When we substitute for the constants D_0, b, C_1, D_0', b', and C_1' from Eqs. (103), (108), and (109), we can easily prove that

$$D_0 b + D_0' b_0' = 0$$

$$\frac{1}{2D} (D_0 C_1 + D_0' C_1') = \frac{1}{2} \frac{r}{mB - rC}$$

Hence

$$\gamma_\ell^{(1)}(\mu) = \frac{1}{2} D_0' P_\ell(\mu) = - \frac{1}{4(mB - rC)} P_\ell(\mu) = - \frac{1}{2} \Delta' D P_\ell(\mu)$$

$$\gamma_r^{(1)}(\mu) = \frac{1}{2} D_0 P_r(\mu) + \frac{1}{2} \frac{r}{mB - rC} \mu Q_r(\mu) \qquad (150b)$$

$$= \frac{m P_r(\mu) + r \mu Q_r(\mu)}{2(mB - rC)} = \Delta' D [m P_r(\mu) + r \mu Q_r(\mu)]$$

From Eqs. (123a) and (129)

$$-D = -(\beta_0' + \beta_2') = \frac{3}{8} (d_0 - d_2)$$

$$\Delta' = \frac{\nu_2 - \nu_1}{-G} = 2Q(\nu_2 - \nu_1)$$

which on substitution in Eq. (150b) gives

$$\gamma_\ell^{(1)}(\mu) = \frac{3}{8} Q(\nu_2 - \nu_1)(d_0 - d_2)[X_\ell(\mu) + Y_\ell(\mu)]$$

which is the exact expression derived by Chandrasekhar,[3] (p. 280).

Since from Eq. (124)

$$\nu_2 - \nu_1 = \frac{16}{9} \Delta_2$$

and Chandrasekhar's function u_5 is defined as

$$u_5 = \Delta_2(c_0\varkappa_1 - c_1\varkappa_2) = \Delta_2 \frac{32}{9} (\alpha_0\alpha_1' - \alpha_1\alpha_0') = \frac{32}{9} \Delta_2 r \tag{151}$$

therefore

$$\Delta' = 2Q(\nu_2 - \nu_1) = Q \frac{32}{9} \Delta_2, \quad \Delta' r = Qu_5$$

and

$$\Delta' Dr = - \frac{3}{8} Q(d_0 - d_2)u_5$$

From Eq. (124)

$$m\Delta' = \frac{1}{2G} (u_4 - u_3) = - Q(u_4 - u_3)$$

and

$$m\Delta' D = \frac{3}{8} Q(d_0 - d_2)(u_4 - u_3)$$

Substituting in Eq. (150), we get

$$\gamma_r^{(1)}(\mu) = \frac{3}{8} Q(d_0 - d_2)\{(u_4 - u_3)[X_r(\mu) + Y_r(\mu)] - u_5\mu[X_r(\mu) - Y_r(\mu)]\}$$

which is again the expression derived by Chandrasekhar[3] for $\gamma_r^{(1)}$ (Eq. (241) p. 280). In this way we obtain a much simpler proof for Chandrasekhar's formulas for the functions $\gamma_\ell^{(1)}$ and $\gamma_r^{(1)}$. Comparing the expressions in Eqs. (145) and (150), we see that

$$\begin{aligned} s_\ell(\mu) &= \mu[1 - \gamma_\ell^{(1)}(\mu)] \\ s_r(\mu) &= \mu[1 - \gamma_r^{(1)}(\mu)] \end{aligned} \tag{152}$$

which are the relations given by Chandrasekhar[3] (Eq. (242), p. 280).

To get the expression for the flux components, let us perform the transposition of Eq. (142),

$$\underset{\sim}{\tilde{F}}^{\uparrow} = \frac{1}{2} F_0 \underset{\sim}{\tilde{\mathcal{E}}} \int_0^1 \underset{\sim}{\tilde{S}}_0(\tau; \mu, \mu_0)\, d\mu$$

$$\underset{\sim}{\tilde{F}}^{\downarrow} = \frac{1}{2} F_0 \underset{\sim}{\tilde{\mathcal{E}}} \cdot \int_0^1 \underset{\sim}{\tilde{T}}_0(\tau; \mu, \mu_0)\, d\mu$$

However, from the reciprocity theorem,[1] we have

$$\underset{\sim}{\tilde{N}}_0(\tau; \mu, \mu_0) = \underset{\sim}{N}_0(\tau; \mu_0, \mu) \quad (N \equiv S, T)$$

Hence, from Eq. (143)

$$\underset{\sim}{\tilde{F}}^{\uparrow} \cdot \underset{\sim}{\mathcal{E}} = F_0 \underset{\sim}{\tilde{\mathcal{E}}} \cdot \underset{\sim}{s}(\mu_0)$$

or

$$\underset{\sim}{\tilde{F}}^{\downarrow} \cdot \underset{\sim}{\mathcal{E}} = F_0 \cdot \underset{\sim}{\tilde{\mathcal{E}}} \cdot \underset{\sim}{t}(\mu_0)$$

It is easy to see that

$$\underset{\sim}{\tilde{F}} \cdot \underset{\sim}{\mathcal{E}} = \underset{\sim}{\tilde{\mathcal{E}}} \cdot \underset{\sim}{F}$$

and hence

$$\underset{\sim}{F}^{\uparrow} = F_0 \underset{\sim}{s}(\mu_0)$$

$$\underset{\sim}{F}^{\downarrow} = F_0 \underset{\sim}{t}(\mu_0) \qquad (153)$$

X. CONCLUDING REMARKS

There are several reasons why the reduction of the singular linear equations for the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-matrices to the corresponding scalar equations is presented, in the preceding sections, in detail.

First, the paper is a demonstration of the advantages of the use of the singular linear equations in the reduction to scalar form that can serve as an example for application to problems other than radiative transfer in a homogeneous atmosphere with Rayleigh scattering. The main advantage is, indeed, the possibility of deriving the constraints for nonunique solutions in general matrix form as a consequence of the requirement for analytical behavior of the solution at the double roots of the discriminant matrix.

The derivation of the expressions for the elements of the matrices $\underset{\sim}{K}$ and $\underset{\sim}{L}$ in terms of the $\underset{\sim}{X}$- and $\underset{\sim}{Y}$-functions for the characteristic functions ψ_ℓ and ψ_r is presented in greater detail. This derivation can be easily followed and leads to the expressions derived in a slightly more complicated way by Chandrasekhar.[6]

Finally, the proof of the complete equivalence of the two methods used for the derivations of these expressions, namely the selection of the unique solutions by requirements of analytical behavior (Busbridge's solution) or by requirements of "standard solution," defined by special moment conditions and by determination of remaining constants from the flux equation, deserves a detailed discussion because of its great importance in the reduction of the equations of radiative transfer.

In conclusion, it should be mentioned also that the final expressions,

derived in this paper for perfect scattering, can be easily extended to imperfect scattering, with the albedo of single scattering, ω_0, constant within the atmosphere. As shown in the previous paper,[1] for imperfect scattering all formulas remain unchanged except for the characteristic functions, which have to be multiplied by ω_0, the constant albedo of single scattering.

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