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RM-5084-NASA
AUGUST 1966

A BAYESIAN APPROACH TO RELIABILITY ASSESSMENT

B. L. Fox

PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

The research described here was performed for the National Aeronautics and Space Administration, and deals with the APOLLO Mission Reliability Assessment Study. In this Memorandum, the author uses Bayesian analysis to specify parameters of a prior distribution for two cases: (1) reliability of a unit that either performs satisfactorily throughout a mission or does not, and (2) failure rate of a unit that fails according to the exponential distribution. Prediction of demand for spares is considered in each case. The cases can be read independently.

An estimate of reliability is the posterior mean. Alternatively, the posterior distribution can be used to obtain a (subjective) confidence interval for reliability. The posterior distribution is also useful in a decision-theoretic approach to resource allocation for maximal system reliability; such a study is planned as a sequel to the present work.

This Memorandum should be of interest to those working on reliability estimation; allocation of investment among system components to achieve maximum system reliability; and stockage applications.

SUMMARY

This Memorandum specifies the parameters of a prior distribution for two cases: the reliability of a unit that either performs satisfactorily throughout a mission or does not; and the failure rate of a unit that fails according to the exponential distribution.

Bayesian analysis is an obvious approach in estimating reliability parameters from mixed data sources such as: (1) test results; (2) information on analogous components; and (3) engineering estimates. The prior distribution, of necessity subjective, is (ideally) based on (2) and (3) alone. Test results are then merged with the prior via Bayes' rule to obtain a posterior distribution. Roughly, the spread of the prior distribution is inversely proportional to the degree of prior belief, and determines how heavily it will be weighted when combining it with test data.

A topic that most writers on Bayesian analysis avoid is how to specify the parameters of the prior distribution based on (2) and (3). We give a method for specifying these parameters that requires only information corresponding (i) to the most likely value of reliability and (ii) to the subjective odds that the error in this estimate is less than a given percent. We have computed tables of parameters of the prior distribution corresponding to these subjective inputs. These appear in the Appendix.

As an application of Bayesian analysis, we consider prediction of demand for spares in both the GO NO-GO and constant failure rate cases.

CONTENTS

1. SPECIFYING THE PARAMETERS OF A PRIOR DISTRIBUTION FOR RELIABILITY OF A GO NO-GO UNIT

Suppose we have a unit that works with probability p but that the precise value of p is unknown. If we were totally ignorant about the value of p, then our prior belief would be reflected by a uniform distribution over [0,1]. However, intuition tells us that total ignorance is an anomaly; that is, our prior distribution is really not flat. A smooth, unimodal prior distribution having support (0,1) with peak over what we believe to be the most likely value of p seems appropriate. In addition, the beta distribution is a natural conjugate [5] prior distribution; i.e., the posterior distribution is again a beta distribution (with parameters transformed according to Bayes' rule). The beta density with positive parameters (a,b) is given by

$$(1.1) \qquad \beta(p|a,b) = \begin{cases} c\ p^{a-1}(1-p)^{b-1}, & 0 < p < 1 \\ 0, & \text{elsewhere} \end{cases},$$

with c as a normalizing factor. The mean and variance are, respectively:

$$(1.2) \qquad \mu = a/(a+b),$$

$$(1.3) \qquad \sigma^2 = ab/[(a+b)^2(a+b+1)],$$

and, for a, b > 1, there is a unique mode at

$$(1.4) \qquad \theta = (a-1)/(a+b-2).$$

After observing test data, say a sample with m successes and n failures, the posterior density is $\beta(p|a+m, b+n)$ from the Bayes' rule relation: posterior density = prior density x likelihood function x a normalizing factor independent of p. As more test data are observed, the posterior distribution is updated. (The updating procedure is valid only if all units are stochastically identical. If, for example, design changes are made, as a result of failure mode analysis, a new prior distribution should be constructed from scratch).

It remains to specify the parameters (a,b). The procedure we give is heuristic and, while not the simplest mathematically, uses information that corresponds to subjective notions.* For example, one is less likely to have a feel for the variance of the prior distribution than for the error in his estimate of the most likely value of p. Of course, if we were interested in psychological consistency, we could ask for an estimate of the variance as well -- but we shall ignore such considerations here. If the designer/ engineer being asked these questions has seen any test data, it is probably impossible for him to ignore them. Therefore, in this case, it is suggested that the prior distribution be based on all information the designer knows. On the other hand, if the designer has not seen any test data, this is all to the good; test results are then accounted for in the posterior distribution. Whenever possible, the parameters of the prior distribution should be specified before any tests are performed.

Suppose that our subjective assessment of the most likely value** of p is $\hat{p}$; then we set

$$(\hat{a}-1)/(a+\hat{b}-2) = \hat{p}. \tag{1.5}$$

---

*For example, if $\hat{\mu}$ and $\hat{\sigma}$ were subjective estimates of the mean and variance, respectively, of the prior distribution, then solving the equations (1.2) and (1.3) yields

$$\hat{a} = \hat{\sigma}^{-2}\,\hat{\mu}^2(1-\hat{\mu}) - \hat{\mu}$$

$$\hat{b} = \hat{\sigma}^{-2}\,\hat{\mu}(1-\hat{\mu})^2 - (1-\hat{\mu}).$$

**The analysis of the case where one estimates the mean rather than the mode is analogous. We give no details for the former case, except that numerical results for both cases are given in Tables 1 and 2 in the Appendix.

Next, we ask what odds we would give that the true value of p lies in $(\hat{p}-k\hat{p},\ \hat{p}+k\hat{p})$, where $0 < k < 1$. For example, if $k = .1$, then we ask what the chance is that the error in our estimate is less than 10 percent. Suppose that the subjective odds are x to y; then, setting $v = x/(x+y)$, we have

$$(1.6) \qquad \int_{\hat{p}-k\hat{p}}^{\hat{p}+k\hat{p}} \beta(p|\hat{a},\hat{b})\ dp = v .$$

Thus, to find $\hat{a}$ and $\hat{b}$, it suffices to specify $\hat{p}$, k, and v. If the views of several people are solicited, it is suggested that the decision-maker take weighted averages, the weights $\{\alpha_i\}$ depending on the technical backgrounds and personalities of the people involved. Some may be conservative in their estimates, while others are optimistic. It is suggested that in asking the questions the decision-maker fix the value of v. If person i responds $(\hat{p}_i, k_i)$, then $\hat{p} = \Sigma\alpha_i\hat{p}_i/\Sigma\alpha_j$ and $k = \Sigma\alpha_i k_i/\Sigma\alpha_j$.

Equations (1.5) and (1.6) can be resolved by using the tables of the incomplete beta function [3], but to expedite matters we have provided a table of $(\hat{a},\hat{b})$ in the Appendix corresponding to selected values of (u,v,k), where $u = \hat{p}$.

Defining

$$(1.7) \qquad \phi(t;u,v,k) = v - \frac{\int_{u(1-k)}^{\min(1,u(1+k))} p^{\frac{u(t-1)}{1-u}} (1-p)^{t-1}\ dp}{\int_0^1 p^{\frac{u(t-1)}{1-u}} (1-p)^{t-1}\ dp}$$

and

$$(1.8) \qquad \phi[g(u,v,k);u,v,k] = 0 ,$$

it follows from (1.5) and (1.6) that

$$\hat{b} = g(\hat{p},v,k), \tag{1.9}$$

$$\hat{a} = [\hat{p}(\hat{b}-2) + 1]/(1-\hat{p}) . \tag{1.10}$$

A uniform prior is suggested, if there is not enough prior information to quantify sensibly; however, it is felt that introspection will generally reveal the contrary.

In base stockage application [2], appropriate levels of spares inventory must be determined. To provision spares properly, an estimate of the demand distribution is required. For this, the Poisson approximation may be useful. Assuming that p is near one and the sample size n (say, aircraft landings or space vehicle launchings) is large, the probability of k failures,* corresponding to demands for spares of a given type, is closely approximated by

$$f(k|p,n) = [n(1-p)]^k \, e^{-n(1-p)}/k!. \tag{1.11}$$

Removing the conditioning on p, the demand distribution is

$$g(k|a,b,n) = \int_0^1 f(k|p,n)\, \beta(p|a,b)\, dp. \tag{1.12}$$

An approximation to $g(k|a,b,n)$ is obtained by using the mean of the prior distribution $[a/(a+b)]$ in place of p in (1.11). We do not know how good this approximation is.

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*We assume that the failure distribution over successive missions is geometric (i.e., no memory). The part in question is assumed stressed (used) exactly once per mission -- or, with obvious modifications, twice per mission. If it is stressed continuously, the results of Sec. 2 can be applied; of course, if all missions have the same length, we get a reduction back to the case considered here.

If the distribution of n, say $\phi(n)$, is known,* then the distribution of the number of failures is

$$h(k|a,b) = \sum_n g(k|a,b,n)\ \phi(n) . \tag{1.13}$$

To the author this indirect route to demand prediction seems preferable to a direct attack because the former is more physically motivated.

A device sometimes used is to inflate the estimate of demand deliberately in order to cause a larger provisioning of buffer stocks, with the object of reducing the incidence of stockouts due to demand fluctuation. The author feels that the approach outlined in the next paragraph is more rational.

With an unbiased estimate of demand, the proper inventory level can be determined by a decision-theoretic approach. Let L(k,s) be the loss when k units are demanded and s units are stocked.** The minimal expected loss L^o is

$$L^o = \min \sum_{k=0}^{\infty} L(k,s)\ h(k|a,b) . \tag{1.14}$$

Minimizing $L(\bar{k},s)$, where $\bar{k}$ is the estimate of mean demand, may be grossly incorrect.

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*Predicting n via a Bayesian approach -- perhaps in conjunction with spectral analysis of time series -- may be appropriate. This involves simply one more conditioning-unconditioning in (1.13). Since the values of n over successive time periods may be autocorrelated, spectral analysis may be useful in finding a suitable parametric form for the distribution of n. For a treatment of spectral analysis, see Yaglom [6].

**For example,

$$L(k,s) = c_1\ s + c_2 \max(0,s-k) + c_3 \max(0,k-s),$$

where c_1 is the unit purchase cost, c_2 is the unit holding cost, and c_3 is the unit stockout cost.

2. SPECIFYING THE PARAMETERS OF A PRIOR DISTRIBUTION FOR FAILURE RATE

Suppose that a unit has constant failure rate θ, fixed but unknown. We assume a (natural conjugate [5]) prior distribution with density of the form

(2.1) $$g(\theta|a,b) = a^b \theta^{b-1} e^{-a\theta}/\Gamma(b),$$

where the parameters (a,b) are to be specified. Its mean and variance are:

(2.2) $$\mu = b/a,$$

(2.3) $$\sigma^2 = b/a^2,$$

respectively. There is a unique mode at $(b-1)/a$, $b \geq 1$, but it is felt that the mean time to failure is more amenable to subjective assessment in this case.

If the subjective estimate of the mean time to failure is $\hat{\nu}$, then using (2.2) we set

(2.4) $$\hat{b}/\hat{a} = 1/\hat{\nu}.$$

Based on subjective odds, let v be the chance that the failure rate exceeds $k/\hat{\nu}$. This yields

(2.5) $$\int_{k/\hat{\nu}}^{\infty} g(\theta|\hat{a},\hat{b})\, d\theta = v.$$

Equations (2.4) and (2.5) could be resolved by using tables of the incomplete gamma function [4], but this would be a tedious job.*
To save time, for selected values of k and v, Table 3 of the Appendix provides the corresponding $\hat{h}$, where

$$\hat{a} = \hat{h}\hat{\nu}, \tag{2.6}$$

$$\hat{b} = \hat{h}. \tag{2.7}$$

Defining

$$\delta(h;k,v) = v - \int_k^{\infty} g(\theta|h,h)\, d\theta\ , \tag{2.8**}$$

$$\delta[\rho(k,v);k,v] = 0\ , \tag{2.9}$$

we see that

---

*If we had used a subjective estimate, say $\hat{\sigma}^2$, of the variance of the prior distribution instead of (2.5), then using (2.3) and (2.4) we would have the explicit expressions

$$\hat{a} = 1/\hat{\nu}\hat{\sigma}^2$$

$$\hat{b} = 1/(\hat{\nu}\hat{\sigma})^2.$$

However, it is felt that intuition for σ^2 would be poor.

**Note that

$$\int_k^{\infty} g(\theta|h,h)\, d\theta = \Gamma(h,hk)/\Gamma(h,0),$$

where

$$\Gamma(a,x) = \int_x^{\infty} e^{-u}\, u^{a-1}\, du.$$

A standard subroutine for computing $\Gamma(a,x)$ is available.

$$\hat{h} = \rho(k,v). \tag{2.10}$$

In specifying the parameters of the prior distribution, we refer to the suggestions given in Sec. 1 for handling data already on hand. Failure data (except that used in forming the prior distribution) are incorporated in the posterior distribution by Bayes' rule. Having observed failures at ages $t_1, \ldots, t_k$, and a nonfailed group with ages $t_{k+1}, \ldots, t_m$, the posterior density* is

$$g(\theta | a + \sum_{i=1}^{m} t_i, b+k).$$

This gives us our current prior distribution, which is updated as more observations are recorded. If, for example, the unit corresponding to t_j, $j > k$, fails at t', updating yields

$$g(\theta | a + \sum_{i=1}^{m} t_i + (t'-t_j), b+k+1).$$

We now give an application to demand prediction. Suppose each of n units operates continuously until failure, at which time it is replaced instantaneously (for practical purposes) by a unit as good as new. These failures generate the demands for spares and/or repair. If each unit has constant failure rate θ, the probability of k demands in time T is $p(k|n\theta T)$, where

$$p(k|\lambda) = \lambda^k e^{-\lambda k}/k!; \tag{2.11}$$

---

*If the failure distribution were $1-e^{-\theta x^{\alpha}}$ (Weibull with known shape parameter α), replace t_i by t_i^{α}, $i = 1, \ldots, m$.

that is, demand is Poisson with rate $n\theta$. Removing the conditioning on θ, the probability of k demands is

$$(2.12) \qquad f(k|a,b,n,T) = \int_0^\infty p(k|n\theta T)\, g(\theta|a,b)\, d\theta.$$

If n and/or T is a random variable, we can further uncondition in a similar manner.

REMARKS. If, in fact, the life distribution of the jth unit has mean μ_j and is nonlattice but not necessarily exponential, then [1], with

$$\theta = (1/n) \sum_{j=1}^{n} \mu_j^{-1} \quad \text{and } n \to \infty ,$$

the stationary demand distribution becomes $p(k|n\theta T)$. If planned replacement takes place at age τ, the same result holds if we replace μ_j by the mean of the distribution truncated at τ. (In a more sophisticated replacement policy, the planned replacement age should ideally depend on the current inventory level.) If replacement can take a significant amount of time (due, for example, to stockouts or non-negligible repair times), then the replacement time distribution should be convolved with the failure distribution, and the mean of the resulting distribution used in place of μ_j.

For aircraft spares provisioning, a somewhat different model of the demand process may be appropriate. Suppose that a part, used only when the aircraft is flying, has constant failure rate θ during flight* and failure rate 0 on the ground. Let flights to the base originate from points $\{1, \ldots, m\}$, with respective flying times $\{w_1, \ldots, w_m\}$.

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*See Sec. 1 for the case where the unit is not stressed continuously during flight.

During a period of length T, let n_i be the number of flights to the base from point i. The probability of k demands during time T is

$$D(k|T) = \sum_{(k_1,\ldots,k_m)\in S_k} \binom{n_i}{k_i} v_i^{k_i}(1-v_i)^{n_i-k_i}, \tag{2.13}$$

where

$$v_i = 1-e^{-\theta w_i}, \tag{2.14}$$

$$S_k = \left\{(k_1,\ldots,k_m): \sum_{i=1}^{m} k_i = k\right\}. \tag{2.15}$$

If all the v_i's are near 0 and all the n_i's are large, then we have the Poisson approximation

$$D(k|T) \approx p\left(k\Big|\sum_{i=1}^{m} n_i v_i\right). \tag{2.16}$$

When the n_i's and θ are unknown, we condition and then uncondition in the usual way. If θ has a prior distribution $g(\theta|a,b)$ given by (2.1), then

$$p\left(k\Big|\sum_{i=1}^{m} n_i v_i; a,b\right) = p\left(k\Big|\sum_{i=1}^{m} n_i\left[1-\left(\frac{a}{a+w_i}\right)^b\right]\right). \tag{2.17}$$

APPENDIX

In this Appendix we give tables of parameters of prior distributions corresponding to subjective assessments of reliability, as described in Secs. 1 and 2. The entries in the tables were computed using a program written by Mrs. Sarah Higgins, with the assistance of Robert Mobley and the author. Several test cases were computed by hand (using tables) for each program, with agreement to more than four significant figures throughout. The programs are believed to be completely debugged and are listed here for the convenience of those who may want values that are not tabulated. Tables 1 and 2 refer to Sec. 1 (beta prior). Table 3 refers to Sec. 2 (gamma prior). Asterisks in the tables indicate that the rootfinder did not locate a root within the allotted time.

Table 1

PARAMETERS OF BETA PRIOR DISTRIBUTION FOR SELECTED VALUES OF u, v, AND k

A Priori Estimates			Mode		Mean	
u	v	k	a	b	a	b
.9	.5	.1	5.510	1.501	15.311	1.701
		.075	8.995	1.888	18.856	2.095
		.050	19.067	3.007	29.012	3.224
		.025	73.625	9.069	83.657	9.295
	.67	.1	11.233	2.137	21.405	2.378
		.075	18.352	2.928	28.576	3.175
		.050	39.311	5.257	49.576	5.508
		.025	*	*	163.424	18.158
	.75	.1	15.862	2.651	26.222	2.914
		.075	25.727	3.748	36.173	4.019
		.050	54.915	6.991	65.395	7.266
		.025	*	*	*	*
	.95	.1	50.195	6.466	61.012	6.779
		.075	80.184	9.798	91.500	10.167
		.050	165.106	19.234	176.772	19.641
		.025	*	*	*	*
	.99	.1	90.232	10.915	101.161	11.240
		.075	145.467	17.052	157.047	17.450
		.050	*	*	*	*
		.025	*	*	*	*
.95	.5	.1	5.910	1.258	25.348	1.334
		.075	7.954	1.366	27.555	1.450
		.050	12.165	1.588	32.062	1.687
		.025	39.270	3.014	59.320	3.122
	.67	.1	10.828	1.517	30.424	1.601
		.075	15.145	1.744	34.972	1.841
		.050	24.645	2.244	44.926	2.365
		.025	80.430	5.181	100.846	5.308
	.75	.1	14.503	1.711	34.177	1.799
		.075	20.667	2.035	40.602	2.137
		.050	34.680	2.773	55.141	2.902
		.025	112.251	6.855	132.911	6.995
	.95	.1	39.402	3.021	59.307	3.121
		.075	59.151	4.061	79.382	4.178
		.050	107.748	6.618	128.632	6.770
		.025	339.228	18.801	361.167	19.009
	.99	.1	67.365	4.493	87.362	4.598
		.075	102.944	6.366	123.272	6.488
		.050	192.310	11.069	213.312	11.227
		.025	*	*	*	*
.975	.5	.1	6.239	1.134	45.458	1.166
		.075	8.411	1.190	47.710	1.223
		.050	12.734	1.301	52.197	1.338
		.025	25.695	1.633	65.637	1.683
	.67	.1	10.689	1.248	49.979	1.282
		.075	14.711	1.352	54.111	1.387
		.050	23.166	1.568	62.790	1.610
		.025	51.772	2.302	92.096	2.361

Table 1 -- continued

A Priori Estimates			Mode		Mean	
u	v	k	a	b	a	b
.975	.75	.1	13.847	1.329	53.174	1.363
		.075	19.258	1.468	58.709	1.505
		.050	30.927	1.767	70.631	1.811
		.025	72.607	2.836	113.105	2.900
	.95	.1	33.943	1.845	73.393	1.882
		.075	48.848	2.227	88.461	2.268
		.050	83.263	3.109	123.202	3.159
		.025	223.049	6.694	263.966	6.768
	.99	.1	55.718	2.403	147.780	2.442
		.075	81.364	3.061	167.403	3.104
		.050	141.906	4.613	181.930	4.665
		.025	396.793	11.149	437.834	11.227
.99	.5	.1	6.441	1.055	105.529	1.066
		.075	8.695	1.078	107.815	1.089
		.050	13.190	1.123	112.374	1.135
		.025	26.601	1.259	125.981	1.273
	.67	.1	10.592	1.097	109.706	1.108
		.075	14.423	1.136	113.580	1.147
		.050	*	*	121.496	1.227
		.025	47.093	1.466	146.599	1.481
	.75	.1	13.437	1.126	112.565	1.137
		.075	18.381	1.176	117.556	1.187
		.050	28.610	1.279	127.882	1.292
		.025	62.041	1.617	161.611	1.632
	.95	.1	30.642	1.299	129.821	1.311
		.075	42.607	1.420	141.851	1.433
		.050	68.396	1.681	167.770	1.695
		.025	160.581	2.612	260.346	2.630
	.99	.1	48.573	1.480	147.780	1.493
		.075	68.123	1.678	167.403	1.691
		.050	111.020	2.111	210.442	2.126
		.025	269.774	3.715	369.615	3.733
.999	.5	.1	6.565	1.006	1005.572	1.007
		.075	8.871	1.008	1007.881	1.009
		.050	*	*	1012.496	1.014
		.025	27.293	1.026	*	*
	.67	.1	10.530	1.010	*	*
		.075	14.241	1.013	1013.255	1.014
		.050	*	*	1020.678	1.022
		.025	44.127	1.043	1043.174	1.044
	.75	.1	13.186	1.012	*	*
		.075	17.842	1.017	1016.857	1.018
		.050	*	*	*	*
		.025	55.493	1.055	1054.545	1.056

Table 2

PARAMETERS OF BETA PRIOR DISTRIBUTION, VARYING u
(v = .5, k = .2)

A Priori Estimate	Mode		Mean	
u	a	b	a	b
.990	3.049	1.021	102.089	1.031
.991	3.055	1.019	113.201	1.028
.992	3.060	1.017	127.091	1.025
.993	3.066	1.015	144.950	1.022
.994	3.072	1.013	168.762	1.019
.995	3.078	1.010	202.097	1.016
.996	3.083	1.008	252.099	1.012
.997	3.089	1.006	335.434	1.009
.998	3.095	1.004	502.103	1.006
.999	3.100	1.002	1001.997	1.003

Table 3

PARAMETERS OF GAMMA PRIOR DISTRIBUTION FOR SELECTED VALUES OF k AND v

k	v	$\hat{h}$
.1	.5	*
	.67	.3683137
	.75	.4964839
	.90	.9725192
.5	.5	.5602821
	.67	1.4046943
	.75	2.1610065
	.90	5.3209309
.9	.5	3.2660242
	.67	*
	.75	*
	.90	*

```
C         PROGRAM 1      BETA PRIOR, MODE ESTIMATED

$IBFTC MODE     REF
C                  PARAMETERS OF BETA PRIOR A,B
C                  A = (U(B-2)+1)/(1-U)
C                  B = ROOT
C                  U = MODE, OR MOST LIKELY VALUE OF THE PROBABILITY THAT A SYSTEM
C     WORKS
C                  XK = PERCENTAGE ERROR IN U
C                  V = CHANCE THAT ERROR IS LESS THAN XK
C                  II = NUMBER OF INPUT VALUES OF U
C                  JJ = NUMBER OF INPUT VALUES OF V
C                  KK = NUMBER OF INPUT VALUES OF XK
C                  TX = INITIAL ESTIMATE OF ROOT TO BE USED BY ROOTFINDER SUBROUTINE
C                  N = NUMBER OF ROOTS DESIRED
C                  IN = NUMBER OF ITERATIONS USED
C                  IF = 0 - NO RETURN AFTER EACH ITERATION
C                  RT = LOCATION OF THE ITERANT
C                  FRT = LOCATION WHERE THE VALUE OF THE ITERANT IS STORED

      COMMON /INPUT/U(9),V(5),XK(4),UPPER,XLOWER,TX,I,PHI,J
  850 FORMAT (I2)
  870 FORMAT (F10.4,2(5X,F5.3),2(5X,F10.3),5X,F10.8)
  900 FORMAT (6F10.4)
  901 FORMAT (5F5.3)
  902 FORMAT (4F5.3)
  903 FORMAT (F5.1)
      READ (5,850) II
      READ (5,850) JJ
      READ (5,850) KK
      READ (5,900) (U(I),I=1,II)
      READ (5,901) (V(J),J=1,JJ)
      READ (5,902) (XK(K),K=1,KK)
      READ (5,903) TX
      N=1
      IN=20
      IF=0
      DO 600 K=1,KK
      DO 599 I=1,II
      DO 598 J=1,JJ
      TEST=U(I)*(1.+XK(K))
      UPPER=AMIN1(1.,TEST)
   25 XLOWER=U(I)*(1.-XK(K))
      CALL GRT (N,TX,IN,IF)
      A=(U(I)*(TX-2.)+1.)/(1.-U(I))
      WRITE (6,870) U(I),V(J),XK(K),A,TX,PHI
  598 CONTINUE
  599 CONTINUE
  600 CONTINUE
      CALL EXIT
      END
```

```
$IBFTC GRTF     LIST                                                    W0080010
C             GENERAL ROOTFINDER WRITTEN BY WERNER L. FRANK - OCTOBER 20, 1958
      SUBROUTINE GRT  (N,C,IN,IF)                                       W0080020
      DIMENSION C(50)                                                   W0080030
      DO 100 L=1,N                                                      W0080040
      JK=0                                                              W0080050
      IF (C(L))45,46,45                                                 W0080060
   45 RT=.9*C(L)                                                        W0080070
      ASSIGN 1 TO NN                                                    W0080080
      GO TO 80                                                          W0080090
    1 X0=FPRT                                                           W0080100
      RT=1.1*C(L)                                                       W0080110
      ASSIGN 2 TO NN                                                    W0080120
      GO TO 80                                                          W0080130
    2 X1=FPRT                                                           W0080140
      RT=C(L)                                                           W0080150
      ASSIGN 3 TO NN                                                    W0080160
      GO TO 80                                                          W0080170
    3 X2=FPRT                                                           W0080180
      GO TO 50                                                          W0080190
   46 RT=-1.0                                                           W0080200
      ASSIGN 4 TO NN                                                    W0080210
      GO TO 80                                                          W0080220
    4 X0=FPRT                                                           W0080230
      RT=1.0                                                            W0080240
      ASSIGN 5 TO NN                                                    W0080250
      GO TO 80                                                          W0080260
    5 X1=FPRT                                                           W0080270
      RT=0.0                                                            W0080280
      ASSIGN 6 TO NN                                                    W0080290
      GO TO 80                                                          W0080300
    6 X2=FPRT                                                           W0080310
   50 H=-1.0                                                            W0080320
      D=-.5                                                             W0080330
   49 DD=1.0+D                                                          W0080340
      BI=(X0*D*D)-(X1*DD*DD)+(X2*(DD+D))                                W0080350
      DEN=BI*BI-(4.0*X2*D*DD)*(X0*D-(X1*DD)+X2)                         W0080360
      IF (DEN)36,36,51                                                  W0080370
   36 DEN=0.0                                                           W0080380
   51 DEN=SQRT(DEN)                                                     W0080390
   53 DN=BI+DEN                                                         W0080400
      DM=BI-DEN                                                         W0080410
      IF (ABS(DN)-ABS(DM))57,57,56                                      W0080420
   56 DEN=DN                                                            W0080430
      GO TO 58                                                          W0080440
   57 DEN=DM                                                            W0080450
   58 IF (DEN)55,54,55                                                  W0080460
   54 DEN=1.0                                                           W0080470
   55 DI=(-2.0*X2*DD)/DEN                                               W0080480
      H=DI*H                                                            W0080490
      RT=RT+H                                                           W0080500
      IF (ABS(H/RT)-1.0E-6)75,75,60                                     W0080510
   60 ASSIGN 7 TO NN                                                    W0080520
      GO TO 80                                                          W0080530
    7 IF (ABS(FPRT)-ABS(X2*10.0))62,61,61                               W0080540
   61 DI=DI*.5                                                          W0080550
      H=H*.5                                                            W0080560
      RT=RT-H                                                           W0080570
      GO TO 80                                                          W0080580
   62 X0=X1                                                             W0080590
```

```
      X1=X2
      X2=FPRT
      D=DI
      GO TO 49
   75 CALL AUX (RT,FRT)
   76 C(L)=RT
  100 CONTINUE
      IN=JK
   33 RETURN
   80 JK=JK+1
      IF (100-JK)75,75,86
   86 CALL AUX (RT,FRT)
      FPRT=FRT
      IF (L-1)81,89,81
   81 DO 82 I=2,L
      TEM=RT-C(I-1)
      IF (ABS(TEM)-1.0E-20)85,82,82
   82 FPRT=FPRT/TEM
   89 IF (ABS(FRT)-1.0E-20)90,91,91
   90 IF (ABS(FPRT)-1.0E-20)76,91,91
   91 IF(IF) 33,84,33
   84 GO TO NN,(1,2,3,4,5,6,7)
   85 RT=RT+.001
   88 GO TO 80
      END


$IBFTC AUX
C            AUXILIARY PROGRAM CALLED BY ROOTFINDER SUBROUTINE
C            THIS SUBROUTINE RETURNS CONTROL TO THE ROOTFINDER ROUTINE
      SUBROUTINE AUX(RT,FRT)
      EXTERNAL FNC
      COMMON/INPUT/U(9),V(5),XK(4),UPPER,XLOWER,FX,I,PHI,J,AX
  598 FORMAT (4H W1=,1PE15.7,4H W2=,1PE15.7)
  599 FORMAT (6H XNUM=,1PE15.7)
  600 FORMAT (5H DEN=,1PE15.7)
  601 FORMAT (4H Z1=,1PE15.7)
  602 FORMAT (4H Z2=,1PE15.7)
  803 FORMAT (5H PHI=,1PE15.7)
  804 FORMAT (4HORT=,1PE15.7)
      AX=RT
      DIF=.0001
      WRITE (6,804) RT
      IF (RT-1.) 25,28,35
   25 PHI=V(J)-(UPPER-XLOWER)+1.-RT
C     ARTIFICIAL VALUE INTENDED TO DRIVE ROOTFINDER TOWARD A ROOT
      GO TO 46
   28 PHI=V(J)-(UPPER-XLOWER)
      GO TO 46
   35 CALL RINT2 (U(I),UPPER,DIF,FNC,W1,IND)
      CALL RINT2 (XLOWER,U(I),DIF,FNC,W2,IND)
      WRITE (6,598) W1,W2
      XNUM=W1+W2
      WRITE (6,599) XNUM
      TOL=0.0001*XNUM
      X=0.0025
    5 VALUE=X**(U(I)*(RT-1.)/(1.-U(I))) * (1.-X)**(RT-1.)
      IF (VALUE.GE.TOL.AND.X.EQ.0.0025) GO TO 50
```

```
      IF (VALUE.LT.TOL.AND.X.EQ.0.0025) GO TO 51
      IF (VALUE.LT.TOL) GO TO 39
      X=XLOWER-0.1
   27 IF (X.LE.0.0) GO TO 50
   29 VALUE=X**(U(I)*(RT-1.)/(1.-U(I))) * (1.-X)**(RT-1.)
      IF (VALUE.LT.TOL) GO TO 60
      X=X-0.1
      GO TO 27
   32 CALL RINT2 (YLOWER,XLOWER,DIF,FNC,Z1,IND)
   30 WRITE (6,601) Z1
      IF (UPPER.EQ.1.0) GO TO 40
      CALL RINT2 (UPPER,1.0,DIF,FNC,Z2,IND)
      WRITE (6,602) Z2
   41 DEN=XNUM+Z1+Z2
      WRITE (6,600) DEN
      IF (ABS(XNUM).GT.1.E10*DEN) GO TO 3
      GO TO 45
   40 Z2=0.0
      GO TO 41
   45 PHI=V(J)-(XNUM/DEN)
      GO TO 46
    3 PHI=-.001*RT
   46 WRITE (6,803) PHI
      GO TO 80
   39 Z1=0.0
      GO TO 30
   50 YLOWER=0.0
      GO TO 32
   51 X=XLOWER
      GO TO 5
   60 YLOWER=X
      GO TO 32
   80 FRT=PHI
      RETURN
      END


$IBFTC RINT2   LIST                                                     W0570010
C              INTEGRATION SUBROUTINE WRITTEN BY ROBERT L. MOBLEY
C              DATE OF WRITE-UP - 2-2-65                                W0570020
C              DATE OF SOURCE DECK - 2-2-65                             W0570030
      SUBROUTINE RINT2 (A,B,E,FNC,F,IND)                                W0570040
C                                                                       W0570050
C                                                                       W0570060
C           A   =  ONE LIMIT OF THE INTEGRATION.                        W0570070
C           B   =  OTHER LIMIT OF THE INTEGRATION.                      W0570080
C           E   =  ERROR BOUND (NON-DIMENSIONAL).                       W0570090
C         FNC   =  FUNCTION SUBPROGRAM.                                 W0570100
C                     THE FUNCTION STATEMENT MUST BE - FUNCTION FNC(X)W0570110
C                     WHERE X IS THE INDEPENDENT VARIABLE.              W0570120
C           F   =  THE VALUE OF THE INTEGRAL IS RETURNED HERE.          W0570130
C         IND   =  AN INDICATOR WHICH IS RETURNED.                      W0570140
C                    ZERO INDICATES THE INTEGRAL DID NOT CONVERGE       W0570150
C                              USING 2**10 INTERVALS.                   W0570160
C                    NON-ZERO INDICATES THE INTEGRAL CONVERGED WITHIN W0570170
C                              THE ERROR BOUND.                         W0570180
C                                                                       W0570190
      DOUBLE PRECISION T(31),AA,BB,SIGMA,FNC,H0,W,P,EMIN,R,FF,ANS,A4K,DA
```

```
       1BS
        EMIN=10000000.                                                  W0570210
        AA=A
        BB=B
        H0=BB-AA
        T=.5*(FNC(AA)+FNC(BB))*H0
        P=1.                                                            W0570240
        DO 100 I=1,12
        SIGMA=0.                                                        W0570260
        O=1.                                                            W0570270
        P=P*2.                                                          W0570280
   10   CONTINUE                                                        W0570290
        SIGMA=SIGMA+FNC(AA+(H0*O)/P)
        O=O+2.                                                          W0570310
        IF (O.LT.P) GO TO 10                                            W0570320
        T(I+1)=(SIGMA*H0)/P+.5*T(I)                                     W0570330
        FF=T
        A4K=1.                                                          W0570350
        DO 20 J=1,I                                                     W0570360
        L=I-J+1                                                         W0570370
        A4K=A4K*4.                                                      W0570380
        T(L)=T(L+1)+(T(L+1)-T(L))/(A4K-1.)                              W0570390
   20   CONTINUE                                                        W0570400
C                                                                       W0570410
        WRITE (6,99) (T(KK),KK=1,I)
   99 FORMAT (1P8 D15.7/8D15.7)
        I2=I+1                                                          W0570420
        R=FF-T
        IF (DABS(T).GT.1.D-10) R=R/T
        EMIN=DABS(R)
        ANS=T                                                           W0570470
   90 IF (DABS(R).LE.E.AND.I.GT.3) GO TO 200
  100   CONTINUE                                                        W0570490
C                                                                       W0570500
        I=I-1                                                           W0570510
  200 IND=15-I
        WRITE (6,98)EMIN
   98 FORMAT (1PD15.7)
        F=ANS                                                           W0570530
        RETURN                                                          W0570540
        END                                                             W0570550


$IBFTC FNC        LIST
C                 FUNCTION SUBPROGRAM AS REQUIRED BY INTEGRATION ROUTINE
C                 THIS ROUTINE RETURNS CONTROL TO THE INTEGRATION SUBROUTINE
      DOUBLE PRECISION FUNCTION FNC(X)
      DOUBLE PRECISION X
      COMMON/INPUT/U(9),V(5),XK(4),UPPER,XLOWER,TX,I,PHI,J,AX
    2 FNC=X**(U(I)*(AX-1.)/(1.-U(I)))*(1.D0-X)**(AX-1.)
    4 RETURN
      END

$ENTRY            MODE
```

```
C         PROGRAM 2     BETA PRIOR, MEAN ESTIMATED

$IBFTC MEAN     REF
C               PARAMETERS OF BETA PRIOR  A, B
C               A = BU/(1-U)
C               B = ROOT

C               U = MEAN, OR AVERAGE VALUE OF THE PROBABILITY THAT A SYSTEM WORKS

C               ALL OTHER VARIABLES ARE DEFINED AS IN PROGRAM 1

      COMMON /INPUT/U(9),V(5),XK(4),UPPER,XLOWER,TX,I,PHI,J
  850 FORMAT (I2)
  900 FORMAT (6F10.4)
  901 FORMAT (5F5.2)
  902 FORMAT (4F5.3)
  903 FORMAT(F5.1)
  870 FORMAT (F10.4,2(5X,F5.3),2(5X,F10.3),5X,F10.8)
      READ (5,850) II
      READ (5,850) JJ
      READ (5,850) KK
      READ (5,900) (U(I),I=1,II)
      READ (5,901) (V(J),J=1,JJ)
      READ (5,902) (XK(K),K=1,KK)
      READ (5,903) TX
      N=1
      IN=20
      IF=0
      DO 600 K=1,KK
      DO 599 I=1,II
      DO 598 J=1,JJ
      TEST=U(I)*(1.+XK(K))
      UPPER=AMIN1(1.,TEST)
   25 XLOWER=U(I)*(1.-XK(K))
      CALL GRT (N,TX,IN,IF)
      A=(U(I)*TX)/(1.0-U(I))
      WRITE (6,870) U(I),V(J),XK(K),A,TX,PHI
  598 CONTINUE
  599 CONTINUE
  600 CONTINUE
      CALL EXIT
      END



C               GENERAL ROOT FINDER FOLLOWS AS ABOVE



C               AUXILIARY SUBROUTINE FOLLOWS AS ABOVE



C               INTEGRATION ROUTINE FOLLOWS AS ABOVE
```

```
$IBFTC FNC        LIST
C                 FUNCTION SUBPROGRAM REQUIRED BY INTEGRATION ROUTINE
C                 THIS ROUTINE RETURNS CONTROL TO THE INTEGRATION SUBROUTINE
      DOUBLE PRECISION FUNCTION FNC(X)
      DOUBLE PRECISION X
      COMMON/INPUT/U(9),V(5),XK(4),UPPER,XLOWER,TX,I,PHI,J,AX
      FNC=X**(((U(I)*AX)-1.0)/(1.0-U(I)))*(1.00-X)**(AX-1.0)
    4 RETURN
      END

$ENTRY            MEAN



C         PROGRAM 3      GAMMA PRIOR, MTBF ESTIMATED

$IBFTC ROOT-2

C                 PARAMETERS OF GAMMA PRIOR  A, B
C                 A = ROOT X ESTIMATED MTBF
C                 B = ROOT

C                 H = INITIAL ESTIMATE OF ROOT TO BE USED BY ROOTFINDER SUBROUTINE
C                 ALL OTHER VARIABLES ARE DEFINED AS IN PROGRAM 1
      COMMON/INPUT/V(9),XK(3),J,K,H,THETA
  900 FORMAT (I2)
  901 FORMAT (9F5.2)
  910 FORMAT (F6.2,5X,F5.2,5X,F10.7,5X,F10.7)
      READ (5,900) JJ
      READ (5,900) KK
      READ (5,901) (V(J),J=1,JJ)
      READ (5,901) (XK(K),K=1,KK)
      N=1
      IF=0
      DO 599 K=1,KK
      H=1.0
      DO 600 J=1,JJ
      CALL GRT (N,H,IN,IF)
      WRITE (6,910) V(J),XK(K),H,THETA
  600 CONTINUE
  599 CONTINUE
      CALL EXIT
      END


C                 ROOTFINDER SUBROUTINE FOLLOWS AS ABOVE
```

```
$IBFTC AUX
      SUBROUTINE AUX (RT,FRT)
      COMMON/INPUT/V(9),XK(3),J,K,B,THETA
  799 FORMAT (4HORT=,F10.7)
  800 FORMAT (10X,E12.8)
  801 FORMAT (5X,E12.8)
  802 FORMAT (26HODENOMINATOR TOO NEAR ZERO)
  803 FORMAT (3(5X,E12.8))
      WRITE (6,799) RT
      IF (RT.LE.0.) GO TO 5
      IF (RT.GE.20.0) GO TO 10
      B=RT*XK(K)
      Z=GAMMA (RT,B)
      WRITE (6,800) Z
      Y=GAMMA (RT,0.)
      WRITE (6,801) Y
      IF (Y.EQ.0.) GO TO 3
      THETA=V(J)-(Z/Y)
      WRITE (6,803) THETA,Z,Y
      GO TO 4
    3 WRITE (6,802)
    5 THETA = V(J)-1.0+RT
C     ARTIFICIAL VALUE INTENDED TO DRIVE ROOTFINDER TOWARD A ROOT
      GO TO 4
   10 THETA=-1.0-20.0*RT
    4 FRT=THETA
      RETURN
      END




C                    NYU GAMM (FAP CODED ROUTINE) FOLLOWS
C                    SHARE PROGRAM (NYU GAMM, 3218) IS CALLED BY AUXILIARY ROUTINE




$ENTRY               ROOT-2
```

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