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GAMES WITH UNIQUE SOLUTIONS WHICH ARE NONCONVEX

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PREFACE

A solution concept for n-person games in characteristic function form was defined by von Neumann and Morgenstern. It is not yet known whether every game has a solution. Many games have been known to have unique solutions, but any such solution has always been a convex polyhedron. This Memorandum describes a game with a polyhedral solution which is unique and nonconvex. The existence of such solutions may prove useful in determining whether solutions exist for every game.

ABSTRACT

This Memorandum describes an eight—person game in characteristic function form which has a unique von Neumann—Morgenstern solution that is nonconvex. A method for constructing other games with similar solutions is discussed briefly.

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1. INTRODUCTION

This Memorandum describes an eight-person game in characteristic function form which has a von Neumann-Morgenstern solution that is unique and nonconvex. The existence of such games may give some insight into the unsolved problem of whether a solution exists for every game.

First, the essential definitions will be stated. Then, a particular eight-person game will be described. Finally, there is a brief discussion on how to construct additional games with unique and nonconvex solutions.

2. DEFINITIONS

An n-person game is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ and v is a real valued characteristic function on 2^N, that is, v assigns the real number v(S) to each subset S of N and $v(\varphi) = 0$. The set of all imputations is

$$A = \{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components. If x and y are in A and S is a nonempty subset of N, then $x \text{ dom}_S y$ means $\sum_{i \in S} x_i \leq v(S)$ and $x_i > y_i$ for all $i \in S$.

For $B \subset A$ let $\text{Dom}_S\ B = \{y \in A:$ there exists $x \in B$ such that $x\ \text{dom}_S\ y\}$ and let $\text{Dom}\ B = \bigcup_{S \subset N} \text{Dom}_S\ B$. A subset K of A is a solution if $K \cap \text{Dom}\ K = \varphi$ and $K \cup \text{Dom}\ K = A$. The core of a game is

$$C = \{\ x \in A: \sum_{i \in S} x_i \geq v(S) \quad \text{for all } S \subset N\}.$$

The core consists of those imputations which are maximal with respect to all of the relations dom_S, and hence it is contained in every solution.

3. EXAMPLE

Consider the game (N, v) where $N = \{1,2,3,4,5,6,7,8\}$ and where v is given by: $v(N) = 4$, $v(\{1,4,6,7\}) = 2$, $v(\{1,2\}) = v(\{3,4\}) = v(\{5,6\}) = v(\{7,8\}) = 1$, and $v(S) = 0$ for all other $S \subset N$. For this game

$$A = \{x: \sum_{i \in N} x_i = 4 \text{ and } x_i \geq 0 \text{ for all } i \in N\}$$

and

$$C = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1$$
$$\text{and } x_1 + x_4 + x_6 + x_7 \geq 2\}.$$

Also define the four-dimensional hypercube

$$H = \{x \in A: \quad x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\}.$$

Three traces of H as well as its 16 vertices are pictured in Fig. 1.

The unique solution for this game is

$$K = C \cup F_1 \cup F_4 \cup F_6 \cup F_7$$

where the cube F_i is the face of H given by

$$F_i = H \cap \{x: x_i = 1\} \qquad i = 1, 4, 6, 7.$$

Each $F_i - C$ is a tetrahedron with one face meeting C. In the three traces of H illustrated in Fig. 1, the traces of C are shown in heavy solid lines and the traces of the $F_i - C$ are shown in heavy broken lines.

The proof that K is the unique solution follows readily from two observations. First, K is just those imputations in H which are maximal in H with respect to the relation $\text{dom}_{\{1,4,6,7\}}$. Second, the closed line segment L joining the imputations (0,1,0,1,0,1,0,1) and (1,0,1,0,1,0,1,0) has the properties $L \subset C$ and $\text{Dom } L = A - H$.

To see that K is nonconvex, note the lower trace $F_8 = H \cap \{x: x_8 = 1\}$ in Fig. 1. The heavy lines (solid and broken) in this trace show $K \cap F_8$, which is clearly not convex. For example, the imputation $\frac{1}{3}(1,2,2,1,2,1,0,3) = \frac{1}{3}(0,1,1,0,0,1,0,1) + \frac{1}{3}(0,1,0,1,1,0,0,1) + \frac{1}{3}(1,0,1,0,1,0,0,1)$ is a linear combination of points in K, but it is not itself in K.

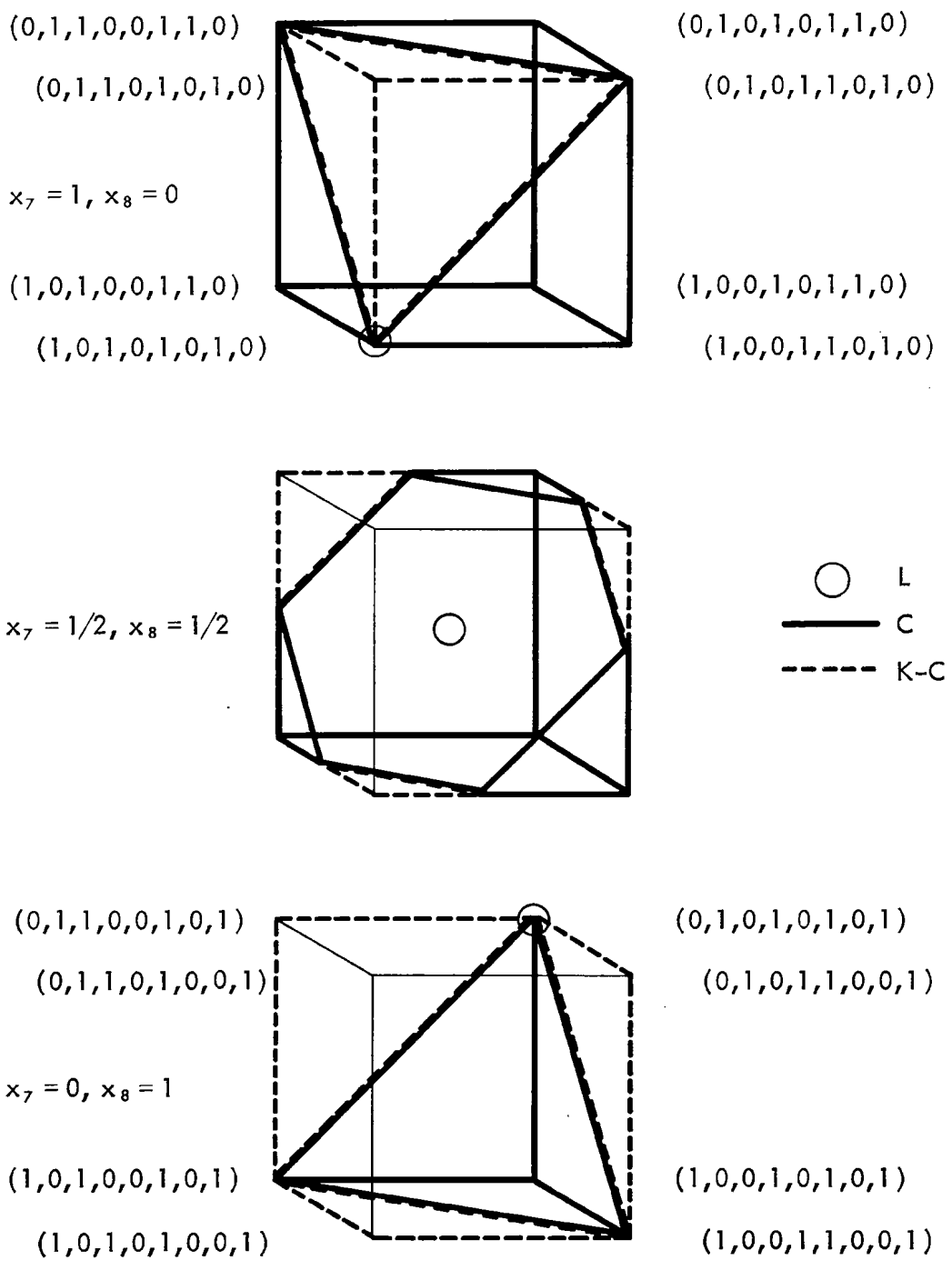

Fig. 1—Traces in H of L, C and K-C

4. REMARKS

The original von Neumann—Morgenstern theory assumed that the characteristic function of a game is superadditive, that is, $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \varphi$. This example can easily be made into a game with a super—additive characteristic function without changing the unique solution.

The essential idea in the example above is that Dom L = A − H. One can generalize this property to get many games in other dimensions which have a similar property. He can then introduce into these games addi—tional $S \subset N$ with $v(S) > 0$, but in such a way as to main—tain the corresponding L as a subset of the core. As a result he will obtain large classes of interesting solutions, many of which are unique and nonconvex.