A GAME WITH NO SOLUTION

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PREFACE

This Memorandum reports a theoretical result in the mathematical theory of n-person games. It is a product of a continuing study on game theory sponsored by Project RAND.

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SUMMARY

A solution concept for n-person games in characteristic function form was defined by von Neumann and Morgenstern, and it has been conjectured that every game has a solution.

This Memorandum states the definitions for an n-person game. It then describes a ten-person game which has no solution, thus providing a counterexample to the conjecture on existence.

A proof that this game has no solution is outlined.
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1. INTRODUCTION

In 1944 von Neumann and Morgenstern [2] introduced a theory of solutions for n-person games in characteristic function form. The main mathematical question concerning their model is whether every game has at least one solution. This announcement describes a ten-person game which has no solution. The essential definitions for an n-person game will be reviewed briefly before the particular example is given. The proof that the game has no solution will then be sketched; a detailed proof will be published elsewhere.

2. DEFINITIONS

An n-person game is a pair \((N, v)\) where \(N = \{1, 2, \ldots, n\}\) is the set of players and \(v\) is a characteristic function on \(2^N\), i.e., \(v\) assigns the real number \(v(S)\) to each subset \(S\) of \(N\) and \(v(\emptyset) = 0\). The set of imputations is
$$A = \{ x : \sum_{i \in N} x_i = v(N) \text{ and } x_i > v(i) \text{ for all } i \in N \}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components. For any $X \subseteq A$ and nonempty $S \subseteq N$, define $\text{Dom}_S X$ to be the set of all $x \in A$ such that there exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $\sum_{i \in S} y_i \leq v(S)$. Let $\text{Dom}^{-1} X = \bigcup_{S \subseteq N} \text{Dom}_S X$. Also let $\text{Dom}^{-1} X$ be the set of all $y \in A$ such that there exists $x \in X$ with $x \in \text{Dom} \{ y \}$. A subset $K$ of $A$ is a solution if $K \cap \text{Dom} K = \emptyset$ and $K \cup \text{Dom} K = A$.

If $X \subseteq A$ and $K' \subseteq X$, then $K'$ is a solution for $X$ if $K' \cap \text{Dom} K' = \emptyset$ and $K' \cup \text{Dom} K' \supseteq X$. The core of a game is

$$C = \{ x \in A : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \}.$$ 

For any solution $K$, $C \subseteq K$ and $K \cap \text{Dom} C = \emptyset$.

A characteristic function $v$ is superadditive if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \emptyset$. The game listed below does not have a superadditive $v$ as assumed in the classical theory. However, it is equivalent solutionwise to a game with a superadditive $v$.

(See Gillies [1: p. 68].)

3. **EXAMPLE**

Consider the game $(N, v)$ where $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

and $v$ is given by:
v(N) = 5, v({1, 3, 5, 7, 9}) = 4
v({1, 2}) = v({3, 4}) = v({5, 6}) = v({7, 8}) = v({9, 10}) = 1
v({3, 5, 7, 9}) = v({1, 5, 7, 9}) = v({1, 3, 7, 9}) = 3
v({3, 5, 7}) = v({1, 5, 7}) = v({1, 3, 7}) = 2
v({3, 5, 9}) = v({1, 5, 9}) = v({1, 3, 9}) = 2
v({1, 4, 7, 9}) = v({3, 6, 7, 9}) = v({5, 2, 7, 9}) = 2
v(S) = 0 for all other S ⊂ N.

For this game

A = {x: Σ x_i = 5 and x_i ≥ 0 for all i ∈ N}.

One can also show that C is the convex hull of the six imputations:
(1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0), (0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0),
(1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0), (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0),
and (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0).

4. OUTLINE OF PROOF

Consider the following subsets of A:

B = {x ∈ A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = x_9 + x_{10} = 1}

E_i = {x ∈ B: x_i = 1, x_j < 1, x_7 + x_9 < 1}

E = ∪ E_i

i

F = [ ∪ {x ∈ B: x_j = 1, x_7 + x_9 ≥ 1}

j,k

∪ ∪ {x ∈ B: x_p = 1, x_q < 1, x_3 + x_5 + x_q ≥ 2,

p,q

x_1 + x_5 + x_2 ≥ 2, x_1 + x_3 + x_4 ≥ 2}

∪ {x ∈ B: x_7 = x_9 = 1}

∪ {x ∈ B: x_1 = x_3 = x_5 = 1}] − C
where \((i, j, k) = (1, 3, 5), (3, 5, 1), \) and \((5, 1, 3); \) and \((p, q) = (7, 9) \) and \((9, 7)\). One can verify that the subsets \(A - B, B - (C \cup E \cup F), \) \(C, \) \(E, \) and \(F\) form a partition of \(A\).

To prove that this game has no solution it is sufficient to prove that

\[
\begin{align*}
(1) & \quad \text{Dom } C \supseteq [A - B] \cup [B - (C \cup E \cup F)], \\
(2) & \quad E \cap \text{Dom } (C \cup F) = \emptyset, \text{ and } \\
(3) & \quad \text{there is no solution for } E.
\end{align*}
\]

One can prove (1) and (2) by checking various subsets \(S\) of \(N\). In fact, one can prove in addition that

\[
\begin{align*}
\text{Dom } C & = A - (C \cup E \cup F), \text{ and } \\
F \cap \text{Dom } (C \cup E \cup F) & = \emptyset;
\end{align*}
\]

and thus \(C \cup F\) is contained in every solution.

Now consider the region \(E\). One can check that

\[
E_i \cap \text{Dom}_S E = \emptyset
\]

for all \(S\) except \([i, r, 7, 9]\), and

\[
E_i \cap \text{Dom}_{[i, r, 7, 9]} (E_i \cup E_k) = \emptyset
\]

where \((i, r, k) = (1, 4, 5), (3, 6, 1), \) and \((5, 2, 3)\). Thus the "Dom" pattern in \(E\) is cyclic as illustrated by the diagram:

\[
E_5 \ {3, 6, 7, 9} \ E_3 \ {1, 4, 7, 9} \ E_1 \ {5, 2, 7, 9} \ E_5
\]

To prove (3), assume that \(K' (\neq } \emptyset)\) is a solution for \(E\) and pick any \(y \in K'\). Using the symmetry in \(E\), one can assume \(y \in E_3\).
Define

\[ G_1(y) = \{ x \in E_1 : x_7 > y_7, \ x_9 > y_9, \ x_k + x_r + x_7 + x_9 \leq 2 \} \]

where \((i, k, r) = (1, 5, 2), (3, 1, 4), \text{ and } (5, 3, 6)\). Then one can verify that \(E \cap \text{Dom}^{-1} \{y\} = G_5(y)\), and so \(K' \cap G_5(y) = \emptyset\). However, \(E \cap \text{Dom}^{-1} G_1(y) = G_1(y)\), and so

\[ K' \cap G_1(y) \neq \emptyset. \]

On the other hand, \(G_3(y) \cap \text{Dom} (E_5 - G_5(y)) = \emptyset\), and so \(G_3(y) \subseteq K'\). However, \(G_1(y) \subseteq \text{Dom} G_3(y)\), and so

\[ K' \cap G_1(y) = \emptyset \]

which gives a contradiction. Therefore, there is no solution \(K'\) for \(E\).
REFERENCES

