MEMORANDUM
RM-5518-PR
NOVEMBER 1967

A GAME WITH NO SOLUTION

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is supported by the United States Air Force under Project RAND—Contract No. F44620-67-C-0045—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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Published by The RAND Corporation

PREFACE

This Memorandum reports a theoretical result in the mathematical theory of n-person games. It is a product of a continuing study on game theory sponsored by Project RAND.

The author is grateful to L. S. Shapley for many helpful discussions.

SUMMARY

A solution concept for n–person games in characteristic function form was defined by von Neumann and Morgenstern, and it has been conjectured that every game has a solution. This Memorandum states the definitions for an n–person game. It then describes a ten–person game which has no solution, thus providing a counterexample to the conjecture on existence. A proof that this game has no solution is outlined.

CONTENTS

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1. INTRODUCTION

In 1944 von Neumann and Morgenstern [2] introduced a theory of solutions for n–person games in characteristic function form. The main mathematical question concerning their model is whether every game has at least one solution. This announcement describes a ten–person game which has no solution. The essential definitions for an n–person game will be reviewed briefly before the particular example is given. The proof that the game has no solution will then be sketched; a detailed proof will be published elsewhere.

2. DEFINITIONS

An n–person game is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a characteristic function on 2^N, i. e., v assigns the real number $v(S)$ to each subset S of N and $v(\varphi) = 0$. The set of imputations is

$$A = \{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components. For any $X \subset A$ and nonempty $S \subset N$, define $\mathrm{Dom}_S X$ to be the set of all $x \in A$ such that there exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $\sum_{i \in S} y_i \leq v(S)$. Let $\mathrm{Dom}\, X = \bigcup_{S \subset N} \mathrm{Dom}_S X$. Also let $\mathrm{Dom}^{-1} X$ be the set of all $y \in A$ such that there exists $x \in X$ with $x \in \mathrm{Dom}\,\{y\}$. A subset K of A is a <u>solution</u> if $K \cap \mathrm{Dom}\, K = \varphi$ and $K \cup \mathrm{Dom}\, K = A$. If $X \subset A$ and $K' \subset X$, then K' is a <u>solution for</u> X if $K' \cap \mathrm{Dom}\, K' = \varphi$ and $K' \cup \mathrm{Dom}\, K' \supset X$. The <u>core</u> of a game is

$$C = \{x \in A: \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}.$$

For any solution K, $C \subset K$ and $K \cap \mathrm{Dom}\, C = \varphi$.

A characteristic function v is <u>superadditive</u> if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \varphi$. The game listed below does not have a superadditive v as assumed in the classical theory. However, it is equivalent solutionwise to a game with a superadditive v. (See Gillies [1: p. 68].)

3. EXAMPLE

Consider the game (N, v) where $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and v is given by:

$v(N) = 5, \ v(\{1, 3, 5, 7, 9\}) = 4$

$v(\{1, 2\}) = v(\{3, 4\}) = v(\{5, 6\}) = v(\{7, 8\}) = v(\{9, 10\}) = 1$

$v(\{3, 5, 7, 9\}) = v(\{1, 5, 7, 9\}) = v(\{1, 3, 7, 9\}) = 3$

$v(\{3, 5, 7\}) = v(\{1, 5, 7\}) = v(\{1, 3, 7\}) = 2$

$v(\{3, 5, 9\}) = v(\{1, 5, 9\}) = v(\{1, 3, 9\}) = 2$

$v(\{1, 4, 7, 9\}) = v(\{3, 6, 7, 9\}) = v(\{5, 2, 7, 9\}) = 2$

$v(S) = 0$ for all other $S \subset N$.

For this game

$$A = \{x: \sum_{i \in N} x_i = 5 \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

One can also show that C is the convex hull of the six imputations:

$(1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$, $(0, 1, 1, 0, 1, 0, 1, 0, 1, 0)$, $(1, 0, 0, 1, 1, 0, 1, 0, 1, 0)$,

$(1, 0, 1, 0, 0, 1, 1, 0, 1, 0)$, $(1, 0, 1, 0, 1, 0, 0, 1, 1, 0)$, and $(1, 0, 1, 0, 1, 0, 1, 0, 0, 1)$.

4. OUTLINE OF PROOF

Consider the following subsets of A:

$$B = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = x_9 + x_{10} = 1\}$$

$$E_i = \{x \in B: x_j = x_k = 1, \ x_i < 1, \ x_7 + x_9 < 1\}$$

$$E = \bigcup_i E_i \qquad i = 1, 3, 5$$

$$F = [\bigcup_{(j,k)} \{x \in B: x_j = x_k = 1, \ x_7 + x_9 \geq 1\}$$

$$\cup \bigcup_{(p,q)} \{x \in B: x_p = 1, \ x_q < 1, \ x_3 + x_5 + x_q \geq 2,$$

$$x_1 + x_5 + x_q \geq 2, \ x_1 + x_3 + x_q \geq 2\}$$

$$\cup \{x \in B: x_7 = x_9 = 1\}$$

$$\cup \{x \in B: x_1 = x_3 = x_5 = 1\}] - C$$

where $(i, j, k) = (1, 3, 5)$, $(3, 5, 1)$, and $(5, 1, 3)$; and $(p, q) = (7, 9)$ and $(9, 7)$. One can verify that the subsets $A-B$, $B-(C \cup E \cup F)$, C, E, and F form a partition of A.

To prove that this game has no solution it is sufficient to prove that

(1) $\quad \text{Dom } C \supset [A-B] \cup [B-(C \cup E \cup F)]$,

(2) $\quad E \cap \text{Dom}(C \cup F) = \varphi$, and

(3) $\quad$ there is no solution for E.

One can prove (1) and (2) by checking various subsets S of N. In fact, one can prove in addition that

$$\text{Dom } C = A - (C \cup E \cup F), \text{ and}$$

$$F \cap \text{Dom}(C \cup E \cup F) = \varphi ;$$

and thus $C \cup F$ is contained in every solution.

Now consider the region E. One can check that

$$E_i \cap \text{Dom}_S E = \varphi$$

for all S except $\{i, r, 7, 9\}$, and

$$E_i \cap \text{Dom}_{\{i, r, 7, 9\}} (E_i \cup E_k) = \varphi$$

where $(i, r, k) = (1, 4, 5)$, $(3, 6, 1)$, and $(5, 2, 3)$. Thus the "Dom" pattern in E is cyclic as illustrated by the diagram:

$$E_5 \xrightarrow{\{3, 6, 7, 9\}} E_3 \xrightarrow{\{1, 4, 7, 9\}} E_1 \xrightarrow{\{5, 2, 7, 9\}} E_5 .$$

To prove (3), assume that K' $(\neq \varphi)$ is a solution for E and pick any $y \in K'$. Using the symmetry in E, one can assume $y \in E_3$.

Define

$$G_i(y) = \{x \epsilon E_i : x_7 > y_7,\ x_9 > y_9,\ x_k + x_r + x_7 + x_9 \leq 2\}$$

where $(i, k, r) = (1, 5, 2)$, $(3, 1, 4)$, and $(5, 3, 6)$. Then one can verify that $E \cap \mathrm{Dom}^{-1}\{y\} = G_5(y)$, and so $K' \cap G_5(y) = \varphi$. However, $E \cap \mathrm{Dom}^{-1} G_5(y) = G_1(y)$, and so

$$K' \cap G_1(y) \neq \varphi.$$

On the other hand, $G_3(y) \cap \mathrm{Dom}\,(E_5 - G_5(y)) = \varphi$, and so $G_3(y) \subset K'$. However, $G_1(y) \subset \mathrm{Dom}\ G_3(y)$, and so

$$K' \cap G_1(y) = \varphi$$

which gives a contradiction. Therefore, there is no solution K' for E.

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