MEMORANDUM
RM-6150-PR
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EXTERNAL RADIATION FIELDS FOR ISOTROPICALLY SCATTERING FINITE ATMOSPHERES BOUNDED BY A LAMBERT LAW REFLECTOR

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PREFACE

The determination of diffusely transmitted and reflected radiation fields for an atmosphere of finite thickness is a basic problem in radiative transfer. This Memorandum presents methods of calculating these fields for a model planetary atmosphere, where the surface of the planet is assumed to scatter incident radiation according to Lambert's law of scattering. Scattering in the atmosphere is assumed to be isotropic, and curvature, polarization, and frequency dependence are not considered. The Memorandum also provides numerical results and checks.

SUMMARY

This Memorandum provides formulas for obtaining the diffusely transmitted and reflected radiation fields for a planetary, isotropically scattering atmosphere of finite thickness in terms of the solution to the problem with no planetary surface. Numerical results show that these reflected and transmitted fluxes are essentially the same whether isotropic or Rayleigh scattering laws are assumed.

CONTENTS

I. INTRODUCTION

A fundamental problem in radiative transfer studies of a finite atmosphere illuminated by sunlight is the determination of the intensities of the diffusely reflected and transmitted radiation (i.e., the radiation reaching the surface). In a recent paper [1], Kahle studied this problem for the case of a plane-parallel atmosphere bounded by a reflecting surface obeying a Lambert law of reflection. A Rayleigh law of scattering was assumed in the atmosphere, and reflection and transmission functions were obtained via the solution of the appropriate singular integral equations.

This Memorandum presents an alternate approach to a related planetary problem. Isotropic rather than Rayleigh scattering is assumed in the atmosphere. Aside from the intrinsic interest of the problem, a major goal of this work is to ascertain the differences in reflected and transmitted fluxes caused by these two scattering laws. This Memorandum's conclusion is that virtually no difference exists.

II. DERIVATION OF THE EQUATIONS

To derive an appropriate system of ordinary differential equations for the reflected and transmitted intensities in the case of a plane-parallel atmosphere, define the function

$r(v,u,x) = R(v,u,x)/4v$

= intensity of radiation reflected in a direction arc cos v due to parallel rays of incident radiation of net flux π in a direction arc cos u (with respect to the inward directed normal) for an isotropically scattering atmosphere of optical thickness x having no reflector at the bottom.

The functions $t(v,u,x)$ and $T(v,u,x)$ for the diffusely transmitted intensities are defined similarly. Also introduce the functions $r^*(v,u,x,A)$ and $t^*(v,u,x,A)$ to represent the reflected and transmitted intensities, respectively, for the case of an atmosphere bounded by a Lambert law reflector having albedo A.

Reference 2 shows that the functions R and T satisfy the system of differential equations:

$$\begin{aligned}\frac{d}{dx} R(v,u,x) = &- \left(\frac{1}{u} + \frac{1}{v}\right) R(v,u,x) \\ &+ \lambda \left[1 + \frac{1}{2}\int_0^1 R(v,u',x)\,du'/u'\right] \\ &\times \left[1 + \frac{1}{2}\int_0^1 R(v',u,x)\,dv'/v'\right] \\ &0 \le v,\ u \le 1 \\ &x > 0\ ,\end{aligned} \tag{1}$$

$$R(v,u,0) = 0 \ , \tag{2}$$

$$\frac{d}{dx} T(v,u,x) = -\frac{1}{v} T(v,u,x) \tag{3}$$

$$+ \lambda \left[1 + \frac{1}{2} \int_0^1 R(v,u',x)\, du'/u' \right]$$

$$\times \left[e^{-x/u} + \frac{1}{2} \int_0^1 T(v',u,x)\, dv'/v' \right] ,$$

$$T(v,u,0) = 0 \ . \tag{4}$$

The parameter λ represents the albedo for single scattering.

To obtain equations for r^* and t^*, let $I(u,x,A)$ be the constant intensity of radiation reflected from the bottom surface of albedo A, the incident direction being arc cos u and the optical thickness being x [3]. Then using the conservation law that at the bottom surface

(upward flux) = A · (downward flux) ,

the equation

$$\pi\, I(u,x,A) = A \left[\begin{array}{l} \pi u e^{-x/u} + \displaystyle\int_0^1 \frac{T(v',u,x)}{4v'}\, 2\pi v' dv' \\[2ex] + \displaystyle\int_0^1 I(u,x,A) \int_0^1 \frac{R(v',u',x)}{4v'}\, 2\pi 2\pi du' dv' \end{array} \right] \tag{5}$$

for the function I is obtained. Solving for I yields [illegible] 6150

$$I(u,x,A) = \frac{A\left[ue^{-x/u} + \frac{1}{2}\int_0^1 T(v',u,x)\,dv'\right]}{1 - A\int_0^1\int_0^1 R(v',u',x)\,du'dv'} \; . \tag{6}$$

Viewing the function I(u,x,A) as a new source of radiation incident on the bottom of the atmosphere leads to the relations

$$t^*(v,u,x,A) = T(v,u,x)/4v \tag{7}$$

$$+ \frac{I(u,x,A)}{\pi}\int_0^1 \frac{R(v,u',x)}{4v}\,2\pi du' \; ,$$

$$= T(v,u,x)/4v \tag{8}$$

$$+ \frac{1}{2}\, I(u,x,A)\int_0^1 R(v,u',x)\,du'/v \; ,$$

$$r^*(v,u,x,A) = R(v,u,x)/4v + I(u,x,A)e^{-x/v} \tag{9}$$

$$+ \frac{I(u,x,A)}{\pi}\int_0^1 \frac{T(v,u',x)}{4v}\,2\pi du' \; ,$$

$$= R(v,u,x)/4v \tag{10}$$

$$+ I(u,x,A)\left[e^{-x/v} + \frac{1}{2}\int_0^1 T(v,u',x)\,du'/v\right] .$$

The above equations clearly exhibit the fact that knowledge of the radiation fields for the nonplanetary problem is sufficient for obtaining the radiation fields for the planetary problem with a Lambert-law reflecting surface [3].

III. THE COMPUTATIONAL SCHEME

For the purposes of numerical calculation of the quantities $r^*(v,u,x,A)$, $t^*(v,u,x,A)$, and the reflected and transmitted fluxes, finite sums replace all integrals appearing in Eqs. (1), (3), (6), (7), and (9). If $z_1, z_2, \ldots, z_N$ represent the nodes and $w_1, w_2, \ldots, w_N$ the weights of an N point quadrature scheme, the above differential integral equations reduce to the system of ordinary differential equations

$$\frac{d}{dx} R_{ij}(x) = -\left(\frac{1}{z_i} + \frac{1}{z_j}\right) R_{ij}(x) + \lambda \left[1 + \frac{1}{2} \sum_{k=1}^{N} R_{ik}(x) w_k / z_k\right] \times \left[1 + \frac{1}{2} \sum_{k=1}^{N} R_{kj}(x) w_k / z_k\right], \tag{11}$$

$$R_{ij}(0) = 0 , \tag{12}$$

$$\frac{d}{dx} T_{ij}(x) = -\frac{1}{z_i} T_{ij}(x) + \lambda \left[1 + \frac{1}{2} \sum_{k=1}^{N} R_{ik}(x) w_k / z_k\right] \times \left[e^{-x/z_j} + \frac{1}{2} \sum_{k=1}^{N} T_{kj}(x) w_k / z_k\right], \tag{13}$$

$$T_{ij}(0) = 0 \,, \qquad\qquad i,j=1,2,\ldots,N \,, \quad x \geq 0 \,. \tag{14}$$

Here the convention

$$R_{ij}(x) = R(z_i,z_j,x) \,, \tag{15}$$

$$T_{ij}(x) = T(z_i,z_j,x) \,,$$

is used. Knowledge of the functions R_{ij} and T_{ij} allows calculation of the functions r^* and t^* via the formulas

$$\begin{aligned} r^*_{ij}(x,A) = {} & R_{ij}(x)/4z_i + I(z_j,x,A) \\ & \times \left[e^{-x/z_i} + \frac{1}{2} \sum_{k=1}^{N} T_{ik}(x) w_k/z_i \right] \,, \\ & x > 0,\ 0 \leq A \leq 1 \,, \end{aligned} \tag{16}$$

$$\begin{aligned} t^*_{ij}(x,A) = {} & T_{ij}(x)/4z_i \\ & + \frac{1}{2}\, I(z_j,x,A) \sum_{k=1}^{N} R_{ik}(x) w_k/z_i \,, \\ & x > 0,\ 0 \leq A \leq 1 \,. \end{aligned} \tag{17}$$

The quantity $I(z_j,x,A)$ is calculated by

$$I(z_j,x,A) = \frac{A\left[z_j e^{-x/z_j} + \frac{1}{2}\sum_{k=1}^{N} T_{kj}(x)w_k\right]}{1 - A\sum_{k=1}^{N}\sum_{m=1}^{N} R_{km}(x)w_k w_m} . \tag{18}$$

The reflected, transmitted, and global transmitted fluxes, defined by the equations

$$\rho(u,x,A) = 2\pi\int_0^1 r^*(z,u,x,A)z\,dz , \tag{19}$$

$$\tau(u,x,A) = 2\pi\int_0^1 t^*(z,u,x,A)z\,dz , \tag{20}$$

and

$$\tau_g(u,x,A) = \pi e^{-x/u} + \tau(u,x,A) \tag{21}$$

are computed by replacing the integrals with sums and by using the intensity functions $r^*_{ij}(x,A)$ and $t^*_{ij}(x,A)$ in the formulas

$$\rho(z_j,x,A) = 2\pi\sum_{i=1}^{N} r^*_{ij}(x,A)z_i w_i , \tag{22}$$

$$\tau(z_j,x,A) = 2\pi\sum_{i=1}^{N} t^*_{ij}(x,A)z_i w_i , \tag{23}$$

and

$$\tau_g(z_j,x,A) = \pi e^{-x/u_j} + \tau(u_j,x,A) \ . \qquad (24)$$

The following steps summarize the numerical procedure:

1) Integrate Eqs. (11) and (13) with the initial conditions of Eqs. (12) and (14) from $x = 0$ to $x = x_d$, the desired thickness.

2) At $x = x_d$, for a fixed value of A, compute $I(u,x,A)$ from Eq. (18).

3) Using the computed value of $I(u,x,A)$ and the solutions of step 1, calculate the reflected and transmitted intensities from Eqs. (16) and (17).

4) Calculate reflected and transmitted fluxes from Eqs. (21) and (22) using previously computed intensities of step 3.

Note: Since the parameter A occurs only in the calculation of $I(u,x,A)$, the set of intensities and fluxes for a range of A values may be obtained by recalculating only $I(u,x,A)$ for the desired values of A.

IV. NUMERICAL RESULTS

The basic numerical calculation consisted of producing the reflected and transmitted intensities and fluxes for surface albedos A = 0,.1,.2,...,1.0, optical thickness 0-100, and conservative, isotropic scattering ($\lambda = 1$). The integration step size used was $\Delta = .005$ with Gaussian quadrature of order N = 7. A check calculation involving changing the step size to $\Delta = .01$ and varying the order of the quadrature from N = 3 to N = 5 was run to a thickness of 1 resulting in, at most, a change of one unit on the fourth significant figure. All calculations were performed on a CDC 6600 computer using a fourth-order Adams-Moulton predictor-corrector integration scheme. Execution time for the basic calculations was about 7 min.

One primary objective of the computational study was to evaluate the differences, if any, in the reflected and transmitted intensities and fluxes when both Rayleigh and isotropic scattering laws were assumed. Reference 1 has examined the case of Rayleigh scattering by solving the problem via singular integral equations. Comparison of Figs. 1, 2, and 3 (depicting reflected diffusely transmitted, and global transmitted fluxes) with the analogous graphs in Kahle's paper leads to this conclusion: as far as reflected and transmitted fluxes are concerned, virtually no quantitative difference exists between the reflected and transmitted fluxes for the two scattering laws for any optical thickness in the range 0-100 and any surface albedo $0 \leq A \leq 1$ for normal incidence.

Tables 1 to 9 present the intensities and fluxes of the reflected and diffusely transmitted radiation for slab thicknesses .2, 10, and 100 and surface albedos A = 0, .5, and 1.0. These tables have been excerpted from the main calculations that output intensities and fluxes at 61 optical thicknesses in the range 0-100 for A = 0.0,0.1,0.2,...,1.0. The incident angle is constant across a row in the tables. It takes on one of eight different angles as indicated. The tables give intensities for seven outgoing angles, and list fluxes in the last column.

18.

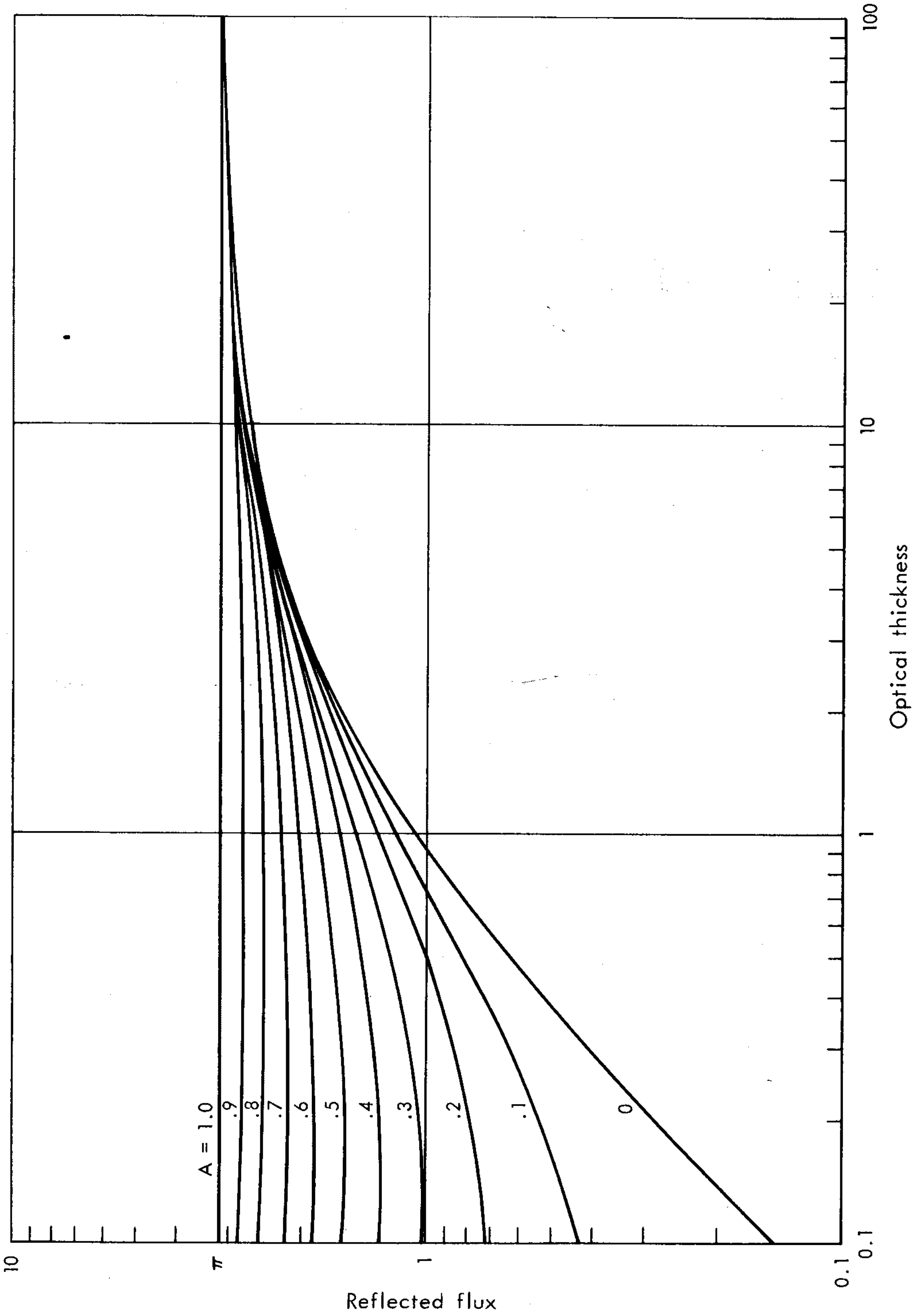

Fig. 1--Reflected Flux at Normal Incidence
for Various Values of A

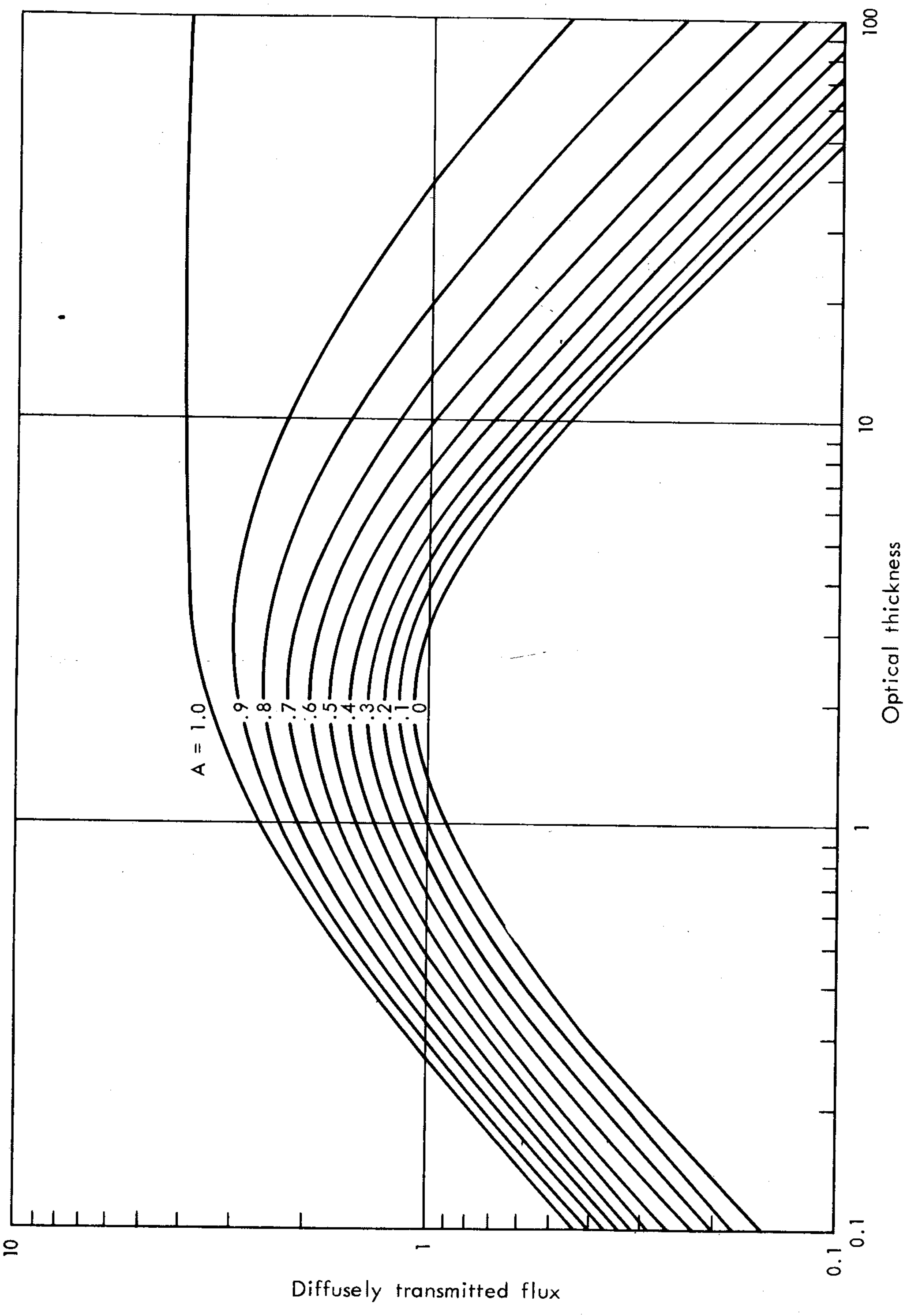

Fig. 2--Diffusely Transmitted Flux at Normal Incidence for Various Values of A

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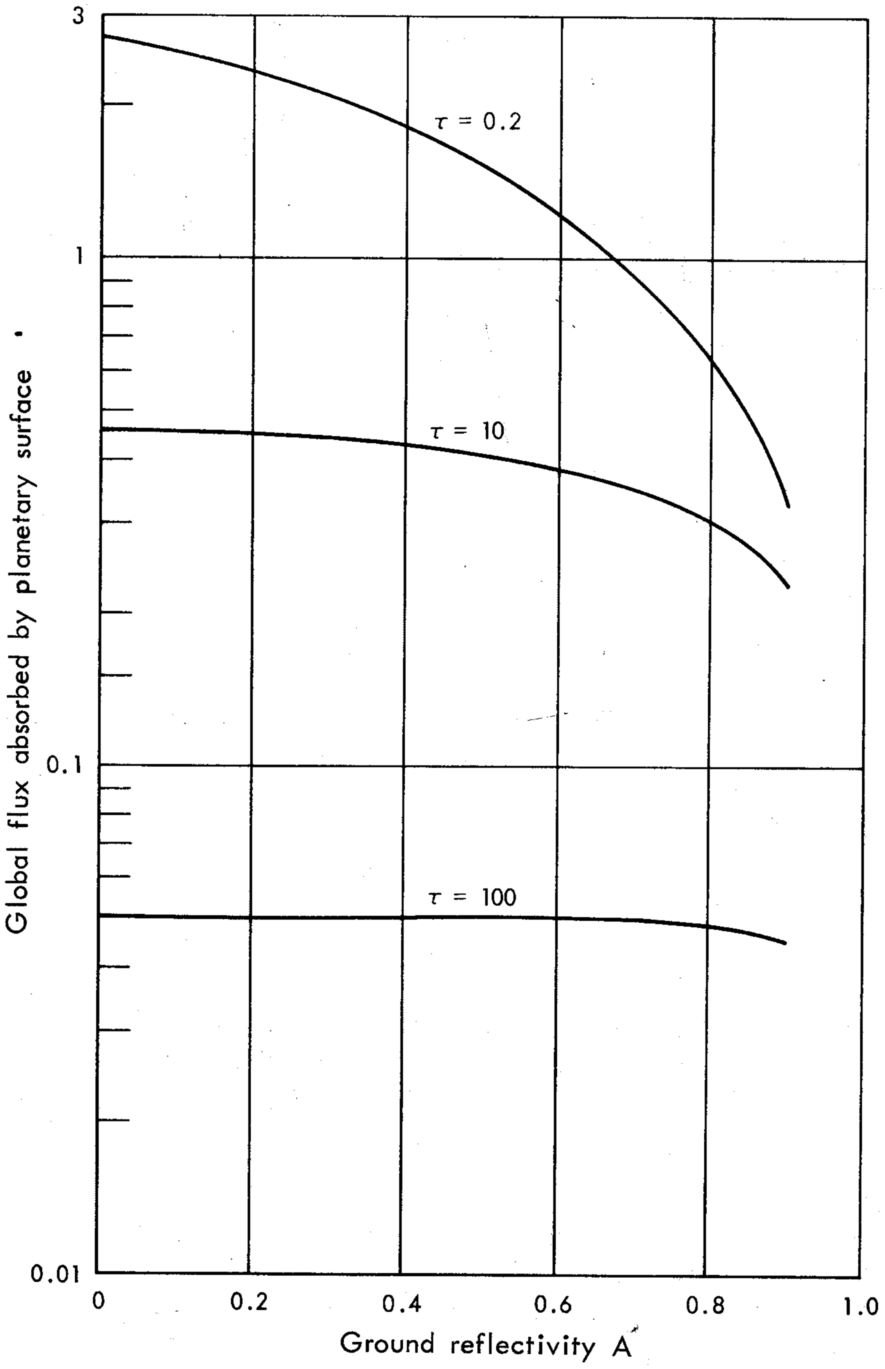

Fig. 3--Global Flux Absorbed versus Ground Reflectivity for Thickness τ = 0.2, 10, and 100, normal incidence

Table 1

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 0.2, AND SURFACE ALBEDO 0.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1394	.0503	.0251	.0156	.0114	.0093	.0083	.0463
2	(82.6)	.2555	.1556	.0897	.0588	.0437	.0360	.0325	.1682
3	(72.7)	.2929	.2062	.1240	.0825	.0616	.0510	.0460	.2340
4	(60.0)	.3073	.2276	.1388	.0928	.0695	.0575	.0519	.2626
5	(45.3)	.3139	.2377	.1459	.0977	.0732	.0606	.0548	.2762
6	(29.5)	.3171	.2427	.1494	.1001	.0751	.0622	.0562	.2830
7	(13.0)	.3186	.2450	.1510	.1012	.0759	.0629	.0568	.2861
Normal	(00.0)	.3189	.2455	.1513	.1015	.0761	.0630	.0570	.2867

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0078	.0202	.0168	.0123	.0096	.0081	.0074	.0336
2	(82.6)	.1027	.1157	.0786	.0544	.0413	.0344	.0312	.1514
3	(72.7)	.1956	.1806	.1169	.0796	.0601	.0499	.0452	.2233
4	(60.0)	.2416	.2103	.1340	.0908	.0684	.0568	.0514	.2553
5	(45.3)	.2645	.2247	.1422	.0962	.0724	.0601	.0543	.2707
6	(29.5)	.2761	.2319	.1464	.0989	.0744	.0617	.0558	.2784
7	(13.0)	.2815	.2352	.1483	.1002	.0753	.0625	.0565	.2820
Normal	(00.0)	.2827	.2360	.1487	.1004	.0755	.0627	.0567	.2827

Table 2

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 0.2, AND SURFACE ALBEDO 0.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1418	.0537	.0294	.0205	.0164	.0145	.0136	.0618
2	(82.6)	.2727	.1795	.1204	.0929	.0795	.0727	.0696	.2774
3	(72.7)	.3435	.2767	.2142	.1827	.1670	.1589	.1551	.5551
4	(60.0)	.4020	.3595	.3075	.2803	.2665	.2594	.2561	.8633
5	(45.3)	.4537	.4325	.3951	.3747	.3642	.3588	.3563	1.1634
6	(29.5)	.4946	.4900	.4657	.4517	.4445	.4407	.4389	1.4093
7	(13.0)	.5194	.5249	.5090	.4992	.4940	.4912	.4900	1.5607
Normal	(00.0)	.5255	.5333	.5195	.5108	.5061	.5036	.5025	1.5978

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0112	.0226	.0182	.0133	.0103	.0087	.0079	.0363
2	(82.6)	.1265	.1327	.0888	.0612	.0464	.0387	.0350	.1708
3	(72.7)	.2653	.2305	.1470	.0997	.0752	.0624	.0564	.2803
4	(60.0)	.3721	.3036	.1904	.1285	.0966	.0801	.0724	.3620
5	(45.3)	.4572	.3625	.2256	.1518	.1140	.0945	.0854	.4283
6	(29.5)	.5208	.4068	.2522	.1695	.1272	.1054	.0953	.4784
7	(13.0)	.5584	.4332	.2681	.1800	.1351	.1119	.1012	.5083
Normal	(00.0)	.5675	.4395	.2719	.1826	.1370	.1135	.1026	.5156

Table 3

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 0.2, AND SURFACE ALBEDO 1.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1447	.0577	.0345	.0261	.0224	.0206	.0197	.0799
2	(82.6)	.2930	.2078	.1565	.1331	.1217	.1159	.1133	.4060
3	(72.7)	.4031	.3598	.3204	.3008	.2910	.2860	.2837	.9333
4	(60.0)	.5135	.5148	.5062	.5012	.4985	.4971	.4965	1.5708
5	(45.3)	.6184	.6619	.6885	.7008	.7069	.7099	.7113	2.2083
6	(29.5)	.7036	.7812	.8382	.8658	.8795	.8864	.8896	2.7356
7	(13.0)	.7560	.8544	.9305	.9677	.9862	.9957	1.0000	3.0616
Normal	(00.0)	.7688	.8723	.9531	.9927	1.0124	1.0225	1.0271	3.1416

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0151	.0254	.0199	.0144	.0111	.0094	.0085	.0395
2	(82.6)	.1544	.1526	.1009	.0693	.0525	.0436	.0395	.1937
3	(72.7)	.3475	.2892	.1826	.1234	.0929	.0770	.0697	.3474
4	(60.0)	.5258	.4135	.2569	.1728	.1298	.1075	.0972	.4876
5	(45.3)	.6842	.5247	.3238	.2173	.1630	.1350	.1220	.6138
6	(29.5)	.8090	.6128	.3769	.2526	.1894	.1568	.1417	.7140
7	(13.0)	.8845	.6662	.4091	.2741	.2055	.1701	.1537	.7749
Normal	(00.0)	.9029	.6793	.4170	.2793	.2094	.1733	.1567	.7897

Table 4

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 10.0, AND SURFACE ALBEDO 0.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1441	.0571	.0338	.0250	.0209	.0189	.0179	.0756
2	(82.6)	.2901	.2092	.1563	.1278	.1121	.1033	.0990	.3791
3	(72.7)	.3941	.3592	.3175	.2856	.2639	.2503	.2432	.8561
4	(60.0)	.4915	.4944	.4807	.4621	.4452	.4326	.4254	1.4110
5	(45.3)	.5782	.6095	.6245	.6259	.6205	.6136	.6087	1.9427
6	(29.5)	.6451	.6962	.7338	.7535	.7601	.7601	.7585	2.3652
7	(13.0)	.6847	.7469	.7979	.8291	.8440	.8489	.8496	2.6186
Normal	(00.0)	.6943	.7590	.8133	.8474	.8642	.8704	.8717	2.6798

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0006	.0008	.0010	.0012	.0014	.0016	.0017	.0043
2	(82.6)	.0040	.0048	.0061	.0075	.0088	.0099	.0106	.0269
3	(72.7)	.0115	.0139	.0174	.0214	.0253	.0285	.0305	.0772
4	(60.0)	.0237	.0288	.0361	.0443	.0524	.0590	.0631	.1598
5	(45.3)	.0394	.0479	.0599	.0737	.0871	.0981	.1048	.2656
6	(29.5)	.0549	.0668	.0836	.1028	.1215	.1368	.1461	.3704
7	(13.0)	.0657	.0799	.1000	.1229	.1453	.1635	.1747	.4430
Normal	(00.0)	.0685	.0833	.1042	.1281	.1514	.1704	.1821	.4617

Table 5

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 10.0, AND SURFACE ALBEDO 0.5

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1441	.0572	.0339	.0251	.0211	.0190	.0181	.0760
2	(82.6)	.2905	.2097	.1569	.1286	.1130	.1044	.1002	.3820
3	(72.7)	.3953	.3607	.3194	.2878	.2666	.2533	.2464	.8641
4	(60.0)	.4940	.4974	.4844	.4667	.4507	.4388	.4320	1.4277
5	(45.3)	.5824	.6145	.6308	.6336	.6296	.6239	.6197	1.9705
6	(29.5)	.6509	.7032	.7425	.7642	.7728	.7744	.7738	2.4039
7	(13.0)	.6916	.7553	.8084	.8420	.8591	.8660	.8679	2.6649
Normal	(00.0)	.7014	.7678	.8242	.8607	.8800	.8882	.8908	2.7281

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0018	.0019	.0021	.0023	.0025	.0027	.0028	.0078
2	(82.6)	.0112	.0120	.0131	.0143	.0155	.0165	.0171	.0481
3	(72.7)	.0323	.0345	.0376	.0412	.0447	.0475	.0493	.1383
4	(60.0)	.0668	.0714	.0778	.0853	.0925	.0984	.1020	.2862
5	(45.3)	.1110	.1186	.1294	.1417	.1537	.1635	.1695	.4757
6	(29.5)	.1548	.1654	.1804	.1976	.2144	.2280	.2364	.6634
7	(13.0)	.1851	.1978	.2158	.2364	.2564	.2727	.2827	.7934
Normal	(00.0)	.1930	.2062	.2249	.2463	.2672	.2842	.2947	.8269

Table 6

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 10.0, AND SURFACE ALBEDO 1.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1447	.0579	.0347	.0262	.0224	.0205	.0196	.0799
2	(82.6)	.2941	.2140	.1623	.1352	.1209	.1132	.1096	.4060
3	(72.7)	.4056	.3732	.3350	.3070	.2893	.2789	.2737	.9333
4	(60.0)	.5152	.5232	.5167	.5064	.4976	.4917	.4885	1.5708
5	(45.3)	.6176	.6574	.6844	.6996	.7076	.7117	.7135	2.2083
6	(29.5)	.7001	.7630	.8174	.8562	.8816	.8969	.9046	2.7356
7	(13.0)	.7505	.8268	.8979	.9521	.9893	1.0125	1.0244	3.0617
Normal	(00.0)	.7628	.8423	.9175	.9755	1.0157	1.0409	1.0540	3.1416

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0119	.0119	.0119	.0119	.0119	.0119	.0119	.0372
2	(82.6)	.0732	.0732	.0732	.0732	.0732	.0732	.0732	.2300
3	(72.7)	.2106	.2106	.2106	.2106	.2106	.2106	.2106	.6616
4	(60.0)	.4358	.4358	.4358	.4358	.4358	.4358	.4358	1.3690
5	(45.3)	.7242	.7242	.7242	.7242	.7242	.7242	.7242	2.2751
6	(29.5)	1.0100	1.0100	1.0100	1.0100	1.0100	1.0100	1.0100	3.1730
7	(13.0)	1.2081	1.2081	1.2081	1.2081	1.2081	1.2080	1.2080	3.7952
Normal	(00.0)	1.2591	1.2591	1.2591	1.2591	1.2591	1.2590	1.2590	3.9555

Table 7

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 100.0, AND SURFACE ALBEDO 0.0

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1446	.0578	.0346	.0261	.0222	.0203	.0194	.0795
2	(82.6)	.2936	.2135	.1616	.1344	.1199	.1121	.1084	.4030
3	(72.7)	.4043	.3716	.3330	.3046	.2864	.2757	.2703	.9246
4	(60.0)	.5125	.5199	.5127	.5014	.4917	.4850	.4814	1.5528
5	(45.3)	.6132	.6520	.6777	.6913	.6978	.7006	.7017	2.1784
6	(29.5)	.6939	.7555	.8080	.8447	.8679	.8815	.8882	2.6939
7	(13.0)	.7431	.8178	.8866	.9382	.9729	.9940	1.0047	3.0117
Normal	(00.0)	.7550	.8329	.9057	.9610	.9986	1.0217	1.0334	3.0896

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0005
2	(82.6)	.0004	.0005	.0007	.0008	.0010	.0011	.0012	.0030
3	(72.7)	.0013	.0016	.0020	.0024	.0029	.0032	.0034	.0087
4	(60.0)	.0027	.0032	.0041	.0050	.0059	.0066	.0071	.0180
5	(45.3)	.0044	.0054	.0067	.0083	.0098	.0110	.0118	.0299
6	(29.5)	.0062	.0075	.0094	.0116	.0137	.0154	.0165	.0417
7	(13.0)	.0074	.0090	.0113	.0138	.0138	.0184	.0197	.0499
Normal	(00.0)	.0077	.0094	.9117	.0144	.0171	.0192	.0205	.0520

Table 8

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 100.0, AND SURFACE ALBEDO 0.5

REFLECTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Reflected Flux
1	(88.5)	.1446	.0578	.0346	.0261	.0222	.0203	.0194	.0795
2	(82.6)	.2936	.2135	.1617	.1344	.1199	.1121	.1085	.4030
3	(72.7	.4043	.3716	.3330	.3046	.2864	.2757	.2703	.9247
4	(60.0)	.5126	.5200	.5127	.5015	.4918	.4851	.4815	1.5530
5	(45.3)	.6132	.6521	.6778	.6914	.6979	.7008	.7019	2.1788
6	(29.5)	.6940	.7556	.8081	.8448	.8681	.8817	.8884	2.6944
7	(13.0)	.7432	.8179	.8868	.9384	.9731	.9943	1.0050	3.0124
Normal	(00.0)	.7551	.8331	.9059	.9612	.9988	1.0219	1.0337	3.0902

TRANSMITTED INTENSITIES

Inc. Angle	(Deg.)	1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	Transmitted Flux
1	(88.5)	.0002	.0002	.0003	.0003	.0003	.0003	.0003	.0010
2	(82.6)	.0014	.0015	.0016	.0018	.0019	.0021	.0021	.0060
3	(72.7)	.0040	.0043	.0047	.0051	.0055	.0059	.0061	.0172
4	(60.0)	.0083	.0089	.0097	.0106	.0115	.0122	.0127	.0355
5	(45.3)	.0138	.0147	.0161	.0176	.0191	.0203	.0210	.0590
6	(29.5)	.0192	.0205	.0224	.0245	.0266	.0283	.0293	.0823
7	(13.0	.0230	.0246	.0268	.0293	.0318	.0339	.0351	.0985
Normal	(00.0)	.0239	.0256	.0279	.0306	.0332	.0353	.0366	.1026

Table 9

REFLECTED AND DIFFUSELY TRANSMITTED INTENSITIES AND FLUXES FOR A SLAB WITH CONSERVATIVE ISOTROPIC SCATTERING, THICKNESS 100.0, AND SURFACE ALBEDO 1.0

Inc. Angle	(Deg.)	REFLECTED INTENSITIES							Reflected Flux
		1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	
1	(88.5)	.1447	.0579	.0347	.0262	.0224	.0205	.0196	.0799
2	(82.6)	.2941	.2140	.1623	.1352	.1209	.1132	.1096	.4060
3	(72.7)	.4056	.3732	.3350	.3070	.2893	.2789	.2737	.9333
4	(60.0)	.5152	.5232	.5167	.5064	.4976	.4917	.4885	1.5708
5	(45.3)	.6176	.6574	.6844	.6996	.7076	.7117	.7135	2.2083
6	(29.5)	.7001	.7630	.8174	.8562	.8816	.8969	.9046	2.7356
7	(13.0)	.7505	.8268	.8979	.9521	.9892	1.0125	1.0244	3.0616
Normal	(00.0)	.7628	.8423	.9175	.9755	1.0156	1.0409	1.0539	3.1416

Inc. Angle	(Deg.)	TRANSMITTED INTENSITIES							Transmitted Flux
		1(88.5)	2(82.6)	3(72.7)	4(60.0)	5(45.3)	6(29.5)	7(13.0)	
1	(88.5)	.0119	.0119	.0119	.0119	.0119	.0119	.0119	.0372
2	(82.6)	.0732	.0732	.0732	.0732	.0732	.0732	.0732	.2300
3.	(72.7)	.2105	.2105	.2105	.2105	.2105	.2105	.2105	.6613
4	(60.0)	.4356	.4356	.4356	.4356	.4356	.4356	.4356	1.3686
5	(45.3)	.7239	.7239	.7239	.7239	.7239	.7239	.7239	2.2743
6	(29.5)	1.0096	1.0096	1.0096	1.0096	1.0096	1.0096	1.0096	3.1719
7	(13.0)	1.2077	1.2077	1.2077	1.2077	1.2077	1.2077	1.2077	3.7940
Normal	(00.0)	1.2587	1.2587	1.2587	1.2587	1.2587	1.2587	1.2587	3.9542

V. COMPUTATIONAL CHECKS

Checks on the numerical results may be divided into two categories: 1) internal checks on the consistency of the numerical scheme; and 2) external checks using other sources and independent methods.

As mentioned previously, the internal checks consist of changing the integration step size from $\Delta = .005$ to $\Delta = .01$ and varying the order of the quadrature scheme from $N = 7$ to $N = 3$ and 5. In all cases, the optical thickness ranged from 0 to 1; and the results changed by no more than one unit in the fourth significant figure.

The first external check considered is restricted to the case $A = 0$. Reference 2 presents tables similar to Tables 1 through 9 for optical thicknesses 1-50. In all cases, the current results agree perfectly with those earlier calculations.

A second check, for the case $A = 1$, normal incidence, is the conservation relation that the reflected flux must equal the incident flux. At normal incidence, the reflected flux is 3.1416 and at an angle of incidence of 60° it is $1.5708 \cong \pi/2$. As Tables 3, 6, and 9 exhibit, this is the case for all optical thicknesses.

An independent check is the calculation of reflected and transmitted intensities by generalized X and Y functions as outlined in Ref. 3. Both the cases $u = 0.0$, $v = \cos 88.5^\circ$, and $u = 0.5$, $v = \cos 88.5^\circ$ for $A = .8$ and optical thickness $x = 0.8$ have been computed by this method giving results that agree exactly with those calculated by the initial-value procedure.

The final check compares qualitatively the results of the initial-value method to Kahle's results [1]. Even though Kahle assumed Rayleigh scattering, both sets of calculations should be qualitatively the same. Comparing Figs. 1 through 3 to the analogous figures in Kahle's paper bears out this conjecture.

An interesting property of the solution of the planetary problem with a Lambert-law reflector is obtained by comparing this solution with that presented in Ref. 4 for the case when the reflecting surface is a perfect specular reflector. It is observed for the conservative case $\lambda = 1$, $A = 1$ that the transmitted and reflected fluxes are virtually the same (to three figures) for all optical thicknesses ≥ 3. Thus, even the properties of the reflecting surface have little effect on fluxes for moderately thick media.

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