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ON A GENERAL CLASS OF PROBLEMS INVOLVING SEQUENTIAL ANALYSIS

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Summary

The purpose of this paper is to delineate a general, important class of multi-stage problems occurring in widely different fields, such as strategic bombing, transportation routing, and information testing.

§1. Introduction.

We wish to discuss a general class of multi-stage problems involving a sequence of operations between each of which information is acquired which can be used to guide the subsequent operations. This class of problems is characterized by the fact that at each time the problem may be described by a set of parameters which change from operation to operation, which is to say that each operation performs a mapping of the parameter space upon itself, and secondly, that the purpose of the operations is to optimize according to a criterion which has the important property that after any initial number of operations, starting from the state one finds oneself in, one optimizes according to the same criterion.

The importance of this last point is two-fold. It allows a description of the optimum sequences of operations to be given very simply in terms of best first moves, and it allows a mathematical formulation by means of recurrence relations which are very useful both theoretically and computationally.

This investigation may be subsumed under the general topic of multi-move games. At the moment, we consider not two-person games, but one-person games of the man-vs.-nature type.

The above comments have been necessarily vague. In place of precising them by using an abstract formulation, let us illustrate the wide scope and importance of this class of problems by stating four related problems occurring in different settings.

§2. A strategic bombing model.

The general problem is that of determining the maximum expected damage that can be inflicted upon T targets by a force of B bombers supplied with b bombs. The probability of success against a given target depends upon the defenses of the target, the number of planes allocated to attack the target and the total number of planes sent out on a mission. The probability of the return of a plane depends upon the targets attacked and the number of planes sent out. The bombers are to be used until the targets are destroyed or the supply of bombs or bombers is depleted.

In this case the problem is to determine on each mission which targets are to be attacked and with how many planes. The state of the system under consideration is specified by the values of the targets, the à priori probabilities of their existence, the probabilities of success and return when attacking the targets, and the supply of bombs and bombers at any time.

To illustrate the mathematical technique used, let us consider the simplest case where there is one bomber, an unlimited supply of bombs and only two targets described by the probabilities:

(1) π_i = à priori probability of existence

v_i = value of target

p_i = probability of plane returning from an attack against the i-th target

s_i = à priori probability target is destroyed if plane returns.*

*Consider that one is consistently bombing these two targets in overcast weather.

Let us assume that all other probabilities are zero and that a plane may attack only one target at a time. An optimal sequence of operations is one yielding the maximum expected damage, which, assuming all other parameters are taken to be constants, may be written $f(\pi_1, \pi_2)$. It is clear that under the previous assumptions only π_1, π_2 change from stage to stage.

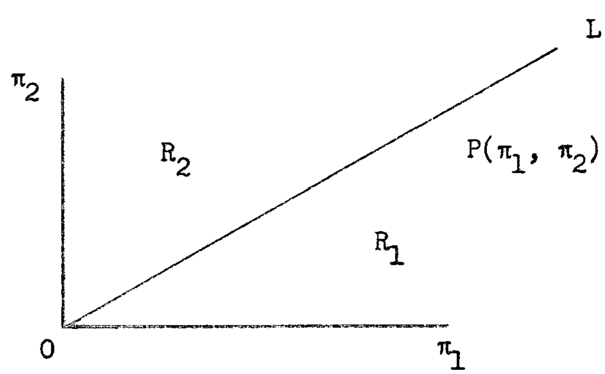
We may describe this optimal sequence either in terms of the complete set of operations -- which is not practicable since it may be non-enumerable in some cases -- or in terms of best first moves.

It is easily seen that whatever the best first move is, the subsequent moves must be optimal as far as the new values of π_1 and π_2 are concerned. The mathematical formulation of this statement is

(2) $$f(\pi_1, \pi_2) = \operatorname*{Max}_{i} \begin{Bmatrix} p_1\left(s_1\pi_1 v_1 + f\left((1 - s_1)\pi_1, \pi_2\right)\right) \\ p_2\left(s_2\pi_2 v_2 + f\left(\pi_1, (1 - s_2)\pi_2\right)\right) \end{Bmatrix} .$$

From this recurrence relation one can determine the optimal procedure and $f(\pi_1, \pi_2)$ itself in many cases. For example, in the simple case if the positive π_1, π_2 quadrant is divided into two regions

(3)



R_1 and R_2 by the line

$$(4) \qquad L: \qquad \frac{\pi_1 v_1 p_1 s_1}{1 - p_1} = \frac{\pi_2 v_2 p_2 s_2}{1 - p_2} ,$$

then if $P(\pi_1, \pi_2) \in R_1$, target one is always attacked first, and target two is always attacked first in R_2.

The more general problem is much more complicated, but preserves many of the features of this simple case. A complete solution has not yet been given, although many important special cases have been treated.

§3. A transportation problem.

Suppose that we have n ports, $A_1, A_2, \cdots, A_n$, with the i-th port having a_i ships available initially and a_{ij}, $j = 1, 2, \cdots, n$, $a_{ii} = 0$, shiploads to be sent to the j-th port. How does one route the ships so as to minimize the total transit time?

For the sake of simplicity, consider the case of three equi-distant ports, A_1, A_2, A_3, where no time is lost in loading or unloading.

The system is described by the nine parameters

$$(1) \qquad \begin{array}{l} A_1(a_1, a_{12}, a_{13}) \\ A_2(a_2, a_{21}, a_{23}) \\ A_3(a_3, a_{31}, a_{32}) \ . \end{array}$$

It is clear that minimum time is a criterion having the desired general property. Consequently, if

$$(2) \qquad f = f(a_1, a_{12}, a_{13}, a_2, a_{21}, a_{23}, a_3, a_{31}, a_{32})$$

$$= \text{time consumed using optimal procedure},$$

we obtain the recurrence relation

$$(3) \qquad f(a_1, a_{12}, a_{13}, a_2, a_{21}, a_{23}, a_3, a_{31}, a_{32}) = 1$$
$$+ \operatorname*{Min}_{x}\Big(f(x_{21}+x_{31}, a_{12}-x_{12}, a_{13}-x_{13}, x_{12}+x_{32},$$
$$a_{21}-x_{21}, a_{23}-x_{23}, x_{13}+x_{23}, a_{31}-x_{31}, a_{32}-x_{32}\Big),$$

where x_{ij} is the number of ships sent from A_i to A_j, $i \neq j$, at the first stage, and we have the constraints

$$(4) \qquad \begin{aligned} x_{12} + x_{13} &= a_1 \\ x_{21} + x_{23} &= a_2 \\ x_{31} + x_{32} &= a_3 \end{aligned} \quad .$$

Using (3) we can determine f recursively.

§4. A missiles model.

A model very similar to that conceived for strategic bombing may be constructed for the missiles problem with the very important difference that there is no return of a missile.

It is clear that again the maximum expected damage will be a criterion of the desired type, and that the parameters specifying the system will undergo a change after every bombardment. Between bombardments, reconnaissance is performed.

§5. An information pattern.

Let us consider the following type of problem which arises in experimental work. Given T objects to be tested and A pieces of apparatus which may be used singly or in conjunction to obtain information concerning the T objects. These probabilities of obtaining information depend upon the objects tested and the number of testing devices used. The probability of a piece of apparatus being rendered defective by the test depends upon the objects tested and the total number of devices used at any time.

It is clear that this problem is a rewording of the strategic bombing model discussed previously so that there is no need to discuss the mathematical procedure any further.

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