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NOTES ON THE  $n$ -PERSON GAME -- I:  
CHARACTERISTIC-POINT SOLUTIONS  
OF THE FOUR-PERSON GAME

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Summary: A family of solutions covering all zero-sum four-person games is presented which behaves continuously under perturbation of the characteristic function. The solutions are typically 1-dimensional; they consist either of at most four line-segments joined at a point (the "characteristic point"), or of three disconnected segments or points.

## NOTES ON THE n-PERSON GAME I

### CHARACTERISTIC-POINT SOLUTIONS OF THE FOUR-PERSON GAME

L. S. Shapley

#### §1. Introduction

The question of solutions for all zero-sum three-person games has been completely settled by von Neumann and Morgenstern in Theory of Games and Economic Behavior (TGEB) Chapter V. For the four-person case the results they obtain are scattered (Ch VII, see p. 304): they do not include any solutions for large classes of games, and give all the solutions in only a few isolated cases.<sup>1</sup> Most of the known solutions consist of finite sets of imputations, and behave very irregularly under perturbation of the characteristic function. The solutions are typically 1-dimensional; they consist either of at most four line segments joined at a point (the "characteristic point"), or of three disconnected segments or points.

The "characteristic point" treatment can be extended to certain restricted classes of games with more players; we shall describe this extension in a later note.

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<sup>1</sup> At a later point in TGEB (p.555), the possibility of producing solutions to any four-person constant-sum game by means of the three-person general-sum theory, is suggested but not carried out.

## 2. Basic definitions and notations.

We give an unadorned review of the basic definitions, referring the reader to TGEB for elucidation if he requires it. An essential, zero-sum,  $n$ -person game in reduced form is represented by its characteristic function  $v(S)$ , a real-valued set-function defined on the subsets  $S$  of

$$I = \{1, 2, \dots, n\},$$

satisfying:

$$(C1) \quad v(S) + v(I-S) = v(I) = 0 \quad \text{for all } S \subseteq I,$$

$$(C2) \quad v(T \cup S) + v(T \cap (I-S)) \leq v(T) \quad \text{for all } S, T \subseteq I,$$

$$(C3) \quad v(\{i\}) = -1 \quad \text{for } i = 1, 2, \dots, n.$$

Conversely, any function satisfying (C1) - (C3) is the characteristic function of some game in reduced form.<sup>1</sup>

We let  $A$  denote the set of  $n$ -tuples ("imputations")<sup>2</sup>

$$A = \{ \langle x_1, x_2, \dots, x_n \rangle \}$$

which satisfy

$$(I1) \quad \sum_{i=1}^n x_i = 0$$

$$(I2) \quad x_i \geq -1 \quad \text{for } i = 1, 2, \dots, n.$$

We say that  $x$   $S$ -dominates  $y$  :

$x \succ_S y$

<sup>1</sup> In many respects, the normalization,  $v(\{i\}) = 0$ ,  $v(I) = 1$  is more convenient. But we shall stick to the form used in TGEB.

<sup>2</sup> Geometrically they form a simplex.

provided that

(D1)  $\alpha, \beta \in A$

(D2)  $\alpha_i > \beta_i, \quad \text{all } i \in S$

(D3)  $\frac{\sum_{i \in S} \alpha_i}{|S|} \leq v(S)$

S-domination cannot occur unless S has more than one element and less than n elements.

The dominion of  $\alpha$  is the set of  $\beta$  S-dominated by  $\alpha$  for some  $S \subseteq I$ ; we write "dom  $\alpha$ ".<sup>1</sup> If K is any subset of A, we define

$$\text{dom } K = \bigcup_{\alpha \in K} \text{dom } \alpha .$$

Finally, we say that K is a solution of the game represented by v(S) if and only if

$$\text{dom } K = A - K .$$

Following are some easily established properties of a solution K :

(S1) If  $\alpha, \beta \in K$ , then never  $\beta \succ_S \alpha$  (internal stability).

(S2) If  $\alpha \in A - K$ , then for some  $\beta \in K, S \subseteq I, \beta \succ_S \alpha$  (external stability).

Conditions (S1) and (S2) together are necessary and sufficient for a set K to be a solution. Further properties of a solution K :

<sup>1</sup> Geometrically dom  $\alpha$  is an open, polyhedral cone (not convex) with vertex at  $\alpha$ , intersected with the simplex A.

- (S3)  $K$  is closed.  
 (S4)  $K$  has more than one element.  
 (S5) Every  $\alpha$  in  $K$  is dominated:  $K \subseteq \text{dom}(A-K)$ .

### §3. The characteristic point.

If we specialize to  $n = 4$ , the conditions (C1) - (C3) leave three degrees of freedom in constructing characteristic functions. The possibilities correspond directly to the points of a cube  $Q$  :

$$Q: \begin{cases} -2 \leq v_{14} \leq 2 \\ -2 \leq v_{24} \leq 2 \\ -2 \leq v_{34} \leq 2 \end{cases}$$

where we have written  $v_{ij}$  for  $v(\{i,j\})$ . The remaining values of the characteristic function are then determined by

$$\begin{cases} v_i = -1, \\ v_{ij} = -v_{kl}, \quad (i,j,k,l \text{ all distinct}). \\ v_{ijk} = 1 \end{cases}$$

We define a certain characteristic point for the four-person game:

$$w = \langle w_1, w_2, w_3, w_4 \rangle$$

by

$$(W1) \quad w_i = \frac{1}{2} \sum_{j \neq i} v_{ij}, \quad i = 1, 2, 3, 4.$$

Two simple properties of  $w$  are

$$(W2) \quad w_i + w_j = v_{ij}, \quad i \neq j,$$

$$(W3) \quad \sum_{i=1}^4 w_i = 0.$$

Our subsequent considerations will all be based on this characteristic point. There is no guarantee that  $w$  belongs to  $A$ , — that is, that

$$w_i \geq -1$$

(Condition (I2)). In fact, it is not hard to verify that  $w$  falls in  $A$  if and only if the point  $\square$  of the cube  $Q$  corresponding to the game in question lies in a certain tetrahedron  $T_0$ ,<sup>1</sup> having the vertices:

$$\begin{aligned} (v_{14}, v_{24}, v_{34}) = & (+2, -2, -2) \\ & (-2, +2, -2) \\ & (-2, -2, +2) \\ & (+2, +2, +2)^2. \end{aligned}$$

The set  $Q - T_0$  consists of four other tetrahedra (open on one face), which we may call  $T_1, T_2, T_3, T_4$ . The constituents of  $T_1$  are games in which the first player is at a definite disadvantage: the defining condition is

$$T_1: v_{12} + v_{13} + v_{14} < -2;$$

<sup>1</sup> In fact, the position of  $\square$  in  $T_0$  is exactly the position of  $w$  in  $A$ , if the two tetrahedra are put into correspondence (linearly) with  $\alpha = \langle 3, -1, -1, -1 \rangle$  going into  $(2, -2, -2)$ , etc.

<sup>2</sup> In TGEB (page 295) these vertices are denoted V, VI, VII, and I respectively.

and the vertices are  $(-2, 2, 2)^1$  and the last three of the four listed above. The other three tetrahedra are similar, the general defining condition being

$$T_j: \sum_{i \neq j} v_{ji} < -2 .$$

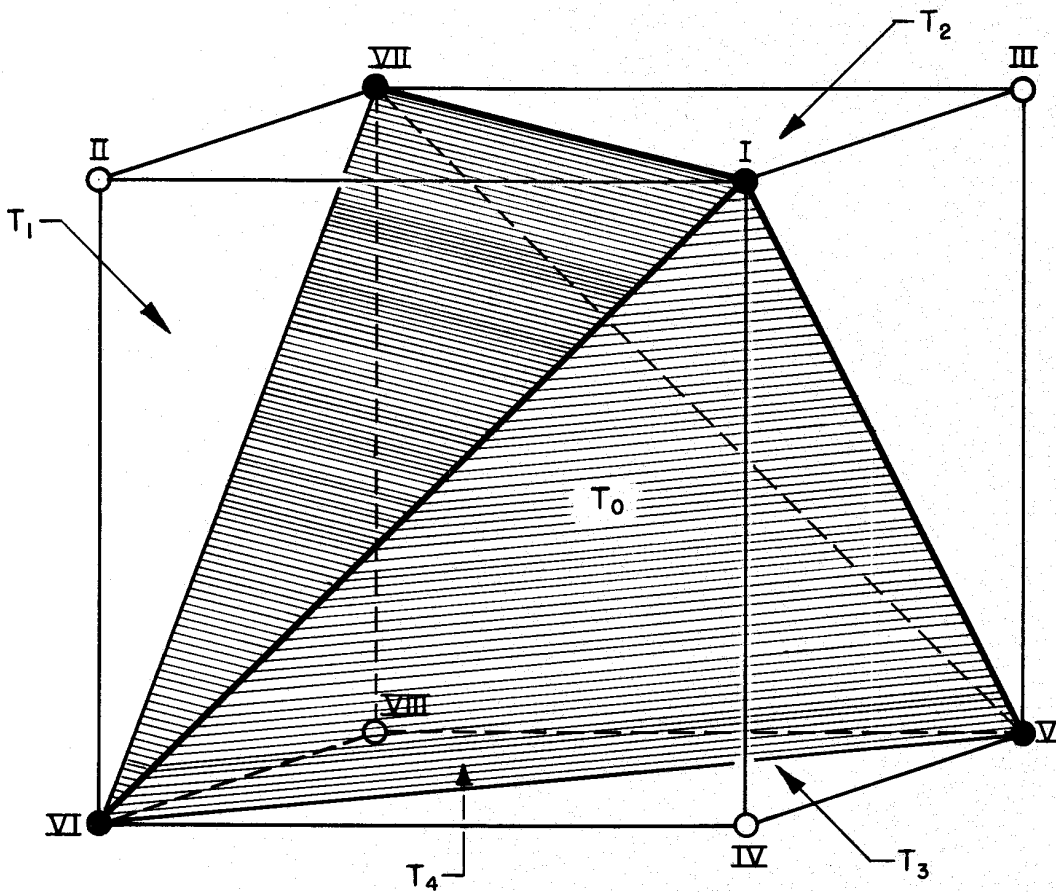


Fig. I — The cube Q

<sup>1</sup> Vertex II in TGEB (loc.cit). The figure has been drawn to correspond with Figures 61 - 63 of TGEB.

From this we can derive:

$$w \text{ in } T_j \iff w_j < -1, \text{ and } w_i > -1, i \neq j.$$

We conclude that when  $w$  fails to fall in  $A$ , one and only one component of the characteristic point violates condition (I2).

§4.  $w$ -Solutions, case I:  $w \in A$ .

Consider the six lines through  $w$  parallel to the edges of the simplex  $A$ . There are in all twelve segments  $L_{ij}$ ,  $i \neq j$ , with endpoints  $w$  and  $\beta^{(ij)}$ , where

$$\begin{cases} \beta_i^{(ij)} = -1 \\ \beta_j^{(ij)} = w_j + w_i + 1 \\ \beta_k^{(ij)} = w_k, & k \neq i, j. \end{cases}$$

$L_{ij}$  and  $L_{ji}$  are colinear, on opposite sides of  $w$ . Each  $L_{ij}$  extends to the face on which  $\alpha_i = -1$ . If  $w_i = -1$ , then the three segments  $L_{ij}$  collapse onto the point  $w$ .

THEOREM 1. If  $w$  is in  $A$ , then a solution of the game is

$$K = \bigcup_{i=1}^4 L_{ij_i},$$

where the  $j_i$  are arbitrary, subject to  $j_i \neq i$ .

This gives, if  $w$  is interior to  $A$ , a total of 81 different " $w$ -solutions" to the game.



PROOF. We first show that every  $\alpha \in A$  is either in  $K$  or in  $\text{dom } K$  (condition (S2)). Suppose that the relation

$$(R) \quad \alpha_i < w_i$$

holds for no  $i$ . Then, by (W3) and (I1) we immediately have  $\alpha = w$ . Hence  $\alpha$  is in  $K$ . Next suppose that (R) holds for just one  $i$ . Then we have

$$\alpha_i + \alpha_{j_i} = -\alpha_k - \alpha_l \leq -w_k = w_i = w_i + w_{j_i},$$

where  $i, j_i, k,$  and  $l$  are all distinct. If the equality holds:

$$\alpha_i + \alpha_{j_i} = w_i + w_{j_i}$$

then we have  $\alpha \in L_{ij_i}$  and hence  $\alpha \in K$ . If the inequality holds:

$$\alpha_i + \alpha_{j_i} < w_i + w_{j_i}$$

then we can choose  $\beta \in L_{ij_i}$  with  $\beta_i$  slightly greater than  $\alpha_i$

and with  $\beta_{j_i}$  greater than  $\alpha_{j_i}$ , so that

$$\beta \in \underset{S}{\alpha} \quad S = \{i, j_i\};$$

whence  $\alpha \in \text{dom } K$ . Finally, suppose that (R) holds for two or more values of  $i$ . Then we have

$$\alpha \in \text{dom } w \subseteq \text{dom } K$$

since, by (W2), condition (D3) for  $S$ -domination holds for every two-element set  $S$ . This completes the proof that  $K$  is externally stable, (condition (S2)). To establish internal

stability (S1), we suppose, on the contrary, that some  $\alpha \in L_{ij_i}$  S-dominates some  $\beta \in K$ . It is easy to see, by (D2), that  $\beta$  cannot be in the segment  $L_{ij_i}$ , and that the set  $S$  must consist of  $j_i$  and one other element,  $k$ , distinct from  $i$ . Considering the effect of  $\alpha$  in the set  $S$ , we have

$$\alpha_{j_i} + \alpha_k = \alpha_{j_i} + w_k \geq w_{j_i} + w_k = v_{j_i k} = v(S).$$

But to fulfill (D3), we must have equality here, and hence  $\alpha = w$ . Since  $\alpha$  is then in every  $L_{ij_i}$ , it becomes impossible to choose  $\beta \in K$  to satisfy (D2). This contradiction completes the proof of the theorem.

### §5. Interpretation of Theorem 1.

We shall now attempt a heuristic interpretation of our solution, as a standard of behavior. To each participant (the player  $i$ ) a "quota" (the amount  $w_i$ ) is assigned by the standard of behavior, and also a "beneficiary" (the player  $j_i$ ). In a particular play of the game, all four participants may receive exactly their quotas (the imputation  $w$ ). If not, then one of the participants, and only one, will receive less than his quota — possibly as little as what he can guarantee himself (the amount  $-1$ ), without help. What he fails to get goes to his beneficiary, while the two other participants receive their quotas.

### §6. A special case.

An interesting particular case occurs when the players pair off, each making his benefactor his beneficiary:

$$i = j_{j_i}.$$

This happens in three of the 81 solutions of the theorem just proved. Geometrically, the solution  $K$  here consists of two straight lines, intersecting at right angles at the point  $\omega$ .

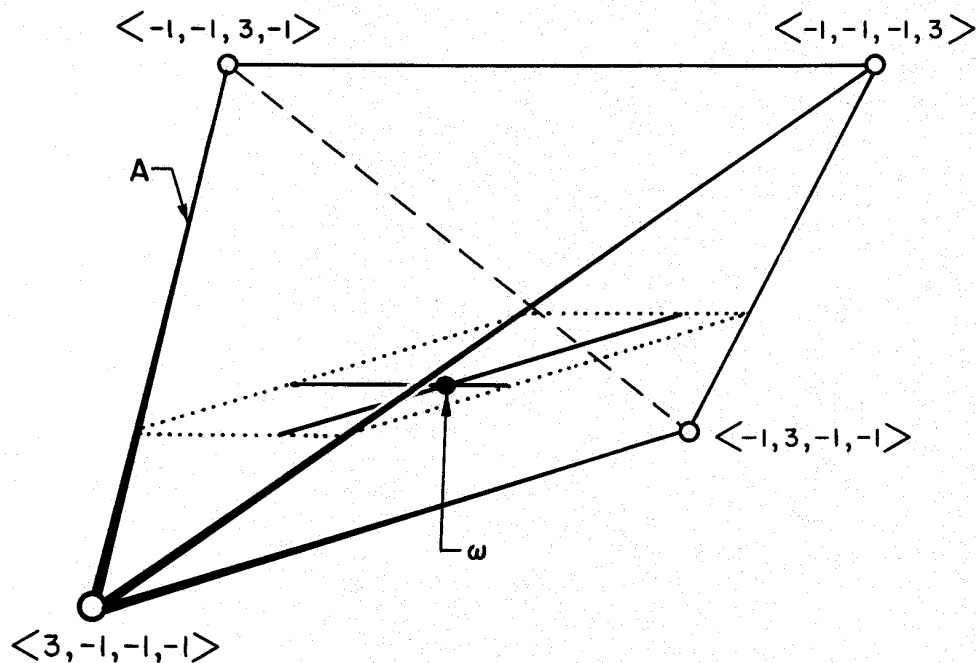


Fig. 2 —  $\omega$  - solution with pairing (12)(34)

A heuristic description of this solution might run as follows: the two couples,  $S$  and  $T$ , first play the game as coalitions, winning the amounts  $v(S)$  and  $v(T)$  respectively. They then bargain among themselves over the division of the winnings (or losses!). A conventional bargain point ( $\omega$ ) exists, but one couple can reach an unconventional bargain if the other adheres to the conventional distribution. This is intuitively reasonable, since if both couples departed at the same time from the conventional distribution the two disgruntled players could threaten to form a coalition on their own and ensure themselves their conventional shares.

§7.  $\omega$ -Solutions, case II:  $\omega \in A$ .

With the game in  $T_j$ :

$$\begin{cases} \omega_j < -1 \\ \omega_j > -1, \end{cases} \quad i \neq j,$$

it is evident that just three of the six lines considered at the beginning of §4 can meet the simplex  $A$ , since the other three are parallel to, and outside, the face  $\omega_j = -1$ . The three relevant segments we shall denote by  $L_{ij}^i$ ,  $i \neq j$  ( $j$  fixed).

Their endpoints are  $\beta^{(ij)}$  (as before) and  $\gamma^{(ij)}$ :

$$\begin{cases} \beta_i^{(ij)} = -1 \\ \beta_j^{(ij)} = \omega_j + \omega_i + 1 \\ \beta_k^{(ij)} = \omega_k \end{cases} \quad \begin{cases} \gamma_i^{(ij)} = \omega_i + \omega_j + 1 \\ \gamma_j^{(ij)} = -1 \\ \gamma_k^{(ij)} = \omega_k \quad \text{for } k \neq i, j, \end{cases}$$

schematically:

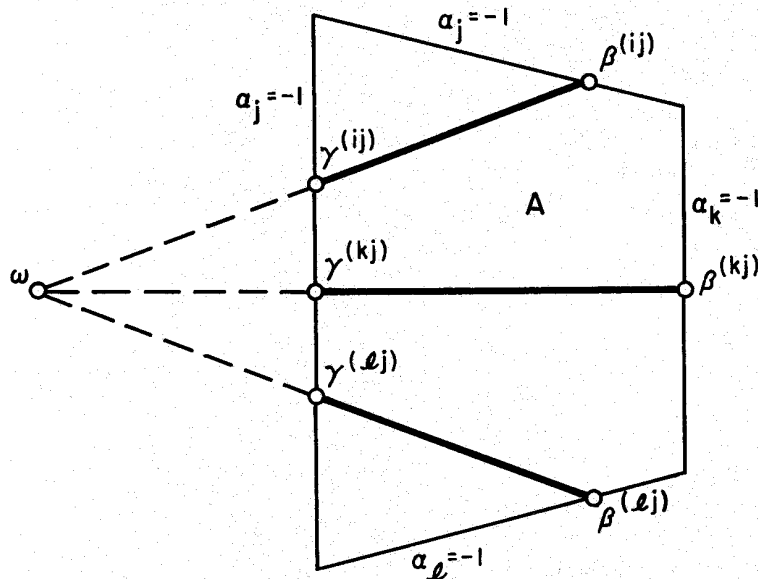


Fig. 3 —  $\omega$ -solution in case II (schematic)

In extreme cases<sup>1</sup> some of these segments will degenerate to points. But none of them ever vanishes completely.

**THEOREM 2.** If  $\omega$  is not in A then a unique "weak player"  $j \in I$  exists with  $\omega_j < -1$ ; and

$$K = \bigcup_{i \neq j} L_{ij}$$

is a solution of the game.

Here there is no arbitrary choice of "beneficiary"  $j_i$ ; the weak player is necessarily beneficiary of all his opponents, and benefactor of none.

The proof is similar to the one in §4, and presents no significant new difficulty.

### §8. Interpretation of Theorem 2.

The heuristic interpretation here can be stated more easily than before. Of the three "strong" players, two always receive exactly their quotas, given by the characteristic point  $\omega$ , while the third receives distinctly less, striking a bargain with the weak player somewhere between their minimum amounts. In extreme cases, this bargain may be uniquely determined; both getting the amount  $-1$  (in normalized units).

### §9. Continuity.

When  $P$  is allowed to vary over the interior of the tetrahedron  $T_0$ , all of the 81  $\omega$ -solutions of Theorem 1 change continuously. The single solution of Theorem 2 also behaves continuously as  $P$  runs over  $T_j$ ,  $j = 1, 2, 3, 4$ . These facts are obvious from

<sup>1</sup> In fact, precisely when  $P$  is in the boundary of  $Q$ .

the definitions of the segments  $L_{ij}$  and  $L'_{ij}$  which make up the  $\omega$ -solutions.

In the boundary of  $T_0$ , Theorem 1 as stated still applies. But as  $\Gamma$  moves into the 2-, 1-, and 0-dimensional boundary cells of  $T_0$ , the 81  $\omega$ -solutions combine by threes, so that there are only 27 distinct solutions on the faces, and nine and three on the edges and vertices respectively<sup>1</sup>. A short argument, which we omit here, shows that the entire system of 81 sets of solutions is connected, by way of these boundary cases. It is also noteworthy that some  $\omega$ -solutions in the boundary of  $A$  are identical point-sets in  $A$ , even though they are based on different characteristic points, and hence refer to different games.

If we carry  $\Gamma$  up to the boundary of  $T_0$  from the outside, we observe that the limiting case of Theorem 2 is one particular solution of Theorem 1<sup>2</sup>. Therefore, we can follow the  $\omega$ -solution continuously from  $T_j$  into  $T_0$ , but we can go the other way only if we are following the right one of the 27 (or 9 or 3) solutions at the boundary. In a future note we shall describe other classes of solutions which fit onto the 26 "loose ends".

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<sup>1</sup> It is easy to see intuitively why this should be so. When  $\omega_i = -1$ , the  $i$ th player has nothing to give; his choice of beneficiary  $j_i$  is therefore irrelevant.

<sup>2</sup> The one in which the weak player is everyone's beneficiary. The three points  $\gamma^{(ij)}$ ,  $i \neq j$ , merge into the characteristic point  $\omega$ .